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ROBUST OPTIMAL FISCAL POLICY

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Preliminary; Comments welcomed.

Abstract:

We introduce a concern for model misspecification in a Lucas and Stokey optimal fiscal policy setting. The representative household in this economy is endowed with the knowledge of one particular model for the government spending process but acknowledges that this model is potentially misspecified. We show how this concern for misspecification affects the competitive equilibrium pricing of state-contingent debt and how this in turn affects the calculation of present value budget constraints. We show how to adapt the Ramsey problem and how the optimal solutions are affected by the size of potential misspecification entertained. This enables us to explore the implications of this setup for the optimal maturity structure for non-contingent debt, paying particular attention to how pricing of these assets is affected by robustness considerations. A linear-quadratic specification serves to implement numerically the economy under study.

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1. Introduction

Criticism of the complete markets Ramsey model as a positive model for the analysis of fiscal policy has led to a spur of recent contributions casting the optimal taxation problem in an incomplete markets setting. This recent optimal fiscal policy literature has mostly made alternative assumptions about the market structure for government debt, be it by exogenously shutting down all trade in state-contingent claims (see for example Aiyagari et al. [1] and Shin [23]) or by endogenizing the set of restrictions to state-contingent trade through limited commitment and incentive compatibility considerations (see Sleet and Yeltekin,[24]). Here, instead, we retain the assumption of complete state-contingent markets but consider the impact of limited reliability of knowledge concerning the process for government expenditures.

Drawing from the contribution of Hansen and Sargent [10], we introduce a concern for robustness in a, otherwise standard, Lucas and Stokey [18] complete markets economy with no capital. The representative household in this economy is endowed with knowledge of one particular model for the government spending process but acknowledges that this model is potentially misspecified. As such the household aims at attaining optimal decision rules for consumption, labor supply and contingent-debt holdings that nevertheless perform well against the potential model misspecifications that it entertains.

We show how this concern for misspecification affects the competitive equilibrium pricing of state-contingent debt and how this in turn affects the calculation of present value budget constraints. Optimal fiscal policy in this economy is, as standard, the solution to a Ramsey planning problem that solves the household objective function subject to an implementability constraint and a resource constraint. However given robustness considerations, both the household's objective function and the implementability constraint - through equilibrium prices - change with respect to the baseline Lucas-Stokey Ramsey problem. We show that the latter is itself a limiting case of our setup - one where the household's concern for robustness vanishes - and, more generally, that the solutions to this robust version of the Ramsey planner's problem are indexed by a parameter governing the size of potential model misspecifications that the household entertains.

Following Angeletos [2] we then explore the implications of this setup for the optimal maturity structure for non-contingent debt paying particular attention to the findings in Buera and Nicolini[5]. The latter argue that 'very extreme positions' of non-contingent debt positions are required to support the state-contingent Ramsey plan. The reason, they argue, is that the returns on bonds of different maturities, though independent, are very highly correlated, and as such spanning requires very large positions.

However, when robustness considerations are involved, the stochastic discount factor pricing these assets does change. To assess the sensitivity of these extreme positions to changes in the specification of the pricing kernel, we first derive expressions for the term structure in the economy and then, following Angeletos, derive an expression linking non-contingent debt positions and state-contingent debt positions. While the structure of the argument is formally the same, the Ramsey model framework allows us to derive expressions that evaluate how state-contingent debt varies with the robustness parameter, how pricing of contingent and non-contingent securities vary and as such how the value of the non-contingent securities that support the Ramsey allocation changes with robustness considerations.

To evaluate this we study a laboratory economy where the representative household's period return function is quadratic. This, combined with a recursive formulation of this problem, enables us to solve explicitly for allocations, policies and prices for this economy. The nature of the exercise is not to calibrate the economy: rather, it is to scrutinize the qualitative properties induced by the free parameter indexing the size of the distortions entertained by the household and to compare them with respect to the existing results for the limiting expected utility complete markets economy.

The analysis below shares objectives and methods with two distinct literatures. The first comprises of the recent effort to formulate and empirically evaluate the optimal fiscal policy predictions delivered by models of dynamic general equilibrium economies. Scott [21] reviews and extends the usage of tax data to test the different tax smoothing properties yielded by two yardstick setups: extensive inter-state smoothing in the complete state-contingent markets setting of Lucas and Stokey versus extensive inter-temporal smoothing through a safe asset in a setting where all state-contingent markets are shut down; see Aiyagari et al.'s [1] rendition of Barro's [4] economy. Scott makes clear that the quality of data available and the difficulties in testing for its properties renders conclusions difficult. Marcet and Scott [19] take on the debt side: the value public debt is found to be much more persistent than other real variables and increasing in response to government spending shocks. Both of these findings agree with the predictions and numerical simulations in Aiyagari et al. [1] and, as the authors show, cannot be supported by a Ramsey plan where access to complete state-contingent bond markets is allowed.

Angeletos [2] introduces the possibility of exploiting term structure variation across states of the world to synthesize the complete state-contingent securities (and its allocations) with a specific non-contingent asset structure: the optimal maturity structure. Buera and Nicolini [5] also shows and exploits this possibility. In calibration exercises, they show that the implied value of the transactions required to achieve the optimal structure of non-contingent assets of different maturities is at least an order of magnitude higher than that observed in the data. The analysis below shows that accounting for robustness considerations and its effects in the pricing of non-contingent assets has the potential to lower the implied equilibrium value of these transactions to observed magnitudes[1]. In this sense, the analysis here as an asset pricing flavor in that in our setup the pricing kernel for the term structure of government-backed debt is altered; in turn, changes in this pricing kernel are a crucial ingredient for the optimal maturity structure and its valuation as shown in Buera and Nicolini. The need for a theory of how the fiscal stance affects the term structure is also echoed in the study of Dai and Philippon[8]. Here we take the robust Ramsey framework derived to study this issue.

This paper shares also the objective of a recent effort in analyzing the implications of fear of model misspecification on the analysis of traditional issues of interest such as asset pricing and the market price of risk in endowment and production economies; see Hansen et al. [13] and Tallarini [25] respectively. Like these, our study draws extensively on the theory developed in Hansen and Sargent [9], [10], [11] and Hansen et al. [14]. The analysis developed here shares much in common with that of Hansen and Sargent [10, , Chapter 18]. Finally, Sargent [20] documents historical episodes and debates in American fiscal policy where the private sector faced ambiguous fiscal policies.

The paper is organized as follows: Section 2 introduces the necessary tools to represent fear of model misspecification. Section 3 uses this to setup a Lucas and Stokey economy where the representative household has a reference model for government expenditures but acknowledges that it is potentially misspecified. Section 4 sets up, describes the Competitive Equilibrium and the Ramsey Planner's problems and applies these to discuss the optimal maturity structure in this economy. Section 5 studies allocations, prices and maturity structure for a laboratory quadratic economy with no capital. Section 6 concludes.

2. Modelling Fear of Model Misspecification

Standard practice in macroeconomic model-building endows a decision-maker with an objective function and a model of the economy, eliciting from this optimal decision rules. Namely, in dynamic stochastic settings, the decision-maker is assumed to *know* the exact specification of the stochastic process driving the economy. Here we will be interested in analyzing a situation where the agents consider the given model as a good approximation - or a reference point - of a true, unknown data generating process; i.e. the decision maker acknowledges model misspecification.

In face of this we are interested in deriving *robust* optimal decisions based on a set models that nevertheless remain *close* to the approximating model. The purpose below is

[1]Any stronger statement here needs to be backed by careful empirical analysis. The analysis here falls short of that but shows the mechanisms in place.

to render operational these concepts of robustness and closeness of models when agents do not know the model generating the stochastic process. To achieve this we draw heavily on the recent contributions of Hansen et al.[14] and Hansen and Sargent [11],[12][2].

2.1. Local Absolute Continuity, Distorted Expectations and Entropy. Fix a complete probability space $(\Omega, \mathcal{F}, \pi)$ and let $\{s_t\}_{t=0}^{\infty}$ denote the sample path of a stochastic process where $s_t : \Omega \to S$ is the time t realization of the process. Let $\{\mathcal{F}_t\}_{t=0}^{\infty}$ denote a filtration generated by a sample path. Finally let π_t be the restriction of the probability measure π to $\mathcal{F}_t$, i.e. the conditional measure defined by $\pi_t(A) \equiv \pi(A|\mathcal{F}_t)$ for $\forall\ A \in \Omega$.

