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The Macroeconomics of Irreversibility

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Abstract: We study aggregate capital dynamics in an investment model with idiosyncratic productivity shocks, fixed capital adjustment costs, and irreversibility driven by a wedge between capital purchase and resale prices. We derive sufficient statistics capturing the role of investment frictions on aggregate capital fluctuations, measure these statistics with investment microdata, and exploit them to discipline the capital price wedge. Irreversibility doubles the persistence of capital fluctuations and is crucial for reconciling micro-level investment behavior with macroeconomic propagation.

JEL classification: D30, D80, E20, E30

Key words: investment frictions, capital price wedge, irreversibility, lumpiness, fixed adjustment costs, capital misallocation, Tobin's q, transitional dynamics, inaction, propagation

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1 Introduction

Capital irreversibility, stemming from a wedge between buying and selling prices of capital, is a pervasive friction in firms' investment decisions. This wedge reflects factors like asset specificity (Ramey and Shapiro, 2001; Lanteri, 2018; Kermani and Ma, 2023), adverse selection and asymmetric information (Akerlof, 1970; Kurlat, 2013; Bigio, 2015; Li and Whited, 2015), intermediary fees (Nosal and Rocheteau, 2011), and obsolescence costs (Caunedo and Keller, 2020).

Exposed to a price wedge, firms adopt cautious investment strategies. During periods of high productivity, firms do not fully scale up their capital stock, fearing future adverse shocks that would force them to sell at a discount and instead make a sequence of gradual purchases. Conversely, during low productivity, firms avoid large sell-offs to limit capital losses and thus sell capital sequentially. The step-by-step nature of the investment process introduces path dependence: Positive investments beget future positive investments; negative investments beget future negative investments. At the aggregate level, investment becomes less responsive to productivity shocks, and business cycle fluctuations persist longer (Pindyck, 1991; Bertola and Caballero, 1994; Abel and Eberly, 1996).

We propose a new perspective for analyzing irreversibility's role in shaping macroeconomic dynamics. Our innovation is leveraging the Cumulative Impulse Response (CIR)—a measure of how firms' capital-productivity ratios respond to aggregate shocks—as both a diagnostic tool and a calibration target. Unlike traditional approaches, which focus on matching steady-state moments of investment rates (Cooper and Haltiwanger, 2006) and treat the CIR as an outcome, we use it as a lens to study irreversibility and as an input for disciplining investment frictions. Applying this framework to Chilean manufacturing data, we estimate a price wedge of 12%, which is necessary to simultaneously match both the CIR and the distribution of investment rates. The calibrated CIR is approximately 2, meaning a 1% aggregate productivity shock generates a 2% cumulative deviation in average capital-productivity ratios. Without irreversibility, the CIR collapses to 1, underscoring the wedge's role in amplifying the persistence of aggregate fluctuations.

To establish this framework, we begin with a parsimonious investment model with idiosyncratic productivity shocks, fixed capital adjustment costs, and a capital price wedge. The optimal investment policy consists of an inaction region and distinct reset points—levels to which firms reset their capital-productivity ratio after adjustments. This ratio would remain constant in a frictionless world, as capital perfectly tracks productivity. Fixed adjustment costs allow the ratio to drift during inaction, but firms reset to the same optimal ratio upon adjustment, erasing the history of shocks. In contrast, a price wedge creates two reset ratios: one for upsizing capital at the buying price and another for downsizing at the selling price. This dual-reset structure introduces persistent heterogeneity, as firms' timing and direction of their future investments differ based on their previous adjustment. We encode this heterogeneity in a tractable way by conditioning behavior on the previous reset point and characterizing a Markov chain across reset points.

Using this theoretical foundation, we define the CIR as the cumulative deviation of average capital-productivity ratios following an aggregate productivity shock. Two challenges arise in its characterization. The first challenge revolves around path dependence, as tracking firms until their first adjustment after an aggregate shock is insufficient. We resolve this by discovering a recursive formulation that splits the CIR into deviations before and after the first adjustment. With restored tractability, we characterize the CIR through three sufficient statistics: (*i*) the dispersion of capital-productivity ratios, (*ii*) the covariance of capital-productivity ratios and their age (i.e., the time elapsed since the last adjustment), and (*iii*) the covariance of the changes in capital-productivity ratios and the expected cumulative deviations when completing the first inaction spell.

The power of sufficient statistics lies in that responsiveness to idiosyncratic shocks, encoded in steady-state moments, informs about responsiveness to aggregate shocks, encoded in the CIR. The first two sufficient statistics describe steady-state behavior during periods of inaction and up to the first adjustment after an aggregate shock, extending insights from Baley and Blanco (2021). The dispersion of capital-productivity ratios reflects how far firms allow their capital to drift from its ideal level; the covariance of these ratios with their age captures how misalignment worsens the longer firms delay action. Irreversibility increases both statistics as firms tolerate larger misalignments and exhibit more significant delays when selling capital. The third statistic, unique to irreversibility, reflects whether firms ultimately choose to buy or sell and how that terminal behavior changes in response to aggregate shocks. For instance, after a negative shock to aggregate productivity, the mass of downsizing firms increases, and due to their sequential selling strategy, the shock's aggregate effects are significantly prolonged.

A second challenge is that capital-productivity distributions are unobservable, making it difficult to compute the CIR directly. To address this, we derive mappings from observable firm actions—such as the size and frequency of adjustments, conditioned on the direction of the previous adjustment—to the reset points and the CIR's sufficient statistics. We then extend the model to a generalized hazard framework, inspired by Caballero and Engel (1999) and Lippi and Oskolkov (2023), incorporating stochastic and asymmetric fixed costs for purchases and sales. This extension enhances the model's ability to replicate the observed investment rate distribution while preserving consistency with the CIR. Crucially, the generalized hazard with irreversibility captures all sufficient statistics, whereas without irreversibility, it fails to generate the second and third.

Our contributions go beyond investment. The CIR, its sufficient statistics, and its measurement in microdata provide a versatile framework for studying lumpy adjustments and path dependence in contexts like inventory management, durable goods consumption, and labor markets with sticky wages. We build on methodologies by Alvarez, Le Bihan and Lippi (2016) and Baley and Blanco (2021), incorporating path dependence (i.e., reinjection). By linking micro-level frictions to macroeconomic fluctuations, our framework provides a foundation for analyzing and quantifying irreversibility's impact on aggregate dynamics.

2 A Parsimonious Investment Model

We study a parsimonious investment model with idiosyncratic productivity shocks, a fixed capital adjustment cost, a wedge between capital purchase and resale prices, and a constant interest rate. We use this model to analyze how irreversibility shapes firms' optimal investment, derive sufficient statistics for aggregate capital dynamics (Section 3), and construct mappings from microdata to parameters and macro outcomes (Section 4). In the quantitative application (Section 5), we consider an extension with a general adjustment cost structure that matches the entire distribution of investment rates in the data that the baseline model misses.

2.1 Firm Investment Problem

Time is continuous, extends forever, and is denoted by s. The future is discounted at a rate of $r > 0$. For any stochastic process x_s, we use the notation $x_{s^-} \equiv \lim_{z \uparrow s} x_z$ to denote the limit from the left. We first present the problem of an individual firm and then consider a continuum of ex-ante identical firms to characterize the aggregate behavior of the economy.

Technology and shocks The firm produces output y_s using capital k_s according to a production function with decreasing returns to scale

$$(1) \qquad y_s \;=\; u_s^{1-\alpha} k_s^{\alpha}, \quad \alpha < 1.$$

Idiosyncratic productivity u_s follows a geometric Brownian motion with drift $\mu > 0$ and volatility $\sigma > 0$,

$$(2) \qquad \log u_s \;=\; \log u_0 + \mu s + \sigma W_s, \qquad W_s \sim Wiener.$$

The capital stock, if uncontrolled, depreciates at a constant rate $\xi > 0$.

Fixed adjustment cost The firm controls its capital stock by buying and selling investment goods. For every active investment, $i_s \equiv \Delta k_s = k_s - k_{s^-} \neq 0$, the firm must pay a fixed cost θ_s proportional to its productivity and measured in output units:

$$(3) \qquad \theta_s \;=\; \theta u_s,$$

where $\theta > 0$ is a deterministic fixed cost equal for positive and negative investments. The fixed cost rationalizes establishment-level data on infrequent and sizeable investment spikes (Doms and Dunne, 1998) and reflects disruptions from installing or uninstalling capital, learning, time-to-build, and other factors independent of the investment size (Cooper and Haltiwanger, 2006).

Price wedge Capital is bought at price p and sold at a discount $p(1-\omega)$. We call ω the price wedge or $1-\omega$ the recovery rate. To simplify notation, we define the pricing function

$$p(\Delta k_s) \equiv p\mathbb{1}_{\{\Delta k_s>0\}} + p(1-\omega)\mathbb{1}_{\{\Delta k_s<0\}}. \tag{4}$$

To the extent that ω is a linear asymmetric cost, the price wedge allows for alternative interpretations, such as installation, transaction, or other fees that scale with the investment size and differ for capital purchases and sales. Moreover, setting $\omega = 1$ eliminates the possibility of disinvesting (Sargent, 1980; Bertola and Caballero, 1994).

Investment problem Let $V(k,u)$ denote the value of a firm with capital stock k and productivity u. Given initial conditions (k_0, u_0), the firm chooses a sequence of adjustment dates $\{T_h\}_{h=1}^{\infty}$ and investments $\{i_{T_h}\}_{h=1}^{\infty}$, where h counts the number of adjustments, to maximize its expected discounted stream of profits. The sequential problem is

$$V(k_0,u_0) = \max_{\{T_h, i_{T_h}\}_{h=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} e^{-rs} y_s \,\mathrm{d}s - \sum_{h=1}^{\infty} e^{-rT_h}\left(\theta_{T_h} + p\left(i_{T_h}\right) i_{T_h}\right)\right], \tag{5}$$

subject to the production technology (1), the idiosyncratic productivity shocks (2), the fixed cost (3), the investment price function (4), and the law of motion for the capital stock

$$\log k_s = \log k_0 - \xi s + \sum_{h:T_h \le s} \log\left(1 + i_{T_h}/k_{T_h^-}\right), \tag{6}$$

which describes a period's capital as a function of its initial value k_0, the physical depreciation rate ξ, and the sum of all adjustments made at prior adjustment dates.

2.2 Capital-Productivity Ratio

To characterize the investment decision, we reduce the state space and recast the firm's problem using a new state variable, the (log) capital-productivity ratio:

$$\hat{k}_s \equiv \log\left(k_s/u_s\right). \tag{7}$$

Without investment frictions, $\hat{k}_s$ is a constant because firms are always at their optimal scale, which is proportional to productivity. Instead, investment frictions lead to prolonged misalignment between firms' capital and productivity levels. Between any two consecutive adjustment dates $[T_{h-1}, T_h]$, the capital-productivity ratio $\hat{k}_s$ follows a stochastic process

$$\mathrm{d}\hat{k}_s = -\nu\,\mathrm{d}s + \sigma\,\mathrm{d}W_s. \tag{8}$$

The drift $\nu \equiv \xi + \mu$ includes the depreciation and productivity growth rates, and the volatility σ is inherited from productivity (the Wiener process is symmetric). At any adjustment date T_h, the capital-productivity ratio changes by the amount

$$\Delta \hat{k}_{T_h} \;=\; \log\left(1 + i_{T_h}/k_{T_h^-}\right), \tag{9}$$

where we use the continuity of the productivity process ($u_{T_h} = u_{T_h^-}$).

2.3 Tobin's q and Optimal Investment Policy

We characterize the optimal investment policy through Tobin's marginal q—the shadow value of installed capital. By definition, q is the marginal valuation of an extra unit of installed capital (the derivative of (5) to k_0) relative to the replacement cost (its purchase price p):

$$q(\hat{k}) \;\equiv\; \frac{1}{p}\frac{\partial V(k,u)}{\partial k}. \tag{10}$$

We pose that q is a function of the capital-productivity ratio $\hat{k}$, defined in (7), not of capital and productivity separately. The reason is that the derivative of the value function is a present discounted value of marginal products—as shown below—and the marginal product of capital can be written as $\alpha e^{\log(k/u)^{\alpha-1}} = \alpha e^{\hat{k}(\alpha-1)}$. The problem admits this reformulation because the production function is homothetic, adjustment costs are proportional to productivity, and idiosyncratic shocks follow a Brownian motion.

Four numbers characterize the optimal investment policy: $\{\hat{k}^- \leq \hat{k}^{*-} \leq \hat{k}^{*+} \leq k^+\}$. The smallest and largest numbers determine an inaction region $\mathcal{R} \equiv (\hat{k}^-, \hat{k}^+)$, which dictates a firm to leave its capital uncontrolled if $\hat{k}$ lies within this region. The two intermediate numbers $\hat{k}^{*-} < \hat{k}^{*+}$ are the reset points to which a firm sets $\hat{k}$ after hitting the corresponding border of inaction. Proposition 1 characterizes q and the optimal policy, defining the user cost of capital $\mathcal{U} \equiv r + \xi$.[1]

Proposition 1. ***(Optimal policy)*** *Marginal $q(\hat{k})$ and the optimal policy $\{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ is characterized by the following sufficient optimality conditions:*

(i) Inside the inaction region $\mathcal{R}$, $q(\hat{k})$ solves the Hamilton-Jacobini-Bellman (HJB) equation:

$$\mathcal{U}q(\hat{k}) = \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} \;-\; \nu q'(\hat{k}) \;+\; \frac{\sigma^2}{2}q''(\hat{k}), \qquad \forall\; \hat{k} \in (\hat{k}^-, \hat{k}^+). \tag{11}$$

[1]Proofs appear in Appendix A, B, and C. All theoretical results are proven for the generalized hazard model introduced in Section 5.

(ii) In the outer inaction regions, $q(\hat{k})$ satisfies the value-matching conditions:

$$\text{(12)} \qquad \frac{\theta}{p} = \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}} \Big(q(\hat{k}) - 1\Big)\, \mathrm{d}\hat{k}, \qquad \forall\ \hat{k} \in [\hat{k}^-, \hat{k}^{*-}],$$

$$\text{(13)} \qquad \frac{\theta}{p} = \int_{\hat{k}^{*+}}^{\hat{k}^+} e^{\hat{k}} \Big((1-\omega) - q(\hat{k})\Big)\, \mathrm{d}\hat{k} \qquad \forall\ \hat{k} \in [\hat{k}^{*+}, \hat{k}^+].$$

(iii) At the borders of the inaction region and reset points, $q(\hat{k})$ satisfies the optimality conditions:

$$\text{(14)} \qquad q(\hat{k}) = 1, \qquad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\},$$

$$\text{(15)} \qquad q(\hat{k}) = 1-\omega, \qquad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}.$$

From these conditions, q's stopping-time formulation is given by

$$\text{(16)} \qquad q(\hat{k}) = \mathbb{E}\left[\int_0^\tau \frac{\alpha e^{-\mathcal{U}s+(\alpha-1)\hat{k}_s}}{p}\, \mathrm{d}s \ + \ e^{-\mathcal{U}\tau} q(\hat{k}_\tau)\right].$$

The optimal policy satisfies (i) the HJB equation in (11) that describes q's evolution during inaction as the flow marginal product of capital expressed in capital units; (ii) two value-matching conditions in (12) and (13) that equalize the value of adjusting to the value of not adjusting; and (iii) four optimality conditions in (14) and (15) at the borders of inaction region and reset points.[2] Later, we use the stopping-time formulation in (16) to estimate the reset points from the data.

