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Monetary Policy and Rational Asset Price Bubbles†

By Jordi Galí*

I examine the impact of alternative monetary policy rules on a rational asset price bubble, through the lens of an overlapping generations model with nominal rigidities. A systematic increase in interest rates in response to a growing bubble is shown to enhance the fluctuations in the latter, through its positive effect on bubble growth. The optimal monetary policy seeks to strike a balance between stabilization of the bubble and stabilization of aggregate demand. The paper's main findings call into question the theoretical foundations of the case for "leaning against the wind" monetary policies. (*JEL* E13, E32, E44, E52, G12)

The spectacular rise (and subsequent collapse) of housing prices experienced by several advanced economies over the past decade is generally viewed as a key factor underlying the global financial crisis of 2007–2009, as well as a clear illustration of the dangers associated with speculative bubbles that are allowed to go unchecked.

The role that monetary policy should play in containing such bubbles has been the subject of a heated debate, well before the start of the recent crisis. The consensus view among most policy makers in the pre-crisis years was that central banks should focus on controlling inflation and stabilizing the output gap, and thus ignore asset price developments, unless the latter are seen as a threat to price or output stability. Asset price bubbles, it was argued, are difficult—if not outright impossible—to identify or measure; and even if they could be observed, the interest rate would be too blunt an instrument to deal with them, for any significant adjustment in the latter aimed at containing the bubble may cause serious "collateral damage" in the form of lower prices for assets not affected by the bubble, and a greater risk of an economic downturn.[1]

* Centre de Recerca en Economia Internacional (CREI), Ramon Trias Fargas 25, 08005 Barcelona, Spain, Universitat Pompeu Fabra, and Barcelona GSE. (e-mail: jgali@crei.cat). I have benefited from comments by Pierre-Olivier Gourinchas, Seppo Honkapohja, Paolo Pesenti, three anonymous referees, and participants at the CREI Workshop on Asset Prices and the Business Cycle, the EABCN Conference on Fiscal and Monetary Policy in the Aftermath of the Financial Crisis, the ECB International Research Forum on Monetary Policy, the NBER Summer Institute and seminars at CREI-UPF, CEMFI, UAB, and the Riksbank. I am grateful to Alain Schlaepfer and Jagdish Tripathy for excellent research assistance. The research leading to these results has been partly funded by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement No. 339656. The author is a regular consultant to the European Central Bank and the Sveriges Riksbank. In addition he is a member of the Board of Directors of BanSabadell Inversión, a Barcelona-based investment fund. To the author's knowledge, none of these institutions is likely to have a financial, ideological, or political stake related to the article. No party had the right to review the paper prior to its circulation.

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[1] See, e.g., Bernanke (2002) and Kohn (2006, 2008) for a central banker's defense of this view. Bernanke and Gertler (1999, 2001) provide a formal analysis in its support.

But that consensus view has not gone unchallenged, with many authors and policy makers arguing that the achievement of low and stable inflation is not a guarantee of financial stability and calling for central banks to pay special attention to developments in asset markets.[2] Since episodes of rapid asset price inflation often lead to a financial and economic crisis, it is argued, central banks should act preemptively in the face of such developments, by raising interest rates sufficiently to dampen or bring to an end any episodes of speculative frenzy—a policy often referred to as "leaning against the wind." This may be desirable—it is argued—even if that intervention leads, as a by-product, to a transitory deviation of inflation and output from target. Under this view, the losses associated with those deviations would be more than offset by the avoidance of the potential fallout from a possible future bursting of the bubble, which may involve a financial crisis and the risk of a consequent episode of deflation and stagnation like the one experienced by Japan after the collapse of its housing bubble in the 1990s.[3]

Independently of one's position in the previous debate, it is generally taken for granted (i) that monetary policy can have an impact on asset price bubbles and (ii) that a tighter monetary policy, in the form of higher short-term nominal interest rates, may help disinflate such bubbles. In the present paper I argue that such an assumption is not supported by economic theory and may thus lead to misguided policy advice, at least in the case of bubbles of the rational type considered here. The reason for this can be summarized as follows: in contrast with the fundamental component of an asset price, which is given by a discounted stream of payoffs, the bubble component has no payoffs to discount. The only equilibrium requirement on its size is that the latter grow at the rate of interest, at least in expectation. As a result, any increase in the (real) rate engineered by the central bank will tend to *increase* the size of the bubble, even though the objective of such an intervention may have been exactly the opposite. Of course, any decline observed in the asset price in response to such a tightening of policy is perfectly consistent with the previous result, since the fundamental component will generally drop in that scenario, possibly more than offsetting the expected rise in the bubble component.

Below I formalize that basic idea by means of a simple asset pricing model, with an exogenous real interest rate. That framework, while useful to convey the basic mechanism at work, fails to take into account the bubble's general equilibrium effects as well as the possible feedback from the bubble to interest rates implied by the monetary policy rule in place. That concern motivates the development of a dynamic general equilibrium model that allows for the existence of rational asset pricing bubbles and where nominal interest rates are set by the central bank according to some stylized feedback rule. The model assumes an overlapping generations structure, as in the classic work on bubbles by Samuelson (1958) and Tirole (1985). This is in contrast with the vast majority of recent macro models, which stick to an infinite-lived representative consumer paradigm, and in which rational bubbles can generally be ruled out under standard assumptions.[4] Furthermore, and in contrast

[2] See, e.g., Borio and Lowe (2002) and Cecchetti et al. (2000) for an early exposition of that view.

[3] See Issing (2009), ECB (2010), and Blanchard et al. (2012) for an account of the gradual evolution of central banks' thinking on this matter as a result of the crisis.

[4] See, e.g., Santos and Woodford (1997). An exception to that statement is given by models with heterogenous infinite-lived agents and borrowing constraints. See Scheinkman and Weiss (1986) and Kocherlakota (1992) for

with the earlier literature on rational bubbles, the introduction of nominal rigidities (in the form of prices set in advance) makes room for the central bank to influence the real interest rate and, through it, the size of the bubble. While deliberately stylized, such a framework allows me to analyze rigorously the impact of alternative monetary policy rules on the equilibrium dynamics of asset price bubbles. In particular, it makes it possible to assess the consequences of having a central bank use its interest rate policy to counteract asset price bubbles in a systematic way, as has been proposed by a number of authors and commentators.[5]

The paper's main results can be summarized as follows:

- Monetary policy cannot affect the conditions for existence (or non-existence) of a bubble, but it can influence its short-run behavior, including the size of its fluctuations.
- Contrary to the conventional wisdom, a stronger interest rate response to bubble fluctuations (i.e., a "leaning against the wind policy") may raise the volatility of asset prices and of their bubble component.
- The optimal policy must strike a balance between stabilization of current aggregate demand—which calls for a positive interest rate response to the bubble—and stabilization of the bubble itself (and hence of future aggregate demand)—which would warrant a negative interest rate response to the bubble. If the average size of the bubble is sufficiently large the latter motive will be dominant, making it optimal for the central bank to lower interest rates in the face of a growing bubble.

The paper is organized as follows. In Section I, I present a partial equilibrium model to illustrate the basic idea. Section II develops an overlapping generations model with nominal rigidities, and Section III analyzes its equilibrium, focusing on the conditions under which the latter may be consistent with the presence of rational bubbles. Section IV describes the impact on that equilibrium of monetary policy rules that respond systematically to the size of the bubble. Section V analyzes the optimal central bank response to the bubble. Section VI discusses some of the caveats of the analysis. Section VII concludes.

I. A Partial Equilibrium Example

The basic intuition behind the analysis below can be conveyed by means of a simple, partial equilibrium asset pricing example. Consider an economy with risk neutral investors and an exogenous, time-varying (gross) riskless interest rate R_t. Let Q_t denote the price in period t of an infinite-lived asset, yielding a dividend stream $\{D_t\}$. In equilibrium the following difference equation must hold:

$$Q_t R_t = E_t\{D_{t+1} + Q_{t+1}\}.$$

early contributions to that literature. Miao and Wang (2012) provide a recent example in that tradition, in which the size of the bubble attached to firms' stocks affects the dividends generated by the latter, through the relaxation of firms' borrowing constraints.

[5] The work of Bernanke and Gertler (1999, 2001) is in a similar spirit. In their framework, however, asset price bubbles are not fully rational, and the optimal policy analysis not fully microfounded.

In the absence of further equilibrium constraints, we can decompose the asset price into two components: a *fundamental* component, Q_t^F, and a *bubble* component, Q_t^B.[6] Formally,

$$Q_t = Q_t^F + Q_t^B,$$

where the fundamental component is defined by the present value relation

$$(1) \qquad Q_t^F = E_t\left\{ \sum_{k=1}^{\infty} \left(\prod_{j=0}^{k-1} (1/R_{t+j}) \right) D_{t+k} \right\}.$$

The bubble component, defined as the deviation between the asset price and its fundamental value, must satisfy

$$(2) \qquad Q_t^B R_t = E_t\{ Q_{t+1}^B \}.$$

It is easy to see that, ceteris paribus, an increase in the interest rate (current or anticipated) will lower Q_t^F, the fundamental value of the asset. On the other hand, the same increase in the interest rate will raise the expected growth of the bubble component, given by $E_t\{Q_{t+1}^B/Q_t^B\}$. Note that the latter corresponds to the bubble's expected return, which must equate the interest rate under the risk neutrality assumption made here. Hence, under the previous logic, any rule that implies a systematic positive response of the interest rate to the size of the bubble, will tend to amplify the movements in the latter—an outcome that calls into question the conventional wisdom about the relation between interest rates and bubbles.

Changes in interest rates, however, may affect the bubble through a second channel: the eventual comovement between the (*indeterminate*) innovation in the bubble with the surprise component of the interest rate. To formalize this, it is convenient to log-linearize (2) (evaluated at $t - 1$) and eliminate the expectational operator to obtain

$$q_t^B = q_{t-1}^B + r_{t-1} + \xi_t,$$

where lowercase letters denote the natural logarithm of the corresponding variable, and where $\{\xi_t\}$ is a zero mean martingale-difference process, i.e., $E_{t-1}\{\xi_t\} = 0$ for all t. Note that $\{\xi_t\}$ may or may not be related to fundamentals, a reflection of the *inherent indeterminacy of the bubble size*. As a result, the contemporaneous impact of an interest rate increase on the size of the bubble depends on the eventual relation between ξ_t and the interest rate innovation, $r_t - E_{t-1}\{r_t\}$. Thus, assuming a stationary environment, one can write without any loss of generality,

$$\xi_t = \xi_t^* + \psi_r(r_t - E_{t-1}\{r_t\}),$$

[6]Transversality conditions generally implied by optimizing behavior of infinite-lived agents are often used to rule out such a bubble component (see, e.g., Santos and Woodford 1997). On the other hand models with an infinite sequence of finite-lived agent types, as the one developed below, lack such transversality conditions.

where $\{\xi_t^*\}$ is a zero-mean martingale-difference process orthogonal to interest rate innovations at all leads and lags, i.e., $E\{\xi_t^* \, r_{t-k}\} = 0$, for $k = 0, \pm 1, \pm 2, \ldots$. Note that neither the sign nor the size of ψ_r, nor its possible dependence on the policy regime, are pinned down by the theory. Accordingly, the impact of an interest rate innovation (or of any other shock) on the bubble is, in principle, indeterminate.

