DYNAMIC PRICING AND IMPERFECT COMMON KNOWLEDGE
SUPPLEMENTARY MATERIAL

KRISTOFFER P. NIMARK

Appendix A. The Optimal Reset Price

Firm j resetting its price in period t maximizes

$$E_t(j)\sum_{i=0}^{\infty}(\theta\beta)^i\left[\frac{P_t(j)}{P_{t+i}}Y_{t+i}(j)-MC_{t+i}(j)Y_{t+i}(j)\right] \tag{1}$$

subject to the demand constraint

$$Y_{t+i}(j)=\left(\frac{P_t(j)}{P_{t+i}}\right)^{-\epsilon}Y_{t+i} \tag{2}$$

where

$$Y_t=\left(\int_0^1 Y_t\left(j\right)^{\frac{\epsilon-1}{\epsilon}}dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{3}$$

and

$$P_t=\left(\int_0^1 P_t\left(j\right)^{1-\epsilon}dj\right)^{\frac{1}{1-\epsilon}}. \tag{4}$$

Substituting (2) into (1) and taking derivatives w.r.t $P_t(j)$ gives the first order condition

$$E_t(j)\sum_{i=0}^{\infty}(\theta\beta)^iY_{t+i}\left[\frac{1-\epsilon}{P_{t+i}}\left[\frac{P_t^*(j)}{P_{t+i}}\right]^{-\epsilon}-MC_{t+i}(j)\frac{\epsilon}{P_{t+i}}\left[\frac{P_t^*(j)}{P_{t+i}}\right]^{-\epsilon-1}\right]=0 \tag{5}$$

Rearranging and simplifying yields

$$P_t^*(j)E_t(j)\left[\sum_{i=0}^{\infty}(\theta\beta)^iY_{t+i}P_{t+i}^{\epsilon-1}\right]=\left(1+\mu\right)E_t(j)\left[\sum_{i=0}^{\infty}(\theta\beta)^iMC_{t+i}(j)P_{t+i}Y_{t+i}P_{t+i}^{\epsilon-1}\right] \tag{6}$$

where

$$(1+\mu)=\frac{\epsilon}{\epsilon-1}.$$

Log linearize

$$\left[\sum_{i=0}^{\infty}(\theta\beta)^i\right](p_t^*(j)-p_t)+\sum_{i=0}^{\infty}(\theta\beta)^i\left[y_{t+i}+(\epsilon-1)p_{t+i}\right] \tag{7}$$

$$=\sum_{i=0}^{\infty}(\theta\beta)^i\left[p_{t+i}+mc_{t+i}+y_{t+i}+(\epsilon-1)p_{t+i}\right]$$

Date: This version March 2007.

Address: Research Department, Reserve Bank of Australia, 65 Martin Place, N.S.W. 2001, Sydney, Australia and Department of Economics, European University Institute, Florence, Italy. e-mail: nimarkk@rba.gov.au.

and simplify

$$p_t^*(j) = (1-\beta\theta)\, E_t(j) \sum_{i=0}^{\infty} (\beta\theta)^i \left(p_{t+i} + mc_{t+i}(j)\right) \tag{8}$$

Appendix B. Deriving a Forward Looking Phillips Curve with Imperfect Common Knowledge

Let the price level follow

$$p_t = \theta p_{t-1} + (1-\theta) p_t^* \tag{9}$$

where p_t^* is the average price chosen by firms resetting their price in period t. The optimal price of firm j is a discounted sum of firm j's current and future nominal marginal costs given by

$$p_t^*(j) = (1-\beta\theta)\, E_t(j) \sum_{i=0}^{\infty} (\beta\theta)^i \left(p_{t+i} + mc_{t+i}(j)\right) \tag{10}$$

Rewrite as

$$p_t^*(j) = (1-\beta\theta)\, E_t(j) \left(p_t + mc_t(j)\right) + E_t(j)\beta\theta p_{t+1}^*(j) \tag{11}$$

To set the optimal price, firm j need to form an estimate of the price level. Substitute (9) into (11) to get

$$p_t^*(j) = (1-\beta\theta)\, E_t(j) \left([\theta p_{t-1} + (1-\theta)p_t^*] + mc_t(j)\right) + E_t(j)\beta\theta p_{t+1}^*(j) \tag{12}$$

where the average reset price p_t^* is

$$p_t^* = (1-\beta\theta)\, \overline{E}_t \left(p_t + mc_t\right) + \overline{E}_t \beta\theta p_{t+1}^* \tag{13}$$

where

$$\overline{E}_t[\cdot] = \int E[\cdot \mid I_t(j)]dj \tag{14}$$

Repeated substitution of (13) and (9) into (12) yields

$$\begin{aligned} p_t^*(j) \quad = \quad & (1-\beta\theta)\, E_t(j)((\theta p_{t-1} + \\ & (1-\theta)\,(1-\beta\theta)\, \overline{E}_t(\theta p_{t-1} + \\ & (1-\theta)\,(1-\beta\theta)\, \overline{E}_t\left(\theta p_{t-1} + (1-\theta)p_t^* + mc_t\right) \\ & + \overline{E}_t\beta\theta p_{t+1}^* + mc_t) + \overline{E}_t\beta\theta p_{t+1}^*) + mc_t(j)) \\ & + E_t(j)\beta\theta p_{t+1}^*(j) \end{aligned} \tag{15}$$