Figure I illustrates $q(\hat{k})$ (solid black line) and the optimal investment policy (four values of $\hat{k}$ in the x-axis). Let us first consider the environment without frictions ($\theta = \omega = 0$). Since firms always have their optimal capital-productivity ratio, then $q(\hat{k}) = 1$ and $q'(\hat{k}) = q''(\hat{k}) = 0$ for all $\hat{k}$. Substituting these values into the HJB in (11), we get that capital's marginal product equals the user cost, and the frictionless optimal capital $\hat{k}^*$ is given by

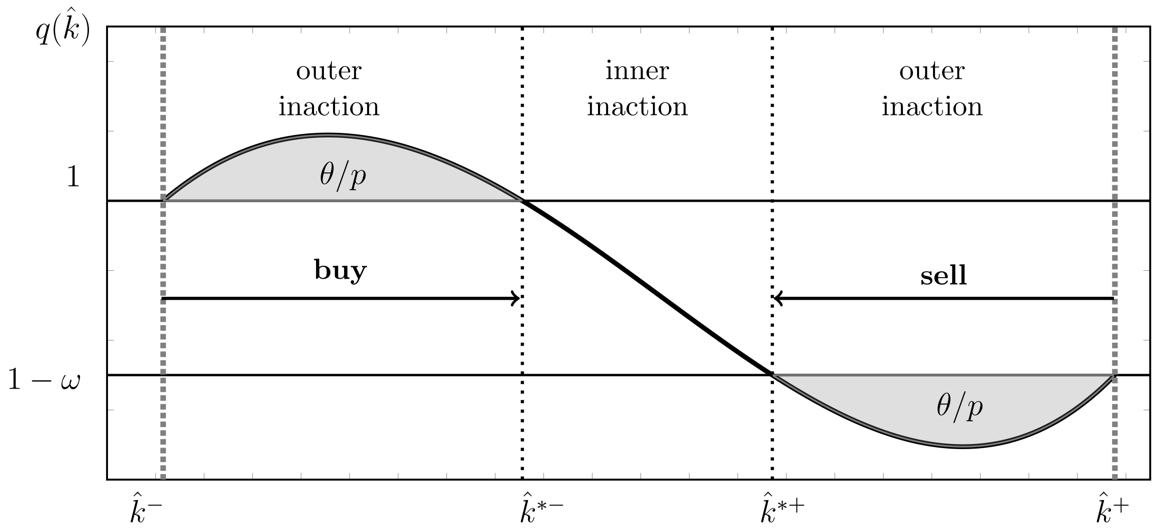
$$\text{(17)} \qquad \frac{\alpha e^{(\alpha-1)\hat{k}^*}}{p} = \mathcal{U} \cdot 1 \iff \hat{k}^* = \frac{1}{1-\alpha} \log\left(\frac{\alpha}{p\mathcal{U}}\right),$$

as in the neoclassical investment theory.

Next, consider an environment with a price wedge ($\omega > 0$) but zero fixed costs ($\theta = 0$). The price wedge gives rise to an "inner" inaction region $[\hat{k}^{*-}, \hat{k}^{*+}]$ where the borders of inaction and reset points coincide, $\hat{k}^{*+} = \hat{k}^+$ and $\hat{k}^{*-} = \hat{k}^-$. Inside this region, $q(\hat{k})$ lies between the two prices,

[2] Given our assumption that the fixed cost scales with productivity ($\theta_s = \theta u_s$), the decision to invest or not encoded in the value matching conditions (12) and (13) depends only on the value of θ. If the fixed cost were scaled with the capital stock $\theta_s = \theta k_s$, as in Miao (2019), an appropriate rescaling of the fixed costs would generate a similar investment policy. The appropriate scaling requires defining $\theta^+ \equiv \theta e^{\hat{k}^+}$ and $\theta^- \equiv \theta e^{\hat{k}^-}$ and letting $\theta_s(i_s) = \theta^- u_s \mathbb{1}_{\{i_s>0\}} + \theta^- u_s \mathbb{1}_{\{i_s<0\}}$. See Baley and Blanco (2021) for the analysis of asymmetric fixed costs.

Figure I: Tobin's $q(\hat{k})$ and Optimal Investment Policy



Notes: This figure illustrates a firm's marginal $q(\hat{k})$ (solid line) and the investment policy $\{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$, expressed in terms of capital to productivity ratios, for $\omega > 0$ and $\theta > 0$. For illustration purposes, we do not rescale q by $e^{\hat{k}}$ in the shaded areas (as the value matching conditions require).

and it falls with $\hat{k}$ due to the decreasing marginal product of capital ($\alpha < 1$); thus, it is optimal to remain inactive. From (14) and (15), a firm purchases capital if $q(\hat{k}) \geq 1$ (or $\hat{k} \leq \hat{k}^{*-}$) and sells capital if $q(\hat{k}) \leq 1 - \omega$ (or $\hat{k} \geq \hat{k}^{*+}$) without any delay.

Finally, consider an environment with fixed costs ($\theta > 0$) and zero price wedge ($\omega = 0$). Without the price wedge, the "inner" inaction region collapses to a unique reset point k^*. However, the fixed cost creates an "outer" inaction region $[\hat{k}^-, \hat{k}^+]$ that prevents firms from adjusting, even if $q(\hat{k})$ lies above the purchase price or below the selling price. From (12) and (13), firms adjust to $\hat{k}^*$ if the value of adjusting (paying the fixed cost in capital units θ/p) is larger than the value of not adjusting (the cumulative deviations of q from one, weighted by capital-productivity ratios).

Investment and duration of inaction When both frictions are active, the investment policy features the "outer" and the "inner" inaction regions, two reset points, and a non-monotonic q function.[3] The firm purchases capital to bring $\hat{k}$ up to $\hat{k}^{*-}$ after hitting the lower border $\hat{k}^-$ and sells capital to bring $\hat{k}$ down to $\hat{k}^{*+}$ after hitting the upper border $\hat{k}^+$. Thus, adjustments occur at dates where the state falls outside the outer inaction region: $T_h = \inf\left\{s \geq T_{h-1} : \hat{k}_s \notin \mathcal{R}\right\}$, with $T_0 = 0$. The duration of a complete inaction spell is the difference between consecutive adjustment dates $\tau_h = T_h - T_{h-1}$. Capital's age is the time elapsed since the last adjustment (the duration of an incomplete spell) $a_s = s - \max\{T_h : T_h \leq s\}$. Since the problem is recursive, we

[3] Although q monotonically decreases in the inner inaction region, it bends as it approaches the inaction thresholds because firms anticipate large adjustments. As $\hat{k}$ approaches the lower threshold $\hat{k}^-$, firms anticipate that a tiny reduction in the state $\mathrm{d}\hat{k} < 0$ would trigger a large positive investment $\Delta\hat{k} > 0$, lowering future and current $q(\hat{k})$ and bending the function down. A similar argument explains why $q(\hat{k})$ bends up as $\hat{k}$ approaches $\hat{k}^+$.

renormalize dates to 0 after each adjustment so that τ is both the subsequent random adjustment date and the duration of inaction and $\hat{k}_{\tau^-}$ the stopped capital-productivity ratio (immediately before adjusting).

For any τ, adjustments equal $\Delta\hat{k} = \hat{k}^*(\hat{k}_{\tau^-}) - \hat{k}_{\tau^-}$, where reset points depend on the stopped capital, denoted with $\hat{k}_{\tau^-}$:

$$\hat{k}^*(\hat{k}_{\tau^-}) \;=\; \begin{cases} \hat{k}^{*-} & \text{if} \quad \hat{k}_{\tau^-} = \hat{k}^- \\ \hat{k}^{*+} & \text{if} \quad \hat{k}_{\tau^-} = \hat{k}^+. \end{cases} \tag{18}$$

Crucially, optimal investment features a *positive serial correlation in the adjustment sign*. A firm is more likely to buy capital if it bought recently and is more likely to sell capital if it sold recently. Positive investments beget future positive investments; negative investments beget future negative investments. The positive serial correlation arises because the price wedge widens the distance between the two borders of inaction but shrinks the distance between each border of inaction and its corresponding reset point. Thus, it is more likely to reach $\hat{k}^-$ from the nearby $\hat{k}^{*-}$ than from the further $\hat{k}^{*+}$. Next, we will condition behavior on the previous reset point (i.e., if the last adjustment was up or down) to handle this path dependence.

2.4 Distribution of Capital-Productivity Ratios

To analyze the macroeconomic effects of irreversibility, we consider an economy populated by a continuum of ex-ante identical firms facing the same investment problem, in which idiosyncratic productivity shocks W_s are independent across firms. The economy has stationary cross-sectional distributions of capital-productivity ratios $\hat{k}$, adjustments $\Delta\hat{k}$, and durations of inaction τ.

Let $g(\hat{k})$ be the stationary density of firms' capital-productivity ratios. Also, let $\mathcal{N}^-$, $\mathcal{N}^+$, and $\mathcal{N} = \mathcal{N}^- + \mathcal{N}^+$ be the frequencies of upward, downward, and non-zero adjustments in the population, which are equal to the mass of firms that upsize to $\hat{k}^{*-}$, downsize to $\hat{k}^{*+}$, or adjust to either point. To avoid confusion with our notation, we emphasize that the sign in the exponent of an object refers to the previous reset point, not to the sign of the adjustment.

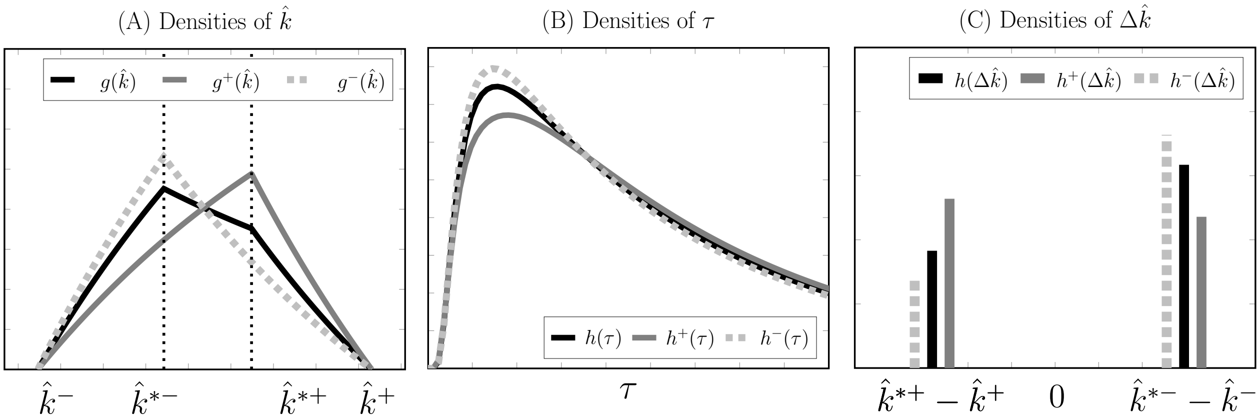
The density $g(\hat{k})$ and adjustment frequencies $\mathcal{N}^-$ and $\mathcal{N}^+$ solve the system that includes:

(i) A Kolmogorov forward equation (KFE) that describes the evolution of $g(\hat{k})$ in the interior of $\hat{k}$'s inaction region

$$\nu g'(\hat{k}) \;+\; \frac{\sigma^2}{2} g''(\hat{k}) \;=\; 0, \quad \text{for all} \quad \hat{k} \in (\hat{k}^-, \hat{k}^+) \setminus \{\hat{k}^{*-}, \hat{k}^{*+}\}; \tag{19}$$

(ii) Two border conditions that set the mass of firms at the inaction thresholds equal to zero

Figure II: Densities of $\hat{k}$, τ and $\Delta\hat{k}$



Notes: These figures plot the conditional and unconditional densities of capital-productivity ratios $\hat{k}$ (Panel A), duration of inaction τ (Panel B), and adjustments $\Delta\hat{k}$ (Panel C) for an illustrative parametrization.

and a condition ensuring that $g(\hat{k})$ is a density:

$$g(\hat{k}^{-}) \;=\; g(\hat{k}^{+}) \;= 0, \tag{20}$$

$$\int_{\hat{k}^{-}}^{\hat{k}^{+}} g(\hat{k})\,\mathrm{d}\hat{k} \;=\; 1; \tag{21}$$

(iii) Two resetting or reinjection conditions that relate the masses of upward and downward adjustments to the discontinuities in the derivative of g at the reset points:[4]

$$\mathcal{N}^{-} \;=\; \frac{\sigma^2}{2}\lim_{\hat{k}\downarrow\hat{k}^{-}} g'(\hat{k}) \;=\; \frac{\sigma^2}{2}\left[\lim_{\hat{k}\uparrow\hat{k}^{*-}} g'(\hat{k}) - \lim_{\hat{k}\downarrow\hat{k}^{*-}} g'(\hat{k})\right], \tag{22}$$

$$\mathcal{N}^{+} \;=\; -\frac{\sigma^2}{2}\lim_{\hat{k}\uparrow\hat{k}^{+}} g'(\hat{k}) \;=\; \frac{\sigma^2}{2}\left[\lim_{\hat{k}\uparrow\hat{k}^{*+}} g'(\hat{k}) - \lim_{\hat{k}\downarrow\hat{k}^{*+}} g'(\hat{k})\right], \tag{23}$$

and two continuity conditions at the reset points.

As anticipated, we study behavior conditional on the previous reset point throughout our analysis. We consider densities conditional on the previous reset point, after upsizing $g^{-}(\hat{k})$ and after downsizing $g^{+}(\hat{k})$, which satisfy the same KFE as $g(\hat{k})$ in (19), except that they only have one kink at the corresponding reset point. Panel A in Figure II plots the three densities g, g^{-}, and g^{+}; these are all proper densities and integrate to 1. We denote expectations computed with these distributions as $\mathbb{E}$, $\mathbb{E}^{-}$, and $\mathbb{E}^{+}$. Recall that the three distributions would be the same without irreversibility, as there would be a unique reset point and no dependence on past adjustments.