In much of what follows I assume that $\{\xi_t\}$ has no systematic relation to interest rate innovations (i.e., $\psi_r = 0$ in the formulation above).[7] While admittedly arbitrary, this seems a natural benchmark assumption. Note that in that case a change in the interest rate does not affect the *current* size of the bubble, but only its expected growth rate. To illustrate this point formally, assume that $\{r_t\}$ follows an exogenous $AR(1)$ process with autoregressive coefficient $\rho_r \in [0, 1)$ and innovation ε_t^r.[8] Then, it can be easily checked that the response of the bubble to a positive interest rate shock at different horizons is given by

$$\frac{\partial q_{t+k}^B}{\partial \varepsilon_t^r} = \frac{1 - \rho_r^k}{1 - \rho_r} > 0,$$

for $k = 0, 1, 2, \ldots$. Thus, we see that a persistent (though transitory) increase in the interest rate does not alter the size of the bubble on impact, but has a positive effect on its subsequent growth rate, leading to a permanent increase in its size, given by

$$\lim_{k \to \infty} \frac{\partial q_{t+k}^B}{\partial \varepsilon_t^r} = \frac{1}{1 - \rho_r} > 0.$$

The previous outcome is clearly at odds with the conventional wisdom regarding the effects of interest rates on a bubble. Of course, the impact on the *observed* asset price may be positive or negative, depending on the relative size of the bubble and fundamental components. In the long run, however, the impact on the fundamental dies out, but the permanent positive effect on the bubble will remain (at least in the partial equilibrium example above).

How does the previous analysis change if we assume an arbitrary value for ψ_r?[9] The resulting response of the bubble to an interest rate shock is now given by

$$\frac{\partial q_{t+k}^B}{\partial \varepsilon_t^r} = \psi_r + \frac{1 - \rho_r^k}{1 - \rho_r},$$

for $k = 0, 1, 2, \ldots$. Thus, the *initial impact* of an interest rate hike on the bubble is just ψ_r. If the latter is negative, the rise in the interest rate will dampen the size of any

[7] $\{\xi_t\}$ being a "pure" sunspot process, i.e., one orthogonal to fundamentals, can be viewed as a particular case of that assumption.

[8] Note that in this case the bubble will follow the process

$$(1 - \rho_r L)\Delta q_t^B = \varepsilon_{t-1}^r + (1 - \rho_r L)\xi_t^*,$$

where $\{\xi_t^*\}$ is exogenous relative to the interest rate process.

[9] In this case the bubble will follow the process

$$\begin{aligned}(1 - \rho_r L)\Delta q_t^B &= \varepsilon_{t-1}^r + (1 - \rho_r L)(\xi_t^* + \psi_r \varepsilon_t^r) \\ &= \psi_r \varepsilon_t^r + (1 - \rho_r \psi_r)\varepsilon_{t-1}^r + (1 - \rho_r L)\xi_t^*.\end{aligned}$$

existing bubble, in a way consistent with the conventional wisdom. But that negative effect may not be permanent. To see this, note that the long term effect is given by

$$\lim_{k\to\infty} \frac{\partial q^B_{t+k}}{\partial \varepsilon^r_t} = \psi_r + \frac{1}{1-\rho_r},$$

which will be negative only if $\psi_r < -1/(1-\rho_r) < 0$, i.e., only if ψ_r is sufficiently negative, relative to the persistence of the interest rate. Otherwise, the effect of an interest rate increase on the size of the bubble will become positive sooner or later.

A. *An Alternative Equilibrium*

As discussed above, the value of ψ_r is, in principle, indeterminate. Though $\psi_r = 0$ seems a natural benchmark (nesting the case of a pure sunspot), other selection criteria may also be plausible.[10] One possible criterion consists in choosing ψ_r so that the (percent) impact of an interest rate innovation on the bubble equals that on the fundamental (which is uniquely determined). This will be the case if investors happen to coordinate their expectations around the belief that the two components of an asset price show an identical response to an interest rate innovation.[11]

For the simple partial equilibrium model of an infinite-lived asset considered above, the response of the fundamental component to an interest rate innovation is given by (see Appendix 1 for a derivation):

$$\frac{\partial q^F_t}{\partial \varepsilon^r_t} = -\frac{R}{R-\rho_r},$$

where $R > 1$ is the steady state gross real interest rate.[12] Thus, under the equilibrium refinement considered here we set $\psi_r = -R/(R-\rho_r)$. Accordingly, the response of the bubble component to an interest rate shock would be given by

$$\frac{\partial q^B_{t+k}}{\partial \varepsilon^r_t} = -\frac{R}{R-\rho_r} + \frac{1-\rho_r^k}{1-\rho_r}.$$

In the long run, the interest rate increase has a permanent positive effect on the bubble, given by

$$\lim_{k\to\infty} \frac{\partial q^B_{t+k}}{\partial \varepsilon^r_t} = \frac{\rho_r(R-1)}{(R-\rho_r)(1-\rho_r)} > 0.$$

[10] Adam (2003) shows how adaptive learning can be used to overcome the multiplicity of equilibria in a monetary overlapping generations model where money itself can be thought of as a bubbly asset.

[11] Interestingly, that assumption would seem to be consistent with the "fundamentals logic" underlying the conventional wisdom about the effects of monetary policy on asset price bubbles, as discussed above.

[12] Note that for the fundamental price of an asset that yields positive (stationary) dividends over an infinite horizon to be well defined (finite) we require that $R > 1$, i.e., the (net) interest rate must be positive in the steady state. As discussed below, that condition is inconsistent with the existence of a bubble *in general equilibrium*. In the present section I ignore these general equilibrium constraints.

Thus, we see that under the refinement proposed here the effect of an interest rate increase on the size of the bubble is negative only over a finite horizon, eventually turning positive.

The simple partial equilibrium example above has illustrated how the relation between monetary policy and asset price bubbles can be potentially at odds with the conventional wisdom, which invariably points to an interest rate increase as the natural way to disinflate a growing bubble. More precisely, the previous analysis makes clear that any case for "leaning against the wind" policies must be based on a systematic negative relation between interest rate and bubble innovations (i.e., a negative value for coefficient ψ_r). Since neither the sign nor the size of that relation is pinned down by economic theory, such a case would rest on extremely fragile grounds, at least under the assumptions made here.

Of course, one might argue that the partial equilibrium nature of the previous example may be misleading in that regard, by not taking into account the existence of aggregate constraints that may impose limits on the size of the bubble and hence on its survival. Furthermore, the type of policy intervention considered (i.e., an exogenous change in the real rate) is arguably less relevant for the issue at hand than a policy rule that describes the systematic response of the interest rate to movements in the size of the bubble. The remainder of the paper seeks to address those potential criticisms by providing an example of possible failure of the conventional wisdom regarding the effects of "leaning against the wind" policies that is grounded on a general equilibrium setting, and in which the central bank follows a well defined interest rate rule allowing for a systematic response to asset price bubbles.

II. Asset Price Bubbles in an OLG Model with Nominal Rigidities

I develop a highly stylized overlapping generations model without capital and where labor is supplied inelastically as a laboratory for the analysis of the impact of monetary policy on asset pricing bubbles. In equilibrium, aggregate employment and output are constant. The assumptions of monopolistic competition and price setting in advance, however, imply that monetary policy is not neutral.[13] In particular, by influencing the path of the real interest rate, the central bank can affect asset prices (including those of bubbly assets) and, as a result, the distribution of consumption across cohorts and welfare.

A. *Consumers*

Each individual lives for two periods. Individuals born in period t seek to maximize expected utility

$$\log C_{1,t} + \beta E_t\{\log C_{2,t+1}\},$$

where $C_{1,t} \equiv \left(\int_0^1 C_{1,t}(i)^{1-\frac{1}{\epsilon}}\, di\right)^{\frac{\epsilon}{\epsilon-1}}$ and $C_{2,t+1} \equiv \left(\int_0^1 C_{2,t+1}(i)^{1-\frac{1}{\epsilon}}\, di\right)^{\frac{\epsilon}{\epsilon-1}}$ are the bundles consumed when young and old, respectively. Note that, in each period, there

[13] See also Adam (2003) for a monetary overlapping generations model with monopolistic competition and sticky prices.

is a continuum of differentiated goods available, each produced by a different firm, and with a constant elasticity of substitution given by ϵ. Henceforth I assume $\epsilon > 1$. Goods (and the firms producing them) are indexed by $i \in [0,1]$. The size of each cohort is constant and normalized to unity.

Each individual is endowed with the "know-how" to produce a differentiated good, and with that purpose he sets up a new firm. That firm becomes productive only after one period (i.e., when its owner is old) *and* only for one period, generating a profit which accrues to its owner.[14] Each individual is also endowed at birth with $\delta \in [0, 1)$ units of an intrinsically worthless asset (a "bubble"), whose price is $Q^B_{t|t} \geq 0$ (with the non-negativity constraint being guaranteed by free disposal). A market is assumed to exist where such bubbly assets, introduced by both current and previous cohorts, can be traded.[15] Each period, a fraction δ of each vintage of bubbly assets is assumed to lose its value (e.g., they are physically destroyed). The latter assumption implies that the total amount of bubbly assets outstanding remains constant and equal to one.

Each young individual sells his labor services inelastically, for a (real) wage W_t. He consumes $C_{1,t}$ and purchases two types of assets: (i) one-period nominally riskless discount bonds yielding a nominal return i_t and (ii) a variety of bubbly assets, introduced by both current and previous cohorts.

Accordingly, the budget constraint for the young at time t is given by

$$\int_0^1 \frac{P_t(i) C_{1,t}(i)}{P_t}\, di \;+\; \frac{Z^M_t}{P_t} \;+\; \sum_{k=0}^{\infty} Q^B_{t|t-k} Z^B_{t|t-k} \;=\; W_t \;+\; \delta Q^B_{t|t},$$

where $P_t \equiv \left(\int_0^1 P_t(i)^{1-\epsilon}\, di\right)^{\frac{1}{1-\epsilon}}$ is the aggregate price index, Z^M_t is the value of one-period bonds purchased, and $Z^B_{t|t-k}$ denotes the quantity purchased of the bubbly asset introduced by cohort born in period $t-k$, and whose current price is $Q^B_{t|t-k}$, for $k = 0, 1, 2, \ldots$.