In the current context we will be terming the π measure as the approximating model the agent in this economy holds. That is, the decision maker regards π as a good approximation to the unknown, true, data generating process. However, recognizing potential misspecification of the true process, we require that agents entertain a set of admissible distorted measures $\widetilde{\Pi}$ on the space $(\Omega, \mathcal{F})$. This set is defined by requiring each element $\widetilde{\pi}$ to be absolutely continuous with respect to the approximating measure over finite intervals of time; i.e. locally absolutely continuous. Formally for all t,$\ell < +\infty$, $\widetilde{\Pi}$ is defined as:

$$\widetilde{\Pi} \equiv \{\widetilde{\pi} : \widetilde{\pi}_t(A) = 0\}, \text{ for } \forall \text{ measurable sets } A \in \mathcal{F}_{t+\ell} \text{ such that } \pi_t(A) = 0$$

that is the approximating and distorted measures agree with respect to zero-measure sets (under the approximating measure). Note that we are silent regarding what happens in the infinite horizon (e.g. events in the tail sigma field). Conversely, if we had insisted on $A \in \mathcal{F}$ we would recover standard absolute continuity; see Kalai and Lehrer[16] and Lehrer and Smorodinsky[17].

The next step is to specify how agents form expectations when they entertain distorted measures in $\widetilde{\Pi}$. The key here is to recall that given local absolute continuity, the Radon-Nikodym theorem ensures the existence of a (non-negative) $\mathcal{F}_t$-measurable function $M_t \equiv \frac{d\widetilde{\pi}_t}{d\pi_t}$. We require that M_t is itself a probability measure, such that $EM_t = 1$. The usefulness of working with the Radon-Nikodym probability measure follows from the fact that we need not work with distorted expectations. Namely, the *conditional distorted expectation,* $\widetilde{E_t}$, of a stochastic, bounded, $\mathcal{F}_t-$measurable function ϕ_t can be represented through the standard conditional expectation operator by noting that[3]:

$$\widetilde{E}\phi_t|\mathcal{F}_t = EM_t\phi_t \tag{1}$$

[2] The field of research is expanding rapidly. See Hansen and Sargent[10] for an exposition of robust control and filtering in macroeconomics. For a variety of related approaches see the review of Backus, Routledge and Zin [3] and references therein.

[3] To see why use the law of iterated expectations and the restriction $EM_t = 1$ to get $EM_t\phi_t = E[M_t(\widetilde{E}\phi_t|\mathcal{F}_t)] = \widetilde{E}\phi_t|\mathcal{F}_t$. This is (half of) claim 6.1 Hansen et al. [14]

Now, a desirable property for the operator $\widetilde{E_t}$ is independency of the conditional distorted expectation today from revelation of additional information in future dates. Hansen and Sargent [11] term this property as consistency, defined as:

$$\widetilde{E}\phi_t|\mathcal{F}_{t+1} = \widetilde{E}\phi_t|\mathcal{F}_t, \mathcal{F}_t \subseteq \mathcal{F}_{t+1}$$

Requiring that the distorted conditional expectation is consistent leads to further restrictions on the process for M_t. To see this note that putting the two conditions above together:

$$\begin{aligned} \widetilde{E}\phi_t|\mathcal{F}_{t+1} &= \widetilde{E}\phi_t|\mathcal{F}_t \Leftrightarrow \\ EM_{t+1}\phi_t|\mathcal{F}_{t+1} &= EM_t\phi_t|\mathcal{F}_t \Leftrightarrow \\ EM_{t+1}|\mathcal{F}_t &= M_t \end{aligned}$$

Thus, consistency under the distorted expectations requires that the sequence of Radon-Nikodym derivatives $\{M_t,\ t = 0, 1, 2, ...\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t,\ t = 0, 1, 2, ...\}$.

As a final step, and in order to operationalize the notion of closeness mentioned above, we need a measure of discrepancy between these distorted densities and the approximating density. The measure of discrepancy is the conditional entropy of the distortion at time t given by:

$$\widetilde{E}\,(\log M_t|\mathcal{F}_t) = E(M_t \log M_t)$$

By defining $m_{t+1} = \frac{M_{t+1}}{M_t}$ and noting that $M_{t+1} = m_{t+1}M_t$, Hansen and Sargent [11, Section 3.4] show that date t conditional entropy, $E(M_t \log M_t|\mathcal{F}_0)$, can be written

$$E(M_t \log M_t|\mathcal{J}_0) = \sum_{j=0}^{t-1} E[M_j E(m_{j+1} \log m_{j+1}|\mathcal{F}_j)|\mathcal{F}_0] \tag{2}$$

where we are initializing M_0 to unity (i.e. only distortions conditioned on date zero events are entertained by the agent). The random variable m_{t+1}, will be the key construct to model distortions from the approximating measure. Note that m_{t+1} is a $\mathcal{F}_{t+1}-$measurable random variable and that $E(m_{t+1}|\mathcal{F}_t) = 1$. Further the distorted conditional expectation of a ϕ_t measurable and bounded function, given the martingale property of M_t, is just

$$E(m_{t+1}\phi_t|\mathcal{F}_t) = \frac{E(M_{t+1}\phi_t|\mathcal{F}_t)}{E(M_{t+1}|\mathcal{F}_t)}$$

2.2. Building a Penalty Robust Control Problem. To derive optimal robust decision rules, Hansen and Sargent [10] show that it is convenient to represent the underlying decision-maker's problem as a fictional zero-sum, two player Nash game between a maximizing player and a malevolent player (thought of as nature). Thus the decision-maker is seen as maximizing its utility by choosing decision rules that perform well against the *minimizing* choices of distortions to the agent's approximating model. This game representation is thus used as a devise to implement a worst-case analysis for an agent who fears misspecification.

To obtain the desired representation consider first the more standard decision problem of the maximizing player. Let $U(c_{t,}y_t)$ denote an increasing, concave and differentiable reward function mapping time t values of (his) control variables $c_t \in \mathcal{C}_t$ and (potentially) state values y_t to current period utility. We assume the state at period zero is known and that its law of motion is given by

$$y_{t+1} = Y(y_t, c_t, w_{t+1}) \tag{3}$$

where $\{w_{t+1} : t \geq 0\}$ is an i.i.d. sequence of random variables. Let π_t be the conditional density implied by this law of motion (thus implicitly dependent on y^t, the history of the state up to time t as induced by the history of controls and shocks). Note that π_t represents the same object as above: the conditional approximating model that the agent holds.

Now, recall that when devising optimal plans the decision-maker forms distorted expectations of future states. The martingale Radon-Nikodym derivative set out above is the key here. Thus, from a time zero standpoint the maximization part of the objective function is just:

$$\max_{\{c_t\}} \widetilde{E} \sum_{t=0}^{\infty} \beta^t U(c_{t,}y_t)|\mathcal{F}_0 = \max_{\{c_t\}} E \sum_{t=0}^{\infty} M_t \beta^t U(c_{t,}y_t)|\mathcal{F}_0 = \max_{\{c_t\}} \sum_{t=0}^{\infty} \sum_{y^t} M_t \beta^t U(c_{t,}y_t)\pi_t$$

where $\beta \in (0, 1)$ is the subjective discount factor.

Recall also that we have a law of motion for M_t. We put this law of motion under the control of a malevolent/minimizing agent by allowing him to choose the sequence of increments m_{t+1}. Disciplining the choices of the minimizing player, i.e. limiting the scope of the worst-case analysis, is achieved by penalizing distortions that are far away from what is considered to be the reference point. Again, our notion of distance is given by the date zero conditional relative entropy of a perturbation to the one-step transition density associated with the approximating model. Denoting this penalty parameter as $\theta \in [\underline{\theta}, +\infty)$, with $\underline{\theta} > 0$, it is possible to express a family of penalty robust control problems whose solutions are functions of θ.

Thus, the penalty robust control problem instructs the household to decision maker over control sequences $\{c_t\}$ while nature is choosing a sequence of perturbations to the household's assessment of conditional probabilities. This problem has the representation:

$$\max_{\{c_t\}} \min_{\{m_{t+1}\}} E\left(\sum_{t=0}^{\infty} M_t \beta^t [U(c_t, y_t) + \beta\theta m_{t+1} \log m_{t+1}] | \mathcal{F}_0\right) \tag{4}$$

subject to the law of motion of the state (3) and

$$\begin{aligned} M_{t+1} &= m_{t+1} M_t \\ M_0 &= 1, \ y_0 \text{ given} \end{aligned} \tag{5}$$

where $Em_{t+1}|\mathcal{F}_t = 1$, $\theta \in [\underline{\theta}, +\infty)$, with $\underline{\theta} > 0$ and both c_t and m_{t+1} are $\mathcal{F}_t-$measurable functions. A number of remarks are in order. First, note that the Radon-Nikodym martingale measure M_t is now a state-variable distorting both the evaluation of the household's utility and the entropy penalty. Second, note that we are discounting equally both the time t utility and entropy penalty. As Hansen and Sargent [11] discuss this ensures that robustness endures, i.e. that the martingale M_t does not converge asymptotically to unity. Third, note that the extremization[4] problem is well posed: the period reward function is concave while the $m_{t+1} \log m_{t+1}$ is convex in m_{t+1}. Fourth, as it will be clear in the economy below, as $\theta \to +\infty$ the penalization for deviating from the approximating model is such that the minimizing player chooses not to distort; i.e. it is as if the agent has no fear of model misspecification.