[4] In a small period $\mathrm{d}s$, the mass $\mathcal{N}^{-}$ that "exits" the inaction region by hitting the lower threshold—equal to $\frac{\sigma^2}{2}\lim_{\hat{k}\downarrow\hat{k}^{-}} g'(\hat{k})$—must coincide with the mass of firms that "enters" at the reset point $\hat{k}^{*-}$—equal to the jump in $\frac{\sigma^2}{2}g'$. This argument is analogous for $\mathcal{N}^{+}$.

It is important to note that, in the steady state, the average capital-productivity ratio of firms that last purchased capital is below the economy's average $\mathbb{E}_s^-[\hat{k}] - \mathbb{E}[\hat{k}] < 0$ (i.e., they don't purchase enough); similarly, the average capital-productivity ratio of firms that last sold capital is above the economy's average $\mathbb{E}_s^+[\hat{k}] - \mathbb{E}[\hat{k}] > 0$ (they do not sell enough). This fact will be critical for understanding why irreversibility amplifies aggregate capital fluctuations in Section 3.

2.5 Distributions of Actions

We denote the joint density over adjustments $\Delta\hat{k}$ and duration of inaction spells τ with $h(\Delta\hat{k}, \tau)$, and the densities conditional on the previous reset point with h^- and h^+, respectively. We also use h and $h^\pm$ to denote the marginal densities of τ and $\Delta\hat{k}$. Expectations conditional on taking action are denoted with bars: $\overline{\mathbb{E}}$, $\overline{\mathbb{E}}^-$, and $\overline{\mathbb{E}}^+$.

Panel B in Figure II plots the three densities of the duration of inaction.[5] We observe that $h^-(\tau)$ is more skewed toward shorter durations than $h^+(\tau)$, which is due to the negative drift. Panel C of Figure II plots the three distributions of adjustments $\Delta\hat{k}$.[6] The simple model generates almost no dispersion of investment rates—$h(\Delta\hat{k})$ only takes two values. However, the distributions conditional on the previous adjustment showcase the serial correlation of the adjustment sign. For instance, $h^-(\hat{k}^{*-} - \hat{k}^-) > h^+(\hat{k}^{*-} - \hat{k}^-)$ means that the probability of upsizing is higher after a previous upsize. Again, we emphasize that, without irreversibility, the three densities of τ and $\Delta\hat{k}$ would be identical. Figure VI in Section 4.3 plots their empirical counterparts.

2.6 Relationship Between Unconditional and Conditional Densities

Unconditional and conditional densities of capital-productivity ratios and adjustments relate in different ways.

Weighing with shares Adjusters' unconditional and conditional densities are related through a simple average using the relative shares of upward $\mathcal{N}^-/\mathcal{N}$ and downward $\mathcal{N}^+/\mathcal{N}$ adjustments: $h(\Delta\hat{k}, \tau) = \frac{\mathcal{N}^-}{\mathcal{N}} h^-(\Delta\hat{k}, \tau) + \frac{\mathcal{N}^+}{\mathcal{N}} h^+(\Delta\hat{k}, \tau)$. Thus unconditional and conditional expectations of adjustments relate according to

$$\overline{\mathbb{E}}[y] = \frac{\mathcal{N}^-}{\mathcal{N}} \overline{\mathbb{E}}^-[y] + \frac{\mathcal{N}^+}{\mathcal{N}} \overline{\mathbb{E}}^+[y], \qquad y \in \{\Delta\hat{k}, \tau\}. \tag{24}$$

Weighing with adjusted shares In contrast, the unconditional density of capital-productivity ratios is a weighted sum of the conditional distributions $g(\hat{k}) = r^- g^-(\hat{k}) + r^+ g^+(\hat{k})$, where the

[5] Let $h(\tau|\hat{k})$ be the stationary density of the duration of inaction given current $\hat{k}$. Using the formulas by Kolkiewicz (2002) evaluated at the two reset points, we obtain the densities of duration conditional on the previous reset point: $h^-(\tau) = h(\tau|\hat{k}^{*-})$ and $h^+(\tau) = h(\tau|\hat{k}^{*+})$.

[6] Since $\Delta\hat{k}$ only takes two values, its distribution consists of two mass points.

renewal weights r^- and r^+ rescale the shares by their relative average duration:

$$r^- \equiv \frac{\mathcal{N}^-}{\mathcal{N}} \frac{\overline{\mathbb{E}}^-[\tau]}{\overline{\mathbb{E}}[\tau]}, \quad \text{and} \quad r^+ \equiv \frac{\mathcal{N}^+}{\mathcal{N}} \frac{\overline{\mathbb{E}}^+[\tau]}{\overline{\mathbb{E}}[\tau]}. \tag{25}$$

Thus, unconditional and conditional expectations of capital-productivity ratios relate as follows:

$$\mathbb{E}[\hat{k}] = r^- \mathbb{E}^-[\hat{k}] + r^+ \mathbb{E}^+[\hat{k}]. \tag{26}$$

Why do we rescale conditional densities by the duration of inaction? The answer is the *fundamental renewal property:* The average behavior in the economy is attributable to firms with more extended periods of inaction (which are observed less frequently). Adjusting the shares with their relative duration corrects this observational bias. Appendix A.5 illustrates the correction with an example.

2.7 A Markov Chain Between Reset Points

We define the remaining objects needed to handle irreversibility. Given a current $\hat{k}_0$, let $\mathbb{P}^-(\hat{k}_0)$ be the probability of a subsequent purchase ($\hat{k}_\tau = \hat{k}^{*-}$) and $\mathbb{P}^+(\hat{k}_0)$ the probability of a subsequent sale ($\hat{k}_\tau = \hat{k}^{*+}$):

$$\mathbb{P}^-(\hat{k}_0) = \Pr[\hat{k}_\tau = \hat{k}^{*-} | \hat{k}_0] \quad \text{and} \quad \mathbb{P}^+(\hat{k}_0) = \Pr[\hat{k}_\tau = \hat{k}^{*+} | \hat{k}_0]. \tag{27}$$

Using these probabilities, we define the transition probability matrix between reset points.[7] Given a current purchase ($\hat{k}_0 = \hat{k}^{*-}$), the probability of making a subsequent purchase is $\mathbb{P}^{--} \equiv \mathbb{P}^-(\hat{k}^{*-})$; and given a current sale ($\hat{k}_0 = \hat{k}^{*+}$), the probability of making a subsequent sale is $\mathbb{P}^{++} \equiv \mathbb{P}^+(\hat{k}^{*+})$. Analogously, we define the off-diagonal elements of the transition matrix between reset points $\mathbb{P}^{-+}$ and $\mathbb{P}^{-+}$. The transition matrix $\mathbb{P}$ has the following four entries:[8]

$$\mathbb{P}^{--} = \Pr[\hat{k}_\tau = \hat{k}^{*-} | \hat{k}_0 = \hat{k}^{*-}], \qquad \mathbb{P}^{-+} = \Pr[\hat{k}_\tau = \hat{k}^{*+} | \hat{k}_0 = \hat{k}^{*-}], \tag{28}$$

$$\mathbb{P}^{+-} = \Pr[\hat{k}_\tau = \hat{k}^{*-} | \hat{k}_0 = \hat{k}^{*+}], \qquad \mathbb{P}^{++} = \Pr[\hat{k}_\tau = \hat{k}^{*+} | \hat{k}_0 = \hat{k}^{*+}]. \tag{29}$$

The following section characterizes aggregate capital fluctuations with irreversibility by conditioning behavior on the previous reset point, which requires the following objects: the conditional densities $g^-(\hat{k})$ and $g^+(\hat{k})$, the adjusting shares $\mathcal{N}^-$ and $\mathcal{N}^+$ in (22) and (23), the renewal weights r^- and r^+ in (25), the upsizing and downsizing conditional probabilities $\mathbb{P}^-(\hat{k})$ and $\mathbb{P}^+(\hat{k})$ in (27), and the Markov chain between reset points $\mathbb{P}^{--}, \mathbb{P}^{-+}, \mathbb{P}^{+-}$ and $\mathbb{P}^{++}$ in (28) and (29).

[7] Our approach complements methodologies by Caballero and Engel (2007) and Lanteri (2018), which diagnose irreversibility through transitions between marginal products of capital.

[8] The adjustment shares in (22) and (23) can also be obtained from the eigenvector of the transition matrix $\mathbb{P}$. The share of current positive investments equals the probability of a future positive investment (analogously for negative investments): $\frac{\mathcal{N}^-}{\mathcal{N}} = \frac{\mathcal{N}^-}{\mathcal{N}} \mathbb{P}^{--} + \frac{\mathcal{N}^+}{\mathcal{N}} (1 - \mathbb{P}^{++})$.

3 Aggregate Fluctuations with Irreversibility

This section theoretically studies how irreversibility affects aggregate fluctuations. Our analysis has four steps. First, we define our notions of capital fluctuations: the impulse response (IRF) and the cumulative impulse response (CIR) of average capital-productivity ratios to an aggregate productivity shock. Second, we analyze the IRF from a time-series perspective, which shows how irreversibility affects aggregate fluctuations by changing the shares of upsizing and downsizing firms. This shift has a persistent effect because of the differential behavior conditional on the previous reset point. Third, we analyze the CIR from a cross-sectional perspective, a preliminary step in finding sufficient statistics for capital fluctuations. In particular, it establishes three groups of firms whose behavior should be tracked: inactive firms until their first adjustment, upsizers, and downsizers. In the fourth and last step, we derive sufficient statistics for the CIR and investigate special cases that illustrate how irreversibility impacts these sufficient statistics.

3.1 Capital Fluctuations

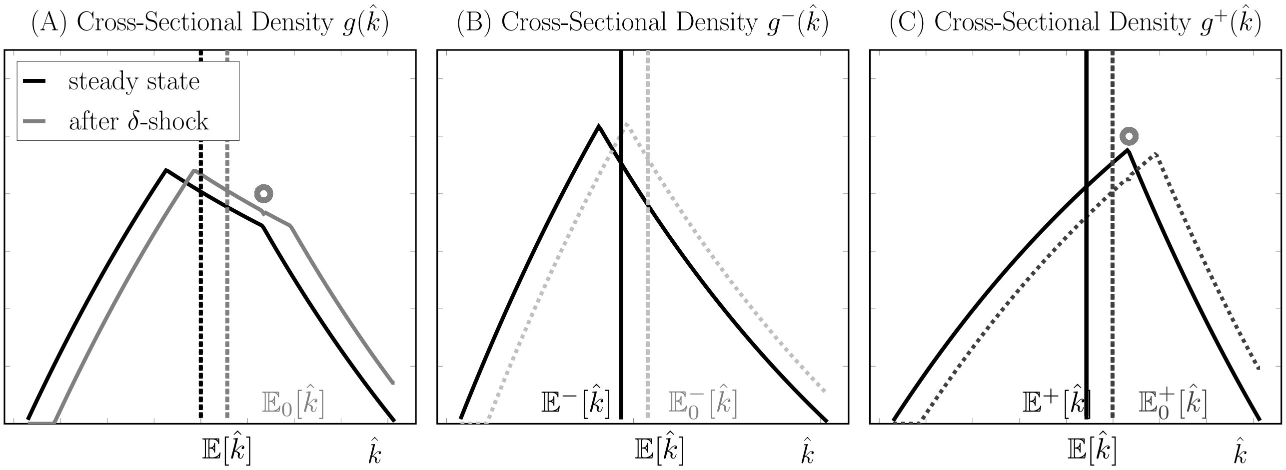
We measure capital fluctuations as the transitional dynamics of the average capital-productivity ratio that follow an aggregate productivity shock. Starting from the steady state at date $s = 0$, we introduce a small, permanent, and unanticipated decrease in the (log) level of productivity of size $\delta > 0$ to all firms. All firms' productivity falls, and capital-productivity ratios increase relative to their pre-shock levels u_{0^-} and $\hat{k}_{0^-}$, as follows:

$$\log(u_0) = \log(u_{0^-}) - \delta; \qquad \log(\hat{k}_0) = \log(\hat{k}_{0^-}) + \delta. \tag{30}$$

Panel A in Figure III plots the steady-state unconditional density $g(\hat{k})$ and the initial density following the productivity shock $g_0 = g(\hat{k} - \delta)$. Panels B and C plot the steady-state densities conditional on the previous reset point $g^{\pm}(\hat{k})$ and after the shock $g_0^{\pm} = g^{\pm}(\hat{k} - \delta)$. The negative productivity shock displaces all the steady-state distributions to the right. Without investment frictions, firms would immediately downsize their capital stock to restore their optimal capital-productivity ratio. With frictions, firms take time to absorb the shock, and the distribution of capital-productivity ratios remains away from the steady state distribution for some time. Until all firms have downsized their capital stock, there will still be positive deviations from the steady-state and persistent effects of the productivity shock. The thick circle in panels A and C corresponds to the mass of firms crossing $\hat{k}^+$ on the shock's impact and downsizing to reach $\hat{k}^{*+}$. Panel B has no thick circle in $g^-(\hat{k})$, as no mass of firms purchase capital after the shock.