Continued substitution and averaging across firms yields

$$\begin{aligned} p_t^* \quad = \quad & (1-\beta\theta) \sum_{k=0}^{\infty} \left((1-\beta\theta)(1-\theta)\right)^k mc_t^{(k)} + \\ & + \frac{(1-\beta\theta)\theta}{1-((1-\theta)(1-\beta\theta))} p_{t-1} + \theta\beta \sum_{k=0}^{\infty} \left((1-\beta\theta)(1-\theta)\right)^k p_{t+1|t}^{*(k+1)} \end{aligned} \tag{16}$$

Substitute (16) into (9) to get

$$\begin{aligned} p_t \quad = \quad & (1-\theta)(1-\beta\theta) \sum_{k=0}^{\infty} \left((1-\beta\theta)(1-\theta)\right)^k mc_t^{(k)} + \\ & + \left(\theta + \frac{(1-\theta)(1-\beta\theta)\theta}{1-((1-\theta)(1-\beta\theta))}\right) p_{t-1} + (1-\theta)\,\theta\beta \sum_{k=0}^{\infty} \left((1-\beta\theta)(1-\theta)\right)^k p_{t+1|t}^{*(k+1)} \end{aligned} \tag{17}$$

First, note that

$$\begin{aligned}\left(\theta+\frac{(1-\theta)(1-\beta\theta)\theta}{1-((1-\theta)(1-\beta\theta))}\right) &= \frac{\theta\left(1-((1-\theta)(1-\beta\theta))\right)+\theta(1-\theta)(1-\beta\theta)}{1-((1-\theta)(1-\beta\theta))} \\ &= \frac{1}{1+\beta-\theta\beta} \qquad (18)\end{aligned}$$

then add $\frac{1}{1+\beta-\theta\beta}p_t$ and subtract $\frac{1}{1+\beta-\theta\beta}p_{t-1}$ and p_t to/from both sides to get

$$\begin{aligned}\frac{1}{1+\beta-\theta\beta}(p_t-p_{t-1}) &= (1-\theta)(1-\beta\theta)\sum_{k=1}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^{k-1}mc_t^{(k)}+ \\ &+\left(\frac{1}{1+\beta-\theta\beta}-1\right)p_t \qquad (19) \\ &+(1-\theta)\,\theta\beta\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^{k}p_{t+1|t}^{*(k+1)}\end{aligned}$$

Divide through by $\frac{1}{1+\beta-\theta\beta}$

$$\begin{aligned}(p_t-p_{t-1}) &= (1-\theta)(1-\beta\theta)\sum_{k=0}^{\infty}(1+\beta-\theta\beta)\left((1-\beta\theta)(1-\theta)\right)^{k}mc_t^{(k)} \\ &+(1-1-\beta+\beta\theta)\,p_t \qquad (20) \\ &+(1-\theta)\,\theta\beta\sum_{k=0}^{\infty}(1+\beta-\theta\beta)\left((1-\beta\theta)(1-\theta)\right)^{k}p_{t+1|t}^{*(k+1)}\end{aligned}$$

simplify and add and subtract

$$\beta\,(\theta-1)\,(1-(1-\beta\theta)(1-\theta))\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^{k}p_t^{(k+1)}$$

and use that $\theta\left((1+\beta-\theta\beta)\right)=1-(1-\beta\theta)(1-\theta)$ to get

$$\begin{aligned}\pi_t &= \frac{(1-\theta)(1-\beta\theta)}{\theta}\,(1-(1-\beta\theta)(1-\theta))\sum_{k=1}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^{k-1}mc_t^{(k)} \\ &+(\theta-1)\,\beta p_t \qquad (21) \\ &+\beta\,(1-\theta)\,(1-(1-\beta\theta)(1-\theta))\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^{k}p_{t+1|t}^{*(k+1)} \\ &+\beta\,(\theta-1)\,(1-(1-\beta\theta)(1-\theta))\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^{k}p_t^{(k+1)} \\ &-\beta\,(\theta-1)\,(1-(1-\beta\theta)(1-\theta))\,E_t\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^{k}p_t^{(k+1)}\end{aligned}$$