While the Appendix further shows that this problem induces a recursive utility setting -namely a version of Whittle's risk-sensitive criterion- the tools set forth here are enough to proceed to analyze optimal fiscal policy when agents fear that their model is misspecified.

3. The Economy

This section sets out the building blocks for the analysis of a simple Lucas and Stokey economy without capital [18]. We introduce fear of model misspecification taking the lead from Hansen and Sargent [10]. Throughout we will be assuming the existence of some commitment technology for the government, thereby avoiding what are arguably important time-consistency issues[5].

3.1. Information and Uncertainty. Time is infinite, discrete and indexed by $t = 0, 1, 2....$ Let the elements of a fixed set $\mathcal{Z} = \{1, ..., Z\}$ be known as shocks. Let $\Omega = \mathcal{Z}^{\infty}$ and $\mathcal{F}$ a $\sigma-$algebra of Ω. At each t, $s_t : \Omega \to \mathcal{Z}$. Let $\mathcal{P}_i$, the probability measure on $(\Omega, \mathcal{F})$ be given by the pair (Π, π_{i0}) respectively, a Markov time invariant transition matrix and an initial condition such that $\pi_{i0}(s_0 = i) = 1$. Further, let $s^t = \{s_0, s_1, ..., s_t\}$

[4] The robust control literature refers to the $\max\min$ operation as extremization.

[5] See Chari and Kehoe [6] for an extensive review of optimal fiscal policy models.

be the information available at any time $t = 0, 1, \ldots.$ We will make use of the convention $s^{t+1} = (s^t, s_{t+1})$ for events at date $t+1$. For a generic dates $t+j$, $j \geq 0$, we denote the set of date j events consistent with an s^t history as $\sigma^{t+j}(s^t) \equiv \{s^t\} \times \mathcal{Z}^j$.

Let government consumption, g, be given by $g(s_t) = \overline{g} + \sigma s_t$. As with government consumption, for any generic function f below, we shall use $f(s^t)$ for measurable functions of s^t. Note that histories s^t constitute the domain of such functions: these will define commodity spaces. Finally let time zero government debt be a known time zero function of the state at $t = 0$, denoted by b_0.

3.2. Preferences. An infinitely lived representative household consumes $c(s^t)$ and enjoys leisure $x(s^t)$ for every time period t. The household orders infinite sequences of consumption-leisure streams according to a specialization of the representation derived in the previous section:

$$E\left(\sum_{t=0}^{\infty} M(s^t)\beta^t[U(c(s^t),x(s^t)) + \beta\theta m(s^{t+1})\log m(s^{t+1})]|s_0\right) \tag{6}$$

where $U(.,.)$ is assumed to be an increasing, strictly concave, twice differentiable function in both arguments, satisfying the Inada conditions. $\beta \in (0,1)$ is the constant subjective discount factor and $\theta \in \Theta \equiv [\underline{\theta}, +\infty)$, with $\underline{\theta} > 0$. Recall also that we preserve the structure above for the law of motion of $M(s^{t+1})$:

$$\begin{aligned} M(s^{t+1}) &= m(s^{t+1})M(s^t) \\ M(s_0) &= 1 \end{aligned}$$

where $Em(s^{t+1})|s^t = 1$ for all t, s^t. For use in the analysis below also define:

$$W(s^t) \equiv U(c(s^t),x(s^t)) + \beta\theta m(s^{t+1})\log m(s^{t+1})$$

3.3. Endowments and Technology. The endowment of leisure time is unity and there is a linear production technology. This implies a resource constraint:

$$c(s^t) + x(s^t) + g(s_t) = 1, \ \forall t, s^t \tag{7}$$

3.4. Market Arrangements. Apart from consumption-leisure decisions, at each period the household can buy or sell complete state-contingent, one-period, zero-coupon, government bonds. Let $b(s^{t+1})$ denote the number of one period maturity debt units paying one unit of the time $t+1$ consumption good when s_{t+1} is realized, s^t given. Similarly, let $p(s^{t+1})$ denote the price of state contingent debt in terms of time t good. Households finance these asset positions from the current debt payback and net-of-taxes labor income. The flat rate tax is $\tau(s^t) \in [0,1]$.

Summarizing, households are subject to the sequence of budget constraints

$$\sum_{s^{t+1}\in\sigma^{t+1}(s^t)} p(s^{t+1})b(s^{t+1}) + c(s^t) \leq (1-\tau(s^t))(1-x(s^t)) + b(s^t), \quad \forall t, s^t... \tag{8}$$

Conversely, at each date t, the government has to finance the realization of the stochastic government expenditure process. As obvious from above, the government has two technologies available for this purpose: linear taxation on labor income and state-contingent government debt. Under these market arrangements, the government budget constraint is given by

$$\sum_{s^{t+1}\in\sigma^{t+1}(s^t)} p(s^{t+1})b(s^{t+1}) \leq \tau(s^t)(1-x(s^t)) - b(s^t) - g(s_t), \quad \forall t, s^t.. \tag{9}$$

4. Ramsey Allocations and Optimal Policy

The objective here is to set out the robust optimal policy allocations according to the solution of a Ramsey planner's problem, as originally proposed in Lucas and Stokey [18]. We assume throughout full commitment to a tax plan at the beginning of time zero and abstract from time-inconsistency matters. We first study the competitive equilibrium under complete markets and then the Ramsey plan. The section concludes by discussing the pricing of non-contingent assets and the construction of the optimal non-contingent maturity structure in the sense of Angeletos [2].

4.1. Competitive Equilibrium under Complete Markets. For a given sample path $\{s^t\}_{t=0}^{\infty}$, split the collection of household controlled infinite sequences $\{c(s^t), x(s^t), M(s^{t+1}), m(s^{t+1}), b(s^{t+1})\}_{t=0}^{\infty}$ into allocations $\{A(s^t)\}_{t=0}^{\infty} \equiv \{c(s^t), x(s^t), b(s^{t+1})\}_{t=0}^{\infty}$ and distortions $\{D(s^t)\}_{t=0}^{\infty} \equiv \{M(s^{t+1}), m(s^{t+1})\}_{t=0}^{\infty}$. Also, given the simplified structure of the economy here, note that a (fiscal) policy is simply $\{\tau(s^t)\}_{t=0}^{\infty}$.

Definition 1. *Fix a sample path $\{s^t\}_{t=0}^{\infty}$, an initial level of debt b_0, an initial condition for the distortion process $M(s_0)$, and a penalty parameter $\theta \in \Theta$. A* Robust Competitive Equilibrium *is a collection of $(\sigma^t(s^t), b_0, M_0)-$measurable stochastic processes of state-contingent debt prices $\{p(s^{t+1})\}_{t=0}^{\infty}$, fiscal policies $\{\tau(s^t)\}_{t=0}^{\infty}$, allocations $\{A(s^t)\}_{t=0}^{\infty}$ and distortions $\{D(s^t)\}_{t=0}^{\infty}$, such that:*

i) given distortions, prices and fiscal policy, the allocation solves the time zero household's maximization problem subject to its sequence of budget constraints;

ii) given allocations, prices and fiscal policy, the distortions solve the time zero household's minimization problem subject to the law of motion for $M(s^{t+1})$ and its martingale restriction.

iii) given allocations, distortions and prices, the tax policy and the contingent debt plan satisfy the government budget constraint.

The equilibrium concept thus requires that, for every fiscal policy and price schedule, the allocation rule has to satisfy the household's first order conditions and the sequence of budget constraints. Thus, at interior allocations, the maximizing player's first-order conditions with respect to consumption and labor imply the familiar condition:

$$\frac{U_x(s^t)}{U_c(s^t)} = (1 - \tau(s^t)), \quad \forall t, s^t \tag{10}$$

which is just the standard wedge imposed by distortionary taxation affecting the intra-period margin between consumption-leisure. Notice that the minimizing player's distortion does not enter directly in this condition.

Now, acknowledging that government debt is simply an Arrow security paying one unit of consumption if next period state is s_{t+1}, one-period assets will be priced through the intertemporal marginal rate of substitution evaluated at the planner's choice of allocations. Let $\pi(s^{t+1})$ denote the relevant element of Π, given time t state, s_t. Given fear of model misspecification, the valuation of state-contingent bonds will be given by:

$$\begin{aligned} p(s^{t+1}) &= \beta\pi(s^{t+1})\frac{U_c(s^{t+1})}{U_c(s^t)}\frac{M(s^{t+1})}{M(s^t)} \\ &= \beta\pi(s^{t+1})\frac{U_c(s^{t+1})}{U_c(s^t)}m(s^{t+1}) \end{aligned} \tag{11}$$

where the stochastic discount factor reflects a multiplicative adjustment to the familiar consumption growth component.