Throughout the transition, the interest rate remains constant, and the steady-state investment policies hold, so there is no feedback from distributional changes in policies. Thus, our analysis

Figure III: Cross-Sectional Densities Before and After the Aggregate Shock



Notes: This figure illustrates the effects of an aggregate shock. Panel A shows the steady-state distribution $g(\hat{k})$ and the initial distribution following a productivity shock $g_0(\hat{k}) = g(\hat{k} - \delta)$. Panels B and C show analogous figures for the distributions conditional on the previous reset point, $g^-(\hat{k})$ and $g^+(\hat{k})$. Probability masses corresponding to entry points are shown in thick circles.

measures the strength of the partial equilibrium response to aggregate shock.[9]

IRF and CIR We define two measures of capital fluctuations. First, we define the impulse-response function, $\text{IRF}(\delta, s)$, as the deviation of the mean of $\hat{k}$ after s periods of the arrival of the aggregate productivity shock and the steady-state mean

$$\text{IRF}(\delta, s) \equiv \mathbb{E}_s[\hat{k}] - \mathbb{E}[\hat{k}], \tag{31}$$

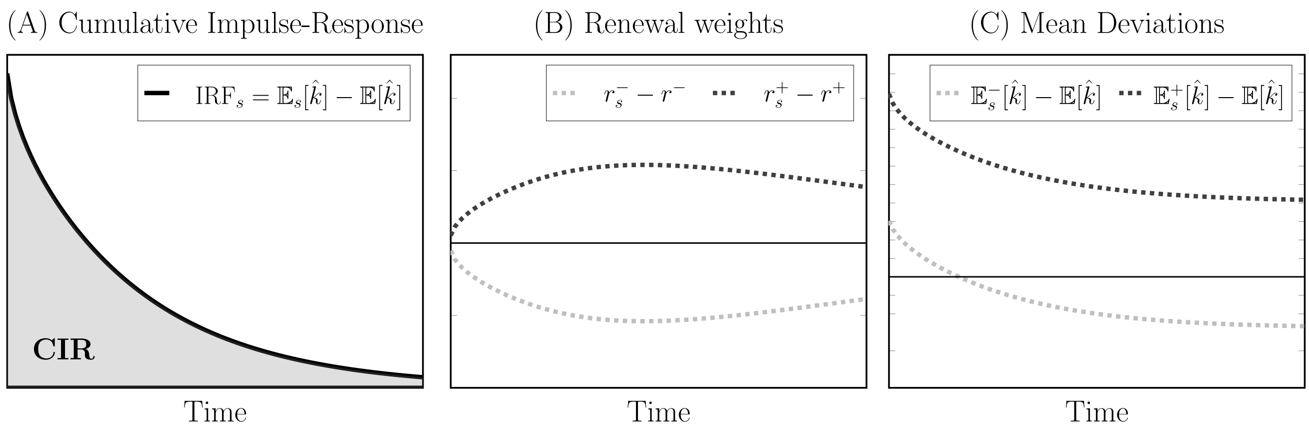
where $\mathbb{E}_s[\cdot]$ denotes expectations with the time-s distribution. Second, we define the cumulative impulse response of the mean of $\hat{k}$, $\text{CIR}(\delta)$, as the area under the $\text{IRF}_s(\delta)$ across all dates $s \in (0, \infty)$:

$$\text{CIR}(\delta) \equiv \int_0^\infty \text{IRF}(\delta, s) \, \mathrm{d}s = \int_0^\infty \left(\mathbb{E}_s[\hat{k}] - \mathbb{E}[\hat{k}]\right) \, \mathrm{d}s. \tag{32}$$

The CIR is a helpful metric of aggregate fluctuations, and it lies at the core of our strategy to discipline the price wedge and the aggregate effects of irreversibility. It summarizes the impact and persistence of the response in one scalar and eases comparison across different models.[10] Without investment frictions, firms respond instantly to the aggregate shock, and the CIR is zero. With investment frictions, the larger the CIR, the longer firms take to respond to the aggregate shock

[9] Appendix D presents a general equilibrium model that delivers constant prices as an equilibrium outcome.
[10] Alvarez, Le Bihan and Lippi (2016), Baley and Blanco (2019); Alvarez, Lippi and Oskolkov (2022); and Alexandrov (2021) use the CIR in the context of price-setting models to assess the real effects of monetary shocks.

Figure IV: Cumulative Impulse Response and Its Components



Notes: The figure shows paths after an aggregate productivity shock for an illustrative parametrization. Panel A shows the $\text{IRF}(\delta, s)$ and the CIR (area under the curve). Panel B shows the evolution of renewal weights r^- and r^+, relative to their steady-state values. Panel C shows average deviations from the steady-state mean, conditional on the previous reset point. In B and C, the lighter lines correspond to upsizing firms (previous reset to $\hat{k}^{*-}$); the darker lines correspond to downsizing firms (previous reset to $\hat{k}^{*+}$).

and the slower the transitional dynamics. Panel A in Figure IV plots the IRF (solid line) and the CIR (the area underneath the IRF).

3.2 The Time-Series Perspective

We begin analyzing aggregate fluctuations from a time-series perspective. This viewpoint teaches us that irreversibility's main contribution to aggregate fluctuations following a negative productivity shock is increasing the share of downsizing firms during the transition to the new steady state, precisely the set of firms that contribute most to persistent deviations from the steady state.

Conditional deviations and renewal weights We decompose deviations from the steady state into deviations conditional on the previous reset point. To do this, we use expectations conditional on the last rest point $\mathbb{E}_s^-$ and $\mathbb{E}_s^+$ and the renewal weights r_s^- and r_s^+ in (25) to rewrite the IRF in (31) as:

$$\text{IRF}(\delta, s) \;=\; r_s^- \left(\mathbb{E}_s^-[\hat{k}] - \mathbb{E}[\hat{k}]\right) \;+\; r_s^+ \left(\mathbb{E}_s^+[\hat{k}] - \mathbb{E}[\hat{k}]\right). \tag{33}$$

Figure IV tracks the various components that give rise to deviations from the steady state. Panel B shows the path of the renewal weights r_s^- and r_s^+ relative to their steady-state values, which tell us the number of adjustments stemming from each reset point. After the negative aggregate productivity shock, the share of downsizing firms r_s^+ increases (darker line), whereas the share of upsizing firms r_s^- falls (lighter line).[11] Panel C shows the path of average capital-productivity ratios conditional on the last rest point relative to the economy's average ($\mathbb{E}_s^{\pm}[\hat{k}] - \mathbb{E}[\hat{k}]$). Both means go up after the shock.

The time-series decomposition in (33) highlights two channels that slow the shock absorption and increase the CIR. First, keeping the mean deviations at their steady-state values ($\mathbb{E}_s^-[\hat{k}] = \mathbb{E}^-[\hat{k}] < \mathbb{E}^+[\hat{k}] = \mathbb{E}_s^+[\hat{k}]$ for all s), consider the change in renewal weights. The population tilts toward more downsizing firms, which enter their typical persistent downsizing phase with capital-productivity ratios above the economy's average ($\mathbb{E}^+[\hat{k}] - \mathbb{E}[\hat{k}] > 0$). These firms, which are now dominant, increase deviations above the steady state and slow convergence. Upsizing firms also enter their typical upsizing phase with average capital-productivity ratios below the average ($\mathbb{E}^-[\hat{k}] - \mathbb{E}[\hat{k}] < 0$). However, since there are fewer of them, they also contribute to slowing the convergence back to the steady state. Thus, the change in renewal weights generates persistence even if the mean deviations remain at their steady-state values.

Second, besides the changes in weights, positive deviations $\mathbb{E}_s^+[\hat{k}] - \mathbb{E}[\hat{k}]$ become larger and converge slower, and negative deviations $\mathbb{E}_s^-[\hat{k}] - \mathbb{E}[\hat{k}]$ become smaller (and even positive on the shock's impact) and converge faster. This change in means brings an additional kick. In sum, the evolution in shares and conditional means following an aggregate shock increases the persistence of aggregate capital fluctuations when capital is partially irreversible.

3.3 The Cross-Sectional Perspective

Next, we formally analyze capital fluctuations from a cross-sectional perspective. The central insight is to express the aggregate dynamic response to the shock as the cross-sectional average of the expected cumulative deviations from the mean. Three steps lie behind this characterization: (*i*) exchanging the integrals over time and over firms, (*ii*) decomposing average cumulative deviations between the first and subsequent adjustments, and (*iii*) summarizing subsequent adjustments with two numbers reflecting upsizing and downsizing behavior.

Exchanging integrals Let us start from the CIR's definition in (32), which tracks the economy's average investment behavior along the transition, that is, integrating first over firms and then over time. We exchange orders of integration (i.e., integrating first over time and then over firms) to track each firm's investment and then average across them. The analytical gain stemming from exchanging the orders of integration arises because we can decompose the infinite horizon $[0, \infty]$ into two intervals: from the arrival of the aggregate shock to the (random) first adjustment $[0, \tau]$, and the first adjustment onward $[\tau, \infty]$.

[11] Naturally, the change in shares also happens without a price wedge and inaction purely generated by fixed costs; however, in that case, firms become identical *after their first adjustment* ($\mathbb{E}_s^-[\hat{k}] = \mathbb{E}_s^+[\hat{k}]$), deviations from the economy's average can be ignored and there is no additional persistence (Baley and Blanco, 2021).

To implement this decomposition, in the first step, we use the law of iterated expectations to condition on the initial capital-productivity ratio $\hat{k}_0$ (right after the δ shock) and integrate over the initial distribution of agents $g_0 = g(\hat{k} - \delta)$. Then, we exchange the order of integration over agents and over time:

$$\text{(34)}\quad \text{CIR}(\delta) \;=\; \mathbb{E}_{g_0}\left[\int_0^\infty \mathbb{E}_s\Big[(\hat{k} - \mathbb{E}[\hat{k}])\Big]\,\mathrm{d}s\Big|\hat{k}_0\right] \;=\; \mathbb{E}_{g_0}\left[\mathbb{E}\left[\int_0^\infty (\hat{k}_s - \mathbb{E}[\hat{k}])\,\mathrm{d}s\Big|\hat{k}_0\right]\right] + \mathbb{C}.$$

The constant $\mathbb{C}$ arises because Fubini's theorem fails (the double integral of the absolute value of deviations is infinite); thus, exchanging the order of integration does not give the same result.[12]

First and subsequent adjustments In the second step, we define $m(\hat{k}_0)$ as the expected cumulative deviations from the steady-state mean for a firm with current $\hat{k}_0$ as

$$\text{(35)}\qquad m(\hat{k}_0) \;\equiv\; \mathbb{E}\left[\int_0^\infty (\hat{k}_s - \mathbb{E}[\hat{k}])\,\mathrm{d}s\Big|\hat{k}_0\right] \;+\; \mathbb{C}.$$

We decompose $m(\hat{k}_0)$ into two intervals $[0, \tau]$ and $[\tau, \infty]$, where τ is the firm's first stopping time after the aggregate shock:

$$\text{(36)}\qquad m(\hat{k}_0) \;=\; \underbrace{\mathbb{E}\left[\int_0^\tau (\hat{k}_s - \mathbb{E}[\hat{k}])\,\mathrm{d}s\Big|\hat{k}_0\right]}_{\text{up to first adjustment}} + \underbrace{\mathbb{E}\left[\int_\tau^\infty (\hat{k}_s - \mathbb{E}[\hat{k}])\,\mathrm{d}s\Big|\hat{k}_0\right] \;+\; \mathbb{C}.}_{\text{subsequent adjustments}}$$

Without irreversibility, firms fully incorporate the aggregate productivity shock with the first adjustment (to a unique reset point $\hat{k}^*$) and then return to their "steady-state behavior"; this implies that the second term in (36) is independent of $\hat{k}_0$ and thus also independent of δ (Alvarez, Le Bihan and Lippi, 2016; Baley and Blanco, 2021). This is no longer the case with irreversibility, as the initial state $\hat{k}_0$ affects subsequent adjustments. The second term in (36) differs from zero because firms only partially adjust to the aggregate shock. In principle, one should keep track of firms until they fully incorporate the aggregate shock. This is a problem "with reinjection", as labeled by Alvarez and Lippi (2021). Nevertheless, it is enough to remember the first reset point and then condition future behavior because the first adjustment cleans from all heterogeneity *except for the adjustment sign.*

Summarizing subsequent adjustments In the third step, we summarize investment behavior after the first adjustment (the second term in (36)) with two numbers: average behavior after

[12] The constant $\mathbb{C}$ does not affect the sufficient statistics' characterization of the CIR in the next section because we make a first-order approximation for small shocks $\delta \approx 0$, and it drops out. Alexandrov (2021) considers large shocks where the constant becomes relevant.

upsizing $m(\hat{k}^{*-})$ and average behavior after downsizing $m(\hat{k}^{*+})$:

(37)
$$\mathbb{E}\Big[\mathbb{E}\Big[\int_{\tau}^{\infty}(\hat{k}_s-\mathbb{E}[\hat{k}])\,\mathrm{d}s\Big|\hat{k}_\tau\Big]\Big|\hat{k}_0\Big] = \mathbb{E}\Big[m(\hat{k}^*(\hat{k}_\tau))\Big|\hat{k}_0\Big]-\mathbb{C}=\mathbb{P}^-(\hat{k}_0)m(\hat{k}^{*-})+\mathbb{P}^+(\hat{k}_0)m(\hat{k}^{*+})-\mathbb{C}.$$

where the probabilities $\mathbb{P}^-(\hat{k}_0)$ and $\mathbb{P}^+(\hat{k}_0)$, defined in (27), depend on initial conditions after the aggregate shock. This expression shows that steady-state behavior is restored after the first adjustment—once a firm completes its first inaction spell.

Last, we join all results by substituting (37) into (36) and then into (34) and write the CIR as a cross-sectional average of cumulative deviations computed with the initial distribution g_0:

$$\text{CIR}(\delta)=\mathbb{E}_{g_0}[m(\hat{k}_0)], \tag{38}$$

where $m(\hat{k}_0)$ is defined recursively as follows:[13]

$$m(\hat{k}_0) \equiv \underbrace{\mathbb{E}\left[\int_0^{\tau}(\hat{k}_s-\mathbb{E}[\hat{k}])\,\mathrm{d}s\Big|\hat{k}_0\right]}_{\text{incomplete spells}} + \underbrace{\mathbb{P}^-(\hat{k}_0)m(\hat{k}^{*-})}_{\text{complete upsizing spell}} + \underbrace{\mathbb{P}^+(\hat{k}_0)m(\hat{k}^{*+})}_{\text{complete downsizing spell}}. \tag{39}$$

Proposition 2 organizes the previous results.

Proposition 2. ***(CIR)*** *Up to the first order, the CIR equals*

$$\frac{CIR(\delta)}{\delta} = -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}+o(\delta), \tag{40}$$

where $m(\hat{k})$ is a continuously differentiable function equal to the average cumulative deviations of the capital-productivity ratio $\hat{k}$ from the economy's mean $\mathbb{E}[\hat{k}]$, satisfying the HJB

$$0 = \hat{k} - \mathbb{E}[\hat{k}] - \nu m'(\hat{k}) + \frac{\sigma^2}{2}m''(\hat{k}) \qquad \forall \hat{k}\in(\hat{k}^-,\hat{k}^+), \tag{41}$$

with two boundary conditions

$$m(\hat{k}^-) = m(\hat{k}^{*-}), \quad \text{and} \quad m(\hat{k}^+) = m(\hat{k}^{*+}), \tag{42}$$

and a stationarity condition

$$\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0. \tag{43}$$

Expression (40) reexpresses the CIR in (38) using a first-order approximation to the initial

[13] $\mathbb{C}$ disappears as it is inside m.

distribution around a small shock δ, $g_0 = g(\hat{k} - \delta) \approx g(\hat{k}) - \delta g'(\hat{k})$, and the stationarity condition (43) which sets $\mathbb{E}_g[m(\hat{k}_0)] = 0$. We reexpress $m(\hat{k})$ in (39) recursively through the HJB in (41), inherited from $\hat{k}$'s law of motion, together the border conditions in (42). Due to continuity, the two boundary conditions equalize cumulative deviations at the inaction thresholds to their value at the corresponding reset point.