When old, the individual consumes all his wealth, which includes the dividends generated by his firm, the payoff from his maturing bond holdings, and the proceeds from the sale of his bubbly assets. Formally,

$$\int_0^1 \frac{P_{t+1}(i) C_{2,t+1}(i)}{P_{t+1}}\, di \;=\; D_{t+1} \;+\; \frac{Z^M_t(1+i_t)}{P_{t+1}} \;+\; (1-\delta) \sum_{k=0}^{\infty} Q^B_{t+1|t-k} Z^B_{t|t-k}.$$

The optimal allocation of expenditures across goods yields the familiar demand functions:

$$C_{1,t}(i) \;=\; \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} C_{1,t} \tag{3}$$

[14] This is just a convenient device to avoid having infinite-lived firms, whose market value would not be bounded under the conditions that make it possible for a bubble to exist in the present model.

[15] In an earlier version of the paper, the bubble was attached to the stock of firms, which could be traded beyond their (productive) life, thus becoming a pure bubble. The current formulation simplifies the notation considerably, without affecting any of the results.

$$(4) \qquad C_{2,t+1}(i) = \left(\frac{P_{t+1}(i)}{P_{t+1}}\right)^{-\epsilon} C_{2,t+1},$$

for all $i \in [0, 1]$, which in turn imply $\int_0^1 \frac{P_t(i) C_{1,t}(i)}{P_t}\, di = C_{1,t}$ and $\int_0^1 \frac{P_{t+1}(i) C_{2,t+1}(i)}{P_{t+1}}\, di = C_{2,t+1}$.

The remaining optimality conditions associated with the consumer's problem take the following form:

$$(5) \qquad 1 = \beta(1 + i_t) E_t\left\{\left(\frac{C_{1,t}}{C_{2,t+1}}\right)\left(\frac{P_t}{P_{t+1}}\right)\right\}$$

$$(6) \qquad Q^B_{t|t-k} = (1 - \delta)\beta E_t\left\{\left(\frac{C_{1,t}}{C_{2,t+1}}\right) Q^B_{t+1|t-k}\right\},$$

for $k = 0, 1, 2, \ldots$. Equation (5) is a standard Euler equation linking consumption growth to the real interest rate. Equation (6) shows that the market value of the bubbly asset reflects investors' expectations of the (properly discounted) price at which it can be sold in the future.

Finally, and for future reference, I define the (gross) real interest rate as

$$R_t \equiv (1 + i_t) E_t\left\{\frac{P_t}{P_{t+1}}\right\}.$$

B. *Firms*

Each individual, endowed with the "know-how" to produce a differentiated good, sets up a firm that becomes productive after one period (i.e., when its founder is old). When productive, the firm operates under the technology

$$(7) \qquad Y_t(i) = N_t(i),$$

where $Y_t(i)$ and $N_t(i)$ denote firm i's output and labor input, respectively, for $i \in [0, 1]$. After its operational period (i.e., once its founder dies) the firm becomes unproductive (with its index i being "inherited" by a newly created firm).

Each firm behaves as a monopolistic competitor, setting the price of its good in order to maximize its value, subject to the demand constraint $Y_t(i) = (P_t(i)/P_t)^{-\epsilon} C_t$, where $C_t \equiv C_{1,t} + C_{2,t}$.

I introduce nominal rigidities by assuming that the price of each good is set in advance, i.e., before the shocks are realized. Thus, the price of a good that will be produced and sold in period t, denoted by P_t^*, is set at the end of $t - 1$ in order to solve

$$\max_{P_t^*} E_{t-1}\left\{\Lambda_{t-1,t} Y_t\left(\frac{P_t^*}{P_t} - W_t\right)\right\},$$

subject to the demand schedule $Y_t(i) = (P_t^*/P_t)^{-\epsilon} C_t$, where $\Lambda_{t-1,t} \equiv \beta(C_{1,t-1}/C_{2,t})$ is the relevant discount factor. The implied optimal price setting rule is then given by

$$E_{t-1}\left\{\Lambda_{t-1,t} Y_t\left(\frac{P_t^*}{P_t} - \mathcal{M} W_t\right)\right\} = 0, \tag{8}$$

where $\mathcal{M} \equiv \frac{\epsilon}{\epsilon - 1}$.

Note also that if firms could instead set the price of their good after the shocks are realized, they would choose a price P_t^* equal to a constant gross markup $\mathcal{M}$ times the nominal marginal cost $P_t W_t$. Hence, under flexible prices (or in the absence of uncertainty),

$$P_t^* = \mathcal{M} P_t W_t.$$

C. *Monetary Policy*

The central bank is assumed to set the short-term nominal interest rate i_t according to the following rule:

$$1 + i_t = RE_t\{\Pi_{t+1}\}(\Pi_t/\Pi)^{\phi_\pi}(Q_t^B/Q^B)^{\phi_b}, \tag{9}$$

where $\Pi_t \equiv P_t/P_{t-1}$ denotes gross inflation, Π is the inflation target, and Q_t^B is an aggregate bubble index (defined below), with Q^B being its steady state value. Note that under the above rule the *real* interest rate responds systematically to fluctuations in inflation and the size of the aggregate bubble, with a strength indexed by ϕ_π and ϕ_b, respectively.[16] Henceforth I assume $\phi_\pi > 0$, which guarantees the determinacy of the price level, as shown below. Much of the explorations below examine the consequences of alternative ϕ_b settings for the equilibrium behavior of the bubble itself as well as for consumers' welfare.

III. Equilibrium

In the present section I derive the model's remaining equilibrium conditions. The clearing of the market for each good requires that $Y_t(i) = C_{1,t}(i) + C_{2,t}(i)$ for all $i \in [0, 1]$ and all t. Letting $Y_t \equiv \left(\int_0^1 Y_t(i)^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}$ denote aggregate output, we can use the consumer's optimality conditions (3) and (4) to derive the aggregate goods market clearing condition:

$$Y_t = C_{1,t} + C_{2,t}. \tag{10}$$

[16] As an alternative I have also analyzed the specification

$$1 + i_t = R(\Pi_t/\Pi)^{\phi_\pi}(Q_t^B/Q^B)^{\phi_b}.$$

The main qualitative results obtained under (9) carry over to this alternative specification, though the analysis is (algebraically) more cumbersome in the latter case.

Also, from the income side we have

$$Y_t = D_t + W_t. \tag{11}$$

Labor market clearing implies

$$\begin{aligned} 1 &= \int_0^1 Y_t(i)\, di \\ &= Y_t, \end{aligned} \tag{12}$$

where the second equality follows from the fact that all firms set identical prices and produce identical quantities in the symmetric equilibrium. Thus, the supply of aggregate output is constant and equal to unity.

Evaluating the optimal price-setting condition under sticky prices at the symmetric equilibrium we obtain

$$E_{t-1}\{(1/C_{2,t})(1 - \mathcal{M}W_t)\} = 0. \tag{13}$$

Note also for future reference that both in the case of flexible prices and/or in the absence of uncertainty, the optimal price setting implies a constant real wage

$$W_t = 1/\mathcal{M}.$$

Asset markets clearing requires

$$Z_t^M = 0,$$

and

$$Z_{t|t-k}^B = \delta(1 - \delta)^k,$$

for $k = 0, 1, 2, \ldots$.

Define the economy's aggregate bubble index, Q_t^B, and the corresponding index for the "pre-existing" bubbles, B_t, as follows:

$$Q_t^B \equiv \delta \sum_{k=0}^{\infty} (1 - \delta)^k Q_{t|t-k}^B$$

$$B_t \equiv \delta \sum_{k=1}^{\infty} (1 - \delta)^k Q_{t|t-k}^B.$$

It is also convenient to let $U_t \equiv \delta Q_{t|t}^B$ denote the aggregate market value of the newly introduced bubbles. The following equilibrium condition then follows from (6) and the previous definitions:

$$Q_t^B \equiv B_t + U_t = \beta E_t\left\{\left(\frac{C_{1,t}}{C_{2,t+1}}\right) B_{t+1}\right\}. \tag{14}$$

Two exogenous driving forces are assumed. First, the value of the new bubbles brought along by the new cohorts, $\{U_t\}$, which is assumed to follow an exogenous *i.i.d.* process with mean U. Secondly, the innovations in the value of the pre-existing bubbles, $B_t - E_{t-1}\{B_t\}$, are assumed to be exogenous and independent from $\{U_t\}$.

Equations (10), (11), (12), (13), and (14) combined with (5) and (9) introduced earlier, describe the equilibrium dynamics of the model economy. Next I characterize the equilibrium for the deterministic case, for which an exact solution exists. For the (more interesting) stochastic case, analyzed further below, I need to rely instead on the log-linearized equilibrium conditions around a deterministic steady state.

A. *Equilibrium Dynamics: The Deterministic Case*

I start by analyzing the deterministic case, where it is assumed that $U_t = U > 0$ and $B_t - E_{t-1}\{B_t\} = 0$ for all t. As discussed above, in the absence of uncertainty the optimal price setting condition (13) implies $W_t = 1/\mathcal{M}$, for all t. It follows from (11) that $D_t = 1 - 1/\mathcal{M}$, whereas consumption of the young and old are given respectively by $C_{1,t} = 1/\mathcal{M} - B_t$ and $C_{2,t} = 1 - 1/\mathcal{M} + B_t$, for all t. Furthermore, the real interest rate is given by

$$R_t = \left(\frac{1}{\beta}\right)\left(\frac{1 - 1/\mathcal{M} + B_{t+1}}{1/\mathcal{M} - B_t}\right) \equiv R(B_t, B_{t+1}). \tag{15}$$

Note that the previous conditions determine the equilibrium allocation, given an equilibrium path for the (pre-existing) bubble, $\{B_t\}$. The latter must satisfy the deterministic version of (14), given by

$$\frac{B_t + U}{1/\mathcal{M} - B_t} = \frac{\beta B_{t+1}}{1 - 1/\mathcal{M} + B_{t+1}}.$$

Thus a *deterministic bubbly equilibrium* is defined by a sequence $\{B_t\}$ satisfying

$$B_{t+1} = \frac{(1 - 1/\mathcal{M})(B_t + U)}{\beta/\mathcal{M} - (1 + \beta)B_t - U} \equiv H(B_t, U), \tag{16}$$

with $B_t \in (0, 1/\mathcal{M})$ for all t, for some $U \geq 0$. Note that the aggregate bubble along that path is then given by $Q_t^B = B_t + U$. Given $\{B_t\}$, we can determine the equilibrium values for the remaining variables using the expressions above.

Similarly, a *bubbly steady state* is defined by a pair $(B,\ U)$ such that $B = H(B,\ U)$ with $B \in (0, 1/\mathcal{M})$ and $U \geq 0$. Note that a steady state is locally stable (unstable) if $\partial H(B,\ U)/\partial B < 1$ (>1).

The following Lemma establishes the conditions for the existence of such bubbly equilibria and steady states.