Finally, for a given allocation sequence, the first order condition of the minimizing player implies:

$$-\theta(1 + \log m(s^{t+1})) = [U(c(s^{t+1}),x(s^{t+1})) + \beta\theta m(s^{t+2}) \log m(s^{t+2})], \quad \forall s^{t+2} \in \sigma^{t+2}(s^{t+1}) \tag{12}$$

Recognizing that the term on the right of the equality is just the household's utility value in state s_{t+1} given an history s^t and solving for the optimal distortion choice yields:

$$m(s^{t+1}) \propto \exp\left(-\frac{W(s^{t+1})}{\theta}\right)$$

That is, the multiplicative adjustment term alluded to earlier is simply the continuation value $W(s^{t+1})$, relative to its risk adjustment. Note that as $\theta \to \infty$ (i.e. fear of model misspecification vanishes) this term is one and the standard discount factor is recovered. From our discussion in Section 2 we also know that the latter term is always positive and that

$$E[\exp\left(-\frac{W(s^{t+1})}{\theta}\right)|s^t] = 1$$

allowing for an interpretation of a distortion of the probability distribution. That is, by letting $\widetilde{\pi}(s^{t+1}) = \pi(s^{t+1})\exp\left(-\frac{W(s^{t+1})}{\theta}\right)$ and denoting the standard real discount factor by $\Lambda(s^{t+j}) = \beta^j \frac{u_c(s^{t+j})}{u_c(s^t)}$ we can rewrite the pricing equation above as:

$$p(s^{t+1}) = \widetilde{\pi}(s^{t+1})\Lambda(s^{t+1}) \tag{13}$$

Below, we shall have cause for pricing multi-period contingent claims. For that let $q(s^{t+j}, s^t)$ denote the price of a security issued at history s^t and paying one unit of consumption if history s^{t+j} is realized,

$$q(s^{t+j}, s^t) = \prod_{\iota=t}^{t+j-1} p(s^{\iota+1}, s^{\iota}) = \Lambda(s^{t+j}) \prod_{\iota=t}^{t+j-1} \widetilde{\pi}(s_{\iota+1}, s_{\iota}) \tag{14}$$

where the last equality follows from the Markov structure of the problem.

Proposition 1. *Assume the following transversality condition holds:*

$$\lim_{t\to\infty} \sum_{s^{t+1}} p(s^{t+1})b(s^{t+1}) = 0$$

Then, given $b_0, M_0 = 1$, allocations, distortions, policies and price sequences constitute a time-0 competitive equilibrium if they satisfy.

$$c(s^t) + x(s^t) + g(s_t) = 1 \tag{15}$$

$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \mu(s^t)[U_c(s^t)c(s^t) - U_x(s^t)(1 - x(s^t))] = U_c(s_0)b_0, \tag{16}$$

$$\text{where } \mu(s^t) \equiv \prod_{\iota=0}^{t-1}\widetilde{\pi}(s_{\iota+1}, s_{\iota})$$

Further, take feasible allocations and time zero policies satisfying 16. Then we can construct fiscal polices, allocations, distortions and prices that constitute a competitive equilibrium.

Proof *i) Feasibility is insured to hold in equilibrium by combining the household's and the government's budget constraint.*

ii)($\Rightarrow$) Multiply the household's budget constraint at time $t+1$ by $p(s^{t+1})$, sum over s_{t+1} and substitute out $\sum_{s_{t+1}} p(s^{t+1})b(s^{t+1})$ in the time t budget constraint. Proceed by forward recursion and use the transversality condition to get:

$$\sum_{t=0}^{\infty}\sum_{s^t} q(s^t,s_0)c(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} q_t(s^t,s_0)(1-\tau_t(s^t))(1-x(s^t)) + b_0$$

where

$$q(s^t,s_0) = \beta^t \frac{U_c(s^t)}{U_c(s_0)} \prod_{\iota=0}^{t-1} \widetilde{\pi}(s_{\iota+1},s_\iota)$$

Substitute back into equation above to get

$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \mu(s^t)[U_c(s^t)c(s^t) - U_x(s^t)(1-x(s^t))] = U_c(s_0)b_0$$

iii) Conversely, take any feasible allocation-distortion pair and period 0 polices satisfying (16) condition can be implemented by setting tax rates through the first-order condition. (10). Given allocations, distortions and the tax policy, prices are given by(11)■

Thus, just as in Lucas and Stokey [18], equilibrium restrictions imposed by the requirements of intertemporal government budget balance and competitive household behavior are embodied in a single implementability constraint. Notice that, in the current setup, this condition is only altered by altering the probabilities over histories from $\pi(s^t)$ to $\mu(s^t)$. That is, just as standard complete markets implementability constraints incorporate the maximizing behavior of the household in a competitive market, condition (16) incorporates the extremizing of the objective function subject to the restrictions imposed both by the market structure and the set of admissible distortions. Note also, that condition (16) above actually defines a family of implementability constraints, parametrized by $\theta \in \Theta$. These insights will be useful below in setting up the robust version of a Ramsey problem.

Finally, applying the same forward substitution, we can use the household's first order conditions to construct a competitive equilibrium bond allocation. For any period r

$$b(s^r) = \sum_{t=r+1}^{\infty}\sum_{s^t} \beta^{t-r}\mu(s^t)[U_c(s^t)c(s^t) - U_x(s^t)(1-x(s^t))]/\pi(s^r)U_c(s^r) \tag{17}$$

4.2. Robust Ramsey Problem and Optimal Policy. From Proposition 1 above we know that the implementable set of competitive equilibria is defined by conditions (16) and (15). Further we know that given a sample path, a sequence pair $\{c(s^t), x(s^t)\}_{t=0}^{\infty}$ is enough to then solve for competitive equilibrium state-contingent debt sequences; condition (17). To do this, note that from the minimizing side of representative of the household we can compute the optimal distortion sequence from the sequence of consumption-leisure pairs.

Missing thus far is an objective function which ranks this elements of the set of implementable equilibria. Following the standard definition of Ramsey plans in the literature, in the current context we define a Ramsey problem as:

Definition 2. *The* Robust Ramsey Problem *is to solve for a competitive equilibrium that extremizes the household's objective function. The* Robust Ramsey Allocations *are the triplet* $\{c(s^t), x(s^t), b(s^{t+1})\}_{t=0}^{\infty}$ *of* $(\sigma^t(s^t), b_0) -$ *measurable* sequences that are solutions to the Ramsey Problem.

From Definition 2, one of the objects of interest in the Ramsey Problem is the planner's choice for the minimizing distortion sequence. To be clear, as we will discuss below, it is the household's distorted view on the fiscal spending prospects that will be reflected in prevailing equilibrium prices. Nevertheless, under the definition above, offering a complete characterization of the Ramsey plan entails finding the planner's worst case model. This will not, in general,coincide with the representative household chosen distortion: the fact that the planner faces implementability restrictions, as summarized by (16), will affect the planner's minimizing choice.

For reference in the analysis below, label the objects $\{D_P(s^t)\}_{t=0}^{\infty} \equiv \{M_P(s^{t+1}), m_P(s^{t+1})\}_{t=0}^{\infty}$ as the planner's distortion sequences. The assumption throughout will be that the planner and the household agree on the approximating model, i.e. they agree on the initial distribution and the one step ahead Markov transition matrix as good reference models for the stochastic process of government expenditures. Further the planner's law of motion for $M_P(s^{t+1})$ will again follow the framework in Section 2 and is thus given by $M_P(s^{t+1}) = m_P(s^{t+1})M_P(s^t)$ with $Em_P(s^{t+1})|s^t = 1$ and the same normalization $M(s_0) = 1$

Now, from our discussion in the preceding subsection, the object $\mu(s^t)$ in the implementability constraint simply reflects the household's minimizing choice for $M(s^t)\pi(s^t, s_0)$. Thus, in the Ramsey planner's problem we rewrite the implementability constraint as:

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t M_P(s^t)\pi(s^t, s_0)[U_c(s^t)c(s^t) - U_x(s^t)(1 - x(s^t))] = U_c(s_0)b_0$$

With this setup, the Ramsey problem amounts to the following program:

$$\max_{\{c(s^t),x(s^t)\}_{t=0}^{\infty}} \min_{\{m_P(s^{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t}\beta^t M_P(s^t)\pi(s^t, s_0)[U(c(s^t),x(s^t), \lambda) + \beta\theta m_P(s^{t+1})\log m_P(s^{t+1})] \tag{18}$$

$$s.t. \quad \sum_{t=0}^{\infty}\sum_{s^t}\beta^t M_P(s^t)\pi(s^t, s_0)[U_c(s^t)c(s^t) - U_x(s^t)(1 - x(s^t))] = U_c(s_0)b_0 \tag{19}$$

$$c(s^t) + x(s^t) + g(s_t) = 1 \tag{20}$$

$$M_P(s^{t+1}) = m_P(s^{t+1})M_P(s^t) \tag{21}$$

$$M_P(s_0) = 1 \tag{22}$$

where $Em(s^{t+1})|s^t = 1$ for all t, s^t.