The astute reader will notice that equations (41) and (42) have infinite solutions: for any candidate solution $m(\hat{k})$, $m(\hat{k}) + a$ is a solution $\forall a \in \mathbb{R}$. The stationarity condition (43) pins down the unique solution by requiring no fluctuations without shocks (i.e., CIR(0) = 0). Effectively, it imposes a linear relationship between the terminal values $m(\hat{k}^{*-})$ and $m(\hat{k}^{*+})$. Stationarity lies at the core of the steady state and plays a crucial role in the CIR's characterization with partial irreversibility. Next, we explain this condition in detail.

3.4 Stationarity: Balancing Complete and Incomplete Spells

The stationarity condition (43) sets the cross-sectional average of $m(\hat{k})$ in (39) to zero.[14] It implies that average deviations that follow a complete inaction spell (from upsizing or downsizing firms) should "balance" the average deviations from incomplete spells (from inactive firms):

$$(44) \qquad \underbrace{\mathbb{C}ov[\hat{k}, a]}_{\text{avg. incomplete spells}} + \underbrace{\mathbb{E}[\mathbb{P}^-(\hat{k})]m(\hat{k}^{*-})}_{\text{avg. complete upsizing spells}} + \underbrace{\mathbb{E}[\mathbb{P}^+(\hat{k})]m(\hat{k}^{*+})}_{\text{avg. complete downsizing spells}} = 0.$$

Let us discuss this condition in incrementally complex environments.

Symmetric environments With zero drift, symmetric inaction region, and zero price wedge, the unique reset point $\hat{k}^*$ equals the economy's average $\mathbb{E}[\hat{k}]$ and deviations above and below the mean for any age cancel out. The covariance between capital-productivity ratios and age is zero $\mathbb{C}ov[\hat{k}, a] = 0$, and consequently, stationarity requires $m(\hat{k}^*) = 0$.

Asymmetric environment without irreversibility In asymmetric environments (with non-zero drift or asymmetric fixed costs), the unique reset point differs from the economy's average. Deviations above and below do not cancel out and $\mathbb{C}ov[\hat{k}, a] \neq 0$. Stationarity requires:

$$(45) \qquad m(\hat{k}^*) + \mathbb{C}ov[\hat{k}, a] = 0.$$

Deviations after complete inaction spells, $m(\hat{k}^*)$, and deviations during incomplete spells, $\mathbb{C}ov[\hat{k}, a]$, must sum up to zero in a steady state. For example, with negative drift $-\nu$, inactive firms have, on

[14] The integral of incomplete spells (the first term in (39)) equals the covariance between capital-productivity ratios and capital age: $\int_{\hat{k}^-}^{\hat{k}^+} \mathbb{E}\left[\int_0^\tau (\hat{k}_s - \mathbb{E}[\hat{k}])\, \mathrm{d}s \middle| \hat{k}_0\right] g(\hat{k}_0)\, \mathrm{d}\hat{k}_0 = \mathbb{C}ov[\hat{k}, a]$. The integral of the complete spells (the second term in (39)) becomes the sum of average probabilities times deviations.

average, capital-productivity ratios below the economy's mean, and more so the older their capital. In the inactive cross-section, $\mathbb{C}ov[\hat{k}, a] < 0$. Completed spells must revert this force to deliver zero deviations in steady-state, and thus, adjusting firms will overshoot their capital-productivity ratio: $m(\hat{k}^*) = -\mathbb{C}ov[\hat{k}, a] > 0$.

With irreversibility With irreversibility, the balancing argument is similar, but now we must consider two types of complete spells. We rewrite (44) as the weighted average of two numbers:

$$(46) \qquad \underbrace{\left(m(\hat{k}^{*-}) + \mathbb{C}ov[\hat{k}, a]\right)}_{\equiv\ \mathcal{M}(\hat{k}^{*-})\ <\ 0} \mathbb{E}[\mathbb{P}^-(\hat{k})] \ + \ \underbrace{\left(m(\hat{k}^{*+}) + \mathbb{C}ov[\hat{k}, a]\right)}_{\equiv\ \mathcal{M}(\hat{k}^{*+})\ >\ 0} (1 - \mathbb{E}[\mathbb{P}^-(\hat{k})]) \ = \ 0,$$

where $\mathbb{E}[\mathbb{P}^+(\hat{k})] = 1 - \mathbb{E}[\mathbb{P}^-(\hat{k})]$. Recall that $\mathbb{C}ov[\hat{k}, a]$ reflects the deviations of inactive firms. For inaction spells ending in upsizing (with probability $\mathbb{E}[\mathbb{P}^-(\hat{k})]$), we have that average deviations are negative, $\mathcal{M}(\hat{k}^{*-}) \equiv m(\hat{k}^-) + \mathbb{C}ov[\hat{k}, a] < 0$, since resets fall short of the average. For inaction spells ending in downsizing (with probability $1 - \mathbb{E}[\mathbb{P}^-(\hat{k})]$), we have instead that average deviations are positive, $\mathcal{M}(\hat{k}^{*+}) \equiv m(\hat{k}^+) + \mathbb{C}ov[\hat{k}, a] > 0$, since resets succeed the average. The adjustments of upsizing and downsizing firms, weighted by their occurrence, must compensate for the deviations of inactive firms to ensure stationarity.

To better understand what $\mathcal{M}(\hat{k}^{*-})$ and $\mathcal{M}(\hat{k}^{*+})$ reflect, Proposition 3 provides alternative expressions, as a triple product of conditional deviations $(\mathbb{E}^{\pm}[\hat{k}] - \mathbb{E}[\hat{k}])$, the average duration of those deviations $\overline{\mathbb{E}}^{\pm}[\tau]$, and switching probabilities between reset points. These relationships will be exploited in Section 4 to recover sufficient statistics in the microdata, as m's are not observed, but the objects on the right-hand side are.

Proposition 3. ***(Expected sum of deviations)*** *The expected sum of deviations after upsizing $\mathcal{M}(\hat{k}^{*-}) \equiv m(\hat{k}^{*-}) + \mathbb{C}ov[\hat{k}, a]$ and after downsizing $\mathcal{M}(\hat{k}^{*+}) \equiv m(\hat{k}^{*+}) + \mathbb{C}ov[\hat{k}, a]$ are equal to*

$$(47) \qquad \mathcal{M}(\hat{k}^{*-}) \ = \ (\mathbb{E}^-[\hat{k}] - \mathbb{E}[\hat{k}]) \ \overline{\mathbb{E}}^-[\tau] \ \frac{\mathbb{E}[\mathbb{P}^+(\hat{k})]}{\mathbb{P}^{-+}} \ < \ 0$$

$$(48) \qquad \mathcal{M}(\hat{k}^{*+}) \ = \ (\mathbb{E}^+[\hat{k}] - \mathbb{E}[\hat{k}]) \ \overline{\mathbb{E}}^+[\tau] \ \frac{\mathbb{E}[\mathbb{P}^-(\hat{k})]}{\mathbb{P}^{+-}} \ > \ 0,$$

where the average downsizing and upsizing probabilities are equal to

$$(49) \qquad \mathbb{E}[\mathbb{P}^-(\hat{k})] = \frac{\overline{\mathbb{E}}\left[\tau' \mathbb{1}_{\{\hat{k}_{\tau'} = \hat{k}^-\}}\right]}{\overline{\mathbb{E}}[\tau]}, \qquad \mathbb{E}[\mathbb{P}^+(\hat{k})] = \frac{\overline{\mathbb{E}}\left[\tau' \mathbb{1}_{\{\hat{k}_{\tau'} = \hat{k}^+\}}\right]}{\overline{\mathbb{E}}[\tau]}.$$

Relative to inactive firms, upsizing firms (47) expect a negative deviation of size $\mathbb{E}^-[\hat{k}] - \mathbb{E}[\hat{k}]$ during the next inaction spell, which is expected to last $\overline{\mathbb{E}}^-[\tau]$ periods. Since the investment sign is serially correlated, upsizing firms remain in an upsizing phase and contribute to negative deviations for several periods; they would only leave this phase after a series of adverse shocks cause them to downsize. The ratio $\mathbb{E}[\mathbb{P}^+(\hat{k})]/\mathbb{P}^{-+}$ precisely reflects the expected time spent in the transient upsizing phase, where $\mathbb{E}[\mathbb{P}^+(\hat{k})]$ is the expected probability of downsizing and $\mathbb{P}^{-+}$ in (28) is the probability of switching phase from upsizing to downsizing. If they switch, they will enter a persistent downsizing phase with positive deviations as in (48). The explanation for downsizing firms is analogous, mutatis mutandis. The mappings in (49) show that average probabilities of upsizing and downsizing equal truncated expectations of durations.

3.5 Latent Deviations

Finally, we state the last ingredient needed to derive the CIR's sufficient statistics. The two numbers, $\mathcal{M}(\hat{k}^{*-})$ in (47) and $\mathcal{M}(\hat{k}^{*+})$ in (48), summarize the behavior of adjusting firms. However, to study the effects of aggregate shocks, we must consider that some firms will switch their investment strategy after the aggregate shock, from purchasing to selling or vice versa. To do that, we construct a new function $\mathcal{M}(\hat{k})$ that takes these two values in the outer inaction regions (where irreversibility plays no role) and then extends it to the inner inaction (where irreversibility plays a significant role) by imposing continuity. Concretely, we consider $\mathcal{M}(\hat{k}) \in \mathbb{C}^2$ to be *any* twice continuously differentiable function in the domain $[\hat{k}^+, \hat{k}^-]$ that takes two values in the outer inaction regions:

$$(50) \qquad \mathcal{M}(\hat{k}) \equiv \begin{cases} m(\hat{k}^{*-}) + \mathbb{C}ov(\hat{k}, a) < 0 & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ m(\hat{k}^{*+}) + \mathbb{C}ov(\hat{k}, a) > 0 & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+]. \end{cases}$$

In the inner inaction region, $\mathcal{M}(\hat{k})$ takes values that ensure continuity up to the second derivative.

3.6 The CIR's Sufficient Statistics

To recap, we have transformed the CIR of capital-productivity ratios following an aggregate productivity shock in an environment with irreversibility—a complicated dynamic object—into a steady-state cross-sectional average of the recursive function $m(\hat{k})$—a static object. We have also discussed how active and inactive firms must balance in a steady state and defined the function $\mathcal{M}(\hat{k})$ that reflects latent deviations. With all these elements, Proposition 4 derives sufficient statistics—cross-sectional steady-state moments of $\hat{k}$ and a—that characterize the CIR.

Proposition 4. ***(Sufficient statistics)*** *Up to the first order, the CIR of average capital-productivity ratios equals the sum of three steady-state cross-sectional moments:*

$$(51) \qquad \frac{CIR(\delta)}{\delta} = \underbrace{\frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} + \frac{\nu \mathbb{C}ov[\hat{k}, a]}{\sigma^2}}_{\textit{up to first adjustment}} + \underbrace{\frac{1}{\sigma^2}\, \mathbb{E}\left[\frac{1}{\mathrm{d}s} \mathbb{E}_s[\mathrm{d}(\hat{k}_s \mathcal{M}(\hat{k}_s))]\right]}_{\textit{subsequent adjustments}} + o(\delta).$$

According to (51), three sufficient statistics determine the CIR: (i) the cross-sectional variance of capital-productivity ratios $\mathbb{V}ar[\hat{k}]$, (ii) the covariance of capital-productivity ratios and age $\mathbb{C}ov[\hat{k}, a]$, and (iii) the average local drift of the product $\mathcal{M}(\hat{k})\hat{k}$, all divided by the idiosyncratic volatility σ^2. These cross-sectional moments are indeed sufficient statistics because an econometrician equipped with the joint distribution of $(\hat{k}, a)$ could back out the CIR from it.[15]

The three sufficient statistics reflect how, in a steady state, idiosyncratic productivity shocks shape incomplete spells and complete spells. Since idiosyncratic shocks $\mathrm{d}W_s$ and aggregate shocks δ enter $\mathrm{d}\hat{k}$ symmetrically, responsiveness to idiosyncratic shocks, encoded in steady-state moments, informs about responsiveness to aggregate shocks, encoded in the CIR. We apply this "responsiveness" principle to dissect the sufficient statistics.

Insensitivity of incomplete spells to productivity shocks The first two statistics indicate firms' willingness to tolerate deviations from their frictionless optimum and thus remain inactive. We follow Baley and Blanco (2021) to explain why this is the case. At any date s, consider the cumulative productivity shock (sum of innovations) received by a firm while inactive, normalized by volatility $\tilde{W}_s \equiv (W_s - W_{s-a_s})/\sigma$. Define the economy's *insensitivity of incomplete spells to productivity shocks* as the covariance of capital-productivity ratios and cumulative shocks: $\mathbb{C}ov[\hat{k}, \tilde{W}]$. Intuitively, if firms are extremely sensitive to productivity shocks, they continuously adjust their capital-productivity ratio to the reset points, yielding $\mathbb{C}ov[\hat{k}, \tilde{W}] = 0$ in the cross-section. Instead, if firms are insensitive, they allow $\hat{k}$ to move with shocks $\tilde{W}$, and this covariance becomes prominent.

Let us link the insensitivity to productivity shocks to the CIR. For simplicity, let us assume away the price wedge. The capital-productivity ratio of any firm at date s can be written as $\hat{k}_s = \hat{k}^* - \nu a_s + \sigma^2 \tilde{W}_s$. Subtracting the mean $\mathbb{E}[\hat{k}]$ on both sides, multiplying by $(\hat{k}_s - \mathbb{E}[\hat{k}])$, and taking the cross-sectional average, we obtain $\mathbb{V}ar[\hat{k}] = -\nu \mathbb{C}ov[\hat{k}, a] + \sigma^2 \mathbb{C}ov[\hat{k}, \tilde{W}]$. Rearranging yields $\mathbb{C}ov[\hat{k}, \tilde{W}] = (\mathbb{V}ar[\hat{k}] - (-\nu \mathbb{C}ov[\hat{k}, a]))/\sigma^2$, which is exactly the expression for the CIR's first two statistics.

It is tempting to claim that the variance of capital-productivity ratios is a sufficient indication of pervasive investment frictions and insensitivity to productivity shocks. But there is a caveat. The variance also reflects dispersion in capital-productivity ratios generated by the drift, which is unrelated to productivity shocks. The second statistic, the covariance of capital-productivity ratios and age, is, in effect, a bias correction term that ensures we accurately measure the reaction to the Brownian shocks by eliminating the drift effects (clearly, when $\nu = 0$, there is no need for correction and the first statistic is sufficient.)