LEMMA 1: *A necessary and sufficient condition for the existence of a deterministic bubbly equilibrium is given by*

$$\mathcal{M} < 1 + \beta. \tag{17}$$

Furthermore, when (17) is satisfied there exists a continuum of *stable* bubbly steady states, $\{(B^S(U), U) \mid B^S(U) = H(B^S(U), U)$ for $U \in (0, \overline{U})\}$, as well as a continuum of *unstable* bubbly steady states $\{(B^U(U), U) \mid B^U(U) = H(B^U(U), U)$ for $U \in [0,\ \overline{U})\}$, where $\overline{U} \equiv \beta + (1+\beta)(1-W) + 2\sqrt{\beta(1+\beta)(1-W)} > 0$.

PROOF:

See Appendix 2.

Figure 1 illustrates graphically the mapping (16), the two steady states, and the trajectories for the bubble consistent with equilibrium for a given $U > 0$.

Let $\mathcal{R}(B) \equiv R(B,\ B)$ denote the steady state real interest rate. One can easily check that condition (17) is equivalent to $\mathcal{R}(0) < 1$, which corresponds to a negative (net) interest rate in a *bubbleless* steady state. The latter is in turn associated with a Pareto suboptimal allocation since it implies $1/C_1 < \beta/C_2$ and, hence, the possibility of making all cohorts better-off by transferring resources from the young to the old (which is what a bubble does). A similar condition holds in the models of Samuelson (1958) and Tirole (1985).

Given that $Q^B = B + U > B$ it follows from (14) that $\mathcal{R}(B) < 1$ must hold in any bubbly steady state, thus implying a negative (net) real interest rate in the latter. Note that if the interest rate were positive any existing bubble would grow unboundedly, which would be inconsistent with the definition of a steady state.[17] Furthermore, the unbounded growth in the size of the bubble would eventually lead to a violation of the resource constraint, and would thus be inconsistent with equilibrium. The negative interest rate is needed in order for the aggregate bubble to remain constant over time, as the shrinking size of the pre-existing bubble is exactly compensated by introduction of new bubbles.[18]

Note that the previous constraint on the interest rate, together with the fact that $\mathcal{R}'(B) > 0$, imposes an upper bound on the steady state bubble, namely, $B < 1/\mathcal{M} - 1/(1+\beta) \equiv B^U(0)$, where $\mathcal{R}(B^U(0)) = 1$. Hence the upper bound on the size of any bubbly steady state is given by the bubbly (unstable) steady state

[17] As is well known, the introduction of secular productivity growth makes it possible to reconcile the existence of a bubbly steady state with a positive real interest rate (see, e.g., Tirole 1985). See below for further discussion.

[18] A similar property can be found in the model of capital accumulation with bubble creation of Martín and Ventura (2012).

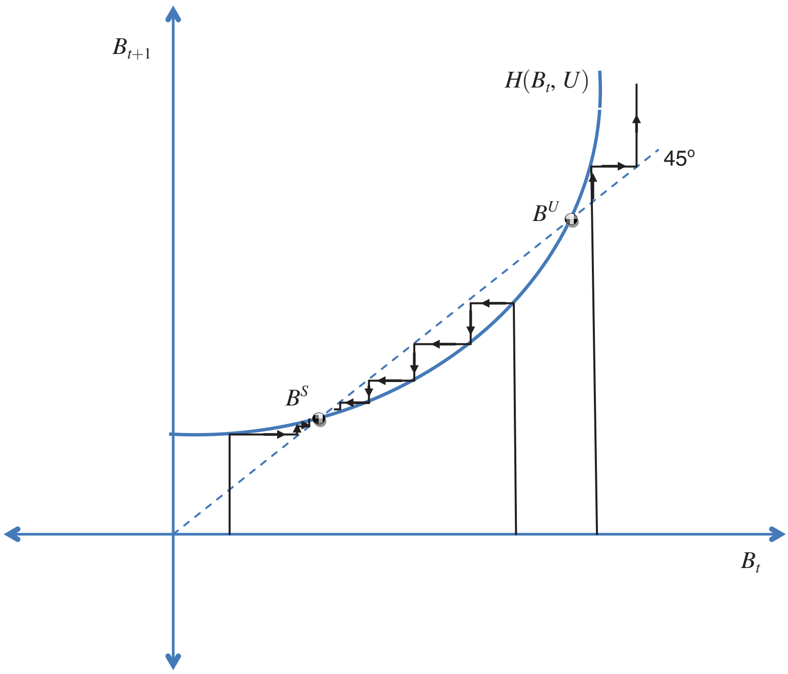

Figure 1. Equilibrium Dynamics

when $U = 0$. Note that the previous upper bound is always smaller than $1/\mathcal{M}$, the wage of the young.

Most importantly for the purposes of the present paper, one should note that neither the existence nor the allocations associated with a *deterministic* bubbly equilibrium are influenced by monetary policy. The intuition behind that result is straightforward: in the absence of uncertainty, the assumed constraint on the timing of price setting is not binding, so the economy behaves as if prices were fully flexible. Hence monetary policy is neutral. In particular, the real interest rate is given by (15), which evolves independently of monetary policy rule. The role of the latter is restricted to pinning down inflation, whose equilibrium path is given by

$$\Pi_t = \Pi \left[(R_t/R)(Q_t^B/Q^B)^{\phi_b}\right]^{\frac{1}{\phi_\pi}}.$$

Extension: The Case of Positive Deterministic Growth.—The analysis above has been conducted under the assumption of a stationary technology. Consider instead a technology $Y_t(i) = A_t N_t(i)$ with constant productivity growth, i.e., $A_t = \Gamma^t$ and $\Gamma > 1$. It is easy to check that under this modified technology the model above implies the existence of an equilibrium with balanced growth. In particular, it can be easily shown that all the equilibrium conditions derived above still hold, with the original real variables (output, consumption, dividend, wage, stock prices, and, eventually, bubble size) now normalized by parameter A_t, and with R_t being replaced with $\tilde{R}_t \equiv R_t/\Gamma$. Accordingly, a bubble can exist along the balanced growth path (i.e., a steady state of the normalized system) only if $\tilde{R} \leq 1$ or, equivalently, $R \leq \Gamma$, i.e., as long as the real interest rate is below the economy's growth rate. Such a bubble would be growing at the same rate as the economy. An analogous result was shown in Samuelson (1958) and Tirole (1985), among others. That extension allows one to reconcile the existence of a bubbly equilibrium with the steady state (net) real interest rate being positive.

Discussion: Robustness to the Introduction of Money.—The previous analysis did not incorporate money explicitly. One may wonder, in particular, whether the existence of bubbly equilibria is robust to the introduction of money. Two cases must be distinguished, corresponding to two different motives for holding money. I briefly discuss them in turn.

The first case is that of pure fiat money, i.e., money is assumed to be an intrinsically worthless asset which can be used as a store of value (see, e.g., Samuelson 1958). In that case, money is just another bubbly asset, one that happens to be used also as a unit of account. Its main distinctive feature is that its net nominal return is zero (by definition) and hence its real return is given by minus the rate of inflation. This has an important consequence in terms of the analysis here: monetary policy can no longer be described by an interest rate rule like (9), since the nominal interest rate must equal zero in any equilibrium in which money is valued. As a result one cannot examine the impact of "leaning against the wind policies" of the sort considered here.[19]

Perhaps a more natural (and realistic) approach to the introduction of money in the framework above consists in assuming that money holdings provide some services (other than "storage of wealth"). In that case the nominal interest rate in any monetary equilibrium is no longer pinned down at zero. Still, a zero lower bound on the nominal interest rate applies. But as in other examples in the literature of monetary models that assume interest rate rules, the zero lower bound can be dealt with whenever it is not the focus of the analysis, by making assumptions that guarantee that it will not be binding. In the context of the present model, those assumptions can take two forms (or a combination thereof), both of which are realistic. First, and as argued above, the assumption of positive trend growth ($\Gamma > 1$) implies that a continuum of bubbly steady states exist involving positive real interest rates. Those would be consistent with positive nominal interest rates in a neighborhood of such steady states, even if a zero steady state inflation ($\Pi = 1$) were to be assumed. Alternatively, one may assume a sufficiently high, positive inflation target ($\Pi > 1$), in which case, even in the absence of trend growth, there will be a continuum of steady states involving positive nominal interest rates. In either case, the assumption that real balances provide services other than "storage of wealth" implies that consumers will be willing to hold money even if the latter is dominated in rate of return by both bonds and bubbly assets, as will be the case whenever the nominal interest

[19] In a model with pure fiat money and a constant growth rate k of the money supply, one can show the existence of bubbly equilibria as long as the condition

$$0 \leq k < \frac{(1+\beta) - \mathcal{M}}{\mathcal{M} - 1}$$

is satisfied. The composition between the monetary and non-monetary components of the bubble is, however, indeterminate. See Galí (2013) for details.

rate is positive. But money will still be valued in that case, and a well defined money demand will determine the amount of steady state real balances as a function of the interest rate. In Appendix 3, I provide an example of an extension of the benchmark model above in which real balances enter separably in the utility function (a common assumption in macro models), and where the analysis can proceed as in the text without any changes (other than the possible "normalization" required by the assumption of positive trend growth, as discussed above).

B. *Equilibrium Dynamics: The Stochastic Case*

The analysis of the deterministic case found above has uncovered the conditions for the existence of a bubbly steady state. My analysis of the stochastic case, as is common in much of the literature on monetary policy rules, focuses on stationary fluctuations in a neighborhood of one such steady state. Thus, and in order to make progress in that direction, I start by log-linearizing the model's equilibrium conditions around a steady state and analyze the resulting system of difference equations. Unless otherwise noted I use lowercase letters to denote the log of the original variable, and the ˆ symbol on top of a variable to indicate the deviation from its steady state value. The resulting equilibrium conditions can be written as

$$0 = \hat{c}_{1,t} + \beta R \hat{c}_{2,t} \tag{18}$$

$$\hat{c}_{1,t} = E_t\{\hat{c}_{2,t+1}\} - \hat{r}_t \tag{19}$$

$$\hat{c}_{2,t} = (1 - \Gamma)\hat{d}_t + \Gamma \hat{b}_t \tag{20}$$

$$\begin{aligned} \hat{q}_t^B &= R\hat{b}_t + (1 - R)\hat{u}_t \\ &= E_t\{\hat{b}_{t+1}\} - \hat{r}_t \end{aligned} \tag{21}$$

$$E_{t-1}\{\hat{w}_t\} = E_{t-1}\{\hat{d}_t\} = 0 \tag{22}$$

$$\hat{r}_t = \phi_\pi \, \hat{\pi}_t + \phi_b \, \hat{q}_t^B, \tag{23}$$

where $R = \mathcal{R}(B)$ (as defined above) and $\Gamma \equiv \epsilon B/(\epsilon B + 1)$.