Let λ be the Lagrange multiplier associated to the implementability constraint and $\{\psi(s^t)\}_{t=0}^{\infty}$ be the sequence of state-date Lagrange multipliers on the feasibility constraint of the economy. Finally define $H(s^t) \equiv U_c(s^t)c(s^t) - U_x(s^t)(1 - x(s^t))$.

With this it is possible to express the first-order conditions of the planner's extremization problem as[6]:

$$\beta^t \pi(s^t, s_0) M_P(s^t)\{U_c(s^t) + \lambda H_c(s^t)\} = \psi(s^t), \tag{23}$$

$$\beta^t \pi(s^t, s_0) M_P(s^t)\{U_x(s^t) + \lambda H_x(s^t)\} = \psi(s^t), \tag{24}$$

for all $s^t, t \geq 1$ and for $\forall s^{t+2} \in \sigma^{t+2}(s^{t+1})$,

$$-\theta(1 + \log m_P(s^{t+1})) = [U(c(s^{t+1}), x(s^{t+1})) + \beta\theta m_P(s^{t+2}) \log m_P(s^{t+2}) \ + \lambda H(s^{t+1})] \tag{25}$$

From the last first order condition, solving for the planner's optimal distortion sequence is immediate and summarized by:

Lemma 1. *Fix any competitive equilibrium consumption-leisure sequence. The Ramsey Planner's minimized distortion sequence is given by:*

$$m_P(s^{t+1}) \propto \exp\left(-\frac{W(s^{t+1}) + \lambda H(s^{t+1})}{\theta}\right) \tag{26}$$

This is as it should be: as, in a different context, Jones et. al. [15] and Chari and Kehoe [6] point out, the Ramsey problem can be seen as an ordinary planning problem where the objective function is augmented by the time t component of the implementability restriction, $\lambda H(s^t)$. As such, the minimizing distortion for the Ramsey planner, simply reflects the different objectives that the planner and the representative household hold.

An important point to note is that we will not use the Ramsey planner worst case model when calculating the present value relationship given by the implementability constraint. Recall the latter is a competitive equilibrium condition, where prices reflect the household's own distortion. Thus the Lagrange multiplier λ determining the allocations are given for the implementability constraint as derived in the previous section, the equilibrium condition under the representative household own worst-case model. As such, in what concerns computing Ramsey allocations, we need not to derive the planner's worst case model[7]. This

[6] As standard these are valid for $t \geq 1$ only as period 0 conditions reflect the initial condition $b_0 \neq 0$.

[7] Note that the planner will still affect the household's distortion by choosing allocations. The problem can now be solved by specifying the objective function of the planner to be the discounted sum of the period utility functions of the household, $U(c, x)$ and confronting the planner with the implementability constraint 16. In this setting, the planner chooses only allocations and the worst-case model of the household is given.

is so since, from the remaining first-order conditions, given any such λ, the consumption-leisure optimality condition for the Robust Ramsey problem is static, yielding:

$$\frac{U_c(s^t)+\lambda H_c(s^t)}{U_x(s^t)+\lambda H_x(s^t)}=1 \tag{27}$$

This can the be solved using standard procedures; see for example Proposition 8 in Chari and Kehoe [6]. Thus for $t \geq 1$ substitute from the resource constraint for $x(s^t)$ into (27) above to get an expression of the form $F(c,g,\lambda)=0$. Now combine this expression with the feasibility condition . Fixing a state, $g(s_t)$ is known and hence this system of equations has a unique solution for consumption-leisure allocations as time invariant rules $c(s^t) = c(s_t,\lambda)$ and $x(s^t)=x(s_t,\lambda)$. Taxes are given by the competitive equilibrium condition 10 as a function of these allocations. One-period ahead debt sequences can be constructed from condition 17. For $b_0=0$, the initial allocations and taxes are given by the same functions.

As such the robustness considerations regarding the government expenditures path has no effect on the Ramsey allocations other than λ, the Lagrange multiplier on the implementability constraint. This immediately yields that the familiar characterization of Ramsey allocations carry over to its robustness counterpart: allocations and taxes are simply a function of the current realization of the state and the Lagrange multiplier on the implementability constraint. In this way the solutions to the Ramsey problem inherit the stochastic properties of the government's expenditure process. This is summarized in the following lemma:

Lemma 2. *The Robust Ramsey allocation sequence preserves the same qualitative features of the Ramsey Allocation: they are a function of* s_t and λ only.

Though we have been treating λ as fixed, note that, from the present-value constraint, the Lagrange multiplier is itself a function of consumption, leisure and (given the minimizing distortion) of θ. Thus robust Ramsey allocations and tax policies are functions parametrized by θ. Also, we know that $\lim_{\theta\to\infty}\exp\left(\frac{-W(s^{t+1})}{\theta}\right)=1$ and therefore we recover the value of the Lagrange multiplier λ that would obtain in a standard Lucas-Stokey setup with no concern for robustness. Thus solutions for the same expression $F(c,g,\lambda)=0$ and necessarily the same feasibility constraint, when evaluating at $\lambda_\infty \equiv \lim_{\theta\to\infty}\lambda(\theta)$, yield the Ramsey allocations for the non-robustness case.

This is as it should be: as $\theta \to \infty$, i.e. as the concern for robustness wears off, the implementability constraint of a Lucas-Stokey economy is recovered. Given Lemmas 1 and 2, we also know that the Ramsey allocations will coincide[8]. The Robust Ramsey problem thus depicts a family of allocations indexed by θ.

[8]Incidentally notice that there is a possibility of observational equivalence in the sense of Hansen et. al. [13].To see this notice that allocations and policies are a function of the current state and λ only, but

4.3. Non-Contingent Debt and the Optimal Maturity Structure. Here we concentrate the implications of the setup above on the pricing implications for government-backed risk free securities. Also, following the discussion in Angeletos [2] and Buera and Nicolini [5], we discuss the optimal maturity structure for the robust version of a Lucas-Stokey economy.

Thus, consider an incomplete markets economy where the only debt instrument available for the government is non-contingent debt - i.e. zero coupon bonds that promise to pay one unit of consumption at maturity for any state of the world. Assume also that the government restructures debt every period by first redeeming all outstanding debt and then issuing new one at all maturities.

Letting $j \in \{1, 2, ..., J\}$ index maturity, define $\overline{b}^j(s^{t-1})$ be the amount of promises held in period t (issued at $t-1$) paying one unit of consumption at $t+j$. The time t price of a promise to deliver the good at time $t+j$, $j \geq 0$ is denoted by $p^b(s^t, j)$.

Now, from Section 4.1., recall the expressions for equilibrium pricing of Arrow securities, 13 and14. Using this, the price of a j-maturity zero-coupon bond is given by:

$$p^b(s^t, j) = \sum_{s^{t+j}} q(s^{t+j}, s^t) = \sum_{s^{t+j}} \left(\Lambda(s^{t+j}) \prod_{\iota=t}^{t+j-1} \widetilde{\pi}(s_{\iota+1}, s_\iota) \right) = \widetilde{E}_t[\Lambda(s^{t+j})] \tag{28}$$

Again using 14 note that $\Lambda(s^{t+j}) = \prod_{\iota=t}^{t+j-1} \Lambda(s^{\iota+1})$. With this we arrive at a recursive formula for deriving prices of zero-coupon bonds for any maturity j [9]:

$$p^b(s^t, j) = \sum_{s^{t+1}} \Lambda(s^{t+1}) p^b(s^{t+1}, j-1) \widetilde{\pi}(s^{t+1}) = \widetilde{E}_t[\Lambda(s^{t+1}) p^b_{j-1,t+1}] \tag{29}$$

Thus, the time t value of outstanding obligations, conditional on the realization of s_t is defined to be:

$$V(s^t) \equiv \sum_{j=0}^{J-1} p^b(s^t, j) \overline{b}^j(s^{t-1}) \tag{30}$$

Note that the value of real non-contingent debt is rendered state-dependent by the dependence of prices on the current realization of the state. The crux of the argument in Angeletos and Buera and Nicolini is whether the government can synthesize the complete markets state-contingent debt allocation 17 using non-contingent debt of different

that the latter is also a function of period zero debt holdings which was left unrestricted in the analysis above. Thus take the no robustness case and increase initial debt holdings. The observational equivalence possibility arises if the value of λ obtained, $\lambda(b_0^*, \infty)$, coincides with that given by a (b_0, θ)