How does the price wedge affect these two statistics? The price wedge ω increases $\mathbb{V}ar[\hat{k}]$ by

[15] Importantly, $\hat{k}$ is not directly observable since it depends on productivity, but under certain assumptions over the production technology, it can be recovered using revenue data (Hsieh and Klenow, 2009). Section 4 proposes an alternative to measure these statistics by exploiting exclusively investment data.

introducing a new layer of heterogeneity linked to the distinct reset points, and it may dramatically change $\mathbb{C}ov[\hat{k}, a]$. By imposing a downward rigidity, unproductive firms tend to have older capital stock and have capital-productivity ratios above the average, which pushes $\mathbb{C}ov[\hat{k}, a]$ to be positive. The sign of the covariance eventually depends on the relative size of the drift (a negative drift favors $\mathbb{C}ov[\hat{k}, a] < 0$) and the price wedge (a positive wedge favors $\mathbb{C}ov[\hat{k}, a] > 0$). When the price wedge dominates, $\mathbb{C}ov[\hat{k}, a] > 0$, the CIR rises, and aggregate fluctuations persist longer.[16]

Insensitivity of complete spells to productivity shocks The third sufficient statistic, exclusive to the irreversibility case, indicates how productivity shocks change the anticipated ending of a complete spell. In other words, if firms anticipated ending their inaction by purchasing $\mathcal{M}(\hat{k}^{*-})$ or selling $\mathcal{M}(\hat{k}^{*+})$ and then a shock makes them anticipate a different ending. If idiosyncratic shocks do not change the anticipated ending in a steady state, then an aggregate shock wouldn't either, and we wouldn't see the dynamics of renewal weights shown in Panel B of Figure IV.

While the expression for the third statistic appears complicated at first sight, we shed light on it by mirroring the explanation of the first two. At any date s, consider a firm's anticipated terminal condition $\mathbb{E}_s[\mathcal{M}(\hat{k}_\tau)]$. We define the economy's *insensitivity of complete spells to productivity shocks* as the covariance between the *change* in capital-to-productivity ratios $\mathrm{d}\hat{k}_s$ and the *change* in anticipated terminal conditions normalized by volatility and averaged across the population: $(1/\sigma^2)\mathbb{E}\left[\mathbb{C}ov_s\left[\mathrm{d}\hat{k}_s, \mathrm{d}\mathbb{E}_s[\mathcal{M}(\hat{k}_\tau)]\right] / \mathrm{d}s\right]$. At the individual level, the covariance is time-varying as it depends on the state. Intuitively, if terminal values are insensitive (e.g., firms always expect to end up buying), productivity shocks do not alter the spell's ending, and this covariance is zero. In contrast, if terminal values are very sensitive, a small $\mathrm{d}\hat{k}$ triggers a big change in the anticipated ending. In such sensitive cases, the renewal weights react strongly to an aggregate shock, generating additional persistence.

As before, there is a caveat. We must be careful in only capturing sensitivity to productivity shocks, not mechanical drift effects. The drift affects both variables, as it reduces capital-productivity ratios and makes upsizing endings at $\mathcal{M}(\hat{k}^{*-})$ more likely. To account for this, we must correct our definition of insensitivity. Let $D_s \equiv \mathbb{E}_s[\mathcal{M}_\tau] - \mathcal{M}_s$ be the expected change in $\mathcal{M}_s$ until the next adjustment. Subtracting the drift of the product $\mathrm{d}(\hat{k}_s D_s)$ from the covariance, which includes $\hat{k}$'s drift ν and D_s's drift, and rearranging, we can express the third sufficient statistic as the unbiased insensitivity of complete spells to productivity shocks in the population:

$$\text{(52)} \qquad \frac{1}{\sigma^2}\mathbb{E}\left[\frac{\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))]}{\mathrm{d}s}\right] = \mathbb{E}\left[\frac{\mathbb{C}ov_s[\mathrm{d}\hat{k}_s, \mathrm{d}\mathbb{E}_s[\mathcal{M}_\tau]] - \mathbb{E}_s[\mathrm{d}(\hat{k}_s D_s)]}{\mathbb{V}ar_s[\mathrm{d}\hat{k}_s]}\right].$$

[16] In Baley and Blanco (2021), we showed that higher fixed costs for downward than upward adjustments also generate a positive covariance and hence increase the CIR. However, asymmetric fixed costs imply one reset point, and the third sufficient statistic remains zero.

Finally, if we write idiosyncratic volatility in the denominator as $\sigma^2 \, \mathrm{d}s = \mathbb{V}ar_s[\mathrm{d}\hat{k}_s]$, expression (52) is just the coefficient of an OLS regression of $\mathrm{d}\mathbb{E}_s[\mathcal{M}_\tau]$ onto $\mathrm{d}\hat{k}_s$, with a bias correction.

3.7 Irreversibility's Role for Sufficient Statistics

Before shifting gears to empirical and quantitative applications, we discuss how irreversibility shapes the CIR's sufficient statistics and establish connections with the literature. Irreversibility plays two different roles in the propagation of aggregate shocks. First, it directly impacts the CIR by shifting the masses of upward and downward adjustments, affecting investment behavior *after* the first adjustment captured by the third sufficient statistic. Second, as a source of downward rigidity, irreversibility indirectly impacts the CIR by turning positive the covariance of capital-productivity ratios and age, affecting investment behavior *before* the first adjustment captured by the first sufficient statistics. We highlight each channel by exploring extreme cases.

Irreversibility's direct effect To showcase irreversibility's direct impact on the CIR, Proposition 5 considers a zero-drift environment where the direct impact is maximal (the covariance is inherently zero in cases (i) and (ii) below) and an infinite-drift limit where the direct impact is minimal (the covariance is negative and largest in absolute value in case (iii)). We derive analytical expressions for the CIR in terms of investment frictions rescaled by the user cost $\mathcal{U}$ and other parameters.

Proposition 5. ***(Extreme cases)*** *Up to the first order, the CIR's sufficient statistics as a function of investment frictions are as follows.*

(i) No drift and only fixed cost: If $\nu = \omega = 0$ *and* $\theta > 0$*, then*

$$(53) \qquad \frac{CIR(\delta)}{\delta} = \frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} = \left(\frac{12\tilde{\theta}}{(1-\alpha)\sigma^6}\right)^{1/4}, \quad where \quad \tilde{\theta} = \frac{\theta}{\alpha}\left(\frac{p\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}.$$

(ii) No drift and only partial irreversibility: If $\nu = \theta = 0$ *and* $\omega > 0$*, then*

$$(54) \qquad \frac{CIR(\delta)}{\delta} = 2 \times \frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} = \left(\frac{12\tilde{\omega}}{(1-\alpha)\sigma^4}\right)^{1/3}, \quad where \quad \tilde{w} = \frac{\omega/2}{\mathcal{U}(1-\omega/2)}.$$

(iii) Large drift: If $\sigma^2 > 0$ *and* $\nu \to \infty$*, then the price wedge is irrelevant and*

$$(55) \qquad \mathbb{E}\left[\frac{\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))]}{\mathrm{d}s}\right] = 0, \qquad \nu\mathbb{C}ov[\hat{k}, a] = -\mathbb{V}ar[\hat{k}], \qquad \frac{CIR(\delta)}{\delta} = 0.$$

As a baseline, case (i) assumes no drift and only a fixed cost so that the only sufficient statistic is the ratio $\mathbb{V}ar[\hat{k}]/\sigma^2$. Expression (53) shows that the rescaled fixed cost $\tilde{\theta}$ increases the inaction region, the cross-sectional variance, and the CIR with an elasticity of 1/4, resembling the results by Barro (1972) and Dixit (1991).

Case (*ii*) assumes no drift and only a price wedge. Expression (54) shows that the rescaled price wedge $\tilde{\omega}$ affects aggregate fluctuations with an elasticity of 1/3, as in Abel and Eberly (1999) and Miao (2019). The CIR equals two times the ratio $\mathbb{V}ar[\hat{k}]/\sigma^2$—obtained in case (*i*)—since the first and third sufficient statistics in (51) are identical. This means that *irreversibility doubles the persistence of aggregate fluctuations* for a given cross-sectional dispersion $\mathbb{V}ar[\hat{k}]$. In other words, the source of inaction (fixed costs vs. price wedge) matters for aggregate fluctuations.

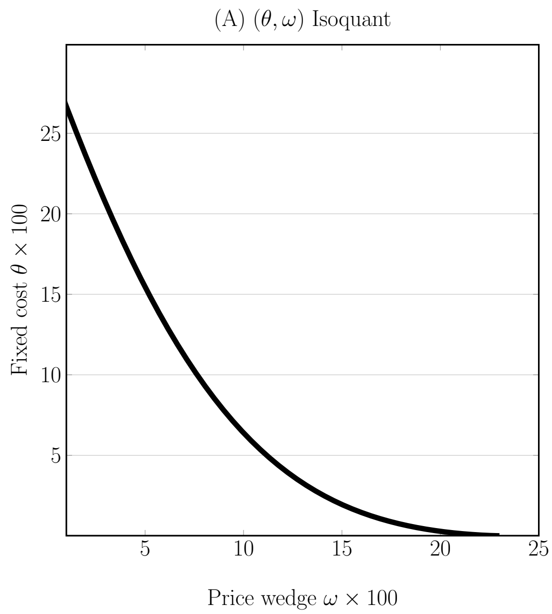
At the other extreme, case (*iii*) considers a huge drift. As the drift unwaveringly depletes capital relative to productivity, firms exclusively make and anticipate positive investments. The price wedge becomes irrelevant as firms never face it and have no bite in the aggregate. This case was first studied by Caplin and Spulber (1987) in a price-setting context. In our framework, this mechanism gets captured in (55), where the covariance becomes so negative that it completely unwinds the variance, and the local drift is zero. Aggregate shocks are immediately absorbed, there are no deviations from the steady state, and the CIR equals zero.[17]

Irreversibility's indirect effect The extreme cases above highlight that irreversibility's direct effect on the CIR depends on the drift. However, irreversibility's indirect impact on the CIR is muted in those cases. Next, we numerically explore cases with a moderate drift of $\nu = 0.07$ and price wedges in the range $\omega \in [0.0, 0.25]$ to showcase both the direct and indirect effects.[18] For a consistent comparison across economies, in the spirit of Hsieh and Klenow (2009), we fix the cross-sectional dispersion of capital-productivity ratios $\mathbb{V}ar[\hat{k}]$ to the value obtained for $\theta = 0$ and $\omega = 0.25$. Then, for other values of the wedge, we find the fixed cost θ that delivers the same level of dispersion. Fixing $\mathbb{V}ar[\hat{k}]$ implicitly fixes the adjustment frequency and the dispersion of adjustments across configurations.[19]

Panel A of Figure V plots the (θ, ω)-isoquant, which is convex. Going from left to right increases the relative importance of the price wedge vis-à-vis the fixed cost while delivering the same cross-sectional dispersion $\mathbb{V}ar[\hat{k}]$. Panel B of Figure V plots the CIR and its three sufficient statistics against the price wedge ω, computed along the isoquant. The CIR (solid black line) increases as the price wedge dominates—capital fluctuations are more persistent. Since $\mathbb{V}ar[\hat{k}]/\sigma^2$ remains fixed (flat dotted line), changes in the other two sufficient statistics generate all the action. The covariance $\mathbb{C}ov[\hat{k}, a]$ (gray dotted line) starts negative at $\omega = 0$ (as case (*iii*) of Proposition 5) as the drift makes old capital-productivity ratios negative. It becomes positive for $\omega > 0.08$ as the downsizing constraints imposed by the price wedge kick in (recall the discussion in Section

[17] Baley and Blanco (2021) shows an analogous result for asymmetric fixed costs and zero price wedge.
[18] Miao (2019) studies the case with full irreversibility ($\omega = 1$) for any drift $\nu \in \mathbb{R}$.
[19] In the quantitative model of Section 5, we match the empirical distribution of investment for each choice of ω. However, the key messages from this section continue to hold.

Figure V: Sufficient Statistics for Different Price Wedges and Fixed Allocation



Notes: Panel A shows the (θ, ω) isoquant, and Panel B the CIR and its components for an illustrative calibration.

3.4). The local drift (dotted gray line) neatly encapsulates the direct irreversibility effect, which is non-monotonic. The local drift dominates the (negative) covariance for low wedges; the (positive) covariance dominates the local drift for high wedges. Overall, we see that the price wedge impacts the CIR's level and the relative contribution of each sufficient statistic. In Section 5 we exploit this result to calibrate the price wedge.

4 Measuring Sufficient Statistics with Microdata

The challenge to analyzing the role of irreversibility on the CIR's sufficient statistics stems from the fact that these moments cannot be directly computed from the data as the distributions of capital-productivity ratios $g(\hat{k})$ and $g^{\pm}(\hat{k})$ are not directly observed. As economists, however, we have available detailed panel data with information on the actions of adjusters: the size of adjustments $(\Delta\hat{k})$ and the duration of complete (τ) and incomplete (a) inaction spells. In this section, we derive mappings from microdata to parameters, optimal investment policies, and sufficient statistics, assuming the price wedge ω as given. Section 5 combines these mappings with a fully-fledged quantitative investment model to discipline the price wedge.

Two-state strategy To measure sufficient statistics from panel microdata, we proceed in two steps. In Stage I, we assume we know the two reset points. We then apply structural relationships to the data linking the behavior of adjusting and non-adjusting firms to reverse-engineer the parameters of the productivity process and steady-state cross-sectional moments.

In Stage II, we use the data and the model's structure to reverse-engineer the reset points. To obtain these mappings, we condition adjusters' behavior on the direction of the previous adjustment so that their actions remain Markovian. Throughout, we exploit the properties of Markov processes and the fact that the two reset points are constant.

4.1 Stage I: Mappings Given the Reset Points

We take the two reset points $\{\hat{k}^{*-}, \hat{k}^{*+}\}$ as given and use the adjusters' expectations conditional on the previous reset point $\overline{\mathbb{E}}^{\pm}[\cdot]$ to back out parameters and cross-sectional moments. For this, we require information on consecutive inaction spells $\{(\Delta\hat{k}, \tau), (\Delta\hat{k}', \tau')\}$ to condition future behavior on the previous reset point.

For each firm's completed inaction spell $(\Delta\hat{k}, \tau)$, if the firm upsized its capital stock $\Delta\hat{k} > 0$ we record the reset point as $\hat{k}^* = \hat{k}^{*-}$ and construct the stopped capital as $\hat{k}_\tau = \hat{k}^{*-} - \Delta\hat{k}$; if the firm downsized its capital stock $\Delta\hat{k} < 0$, we record the reset point as $\hat{k}^* = \hat{k}^{*+}$ and the stopped capital as $\hat{k}_\tau = \hat{k}^{*+} - \Delta\hat{k}$. Similarly, we record the future reset point $\hat{k}^{*\prime}$ and the future stopped capital $\hat{k}_{\tau'}$ using the information from the subsequent spell $(\Delta\hat{k}', \tau')$.