Note that one can rewrite (21) to obtain

$$\hat{b}_t = R\hat{b}_{t-1} + (1 - R)\hat{u}_{t-1} + \hat{r}_{t-1} + \xi_t, \tag{24}$$

where $\{\xi_t\}$ is an arbitrary martingale-difference process (i.e., $E_{t-1}\{\xi_t\} = 0$ for all t). As discussed above, and in order to avoid embedding in the model an arbitrary link between monetary policy and the size of the bubble, I assume in what follows that ξ_t is an exogenous sunspot shock whose variance is independent of the policy rule. By making this assumption I force monetary policy to influence the size of the bubble only through the interest rate channel and not through an (arbitrary) indeterminacy channel.

Flexible Price Equilibrium.—Before I turn to the case of sticky prices, I take a brief detour to analyze the flexible price case. This will help us understand the role played by sticky prices in the analysis below. As discussed above, when firms can adjust freely their prices once the shocks are realized, they optimally choose to maintain a constant gross markup $\mathcal{M}$. This, in turn, implies that the wage and dividend remain constant at their steady state values. Accordingly, equilibrium condition (22) must be replaced by

$$\hat{w}_t = \hat{d}_t = 0. \tag{25}$$

Combined with (19)–(21), the above equilibrium condition implies

$$\hat{r}_t = \epsilon(1 + \beta)BR\,\hat{b}_t + \epsilon B(1 - R)\hat{u}_t. \tag{26}$$

The previous condition makes clear that the real interest rate is, under flexible prices, independent of monetary policy (i.e., of ϕ_π and ϕ_b). Plugging the previous result in (24),

$$\hat{b}_t = \chi\,\hat{b}_{t-1} + (1 - R)(1 + \epsilon B)\hat{u}_{t-1} + \xi_t,$$

where $\chi \equiv R(1 + \epsilon(1 + \beta)B)$. Stationarity of the bubble requires $\chi \in [0, 1)$, which I henceforth assume.[20] As shown in Appendix 4, $\chi = \partial H(B, U)/\partial B_t$. Thus, the condition for (local) stationarity of the bubble around the steady state in the stochastic equilibrium corresponds to the condition of stability of that steady state under the deterministic equilibrium dynamics. The analysis below is restricted to fluctuations around a stable deterministic steady state.[21]

Note that under flexible prices, monetary policy has no influence on the evolution of the bubble, due to its inability to affect the real interest rate. Naturally, though, monetary policy can influence inflation (and other nominal variables). In particular, equilibrium inflation can be derived by combining interest rate rule (23) and (26) to yield

$$\hat{\pi}_t = -(1/\phi_\pi)\big((\phi_b - \epsilon B(1 + \beta))R\hat{b}_t + (\phi_b - \epsilon B)(1 - R)\hat{u}_t\big). \tag{27}$$

Not surprisingly the impact of bubbles on inflation is not independent of the monetary policy rule. In particular, we see that some positive systematic response of the interest rate to the aggregate bubble ($\phi_b > 0$) is desirable from the viewpoint of inflation stabilization. More precisely, the value of ϕ_b that minimizes the variance of inflation under flexible prices is given by $\phi_b = \epsilon B(1 + \lambda\beta) > 0$, where $\lambda \equiv R^2\,\mathrm{var}\{\hat{b}_t\}/\mathrm{var}\{\hat{q}_t^B\}$. Of course, there is no special reason why the central bank

[20] That stationarity assumption also justifies the use of methods based on a log-linear approximation of the equilibrium conditions.

[21] Note that $\mathcal{R}(\tilde{B})(1 + \epsilon\tilde{B}(1 + \beta)) = 1$ implicitly defines an upper bound $\tilde{B} > 0$ on the size of the steady state bubble consistent with stationarity of bubble fluctuations. That upper bound satisfies $\tilde{B} = B^S = B^U$.

would want to stabilize inflation in the present environment, so I do not analyze this issue further here.[22]

C. *Sticky Price Equilibrium*

We can combine (18) through (21) to write the goods market clearing condition as

$$0 = \epsilon B(1+\beta)R\hat{b}_t + \epsilon B(1-R)\hat{u}_t + \beta R\hat{d}_t - \hat{r}_t.$$

As discussed in Section IIIB, in the presence of sticky prices we have

$$E_{t-1}\{\hat{w}_t\} = E_{t-1}\{\hat{d}_t\} = 0 \tag{28}$$

for all t. Note also that the predetermination of prices implies

$$E_{t-1}\{\pi_t\} = \pi_t. \tag{29}$$

Combining the previous equation with the interest rate rule (23) and equilibrium condition (22) one can derive the following closed form solution for the evolution of the bubble (see Appendix 5 for details):

$$\hat{b}_t = \chi\hat{b}_{t-1} + (\phi_b+1)(1-R)\hat{u}_{t-1} + \xi_t + (\phi_b - \epsilon B(1+\beta))R\xi_{t-1}. \tag{30}$$

Note that the persistence of the bubble fluctuations, as measured by the autoregressive coefficient $\chi \equiv R(1+\epsilon B(1+\beta))$, is the same as in the flexible price equilibrium and, hence, independent of monetary policy. The latter, however, can influence the bubble's overall size and volatility through the choice of interest rate rule coefficient ϕ_b, as made clear by (30). Through the influence of the real interest rate on the size of the bubble $\hat{b}_t$ as well as on the incentives to allocate consumption intertemporally, monetary policy will affect the allocation of aggregate consumption across cohorts, thus affecting welfare. This is discussed in detail in the following section.

On the other hand, equilibrium inflation is given by the $AR(1)$ process[23]

$$\hat{\pi}_t = \chi\hat{\pi}_t - (1/\phi_\pi)(\phi_b - \epsilon B(1+\beta)R)(\phi_b+1)\varepsilon_{t-1}, \tag{31}$$

where $\varepsilon_t \equiv R\xi_t + (1-R)\hat{u}_t$ is the innovation in the aggregate bubble. Thus, we see that inflation inherits the persistence of the aggregate bubble, while it fluctuates as a result of innovations in the latter, interacting with the central bank's feedback rule.

[22] It is easy to check that the central bank could fully stabilize inflation in this case if it could identify and respond separately to existing and new bubbles with a rule

$$\hat{r}_t = \phi_\pi\pi_t + \Theta_b\hat{b}_t + \Theta_u\hat{u}_t,$$

where $\Theta_b \equiv \epsilon B(1+\beta)$ and $\Theta_u \equiv \epsilon B$.

[23] See Appendix 5 for details.

IV. The Impact of Monetary Policy on Bubble Dynamics

As made clear by the analysis in the previous section, the *existence* of bubbles in the present model economy is *not* a monetary phenomenon. In other words, the conditions for their existence do not depend on how monetary policy is conducted.

When prices are flexible, monetary policy is neutral vis-à-vis the bubble: it cannot have an effect either on its size or on its persistence. On the other hand, and as shown in Section IIIC, in the presence of nominal rigidities monetary policy can have an effect on the size and volatility of asset price bubbles. In particular, it can influence the *anticipated* component of the bubble, $E_{t-1}\{\hat{b}_t\}$. As shown in Appendix 5, the latter evolves according to the simple $AR(1)$ process,

$$E_{t-1}\{\hat{b}_t\} = \chi E_{t-2}\{\hat{b}_{t-1}\} + (\phi_b + 1)\varepsilon_{t-1}, \tag{32}$$

where, again, $\varepsilon_t \equiv R\xi_t + (1 - R)\hat{u}_t$ is the innovation in the aggregate bubble.

Thus we see that the influence of monetary policy on the anticipated component of the bubble works through the choice of the interest rate coefficient ϕ_b. To see how that choice influences the volatility of the aggregate bubble $\hat{q}_t^B$ note that (32), together with equation

$$\hat{q}_t^B = RE_{t-1}\{\hat{b}_t\} + \varepsilon_t \tag{33}$$

implies

$$\mathrm{var}\{\hat{q}_t^B\} = \left(\frac{R^2(\phi_b + 1)^2}{1 - \chi^2} + 1\right)\sigma_\varepsilon^2, \tag{34}$$

where $\sigma_\varepsilon^2 \equiv R^2\,\sigma_\xi^2 + (1 - R)^2\,\sigma_u^2$ is the variance of the aggregate bubble innovation. That relation is illustrated graphically in Figure 2, which displays the standard deviation of the aggregate bubble as a function of ϕ_b.[24]

An analysis of that relation yields several results of interest (all of which are captured in Figure 2). First, equation (34) implies that a "leaning against the wind" policy (which corresponds to $\phi_b > 0$) generates a larger volatility in the bubble than a policy of "benign neglect" ($\phi_b = 0$). Secondly, and conditional on $\phi_b \geq 0$, the stronger is the interest rate response to the bubble, the larger is the volatility of the latter. Finally, note that the central bank can minimize the bubble volatility by setting $\phi_b = -1 < 0$, a policy which fully stabilizes the *anticipated* component of the bubble (i.e., it implies $E_{t-1}\{\hat{b}_t\} = 0$, for all t). In other words, stabilization of bubble fluctuations requires that the interest rate be *lowered* in response to positive innovations in existing or new bubbles, a finding clearly at odds with the conventional wisdom.

[24] The following parameter settings are assumed in constructing Figure 2: $\beta = 1$, $M = 1.2$, $B = 0.1$, and $\sigma_\xi^2 = \sigma_u^2 = 0.01$. None of the qualitative findings discussed in the text hinge on the specific choice of parameter values, as long as (17) is satisfied.