[9] Note that we can also define forward prices by $p^f_{j,t} = \frac{p^b_{j+1,t}}{p^b_{j,t}}$ and compute yields through the standard formulas $f_{j,t} = -\log(p^f_{j,t})$ and $r_{j,t} = -\frac{1}{n}\log(p^b_{j,t})$.

maturities. For this it has to be the case that:

$$V(s^t) = b(s^t) \tag{31}$$

Writing this in vector notation let the vector of non-contingent debt of different maturities be given by:

$$\overline{B}(s^{t-1}) = \begin{bmatrix} \overline{b}^0(s^{t-1}) \\ \overline{b}^1(s^{t-1}) \\ \vdots \\ \overline{b}^{J-1}(s^{t-1}) \end{bmatrix}$$

The $(Z \times J)$ matrix of returns in period t is

$$P^b(s^t) = \begin{bmatrix} 1 & p^b(1,1) & ... & p^b(1,J-1) \\ 1 & p^b(2,1) & & p^b(2,J-1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p^b(Z,1) & ... & p^b(Z,J-1) \end{bmatrix}$$

and the $(Z \times 1)$ vector of Ramsey contingent debt allocations

$$B(s^t) = \begin{bmatrix} b(s^{t-1},1) \\ b(s^{t-1},2) \\ \vdots \\ b(s^{t-1},Z) \end{bmatrix}$$

Given this, as in Angeletos, the conditions to syntesize state-contingent debt simply require that $J \geq \mathcal{Z}$ and $P^b(s^t)$ is non-singular, that is that the column vectors in $P^b(s^t)$ are linearly independent[10]. Another way to put this is simply that the stochastic variation in the term structure of interest rates must support the contemporaneous variation in the present value surpluses: $P^b(s^t)$ spans $B(s^t)$, $\forall t, s^t$.

The only change brought about by robustness considerations is simply the distortion in probabilities used to calculate the prices of non-contingent assets and changes in $B(s^t)$ [11]. It does not however change the basic mechanism present in Angeletos and Buera and Nicolini: the dependence of prices of assets on the current state of the economy, through changes in tax rates. Exploiting the cross-state variation of the non-contingent maturity structure to synthesize the contingent debt allocation still remains the key.

5. A Linear Quadratic Laboratory

[10] So we can invert (31) and solve for a unique optimal maturity structure.

[11] Recall this is also a function of λ and this in turn varies with θ.

We study a laboratory economy where the representative household's period return function is quadratic. This, combined with a recursive formulation of this problem set out in Appendix A, enables us to solve explicitly for allocations, policies and prices in the economy set out above. The nature of the exercise here is not to calibrate the economy: rather, the aim below is to scrutinize the qualitative properties induced by the free parameter θ, with respect to the existing results for the limiting expected utility Lucas-Stokey economy[12].

5.1. Linear Quadratic Economy. Appendix A and the discussion in Hansen and Sargent [10] show that the extremization problem of the household induces a recursive utility representation for the objective function of the household. Here we specialize the single period return function to be quadratic such that the period t objective function is:

$$W(s^t) = -\frac{1}{2}\left\{[c(s^t) - \overline{c}]^2 + [\ell(s^t)]^2\right\} - \beta\theta \log E\left[\exp\left(-\frac{W(s^{t+1})}{\theta}\right)|s^t\right]$$

where $\ell(s^t)$ is labor at time t, history s^t.

The numerical implementation follows the algorithm set out in Hansen and Sargent [10]. Thus, first solve for the time-0 competitive equilibrium prices and taxes from the household's first order conditions. Substitute these into the time 0 present valued government budget constraint to get the implementability constraint. As described above, from the Ramsey problem thus constructed, we can then solve for consumption-leisure allocations. This enables us to rewrite the single period return function as a quadratic function of the current state, s_t, and the Lagrange multiplier on the implementability constraint λ.

The strategy is then to fix a value for θ and an initial guess for λ. With this, we can solve for the distorted transition probabilities by iterating on the value function state by state. We can then evaluate the distorted expectations version of the implementability constraint and verify whether it is met with equality. We iterate on λ until this is so, checking for admissibility of its values. This gives values for consumption and leisure sequences, given a θ. From the discussion above we know how to compute state-contingent debt sequences (and how to price them), tax sequences and the term structure of interest rates. Further we know how to compute the optimal non-contingent debt structure (provided the conditions discussed above are met). To study how these vary with respect to θ we specify a grid in Θ and apply the same strategy. In all computations below we set $\underline{\theta}= 1$[13].

[12] An attempt to study the quantitative implications of this model would have to take as a baseline case the findings in Chari and Kehoe. This means including productive capital decisions and log utility households; the methods pursued in Tallarini would be the obvious ones to attack this problem. See also the recent discussion in Benigno and Woodford.

[13] This is admittedly an ad-hoc assumption. To improve on this means studying detection error probabilities along the lines of Hansen and Sargent [10].

The yardstick to which these results are measured against is one where $\theta \to \infty$, i.e. the quadratic version of the Lucas-Stokey economy studied by Chari and Kehoe[6], and Buera and Nicolini[5]. We follow the parameterization used by these studies whenever possible. Thus, we specialize to a two-sate economy and parametrize the Markov process so that the approximating model for the representative household coincides with the annual calibration of these authors. Along the same lines we choose their correspondent value for $\beta = 0.98$. We set $\overline{c} = 2$ and take initial debt $b_0 = 0$. All computations below take the approximating model to be the true data generating process.

5.2. Expectations and Allocations. First, for what follows it is informative to see what are the distortions induced by a preference for robustness. We identify state one (two) as a high (low) government expenditures realization. Under the approximating model the Markov process follows a symmetric transition matrix with 5% probability of leaving the current state. The figure below illustrates how the probabilities get distorted as a function of θ.

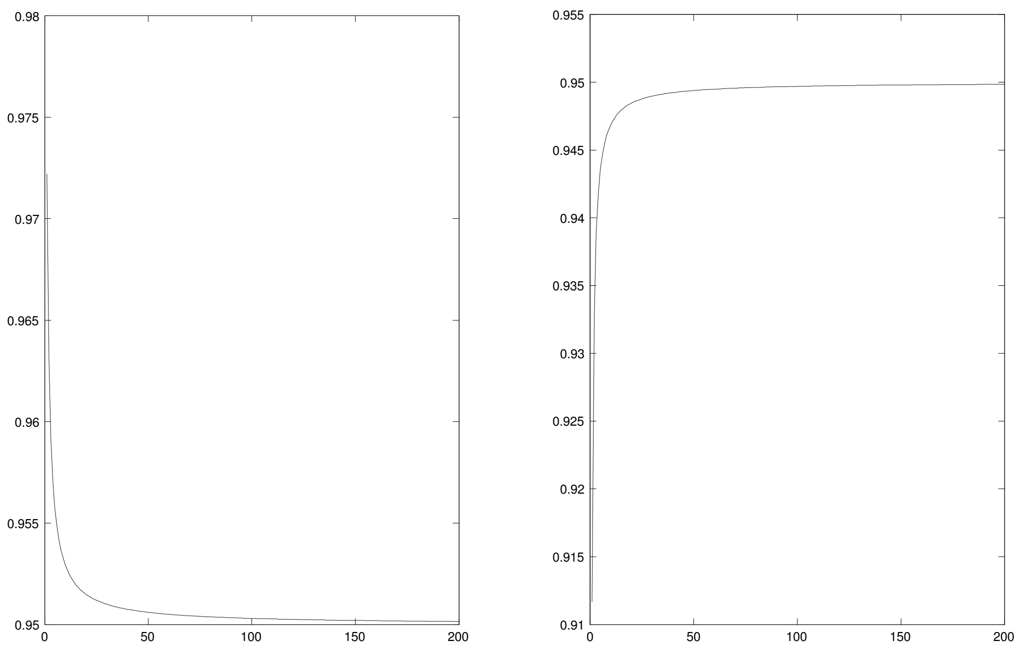

Figure 1: Transition probabilities (y-axis) as a function of θ (x-axis). Left (right) graph depicts probability of next period having a high (low) expenditure realization given that today's state is high (low).

Notice that, as expected, as θ grows large, the transitional probabilities are those given by the approximating model[14]. However, for low values of θ, robustness considerations induce a distorted transition matrix that puts relatively more weight on high expenditure states, breaking the symmetry of the approximating model. That is, introducing a concern over the limited reliability of the approximating model for fiscal expenditures, translates into households that hold pessimistic beliefs about future government outlays. This will of course affect equilibrium pricing of assets in this economy, an issue we study in the next subsection.