We recover the parameters of the capital-productivity process in Proposition 6. Proposition 7 recovers conditional and unconditional means of $\hat{k}$. Proposition 8 recovers the cross-sectional variance of $\hat{k}$ and their covariance with age. Finally, Proposition 9 recovers the irreversibility's term in the CIR. We present the mappings for these objects separately to facilitate exposition, but they should be recovered simultaneously through a non-linear system of equations (details below).

Throughout, we exploit the relationship between conditional and unconditional moments of adjustments presented in (24): $\overline{\mathbb{E}}[y] = \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[y] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[y]$, where $y \in \{\hat{k}_{\tau'}, \tau'\}$.

Proposition 6. ***(Recovering parameters)*** *The drift ν and volatility σ^2 of capital-productivity ratios implied by investment microdata are recovered through the following mappings:*

$$\nu = \frac{\overline{\mathbb{E}}[\Delta\hat{k}]}{\overline{\mathbb{E}}[\tau]}, \tag{56}$$

$$\sigma^2 = \frac{\overline{\mathbb{E}}[(\hat{k}_{\tau'} + \nu\tau')^2 - (\hat{k}^*)^2]}{\overline{\mathbb{E}}[\tau]}. \tag{57}$$

Expression (56) recovers the drift $\nu = \xi + \mu$ using the average adjustment size $\overline{\mathbb{E}}[\Delta\hat{k}]$ times the adjustment frequency—the inverse of the expected duration of inaction $\overline{\mathbb{E}}[\tau] = \mathcal{N}^{-1}$. It uses the fact that, in a stationary environment, the average adjustment size $\overline{\mathbb{E}}[\Delta\hat{k}]$ must compensate for the average drift between adjustments, i.e., $\nu\overline{\mathbb{E}}[\tau]$. Expression (57) recovers idiosyncratic risk σ^2 from the difference in future and past resets squared, also scaled by the adjustment frequency. This difference reflects dispersion in adjustment size, accounting for potential shifts in reset points.[20] In Baley and Blanco (2021), we obtained related mappings from the data to the parameters without irreversibility. Irreversibility does not change the drift mapping, but it changes the volatility mapping because it affects transitions across reset points.

Proposition 7. ***(Recovering means)*** *Let $r^{\pm}$ be the adjusted shares in (25). The unconditional mean $\mathbb{E}[\hat{k}]$ and means conditional on the previous reset $\mathbb{E}^{\pm}[\hat{k}]$ are recovered as:*

$$\mathbb{E}[\hat{k}] = r^{-}\mathbb{E}^{-}[\hat{k}] + r^{+}\mathbb{E}^{+}[\hat{k}], \tag{58}$$

$$\mathbb{E}^{\pm}[\hat{k}] = \overline{\mathbb{E}}^{\pm}\left[\left(\frac{\hat{k}^{*\pm} + \hat{k}_{\tau'}}{2}\right)\left(\frac{\hat{k}^{*\pm} - \hat{k}_{\tau'}}{\overline{\mathbb{E}}^{\pm}[\hat{k}^{*\pm} - \hat{k}_{\tau'}]}\right)\right] + \frac{\sigma^2}{2\nu}. \tag{59}$$

The unconditional mean in (58) is the weighted average of the conditional means using adjusted shares $r^{\pm}$. The conditional means $\mathbb{E}^{\pm}[\hat{k}]$ in (59) are recovered from the middle point between the departing and the ending points of an inaction spell $(\hat{k}^{*\pm} + \hat{k}_{\tau'})/2$, weighed by relative adjustment size.[21] The term $\sigma^2/2\nu$ corrects for the accumulated drift between adjustments.

Proposition 8. ***(Recovering the variance and covariance)*** *The variance $\mathbb{V}ar[\hat{k}]$ and the covariance $\mathbb{C}ov[\hat{k}, a]$ are recovered from the microdata as:*

$$\mathbb{V}ar[\hat{k}] = \frac{1}{3}\frac{\mathbb{E}\left[(\hat{k}^* - \mathbb{E}[\hat{k}])^3\right] - \overline{\mathbb{E}}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^3\right]}{\hat{k}^* - \overline{\mathbb{E}}[\hat{k}_{\tau'}]}. \tag{60}$$

$$\mathbb{C}ov[\hat{k}, a] = \frac{1}{2\nu}\left(\mathbb{V}ar[\hat{k}] + \sigma^2\mathbb{E}[a] - \frac{\overline{\mathbb{E}}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^2\tau'\right]}{\overline{\mathbb{E}}[\tau]}\right). \tag{61}$$

The variance in (60) is recovered from the difference in the *cubes* of the departing and the ending points of an inaction spell, which reflects skewness in adjustments, divided by the difference between departing and ending points to express it in variance units. The covariance $\mathbb{C}ov[\hat{k}, a]$ in (61) is recovered from the sum of the variance $\mathbb{V}ar[\hat{k}]$, average age $\mathbb{E}[a]$, and the product between stopped capital squared and stopping times. We stress that, for a model to match the covariance, it must match that latter dynamic moment, which is critical to correctly identifying the dynamic effects of irreversibility.

[20] Note that $\overline{\mathbb{E}}[(\hat{k}^*)^2] = \frac{\mathcal{N}^-}{\mathcal{N}}(\hat{k}^{*-})^2 + \frac{\mathcal{N}^+}{\mathcal{N}}(\hat{k}^{*+})^2$.

[21] The second term in the product inside the expectation relates to the renewal measure again. Without irreversibility, this term collapses to $\Delta k'/\overline{\mathbb{E}}[\Delta\hat{k}']$, the adjustment size relative to the average adjustment. Midpoints of firms with larger adjustments receive a more prominent weight. With irreversibility, the numerator and denominator consider the different reset points, but effectively, it increases the weight on larger adjustments.

Proposition 9. ***(Recovering the irreversibility term)*** *The CIR's irreversibility term is recovered from the microdata as*

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))\right]\right] \;=\; \frac{\overline{\mathbb{E}}[\hat{k}_{\tau'}\mathcal{M}(\hat{k}_{\tau'})] - \overline{\mathbb{E}}[\hat{k}^*\mathcal{M}(\hat{k}^*)]}{\overline{\mathbb{E}}[\tau]}, \tag{62}$$

where departing deviations $\mathcal{M}(\hat{k}^{*\pm})$ and ending deviations $\mathcal{M}(\hat{k}_{\tau'})$ are recovered in Proposition 3.

According to (62), the third sufficient statistic equals the difference in the expected deviations between departing and ending points. If there was one reset point, both numbers equal $\overline{\mathbb{E}}[\hat{k}^*\mathcal{M}(\hat{k}^*)]$ and the statistic is zero. If there were no fixed costs, then $\hat{k}_{\tau'} = \hat{k}^{*\prime}$, the numerator becomes $\overline{\mathbb{E}}[\hat{k}^{*\prime}\mathcal{M}(\hat{k}^{*\prime})] - \overline{\mathbb{E}}[\hat{k}^*\mathcal{M}(\hat{k}^*)]$, and only consecutive adjustments that switch their ending point (from upsizing to downsizing or vice versa) matter. The larger the difference in resets, transition probabilities, or deviations, the more irreversible the investment, and the larger this statistic.

4.2 Stage II: Recovering the Reset Points

In Stage II, we recover the two reset points. We still take the price wedge ω as given. Evaluating q's stopping-time formulation in (16) at the reset points, we obtain the following conditions linking the optimal stopping policy τ^* and the optimal reset points $\{\hat{k}^{*-}, \hat{k}^{*+}\}$:

$$p \;=\; \overline{\mathbb{E}}^{-}\left[\int_0^{\tau^*} \alpha e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k}')\; e^{-\mathcal{U}\tau^*}\right], \tag{63}$$

$$p(1-\omega) \;=\; \overline{\mathbb{E}}^{+}\left[\int_0^{\tau^*} \alpha e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k}')\; e^{-\mathcal{U}\tau^*}\right]. \tag{64}$$

These expressions say that adjusting firms reset their capital-productivity ratios to equalize marginal costs and expected marginal benefits. The marginal cost is the investment price, either p if buying or $(1-\omega)p$ if selling. The expected marginal benefit is the cumulative marginal product of capital generated during the inaction spell (between date 0 and τ^*) plus the expected value of undepreciated capital upon adjustment. Expectations depend on the past reset. Proposition 10 uses these expressions to derive mappings from microdata to reset points. It extends the formula for the frictionless case in (17) to include a fixed cost and a price wedge.

Proposition 10. ***(Recovering reset points)*** *Let $\Phi \equiv \log\left(\alpha/(\mathcal{U} - (1-\alpha)\nu - (1-\alpha)^2\sigma^2/2)\right)$. The two reset points $\{\hat{k}^{*-}, \hat{k}^{*+}\}$ are recovered from the microdata as:*

$$\hat{k}^{*-} = \frac{1}{1-\alpha}\left(\Phi - \log p \qquad + \log \frac{1-\overline{\mathbb{E}}^{-}\left[e^{-\mathcal{U}\tau^{*} + (1-\alpha)(\hat{k}^{*-}-\hat{k}_{\tau'})}\right]}{1-\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p}e^{-\mathcal{U}\tau^{*}}\right]}\right), \tag{65}$$

$$\hat{k}^{*+} = \frac{1}{1-\alpha}\left(\Phi - \log p(1-\omega) + \log \frac{1-\overline{\mathbb{E}}^{+}\left[e^{-\mathcal{U}\tau^{*} + (1-\alpha)(\hat{k}^{*+}-\hat{k}_{\tau'})}\right]}{1-\overline{\mathbb{E}}^{+}\left[\frac{p(\Delta\hat{k}')}{p(1-\omega)}e^{-\mathcal{U}\tau^{*}}\right]}\right). \tag{66}$$

Recalling $\mathcal{U} = r + \xi$ and $\nu = \mu + \xi$, the first constant term Φ reveals that reset points increase with productivity growth μ but decrease with the discount factor r and the depreciation rate ξ. Higher idiosyncratic risk σ^2 shifts reset points to the right, implying a larger average investment. This effect highlights the fact that firms can expand to exploit good outcomes and contract to insure against bad outcomes, making them potentially risk-loving (Oi, 1961; Hartman, 1972; Abel, 1983). The second term shows that reset points decrease with the corresponding investment price: Firms invest more the lower the purchasing price p and disinvest less the lower the selling price $p(1-\omega)$. Lastly, the third term reflects how irreversibility shapes the reset points through the expected marginal profits accrued during periods of inaction (in the numerator) and the expected user cost $\mathcal{U}$ (in the denominator).

As a measure of irreversibility, consider the difference between reset points: $(\hat{k}^{*+} - \hat{k}^{*-})$:

$$\frac{1}{1-\alpha}\left(\underbrace{\log\frac{1}{1-\omega}}_{\text{exogenous}} + \underbrace{\log \frac{1-\overline{\mathbb{E}}^{+}\left[e^{-\mathcal{U}\tau^{*} + (1-\alpha)(\hat{k}^{*+}-\hat{k}'_{\tau})}\right]}{1-\overline{\mathbb{E}}^{-}\left[e^{-\mathcal{U}\tau^{*} + (1-\alpha)(\hat{k}^{*-}-\hat{k}'_{\tau})}\right]} - \log \frac{1-\overline{\mathbb{E}}^{+}\left[\frac{p(\Delta\hat{k}')}{p(1-\omega)}e^{-\mathcal{U}\tau^{*}}\right]}{1-\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p}e^{-\mathcal{U}\tau^{*}}\right]}}_{\text{endogenous irreversibility}}\right). \tag{67}$$

The constant Φ and the price p cancel out in the difference. The difference naturally increases in the exogenous price wedge ω, further amplified by the output-capital elasticity α.[22] The other two ratios reflect history dependence on the expected marginal product and the user cost. As long as the optimal policy depends on the previous reset, endogenous irreversibility arises.

4.3 Establishment-Level Manufacturing Data

We apply the mappings using yearly investment data on manufacturing establishments in Chile.

Data sources Data comes from the Annual National Manufacturing Survey (*Encuesta Nacional Industrial Anual*) for the period 1980 to 2011. We use information on depreciation rates and price deflators from national accounts and Penn World Tables to construct the capital series. The sample considers plants that appear in the sample for at least ten years (more than 60% of the

[22]In the quantitative section, we discuss the challenge of identifying ω from α.

sample) and have more than ten workers. We keep all pairs of consecutive adjustments $(\Delta k, \tau)$ and $(\Delta k', \tau')$ for each firm. Appendix E presents all the data details.

Capital stock and investment rates We construct the capital stock series using the perpetual inventory method. We include structures, machinery, equipment, and vehicles. A plant's capital stock in year s evolves as

$$k_s = (1-\xi)k_{s-1} + \frac{I_s}{p(I_s)D_s}, \tag{68}$$

where ξ is the physical depreciation rate; I_s is the nominal value of the investment; $p(I_s)$ is the investment pricing function, which considers different prices for capital purchases p and sales $p(1-\omega)$, for a given wedge ω; D_s is the gross fixed capital formation deflator; and k_0 is a plant's self-reported nominal capital stock at current prices for the first year in which it is nonnegative. Note that the ratio $I_s/(p(I_s)D_s)$ is the real investment in capital units (the data counterpart to $i_s = \Delta k_s$ in the model), and it is affected by the price wedge.