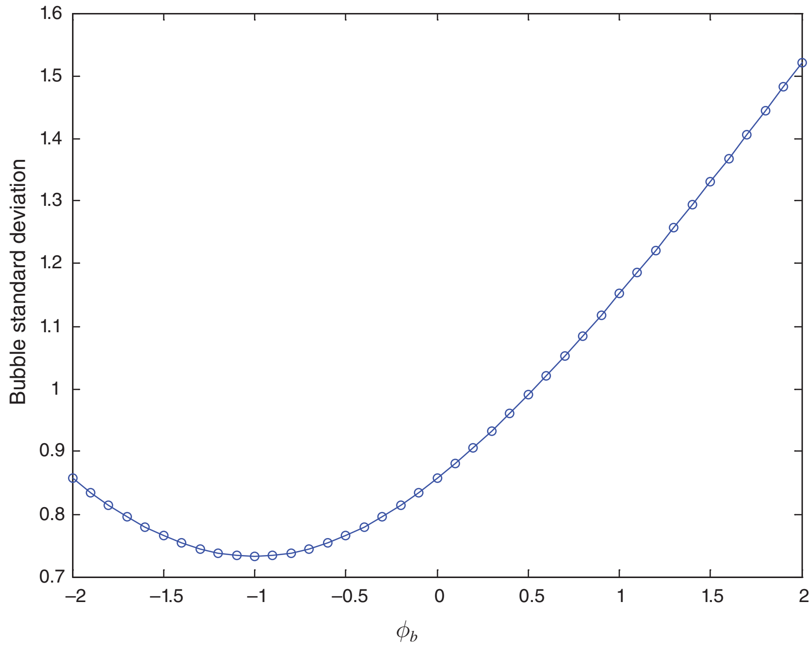

Figure 2. Monetary Policy and Bubble Volatility

As shown above, and as long as $\phi_\pi \neq 0$, equilibrium inflation in the economy with sticky prices is uniquely pinned down and satisfies

$$\hat{\pi}_t = \chi\hat{\pi}_t - (1/\phi_\pi)(\phi_b - \epsilon B(1 + \beta)R)(\phi_b + 1)\varepsilon_{t-1}. \tag{35}$$

The central bank can follow three alternative strategies if it seeks to stabilize inflation. First, it can respond very strongly to inflation itself (by setting ϕ_π arbitrarily large, for any finite ϕ_b). Secondly, it can adjust interest rates in response to fluctuations in the bubble with a strength given by $\phi_b = \epsilon B(1 + \beta)R$ (while setting ϕ_π at a finite value). Doing so exactly offsets the impact of the bubble on (expected) aggregate demand, thus neutralizing its impact on inflation. Note that neither of these policies eliminates fluctuations in the bubble, they just prevent the latter from affecting the aggregate price level. Finally, the central bank may choose to stabilize the *anticipated* component of the bubble (the only one that can affect inflation when prices are set in advance), which can be achieved by setting $\phi_b = -1$, as discussed above. The latter result illustrates how the emergence of an aggregate bubble and the existence of fluctuations in the latter do not necessarily generate a policy trade-off between stabilization of the bubble and stabilization of inflation.[25]

[25] The absence of a trade-off obtains when, as assumed above, bubble shocks are the only source of uncertainty in the economy. Other sources of fluctuations may require interest rate adjustments in order to stabilize inflation, which in turn may induce additional volatility in the size of the bubble.

Note however that in the economy above, with synchronized price-setting and an inelastic labor supply, inflation is *not* a source of welfare losses. Accordingly, and within the logic of the model, there is no reason why the central bank should seek to stabilize inflation. It is also not clear that minimizing the volatility of the aggregate bubble constitutes a desirable objective in itself. In order to clarify those issues, the next section analyzes explicitly the nature of the model's implied optimal policy.

V. Optimal Monetary Policy in the Bubbly Economy

I analyze the optimal response of monetary policy to asset price bubbles in the model economy developed above. I take as a welfare criterion the *unconditional* mean of an individual's lifetime utility. In a neighborhood of the steady state that mean can be approximated up to second order as

$$E\{\log C_{1,t} + \beta \log C_{2,t+1}\} \simeq \log C_1 + \beta \log C_2 - \left(\frac{1}{2}\right)(\mathrm{var}\{\hat{c}_{1,t}\} + \beta \mathrm{var}\{\hat{c}_{2,t}\}).$$

Note that the goods market clearing condition $C_{1,t} + C_{2,t} = 1$ implies that $\mathrm{var}\{\hat{c}_{1,t}\}$ is proportional to $\mathrm{var}\{\hat{c}_{2,t}\}$. Thus, a central bank that seeks to maximize welfare under the criterion set above will minimize the variance of

$$\hat{c}_{2,t} = (1 - \Gamma)\hat{d}_t + \Gamma \hat{b}_t,$$

where, again, $\Gamma \equiv \epsilon B/(\epsilon B + 1) \in [0, 1]$.

That objective poses a dilemma for the central bank. To see this note that, as derived in Appendix 6, dividends are given by[26]

$$\hat{d}_t = (1/\beta R)[(\phi_b - \epsilon B(1 + \beta))R\xi_t + (\phi_b - \epsilon B)(1 - R)\hat{u}_t].$$

Thus, minimizing the volatility of dividends calls for setting $\phi_b = \epsilon B(1 + \beta R^2(\sigma_\xi^2/\sigma_\varepsilon^2)) > 0$. Note that such a policy would require adjusting the interest rate upward in response to positive bubble shocks, in order to stabilize aggregate demand and to prevent upward (downward) pressure on wages (dividends) from emerging. However, as discussed in the previous section, such a policy would amplify the impact of current bubble shocks on the *future* size of the bubble through the effect of interest rates on bubble growth, thus contributing through that channel to the destabilization of cohort-specific consumption. In fact, and as discussed above, minimizing the volatility of cohort-specific consumption *directly* linked to bubble fluctuations calls for setting $\phi_b = -1 < 0$. Note finally that neither the volatility of dividends nor that of the bubble depend on the inflation coefficient ϕ_π.

[26] Accordingly, and given (11) the wage will be given by

$$\begin{aligned} \hat{w}_t &= -(\mathcal{M} - 1)\hat{d}_t \\ &= -(\mathcal{M} - 1)(1/\beta R)[(\phi_b - \epsilon B(1 + \beta))R\xi_t + (\phi_b - \epsilon B)(1 - R)\hat{u}_t]. \end{aligned}$$

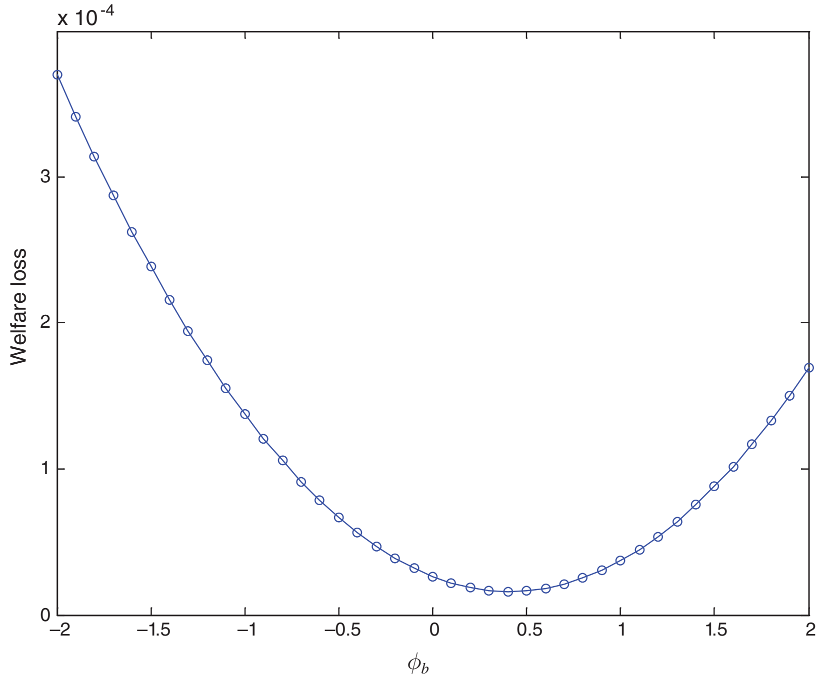

Figure 3. Monetary Policy and Welfare Losses

The welfare-maximizing choice of ϕ_b will naturally seek a compromise between stabilization of dividends and stabilization of the bubble size. Formally, the optimal coefficient minimizes

$$\mathrm{var}\{(1 - \Gamma)\hat{d}_t + \Gamma\hat{b}_t\} \propto \left((\phi_b - \epsilon B)^2 + \frac{(\beta R\epsilon B)^2(\phi_b + 1)^2}{1 - \chi^2}\right)\sigma_\varepsilon^2.$$

Figure 3 displays the expected welfare loss as a function of ϕ_b, under the model's baseline parameter settings. The minimum of that loss function determines the optimal interest rate coefficient. The latter can be written as

$$\phi_b^* = (-1)\Psi + \epsilon B(1 - \Psi), \tag{36}$$

where $\Psi \equiv (\beta R\epsilon B)^2/(1 - \chi^2 + (\beta R\epsilon B)^2) \in [0, 1]$ is an increasing function of B, the steady state size of the bubble (relative to the economy's size, which is normalized to unity).

Thus, the optimal strength of the central bank's response to the bubble is a nonlinear function of the average size of the latter, as well as other exogenous parameters. Figure 4 displays the optimal coefficient ϕ_b^* as a function of B, under the baseline parameter settings. Note that the mapping is non-monotonic: ϕ_b^* is shown to be first increasing, and then decreasing, in the size of the bubble. As the steady state size of the bubble approaches zero, so does the optimal coefficient, i.e., $\lim_{B\to 0} \phi_b^* = 0$, as can be checked using (36). On the other hand, as B approaches its maximum value

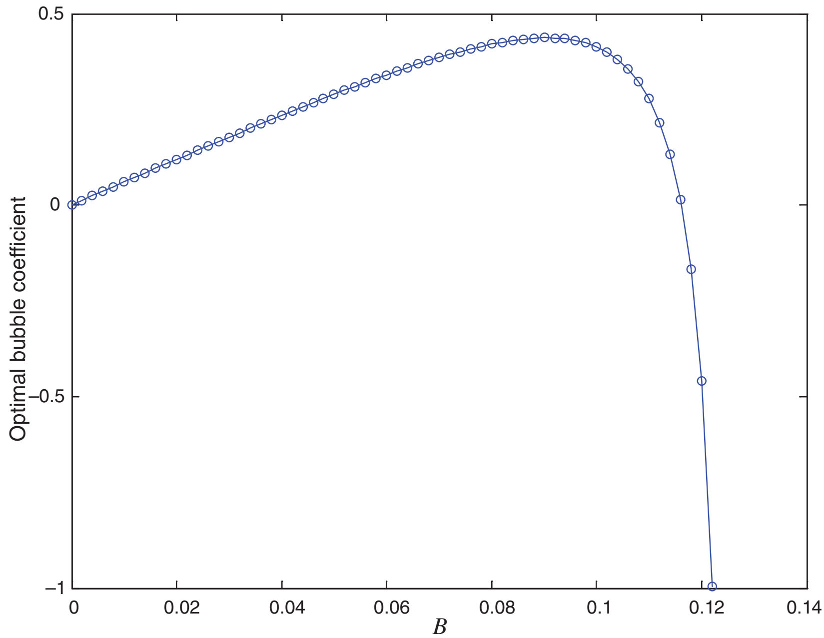

FIGURE 4. OPTIMAL BUBBLE COEFFICIENT

consistent with stationarity (implying $\chi \to 1$), the optimal coefficient converges to (minus) the corresponding interest rate, i.e., $\lim_{B \to \tilde{B}} \phi_b^* = -1 < 0$. Hence, given a sufficiently large average bubble consistent with a stable steady state, it is optimal for the central bank to *lower* interest rates in response to a rise in the size of the bubble.

The latter finding illustrates that the optimal monetary policy strategy in response to asset price bubbles does not necessarily take the form of a "leaning against the wind" policy or one of just "benign neglect."