Further, as discussed in the previous section, varying θ will alter the shadow price on the present value budget constraint given by implementability condition, i.e. it will alter λ. As shown above, the Ramsey allocations are functions of λ and thus varying θ will change the optimal amounts of consumption, labor supplied and taxes on labor.

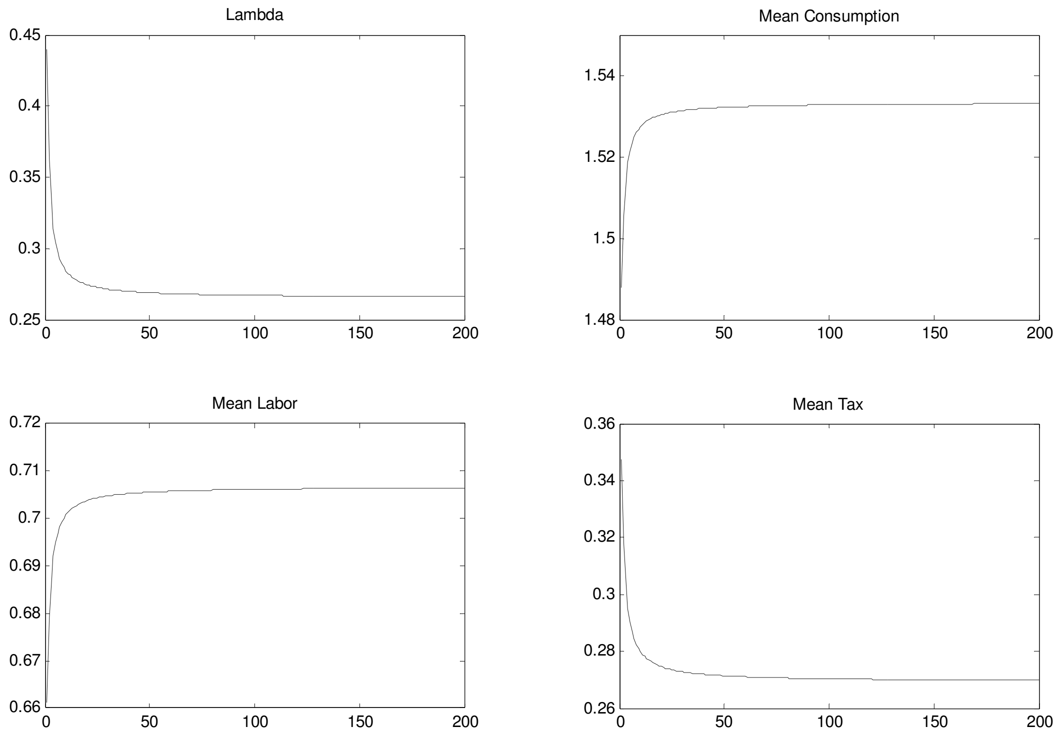

Figure 2: λ, mean consumption, mean labor and mean labor tax rate as functions of θ.

Figure 2 reports how the Lagrange multiplier λ (on the IC constraint), mean consumption, labor and the labor tax rate vary as a function of θ. The results shown are simple

[14]Given the assignment of parameter values, the numerical results here and below yield no qualitative difference in θ for values of $\theta > 200$. Thus in this section we take $\theta = 200$ as a good approximation for $\theta \to \infty$.

averages from 1000 period-lenght simulations for each value of θ. The Lagrange multiplier on the present valued budget constraint is a decreasing function of θ: as uncertainty about the reference model for government expenditures increases (θ decreases), the time zero present value of future government commitments increases[15]. We have seen already that increased uncertainty translates into a more pessimistic assessment of the path of fiscal expenditures (Figure 1): the behavior of λ simply shows that in equilibrium this in turn translates into a tighter constraint on the Ramsey allocations.

Figure 2 also renders clear that increased concern for model misspecification leads to a lower mean consumption, lower amounts of labor supplied and increased labor taxation. The latter feature is interesting insofar as Chari, Christiano and Kehoe [7], in an economy with capital, have found that increasing risk aversion (from a log-utility baseline) leads to decreased labor taxation. Here instead, by disentangling the intertemporal elasticity of substitution from uncertainty aversion and studying the effect of the latter (while keeping the IES fixed) we arrive at the opposite results. These can be explained by the fact that, under our setup, increasing uncertainty aversion (decreasing θ), increases the present valuation of future government commitments: thus a more uncertainty averse planner will set higher tax rates at time zero.

5.3. The Term Structure. Following the discussion in section 4.3., the Ramsey setup can be used to analyze the implications of fiscal policy on the term structure of real interest rates. Dai and Philippon's [8] recent empirical study on the nominal term structure, using an affine term structure setup, shows that fiscal deficits do affect the long term interest rates, partly by increasing the short rates and partly by increasing the risk premia on long term bonds. More generally they show taking into account the fiscal stance improves the forecasting performance of affine models.

Here we take the view of a Ramsey model as giving an explicit model for analyzing the real term structure and the fiscal stance jointly[16]. Thus, using the results derived in section 4.3. we generate yearly real term structures and analyze how these respond to realizations of the expenditures process of the government. We further study how these vary with θ. Figure 3 below provides the main results.

[15] The range of values reported here for λ are consistent with Ramsey plans supporting future government commitments; i.e. government commitments are not too large.

[16] Dai and Phillipon's study is on the nominal term structure where, as they show, other observable and latent factors play a key role in explaining the observed yields. The analysis in this section concentrates on the real term structure implications of fiscal variables through the lenses of a Ramsey model.

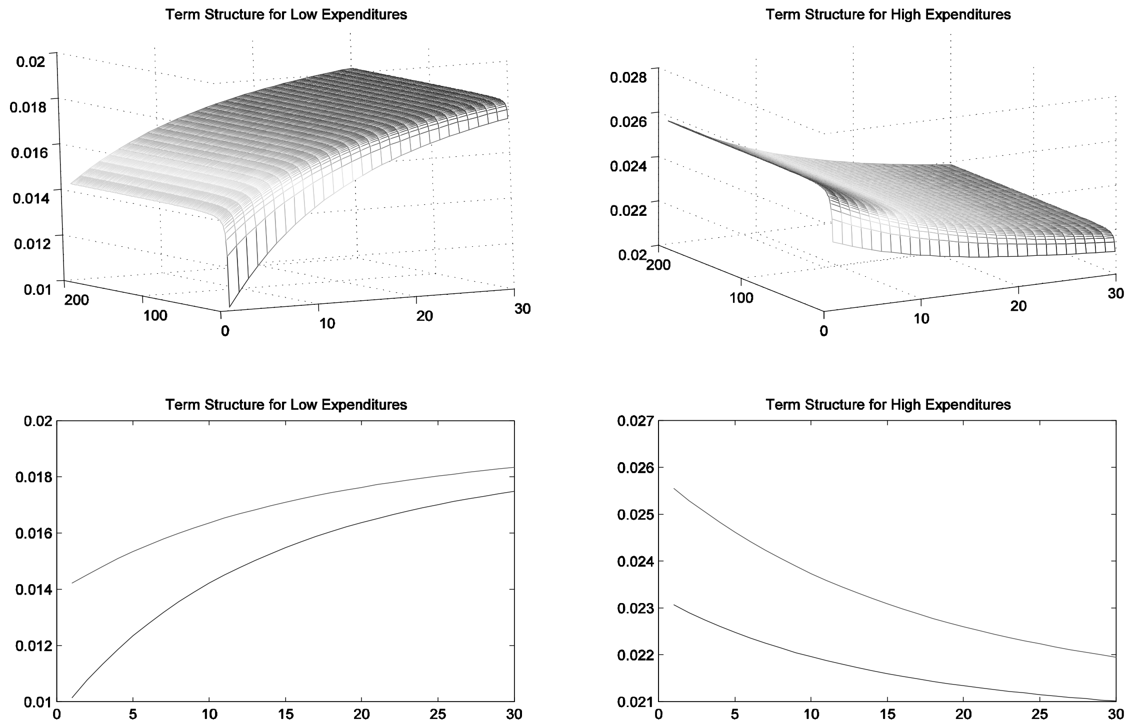

Figure 3: The term structure of risk free government debt (1 to 30 years) as a function of θ. Row above gives term structure surface as a function of θ. Row below gives two cuts: upper (lower) curve is the term structure for $\theta = 200$ ($\theta = 1$).

Figure 3 reveals that the real term structure is upward sloping for periods of low government expenditures and downward sloping for periods of high government expenditures . To relate these findings to the recent analysis of Seppala [22], note that under our setup a positive innovation in fiscal expenditures depresses consumption but expands output: thus the term structure is upward sloping in (output) recessions and downward sloping in expansions as in the Lucas asset pricing model (see Seppala [22]).