Inflation can now be rewritten as a function of higher order expectations of current marginal cost and inflation plus an error term that is a sum of the discrepancies of the higher order beliefs of the price level and the actual price level

$$\begin{aligned}\pi_t &= \frac{(1-\theta)(1-\beta\theta)}{\theta}\left(1-(1-\beta\theta)(1-\theta)\right)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k mc_t^{(k)} \\ &+\beta\left(1-(1-\beta\theta)(1-\theta)\right)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k \pi_{t+1|t}^{(k+1)} \\ &+(\theta-1)\beta\left[p_t-\left(1-(1-\beta\theta)(1-\theta)\right)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k p_t^{(k+1)}\right]\end{aligned} \tag{22}$$

where we used

$$\theta p_t+(1-\theta)E_t p_{t+1}^*-p_t=(\theta-1)\,p_t+(1-\theta)E_t p_{t+1}^*=E_t\pi_{t+1}$$

Substitute

$$p_t=p_{t-1}+\pi_t. \tag{23}$$

into (22) to get

$$\begin{aligned}\pi_t &= \frac{(1-\theta)(1-\beta\theta)}{\theta}\left(1-(1-\beta\theta)(1-\theta)\right)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k mc_t^{(k)} \\ &+\beta\left(1-(1-\beta\theta)(1-\theta)\right)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k \pi_{t+1|t}^{(k+1)} \\ &+(\theta-1)\beta[(\theta p_{t-1}+\theta\pi_t)- \\ &\left(1-(1-\beta\theta)(1-\theta)\right)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k\left(p_{t-1}^{(k+1)}+\pi_t^{(k+1)}\right)].\end{aligned} \tag{24}$$

Collect all terms with actual period t inflation on the left hand side and use that the lagged price level is common knowledge

$$\begin{aligned}&\left(1-(\theta-1)\beta\right)\pi_t= \\ &\frac{(1-\theta)(1-\beta\theta)}{\theta}\left(1-(1-\beta\theta)(1-\theta)\right)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k mc_t^{(k)} \\ &+\beta\left(1-(1-\beta\theta)(1-\theta)\right)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k \pi_{t+1|t}^{(k+1)} \\ &-(\theta-1)\beta\left[\left(1-(1-\beta\theta)(1-\theta)\right)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k \pi_t^{(k+1)}\right].\end{aligned} \tag{25}$$

Divide both sides with $(1-(\theta-1)\beta)$ and simplify to get

$$\begin{aligned}\pi_t &= (1-\beta\theta)(1-\theta)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k mc_t^{(k)} \\ &+\beta\theta\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k \pi_{t+1|t}^{(k+1)} \\ &+\theta\beta(1-\theta)\sum_{k=0}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k \pi_t^{(k+1)}\end{aligned} \tag{26}$$

Common knowledge of rationality implies than any order l of average expectaions of current inflation can be written as

$$\begin{aligned} \pi_t^{(l)} &= (1-\beta\theta)(1-\theta)\sum_{k=l}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k mc_t^{(k)} \\ &+\beta\theta\sum_{k=l}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k \pi_{t+1|t}^{(k+1)} \\ &+\theta\beta\,(1-\theta)\left[\sum_{k=l}^{\infty}\left((1-\beta\theta)(1-\theta)\right)^k \pi_t^{(k+1)}\right] \end{aligned} \tag{27}$$

which implies that the hierarchy of current inflation expectations (including actual current inflation) can be written as

$$\begin{bmatrix} 1 & -\theta\beta\,(1-\theta) & \cdots & & -\theta\beta\,(1-\theta)\left((1-\beta\theta)(1-\theta)\right)^{\infty} \\ 0 & 1 & -\theta\beta\,(1-\theta) & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 1 & -\theta\beta\,(1-\theta) \\ 0 & 0 & 0 & & 1 \end{bmatrix}\pi_t^{(0:\infty)} = \tag{28}$$

$$+\begin{bmatrix} (1-\beta\theta)(1-\theta) & (1-\beta\theta)(1-\theta)^2 & \cdots & & (1-\beta\theta)(1-\theta)^{\infty} \\ 0 & (1-\beta\theta)(1-\theta) & (1-\beta\theta)(1-\theta)^2 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & (1-\beta\theta)(1-\theta)^2 \\ 0 & \cdots & \cdots & 0 & (1-\beta\theta)(1-\theta) \end{bmatrix} mc_t^{(0:\infty)}$$

$$+\begin{bmatrix} \beta\theta & \beta\theta(1-\beta\theta)(1-\theta) & \cdots & & \beta\theta(1-\beta\theta)(1-\theta)^{\infty} \\ 0 & \beta\theta & \beta\theta(1-\beta\theta)(1-\theta) & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & \beta\theta(1-\beta\theta)(1-\theta) \\ 0 & \cdots & \cdots & 0 & \beta\theta \end{bmatrix}\pi_{t+1|t}^{(1:\infty)}$$