VI. Discussion

The analysis above calls into question the theoretical underpinnings of "leaning against the wind" monetary policies with respect to asset price developments. According to those proposals central banks should raise interest rates in the face of a developing asset price bubble, in order to tame it or eliminate it altogether. The analysis above has shown that, at least when it comes to a rational asset pricing bubble, such a policy may be counterproductive and lead instead to larger bubble fluctuations and possibly lower welfare as well. In the model economy developed above, it is generally desirable from the viewpoint of bubble stabilization (and, under some assumptions, from a welfare perspective as well) to pursue the *opposite* policy. That finding, which is a consequence of a basic arbitrage constraint that must be satisfied by a rational bubble, seems to have been ignored (or, at least, swept under the rug) by proponents of "leaning against the wind" policies.

To be clear, it is not my intention to suggest that policies that seek to prevent the emergence of bubbles or its excessive growth are necessarily misguided, but only to point out that certain interest rate policies advocated by a number of economists and policy makers may not necessarily have the desired effects in that regard.

There are at least three assumptions in my model which undoubtedly play an important role in accounting for my findings. I discuss them briefly next.

Firstly, and in the context of the OLG model developed above, I have assumed that there is no systematic impact of interest rate surprises on the "indeterminate" component of the bubble. Some readers may find that assumption arbitrary. But it would be equally arbitrary to assume the existence of a systematic relation of a given size or sign. Furthermore, and as illustrated by the partial equilibrium example of Section I, the possible short run negative impact of an interest rate hike on the size of the bubble when the orthogonality assumption is relaxed may be more than offset by the subsequent higher growth. At the end of the day, whether a systematic relation between interest rate surprises and bubble innovations exists is ultimately an empirical issue, but one that will not be settled easily given the inherent unobservability of bubbles. Thus, and if nothing else, one should view the present paper's contribution as pointing to the *fragility* of the foundations of "leaning against the wind" policies advocated on the basis of such a systematic relation.

Secondly, the asset pricing bubbles introduced in the model economy above are of the *rational* type, i.e., they are consistent with rational expectations on the part of all agents in the economy. In actual economies there may be asset price deviations from fundamentals that are different in nature from the rational bubbles considered here and for which "leaning against the wind" interest rate policies may have more desirable properties. Assessing that possibility would require the explicit modelling of the nature of deviations from fundamentals and how those deviations are influenced by interest rate policy. Of course, one should not rule out the possibility that some models of non-rational bubbles may lead to entirely different implications regarding the desirability of "leaning against the wind" policies.

Thirdly, the analysis above has been conducted in a model economy with no explicit financial sector and no financial market imperfections (other than the existence of bubbles). In fact, the assumption of a representative consumer in each cohort implies that the only financial transactions actually carried out are the sale of bubbly assets by the old to the young, but no credit is needed (in equilibrium) to finance such transactions. By contrast, much of the empirical and policy-oriented literature has emphasized the risks associated with the rapid credit expansion that often accompanies (and helps finance) asset price booms.[27] It is not clear, however, that a tighter monetary policy may be the best way to counter the credit-based speculative bubbles that may arise in this context, as opposed to a stricter regulatory and supervisory framework with the necessary tools to dampen the growth of credit allocated to (potentially destabilizing) speculative activities. Further efforts at modelling explicitly the interaction of credit, bubbles and monetary policy would thus seem highly welcomed.[28]

[27] See, e.g., Schularick and Taylor (2012).

[28] Recent research on non-monetary economies with rational bubbles and credit frictions suggests that such interaction is likely to be a complex one, which may depend on a number of modeling choices. Thus, in a model

Gathering empirical evidence on the impact of monetary policy on asset price bubbles should, of course, be high on the research agenda. It is clear that any empirical analysis of that link faces many challenges. Firstly, the difficulty inherent to the identification of an asset's bubble component does not facilitate the task. Secondly, any observed comovement between asset prices and policy rates can hardly be given a simple causal interpretation since both variables are endogenous and likely to be influenced by numerous factors (including each other). In ongoing research (Galí and Gambetti 2013), we seek to assess the impact of monetary policy shocks on asset price bubbles by estimating time-varying dynamic responses of selected asset price indexes to an exogenous interest rate shock, identified as in Christiano, Eichenbaum, and Evans (2005). In particular, we seek to uncover changes over time in the patterns of response of asset prices to such shocks, which may correspond to changes in the relative size of the bubble component of several asset categories. Further empirical work on this issue, including case studies focusing on specific bubbly episodes, would seem to be highly welcome in order to complement any theoretical efforts.

VII. Concluding Remarks

The present paper should be viewed as part of an effort to enhance our understanding of the relation between monetary policy and bubbles and, more specifically, of the possible underpinnings of "leaning against the wind" policies. Both the simple partial equilibrium example, described in Section I, and the general equilibrium framework analyzed in the remainder of the paper make clear that the predictions of economic theory regarding that relation do not always support the conventional wisdom.

The bulk of my theoretical analysis has made use of a highly stylized overlapping generations model with monopolistic competition and price setting in advance. The overlapping generations structure allows for the existence of asset price bubbles in equilibrium, as in the models of Samuelson (1958) and Tirole (1982). The introduction of nominal rigidities implies that monetary policy is not neutral. In particular, by influencing the path of the real interest rate, the central bank can affect real asset prices (including those of bubbly assets) and, as a result, the distribution of consumption across cohorts and welfare.

Two main results have emerged from the analysis of that model. First, contrary to conventional wisdom, a "leaning against the wind" interest rate policy in the face of bubble fluctuations may raise the volatility of the latter. Secondly, the optimal policy must strike a balance between stabilization of current aggregate demand—which

with capital accumulation and borrowing constraints à la Martín and Ventura (2012), an interest rate increase engineered by the central bank will tighten or relax the borrowing constraint (thus dampening or enhancing investment and growth) depending on its overall impact on the total price (fundament plus bubble) of the assets which are used as collateral. On the other hand, the nature of the borrowing constraints assumed by Miao and Wang (2012), among others, implies that the simple arbitrage relation linking the growth rate of the bubble to the interest rate is broken, since the bubble generates a "dividend" in the form of the extra profits resulting from the implied relaxation of the borrowing constraint. Accordingly, the required expected increase in the bubble resulting from a higher interest rate will be smaller. In addition, the net effect of an interest rate change on aggregate demand is ambiguous since the "conventional" effect may be partly offset or enhanced by the induced effect on borrowing constraints, whose sign may depend on a number of factors.

calls for a positive interest rate response to the bubble—and stabilization of the bubble itself (and hence of future aggregate demand)—which would warrant a negative interest rate response to the bubble. If the average size of the bubble is sufficiently large the latter motive will be dominant, making it optimal for the central bank to lower interest rates in the face of a growing bubble.

Needless to say, the conclusions should not be taken at face value when it comes to designing actual policies. This is so because the model may not provide an accurate representation of the challenges facing actual policy makers. In particular, it may very well be the case that actual bubbles are not of the rational type and, hence, respond to monetary policy changes in ways not captured by the theory above. In addition, the model above abstracts from many aspects of actual economies that may be highly relevant when designing monetary policy in bubbly economies, including the presence of frictions and imperfect information in financial markets. Those caveats notwithstanding, the analysis above may be useful by pointing out a potentially important missing link in the case for "leaning against the wind" policies.

Appendix

Appendix 1

Assuming a stationary environment, the log-linearized difference equation describing the evolution of the fundamental component is

$$\hat{q}_t^F = (1/R)E_t\{\hat{q}_{t+1}^F\} + (1 - 1/R)E_t\{\hat{d}_{t+1}\} - \hat{r}_t,$$

which can be solved forward to yield

$$\hat{q}_t^F = \sum_{k=0}^{\infty}(1/R)^k\big((1 - 1/R)E_t\{\hat{d}_{t+1+k}\} - E_t\{\hat{r}_{t+k}\}\big).$$

Under the $AR(1)$ assumption for the interest rate, $E_t\{\hat{r}_{t+k}\} = \rho_r^k\,\hat{r}_t$ and hence

$$\hat{q}_t^F = -\frac{R}{R - \rho_r}\hat{r}_t + (1 - 1/R)\sum_{k=0}^{\infty}(1/R)^k E_t\{\hat{d}_{t+1+k}\},$$

implying $\partial\hat{q}_t^F/\partial\varepsilon_t^r = -R/(R - \rho_r)$.

Appendix 2

The following properties of the H mapping are stated for future reference:

(P1) $H(B, U) \geq 0$ is twice continuously differentiable for $0 \leq B < \frac{\beta/\mathcal{M} - U}{(1 + \beta)} \equiv \overline{B}(U)$. Note that $H(B,\ U) < 0$ for $B > \overline{B}(U)$.

(P2) $\partial H(B,U)/\partial B_t = \frac{\beta(1-1/\mathcal{M})(1/\mathcal{M}+U)}{[\beta/\mathcal{M}-U-(1+\beta)B]^2} > 0$ and $\partial^2 H(B_t,U)/\partial B_t^2 = \frac{2\beta(1+\beta)(1-1/\mathcal{M})(1/\mathcal{M}+U)}{[\beta/\mathcal{M}-U-(1+\beta)B]^3} > 0$ for $0 \leq B < \overline{B}(U)$ and $\lim_{B\to\overline{B}(U)} H(B,U) = +\infty$.

(P3) $\partial H(B,U)/\partial U_t = \frac{2\beta(1-1/\mathcal{M})(1/\mathcal{M}-B)}{[\beta/\mathcal{M}-U-(1+\beta)B]^2} > 0$ and $\partial^2 H(B,U)/\partial U^2 = \frac{\beta(1-1/M)(1/M-B)}{[\beta/M-U-(1+\beta)B]^3} > 0$ for $0 \leq B < \overline{B}(U)$.

(P4) $\partial^2 H(B,U)/\partial B\partial U > 0$ for $0 \leq B < \overline{B}(U)$.

Consider first the case of $U=0$. A bubbly equilibrium path must then satisfy

$$B_{t+1} = \frac{(1-1/\mathcal{M})B_t}{\beta/\mathcal{M}-(1+\beta)B_t} \equiv H(B_t, 0).$$

Note that $H(0,0)=0$, implying the existence of a bubbleless deterministic steady state in that case, i.e., $B=0$. Given (P2), a necessary and sufficient condition for the existence of a bubbly steady state $B^U > 0$ such that $H(B^U, 0) = B^U$ is $\partial H(0,0)/\partial B_t = \frac{\mathcal{M}-1}{\beta} < 1$, or, equivalently,

$$\mathcal{M} < 1 + \beta. \tag{A1}$$

Note that in that case $H(B_t,0) > B_t$ and $\partial[H(B_t,0)-B_t]/\partial B_t > 0$ for any $B_t > B^U$. Thus, the solution to $B_{t+1} = H(B_t,0)$ given an initial condition $B_0 > B^U$ violates the constraint $B_t < 1/\mathcal{M}$ in finite time and hence is not consistent with equilibrium. On the other hand, $H(B_t,0) < B_t$ for any $B_t < B^U$, implying that the solution to $B_{t+1} = H(B_t,0)$ given an initial condition $B_0 < B^U$ converges asymptotically to the bubbleless steady state $B=0$. Thus, B^U is an unstable steady state.