Also note that the upward sloping curve is uniformly below the downward curve: thus an increase in government expenditures raises interest rates at all maturities, long and short. The short rate is thus pro-cyclical. Note also that long rates rise less than short-rates during expansions and fall less during contractions, yielding a counter-cyclical spread.

Decreasing θ results in lower interest rates across all maturities and states: the more pessimistic is the representative household's view on the government expenditures process the poorer its own consumption growth prospects are (recall Figure 2), thus leading to lower rates. Long-term rates respond less than short rates to θ. Finally notice that decreasing θ results in steeper upward term structures in recessions and flatter downward structures in expansions. Recall that the distorted probabilities put relatively more weight on the events of switching to or staying in high expenditures states. In other words, low expenditure realizations are relatively rarer and, while the short term rate adjusts when they do happen, the right tail of the term structure still reflects the higher likelihood of a return to high expenditures states over the next 30 years. The latter is the result of the Markov structure of this laboratory economy: this implies reversion to a lower, distorted mean in terms of future expected consumption.

5.4. Optimal Maturity Structure. Informed by the behavior of allocations and interest rates we now turn to the optimal maturity structure in an economy with no state-contingent debt, as discussed in section 4.3. The following table gives debt positions in output units for periods of low government expenditures realizations; this is for ready comparison with the results in Buera and Nicolini. Debt positions are defined to be the value of debt issued at maturity j (in years), over current output: $\frac{p^j(g_{low})b^j}{y(g_{low})}$. We report the maturities implementing the Ramsey allocation that minimize the present value of the absolute debt positions.

	θ=1	θ=2	θ=10	θ=200
$j = 1$	-0.189	-0.301	-1.8	-2.244
$j = 30$	0.715	0.922	2.7	3.705

Table 1: Value debt positions in units of current output for periods of low government expenditures. $j = x$ lines give value of positions at maturity x, in years.

The table reveals that despite robustness considerations the optimal maturity structure preserves the same qualitative features of Angeletos and Buera and Nicolini: the government holds short term assets and long term liabilities. The intuition for this is given by the results in the previous subsection and carries over from these earlier studies: 'the interest rate goes up when government expenditures go up, so long term debt prices fall relative to short term debt. At the same time, the government needs resources when government expenditures go up. The way to accrue them is to hold short run assets and long term liabilities whose value is lower when government expenditures go up' (Buera and Nicolini [5, p. 543]).

However, we have also seen that a desire for robust rules by the representative household leads to lower interest rates at all maturities and a steeper upward slope for the real term structure during periods of low government expenditures. Also from the study of the term structure we can back up the variation on short term interest rates across states: for $\theta = 200$ the short term interest rate during periods of high government expenditures is 110 basis points above the one prevailing in periods of low expenditures; for $\theta = 1$ this difference is of 130 basis points. The corresponding value for the thirty year rates are 40 basis points

($\theta = 200$) and 35 basis points ($\theta = 1$). That is, introducing robustness considerations leads to a greater volatility of the short term interest rate but more correlated movements in the long-end of the term structure[17]. The overall effect is that the more important these robustness considerations the smaller the required debt positions.

This same mechanism is at work in an calibrated production economy in Buera and Nicolini [5, Table 3; Section 3.3.]. While the authors there are focusing on the effects of increased dimension of the state-space the discussion here points to the different effects of robustness on the short and long-run valuation of government backed assets.

6. Conclusion

We have set out to construct a dynamic general equilibrium economy with no capital where a fiscal authority faces an exogenous stochastic expenditure process and finances itself by controlling taxation on labor and issuing debt. As standard the representative household in this economy takes decisions on hours worked, output consumed and debt holdings, given its knowledge of the underlying stochastic financing needs of the government. However at this point we endowed the household with less than the knowledge of the true data generating process: the representative household was endowed with the knowledge of a model of the stochastic expenditures process, considered to be a good approximation (or a reference model) but potentially misspecified. Through the use of robust control techniques we then showed how this fear of model misspecification affects the competitive-equilibrium decisions of the household, the pricing of state-contingent assets and the calculation of present value restrictions.

To solve for an optimal policy we adapted the Ramsey specification of Lucas and Stokey to reflect robustness considerations. We showed how to write the planner's problem in this setting and how to compute robust-optimal fiscal policies, where all solutions to the Ramsey problem are indexed by the free parameter governing the size of the set of potential misspecifications entertained (with respect to the approximating model). Given this characterization and the derivation of competitive equilibrium pricing of non-contingent assets of different maturities we then showed how Angeletos' analysis of optimal maturity structure of non-contingent debt carries through in our environment.

A linear-quadratic version of this economy served to scrutinize qualitatively how solutions to the model vary as a function of model uncertainty from part of the representative household. We saw how this twists probabilities with respect to the approximating model

[17] Recall how robustness considerations induce a relatively higher long run interest rate in periods of low government outlays.

putting relatively more weight on high government spending realizations and that it induces higher mean taxation, lower mean hours worked and lower mean consumption. We showed also that accounting for robustness considerations and its effects in the pricing of non-contingent assets has the potential to lower the implied equilibrium value of these transactions to observed magnitudes required to achieve the optimal structure of non-contingent assets of different maturities.

The quantification of these effects appears thus to be the obvious point of departure in order to continue the analysis here. A starting hypothesis is to construct a production economy along the same lines and follow the approximating techniques in Tallarini [25]. A natural yardstick to compare this against exists already and is given in the calibration exercise of Chari et al.[7]. This would also prove useful to compare quantitatively with the optimal maturity structure results in Buera and Nicolini [5]. As a side product this would also yield a quantitative characterization of the effects of fiscal policy on the real term structure offering a model-based study of this question. For this analysis to be meaningful one will also need to be careful establishing the lower bound $\underline{\theta}$; the discussion in Hansen and Sargent [10] serves as a guide here.

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Appendix A

Recalling the penalty robust control problem set out above, note that the minimizing agent's problem, given a sequence of controls chosen by the household is

$$W_0 \equiv \min_{m_{t+1}} E\left(\sum_{t=0}^{\infty} M_t\beta^t[U(c_{t,}y_t) + \beta\theta m_{t+1}\log m_{t+1}]|\mathcal{F}_0\right) \tag{32}$$

subject to (5).Hansen et al.[14] and Hansen and Sargent [11] show how this inner problem has a recursive representation. To see this first solve the problem for a finite T terminal time. The problem starting in time $T-1$ is defined as

$$\begin{aligned}
W_{T-1} &= U(c_{T-1}, y_{T-1}) + \beta E\{M_T U(c_T, y_T) + M_{T-1}\theta m_T \log m_T]|\mathcal{F}_{T-1}\} \Longleftrightarrow \\
W_{T-1} &= U(c_{T-1}, y_{T-1}) + \beta E\{\frac{1}{M_{T-1}}[m_T U(c_T, y_T) + \theta m_T \log m_T]|\mathcal{F}_{T-1}\} \Longleftrightarrow \\
W_{T-1} &= U(c_{T-1}, y_{T-1}) + \beta E\{[m_T U(c_T, y_T) + \theta m_T \log m_T]|\mathcal{F}_{T-1}\} \Longleftrightarrow \\
W_{T-1} &= U(c_{T-1}, y_{T-1}) + \beta\Re_{T-1}(W_T)
\end{aligned}$$

where $\Re_{T-1}(W_T)$ is an operator distorting expectations of time T, $\mathcal{F}_T-$measurable random variables. Now think of solving backwards the sequence of decision problems as $T \to \infty$, using the same logic as above. Hansen and Sargent show that the minimizing choice of m_{t+1} satisfies

$$m_{t+1} \propto \exp\left(-\frac{W_{t+1}}{\theta}\right)$$

and that the optimized value for the operator $\Re_t(W_{t+1})$ is

$$\Re_t(W_{t+1}) = -\theta \log E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{F}_t\right]$$

Note that for $\theta = \infty$ the operator $\Re$ is just the standard conditional expectation operator. For all other admissible values of θ, the operator $\Re$ imposes additional corrections for risk when the household is assessing future distortions.

Finally, notice how this induces the representation of the penalty robust control problem for the household, as a recursive utility problem given by

$$W_t = \max_{c_t} U(c_t, x_t) - \beta\theta \log E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{F}_t\right] \tag{33}$$

subject to (5). The problem is now in the form of a risk-sensitive formulation, namely the one given by Hansen and Sargent's [9] discounted version of Whittle's [26] risk sensitive setting. To see the link note that for the operator $\Re_t$, rewriting $\theta = -2/\sigma$ yields Whittle's risk sensitive criterion.