Pre-multiplying the right hand side of (28) with the inverse of the matrix on the left hand side and picking out the first row gives the desired form of the Phillips curve

$$\begin{aligned} \pi_t &= (1-\beta\theta)(1-\theta)\sum_{k=l}^{\infty}(1-\theta)^k mc_t^{(k)} \\ &+\beta\theta\sum_{k=l}^{\infty}(1-\theta)^k \pi_{t+1|t}^{(k+1)} \end{aligned} \tag{29}$$

Appendix C. Bounded variance of higher order expectations

Common knowledge of rational expectation imply that the unconditional variance of expectations is not increasing as the order of expectation increases, i.e.

$$E\left[x_{t|t}^{(k)}x_{t|t}^{(k)}\right] \geq E\left[x_{t|t}^{(k+1)}x_{t|t}^{(k+1)}\right] \tag{30}$$

Any two orders of expectation obey the identity

$$x_{t|t}^{(k)} \equiv x_{t|t}^{(k+1)} + e_t^{(k+1)} \tag{31}$$

where $e_t^{(k+1)}$ is defined as the $k+1$ order expectation error. The variance of the left hand side of (31) must equal the variance of the right hand side

$$E\left[x_{t|t}^{(k)} x_{t|t}^{(k)}\right] = E\left[x_{t|t}^{(k+1)} x_{t|t}^{(k+1)}\right] + E\left[e_t^{(k+1)} e_t^{(k+1)}\right] + 2E\left[x_{t|t}^{(k+1)} e_t^{(k+1)}\right]$$

The proof follows from the fact that variances are non-negative

$$E\left[e_t^{(k+1)} e_t^{(k+1)}\right] \geq 0 \tag{32}$$

and that the covariance between the estimate and the error must be zero

$$E\left[x_{t|t}^{(k+1)} e_t^{(k+1)}\right] = 0. \tag{33}$$

That the covariance between the estimate and the error is zero is implied by rationality. If the covariance was not zero, a better estimate $\widehat{x}_{t|t}^{(k+1)}$ could be found by subtracting the projection of the error on the estimate from the estimate

$$\widehat{x}_{t|t}^{(k+1)} = x_{t|t}^{(k+1)} - \frac{E\left[x_{t|t}^{(k+1)} e_t^{(k+1)}\right]}{E\left[x_{t|t}^{(k+1)} x_{t|t}^{(k+1)}\right]} x_{t|t}^{(k+1)}$$

so if $x_{t|t}^{(k+1)}$ is an optimal estimate of $x_{t|t}^{(k)}$, then (33) must hold.

Appendix D. The variance of average marginal cost

The economy-wide average marginal cost is given by

$$mc_t = (\gamma + \varphi)\, y_t + \lambda_t \tag{34}$$

or equivalently

$$mc_t = \mathbf{d}\mathbf{x}_{t|t}^{(0:\infty)} + \begin{bmatrix} \mathbf{1}_{1\times 2} & \mathbf{0} \end{bmatrix} \mathbf{x}_{t|t}^{(0:\infty)} + \epsilon_t \tag{35}$$

Define the new state Z_t as

$$Z_t = \begin{bmatrix} \epsilon_t \\ \mathbf{x}_{t|t}^{(0:\infty)} \end{bmatrix} \tag{36}$$

then Z_t follows

$$Z_t = \begin{bmatrix} \mathbf{0} \\ M \end{bmatrix} Z_{t-1} + \begin{bmatrix} e_3' \\ N \end{bmatrix} \begin{bmatrix} \nu_t \\ \eta_t \\ \epsilon_t \end{bmatrix}$$

The unconditional variance of Z_t, Σ_{zz} , is given by the solution of the discrete Lyaponov equation

$$\Sigma_{zz} = \begin{bmatrix} \mathbf{0} \\ M \end{bmatrix} \Sigma_{zz} \begin{bmatrix} \mathbf{0} \\ M \end{bmatrix}' + \begin{bmatrix} e_3' \\ N \end{bmatrix} \begin{bmatrix} e_3' \\ N \end{bmatrix}' \tag{37}$$

which implies that the unconditional variance of average marginal cost, σ_{mc}^2, is given by

$$\sigma_{mc}^2 = \left[\begin{bmatrix} 0 & \mathbf{d} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{1}_{1\times 2} & \mathbf{0} \end{bmatrix} + e_1'\right] \Sigma_{zz} \left[\begin{bmatrix} 0 & \mathbf{d} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{1}_{1\times 2} & \mathbf{0} \end{bmatrix} + e_1'\right]' \tag{38}$$

since

$$mc_t = \left[\begin{bmatrix} 0 & \mathbf{d} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{1}_{1\times 2} & \mathbf{0} \end{bmatrix} + e_1'\right] Z_t \tag{39}$$