Sufficiency: Suppose that (A1) holds. Then it follows from (P3) and the continuity of $H(\cdot,\cdot)$ that there is a non-degenerate set $(0,\overline{U})$ of values for the new bubble U, with $\overline{U} \equiv \beta + (1+\beta)(1-1/\mathcal{M}) + 2\sqrt{\beta(1+\beta)(1-1/\mathcal{M})}$ such that for any $U \in [0,\overline{U})$ the mapping $B_{t+1} = H(B_t,U)$ has two fixed points, denoted by $B^S(U)$ and $B^U(U)$, where $B^S(U) < B^U(U)$ and such that

(i) $H(B_t,U) > B_t$ for $B_t \in [0, B^S(U)) \cup (B^U(U), \overline{B}(U))$,

(ii) $H(B_t,U) < B_t$ for $B_t \in (B^S(U), B^U(U))$,

(iii) $B_t = H(B_t,U)$ for $B_t \in \{B^S(U), B^U(U)\}$.

Thus, given an initial condition $B_0 \in [0, B^U(U))$, the solution to $B_{t+1} = H(B_t,U)$ corresponds to a bubbly equilibrium path, which converges asymptotically to $B^S(U)$. The latter is, thus, a (locally) stable steady state. On the other hand, any solution to $B_{t+1} = H(B_t, U)$ given an initial condition $B_0 > B^U(U)$ violates the constraint $B_t < 1/\mathcal{M}$ in finite time and hence cannot be an equilibrium.

Note also that $B^S(U)$ is a continuous function defined on $[0, \overline{U}]$, such that $B^S(0) = 0$ and $\partial B^S(U)/\partial U > 0$.

Necessity: Suppose $\mathcal{M} \geq 1 + \beta$. Then (P2) implies $\partial H(0,0)/\partial B_t > 1$. Then it follows from (P4) that $\partial H(0, U)/\partial B_t > 1$ for $U > 0$ as well, implying $H(B_t, U) > B_t$ and $\partial[H(B_t, U) - B_t]/\partial B_t > 0$ for any $B_t > 0$. Thus, the solution to $B_{t+1} = H(B_t, 0)$ given an initial condition $B_0 > 0$ violates the constraint $B_t < 1/\mathcal{M}$ in finite time.

Appendix 3

Next I describe a variation on the benchmark model in the text which explicitly introduces money as an additional asset, paying no interest but yielding utility. The consumer's utility function is now given by

$$\log C_{1,t} + \log(M_t/P_t) + \beta E_t\{\log C_{2,t+1}\},$$

where M_t denotes the holdings of money at the end of the period. The modified budget constraints of the young and old are respectively given by

$$\int_0^1 \frac{P_t(i)C_{1,t}(i)}{P_t}\,di + \frac{M_t}{P_t} + \frac{Z_t}{P_t} + Q_t^B = W_t + U_t + \frac{T_t}{P_t}$$

$$\int_0^1 \frac{P_{t+1}(i)C_{2,t+1}(i)}{P_{t+1}}\,di = \frac{M_t}{P_{t+1}} + \frac{Z_t(1 + i_t)}{P_{t+1}} + D_{t+1} + B_{t+1},$$

where T_t represents (lump-sum) monetary injections by the central bank (in the form of transfers to the young).

The optimality conditions of the consumer's problem found in the main text also apply here with no modifications. To those optimality conditions we now have to add an implied money demand equation of the form

$$\text{(A2)} \qquad \frac{M_t}{P_t} = C_{1,t}\left(1 + \frac{1}{i_t}\right),$$

as well as the zero lower bound constraint $i_t \geq 0$ which must hold in equilibrium (otherwise, bonds would be return-dominated by money).

The analysis found in the main text (including the steady state and the log-linearized equilibrium dynamics) goes through without any modifications, once we assume a positive steady state nominal rate, either by having positive trend growth or positive target inflation (see discussion in main text). Given the equilibrium values of $C_{1,t}$, P_t, and M_t, money demand equation (A2) determines the quantity of money M_t, which is supplied passively by the central bank by means of transfers $T_t = \Delta M_t$.

Appendix 4

Note that the $H(B_t, U)$ mapping must satisfy

$$H(B_t, U) = R(B_t, H(B_t, U))(B_t + U),$$

where $R(B_t, B_{t+1}) \equiv \left(\frac{1}{\beta}\right)\left(\frac{1 - 1/\mathcal{M} + B_{t+1}}{1/\mathcal{M} - B_t}\right)$. Differentiating with respect to B_t, noting that $R_1(B, B) = \frac{R}{1/\mathcal{M} - B}$ and $R_2(B, B) = \frac{1}{\beta(1/\mathcal{M} - B)}$ (where R_i denotes the partial derivative with respect to the ith argument), and evaluating the resulting derivative at the steady state, we obtain (after some algebraic manipulation),

$$\frac{\partial H(B, U)}{\partial B} = \mathcal{R}(B)(1 + \epsilon(1 + \beta)B).$$

Appendix 5

Combining (18) through (21) yields the following goods market clearing condition:

$$0 = \epsilon B(1 + \beta)R\hat{b}_t + \epsilon B(1 - R)\hat{u}_t + \beta R\hat{d}_t - \hat{r}_t.$$

Taking expectations, and using (22) we obtain

$$E_{t-1}\{\hat{r}_t\} = \epsilon B(1 + \beta)RE_{t-1}\{\hat{b}_t\}. \tag{A3}$$

Taking expectations on both sides of the interest rate rule (23):

$$E_{t-1}\{\hat{r}_t\} = \phi_\pi \hat{\pi}_t + \phi_b RE_{t-1}\{\hat{b}_t\}. \tag{A4}$$

Combining (A3) and (A4) yields

$$\hat{\pi}_t = -(R/\phi_\pi)(\phi_b - \epsilon B(1 + \beta))\, E_{t-1}\{\hat{b}_t\}.$$

Letting $\varepsilon_t \equiv R\xi_t + (1 - R)\hat{u}_t$, note that

$$\begin{aligned} \hat{r}_t &= E_{t-1}\{\hat{r}_t\} + (\hat{r}_t - E_{t-1}\{\hat{r}_t\}) \\ &= \epsilon B(1 + \beta)RE_{t-1}\{\hat{b}_t\} + \phi_b\, \varepsilon_t \\ &= \epsilon B(1 + \beta)R(R\hat{b}_{t-1} + (1 - R)\hat{u}_{t-1} + \hat{r}_{t-1}) + \phi_b\, \varepsilon_t. \end{aligned} \tag{A5}$$

It follows that

$$(1 - \epsilon B(1 + \beta)RL)\hat{r}_t = \epsilon B(1 + \beta)R(R\hat{b}_{t-1} + (1 - R)\hat{u}_{t-1}) + \phi_b\, \varepsilon_t.$$

Combining the previous result with the bubble difference equation $(1 - RL)\hat{b}_t = (1 - R)\hat{u}_{t-1} + \hat{r}_{t-1} + \xi_t$ yields

$$\hat{b}_t = \chi \hat{b}_{t-1} + (\phi_b + 1)(1 - R)\hat{u}_{t-1} + \xi_t + (\phi_b - \epsilon B(1 + \beta))R\xi_{t-1},$$

where, as above, $\chi \equiv R(1 + \epsilon(1 + \beta)B)$ is assumed to be between zero and one.

Note that the predictable component of the bubble follows the process

$$E_{t-1}\{\hat{b}_t\} = \chi(E_{t-2}\{\hat{b}_{t-1}\} + \xi_{t-1}) + (\phi_b + 1)(1 - R)\hat{u}_{t-1} + (\phi_b - \epsilon B(1 + \beta))R\xi_{t-1}$$
$$= \chi E_{t-2}\{\hat{b}_{t-1}\} + (\phi_b + 1)\varepsilon_{t-1}.$$

Accordingly,

$$\text{var}\{E_{t-1}\{\hat{b}_t\}\} = \frac{(\phi_b + 1)^2}{1 - \chi^2}\,\sigma_\varepsilon^2,$$

where $\sigma_\varepsilon^2 \equiv (1 - R)^2\,\sigma_u^2 + R^2\,\sigma_\xi^2$.

Finally, and using the fact that $\hat{q}_t^B = R(E_{t-1}\{\hat{b}_t\} + \xi_t) + (1 - R)\hat{u}_t = RE_{t-1}\{\hat{b}_t\} + \varepsilon_t$, we have

$$\text{var}\{\hat{q}_t^B\} = \left(\frac{R^2(\phi_b + 1)^2}{1 - \chi^2} + 1\right)\sigma_\varepsilon^2.$$

Note also that we can now write the equilibrium process for inflation as

$$\hat{\pi}_t = -(R/\phi_\pi)\big(\phi_b - \epsilon B(1 + \beta)\big)\,E_{t-1}\{\hat{b}_t\}$$
$$= \chi\,\hat{\pi}_t - (R/\phi_\pi)\big(\phi_b - \epsilon B(1 + \beta)\big)(\phi_b + 1)\varepsilon_{t-1}.$$

Appendix 6

As noted in Appendix 5, goods market clearing implies

$$0 = \epsilon B(1 + \beta)R\hat{b}_t + \epsilon B(1 - R)\hat{u}_t + \beta R\hat{d}_t - \hat{r}_t.$$

Using (A5) to substitute out the real rate $\hat{r}_t$, one can rewrite the above condition as

$$\beta R\hat{d}_t = \phi_b(R\xi_t + (1 - R)\hat{u}_t) + \epsilon B(1 + \beta)RE_{t-1}\{\hat{b}_t\}$$
$$- (\epsilon B(1 - R)\hat{u}_t + \epsilon B(1 + \beta)R\hat{b}_t)$$
$$= (\phi_b - \epsilon B(1 + \beta))R\xi_t + (\phi_b - \epsilon B)(1 - R)\hat{u}_t.$$

Letting $\Gamma \equiv \frac{\epsilon B}{\epsilon B + 1}$ we have

$$\begin{aligned}
\beta R \hat{c}_{2,t} &= \beta R((1-\Gamma)\hat{d}_t + \Gamma \hat{b}_t) \\
&= (1-\Gamma)\beta R(\hat{d}_t + \epsilon B\, \hat{b}_t) \\
&= (1-\Gamma)((\phi_b - \epsilon B(1+\beta))R\xi_t + (\phi_b - \epsilon B)(1-R)\hat{u}_t + \beta R \epsilon B \hat{b}_t) \\
&= (1-\Gamma)((\phi_b - \epsilon B)\varepsilon_t + \beta R \epsilon B E_{t-1}\{\hat{b}_t\},
\end{aligned}$$

implying

$$\operatorname{var}\{\hat{c}_{2,t}\} \propto \left((\phi_b - \epsilon B)^2 + \frac{(\beta R \epsilon B)^2(\phi_b + 1)^2}{1-\chi^2}\right)\sigma_\epsilon^2.$$

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