Constructing investment rates We construct the gross nominal investment i_s with information on purchases, reforms, improvements, and fixed asset sales. We define the investment rate ι_s as the ratio of real gross investment to the capital stock:[23]

$$\iota_s \equiv \frac{I_s/(p(I_s)D_s)}{k_{s-1}}. \tag{69}$$

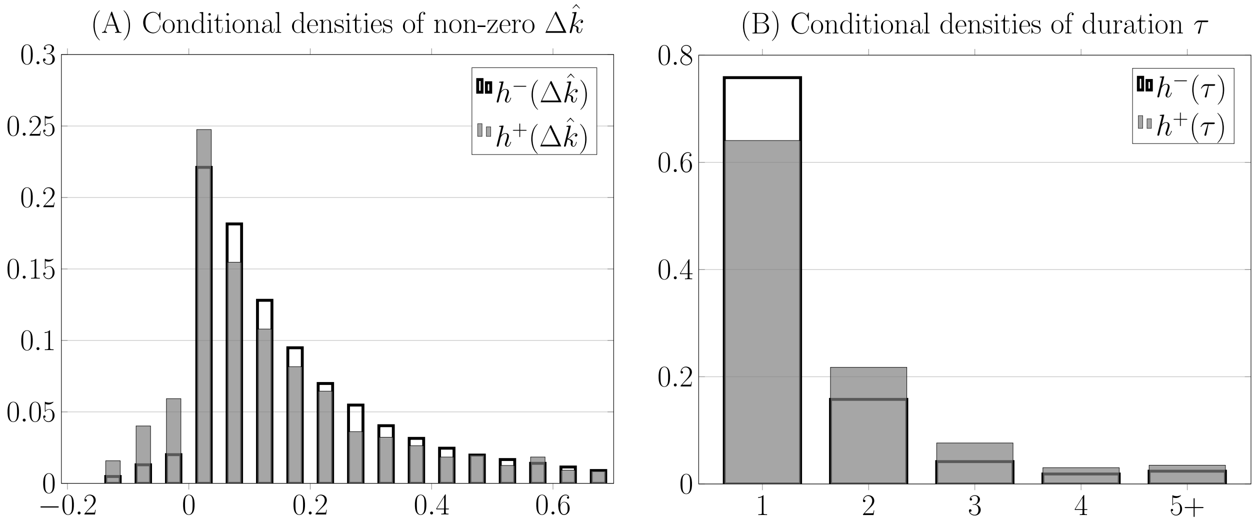
For each plant and each inaction spell h, we record the change in the capital-productivity ratio upon action $\Delta\hat{k}_h$ and the spell's duration τ_h. We construct $\Delta\hat{k}_h$ with investment rates from (69):

$$\Delta\hat{k}_h = \begin{cases} \log(1+\iota_h) & \text{if} \quad |\iota_h| > \underline{\iota}, \\ 0 & \text{if} \quad |\iota_h| < \underline{\iota}. \end{cases} \tag{70}$$

The threshold $\underline{\iota} > 0$ reflects the idea that small maintenance investments should be excluded. Following Cooper and Haltiwanger (2006), we set $\underline{\iota} = 0.01$, such that all investment rates below 1% in absolute value are considered inaction. We define an adjustment date T_h using $\Delta\hat{k}_{T_h} \neq 0$ and compute a spell's duration as the difference between two adjacent adjustment dates: $\tau_h = T_h - T_{h-1}$. Finally, we truncate the investment distribution at the 2^{nd} and 98^{th} percentiles to eliminate outliers.

[23] The investment rate equals $\iota_{T_h} \equiv i_{T_h}/k_{T_h^-} = (k_{T_h} - k_{T_h^-})/k_{T_h^-}$, where $k_{T_h^-} = \lim_{s\uparrow T_h} k_s$. In contrast to the continuous-time model, in which investment is computed as the difference in the capital stock between two consecutive instants, in the data, we compute it as the difference between two consecutive years. Potentially, a bias could arise from time aggregation as we take a continuous time model to annual data. We leave the assessment of this bias for further research.

Figure VI: Empirical Densities of Observable Actions



Notes: Panel A plots the distribution of non-zero changes in capital-productivity ratios, and Panel B plots the duration of inaction spells. Light bars = conditional on previous purchase ($\hat{k}_\tau = \hat{k}^{*-}$); Dark bars = conditional on previous sale ($\hat{k}_\tau = \hat{k}^{*+}$). Sample: Firms with at least ten years of data, truncation at the 2nd and 98th percentiles of investment rate distribution, and inaction threshold of $\underline{\iota} = 0.01$.

Figure VI plots the empirical cross-sectional distribution of non-zero changes in the capital-productivity ratios $\Delta\hat{k}$ in Panel A and completed inaction spells τ in Panel B, conditional on a past sale or purchase. We obtain an inaction rate of 40%.[24] The data shows the same qualitative patterns as in Section 2.5, consistent with irreversibility. Investment distributions have few negative investments, plenty of small positive investments, and few large positive investments, and are convex as they move away from zero. Moreover, the density of investment rates conditional on a sale $h^+(\Delta k)$ is more tilted toward negative values than $h^-(\Delta k)$, which means that the probability of a sale is higher after a sale, and vice versa. The expected duration of inaction is longer after a previous sale than after a purchase. The conditional durations are $\mathbb{E}^-[\tau] = 1.72$ and $\mathbb{E}^+[\tau] = 1.98$.

4.4 Putting the Theory to Work

Before applying the mappings, we need values for a few standard parameters and a price wedge. One period is a year. We set the real interest rate to 6.6% ($r = 0.066$) to match the average real interest rate in Chile reported by the IMF. The productivity growth rate is 2.0% ($\mu = 0.02$). To set the returns to scale α, we consider a Cobb-Douglas production function with frictionless labor input ℓ: $y = u^{1-\eta\tilde{\alpha}} \left(k^{\tilde{\alpha}} \ell^{1-\tilde{\alpha}}\right)^{\eta}$. Static maximization over labor implies $y \propto k^{\frac{\eta\tilde{\alpha}}{1-(1-\tilde{\alpha})\eta}}$. Assuming standard values $\eta = 0.90$ and $\tilde{\alpha} = 0.4$, the output-capital elasticity is $\alpha = (\eta\tilde{\alpha})/(1-(1-\tilde{\alpha})\eta) = 0.85$.[25] We

[24] Table E.1 in Appendix E.6 presents additional investment rate statistics.
[25] For robustness, we conduct comparative statics on α in Appendix E.7.

normalize the price level to $p = 6$. Finally, we set the price wedge to our preferred value $\omega = 0.12$, disciplined in the next section.

We apply the mappings to the Chilean investment data to recover the productivity parameters, the two reset points, and the cross-sectional moments behind the CIR's sufficient statistics. Since all these objects are simultaneously determined, we develop an iterative method to solve the non-linear system of mappings in (56), (57), (65) and (66), substituting the population moments with their sample counterparts. Appendix E.4 provides a step-by-step guide to recovering these objects.

Table I: Mappings from microdata for $\omega = 0.12$

Inputs from microdata		
Size, Duration and Age		
Avg. Size	$\overline{\mathbb{E}}[\Delta\hat{k}]$	0.200
Avg. Duration	$\overline{\mathbb{E}}[\tau]$	1.733
Avg. Age	$\mathbb{E}[a]$	1.677
Dynamic covariances		
Covariance I	$\overline{\mathbb{E}}[(\hat{k}_{\tau'} + \nu\tau')^2]$	0.801
Covariance II	$\overline{\mathbb{E}}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^2\tau'\right]$	0.275
Probabilities		
Adjustment frequency	$\mathcal{N}$	0.482
Purchase frequency	$\mathcal{N}^-$	0.464
Sale frequency	$\mathcal{N}^+$	0.018
Purchase renewal weight	r^-	0.952
Sale renewal weight	r^+	0.047
Purchase – purchase	$\mathbb{P}^{--}$	0.958
Sale – sale	$\mathbb{P}^{++}$	0.124
Avg. purchase prob.	$\overline{\mathbb{E}}[\mathbb{P}^-(\hat{k})]$	0.947
Avg. selling prob.	$\overline{\mathbb{E}}[\mathbb{P}^+(\hat{k})]$	0.052

Outputs from mappings		
Parameters		
Drift	ν	0.115
Volatility	σ^2	0.057
Depreciation	ξ	0.095
User cost	$\mathcal{U}$	0.161
Reset points		
Reset after purchase	$\hat{k}^{*-}$	−0.854
Reset after sale	$\hat{k}^{*+}$	−0.041
Inner inaction	$\hat{k}^{*+} - \hat{k}^{*-}$	0.813
Moments of $\hat{k}$		
Average	$\mathbb{E}[\hat{k}]$	−0.841
Avg. cond on purchase	$\mathbb{E}^-[\hat{k}]$	−0.977
Avg. cond on sale	$\mathbb{E}^+[\hat{k}]$	−0.336
Variance	$\mathbb{V}ar[\hat{k}]$	0.097
Covariance with age	$\mathbb{C}ov[\hat{k}, a]$	0.152
Local drift	$\mathbb{E}\left[\frac{\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))]}{\mathrm{d}s}\right]$	0.035

Notes: Mappings assume a price wedge $\omega = 0.12$, output-capital elasticity $\alpha = 0.85$, real interest rate $r = 0.066$, productivity growth $\mu = 0.02$ and purchase price $p = 6$.

Data inputs The left part of Table I reports the cross-sectional moments of adjusting firms in the microdata, which are inputs into the theory mappings. The average inaction period lasts $\overline{\mathbb{E}}[\tau] = 1.733$ years and ends with an average adjustment of $\overline{\mathbb{E}}[\Delta\hat{k}] = 0.200$. Capital has an average age of 1.677 years.

We also report key dynamic moments, shares, renewal weights, transition probabilities, and average probabilities. On average, half of firms adjust every period ($\mathcal{N} = 0.482$), from which more than 96% of firms upsize and less than 4% downsize. Upsizing is quite persistent, as the likelihood of upsizing after an upsize is $\mathbb{P}^{--} = 0.958$, whereas the probability of downsizing following a downsize is only $\mathbb{P}^{++} = 0.124$.

Theory outputs The right part of Table I reports the values of various model objects. Regarding the productivity process, investment data implies a drift of $\nu = 0.115$, which includes capital depreciation and productivity growth. Given values for μ and r, the implied capital depreciation rate is $\xi = \nu - \mu = 0.095$ and the user cost is $\mathcal{U} = r + \xi = 0.161$. We recover idiosyncratic volatility of $\sigma^2 = 0.058$, consistent with the volatility of productivity used in quantitative models.[26]

The reset points are $\hat{k}^{*-} = -0.854 < -0.041 = \hat{k}^{*+}$ and the average capital-productivity ratio is $\mathbb{E}[\hat{k}] = -0.841$. They imply that firms' capital fluctuates between 0.42 and 0.96 times their idiosyncratic productivity, with an average ratio of 0.43. Conditional on upsizing, on average, firms reset their capital to 0.37 times their productivity ($\mathbb{E}^-[\hat{k}] = -0.986$), and conditional on downsizing, they do it to 0.75 times productivity ($\mathbb{E}^+[\hat{k}] = -0.287$). The width of the inner inaction region—a direct measure of irreversibility—is given by the difference $\hat{k}^{*+} - \hat{k}^{*-} = 0.813$, out of which 45% is generated by the exogenous price wedge and the remaining 55% is generated by the endogenous response to the wedge, according to equation (67).

The dispersion of capital-productivity ratios is $\mathbb{V}ar[\hat{k}] = 0.097$. In turn, the covariance of capital-productivity ratios with capital age is positive $\mathbb{C}ov[\hat{k}, a] = 0.152 > 0$, which means that the price wedge's positive impact on the covariance dominates the drift's negative effects. Finally, the local drift that captures history dependence equals 0.035. The positive covariance and the positive local drift amplify aggregate fluctuations.

Autocorrelation in adjustment sign We assess the serial correlation in the adjustment sign in the data as additional evidence of the effect of irreversibility on plants' investment, complementing the transition probabilities across reset points. Since the adjustment sign is a binary variable, computing a simple correlation is not recommended. The standard Pearson correlation coefficient fails to capture the degree of association between two binary variables meaningfully. Instead, we run a logistic regression of $sign(\Delta\hat{k}')$ on $sign(\Delta\hat{k})$, which yields an odds ratio of 3.3. This ratio suggests it is more than three times more likely to purchase after a previous purchase than after an earlier sale.

5 A Quantitative Investment Model

In this section, we extend the adjustment cost structure of the parsimonious model introduced in Section 2 to better match the empirical investment rate distribution, specifically its dispersion and the coexistence of both large and small investment rates. To achieve this, we incorporate a generalized adjustment hazard, initially proposed by Caballero and Engel (1999, 2007) and further developed in the price-setting context by Alvarez, Lippi and Oskolkov (2022) and in the investment context by Lippi and Oskolkov (2023). By recovering the generalized hazard directly from the data, we discipline the price wedge and quantitatively evaluate the role of irreversibility. As noted earlier, the sufficient statistics outlined in Section 3 and the data mappings discussed in

[26] Irreversibility increases the recovered volatility σ^2 from 0.049 (see Table I in Baley and Blanco (2021)) to 0.058.

Section 4 are valid within this generalized hazard framework.

5.1 The Generalized Hazard Model

The generalization focuses solely on the structure of fixed adjustment costs while preserving the assumptions regarding production technology, the productivity process, and the price wedge.[27] For every non-zero investment, $\Delta k_s \neq 0$, firms incur a *stochastic* fixed adjustment cost, θ_s, proportional to productivity:

$$\theta_s = \Theta_s(\Delta\hat{k})\, u_s. \tag{71}$$

The function $\Theta_s(\Delta\hat{k})$ follows a compound Poisson process, allowing for distributions of fixed costs and free adjustment opportunities (i.e., mass points at zero cost), which may differ between positive and negative adjustments. This adjustment cost structure can alternatively be expressed using an adjustment hazard function, $\Lambda(\hat{k})$, such that for a given capital-productivity ratio, $\hat{k}$, the probability of adjusting within a small interval $\mathrm{d}t$ is $\Lambda(\hat{k})\mathrm{d}t$ in the outer inaction region and zero in the inner inaction region. Unlike the baseline model, which assumes zero adjustments inside the inaction region, this extended model introduces a positive probability of adjustment within that domain.

The extended Kolmogorov Forward Equation characterizing the stationary distribution of $\hat{k}$ includes an additional term, absent in the baseline KFE from (19),

$$\Lambda(\hat{k})g(\hat{k}) = \nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k}), \qquad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+) \setminus \{\hat{k}^{*-}, \hat{k}^{*+}\}, \tag{72}$$

together with the boundary and reinjection conditions that depend on the price wedge, ω.

Our generalized hazard model with irreversibility nests several existing investment models: the standard fixed-cost model (Scarf, 1959); random fixed costs, including those of Thomas (2002), Gourio and Kashyap (2007), and Khan and Thomas (2008); asymmetric fixed costs and time dependent adjustments (Baley and Blanco, 2021); and the generalized hazard model without irreversibility (Lippi and Oskolkov, 2023).

5.2 Recovering the Adjustment Hazard

To recover the hazard function that generates the empirical investment rate distribution shown in Figure VI, we follow the methodology outlined by Lippi and Oskolkov (2023).[28]

[27] See Baley and Blanco (2021) for an analysis of ex-ante heterogeneity in firms' production and adjustment technologies.

[28] Lippi and Oskolkov (2023) propose a detailed framework to recover the underlying distribution of fixed adjustment costs responsible for the observed investment patterns. This approach is critical for understanding heterogeneity and the origins of inaction. For our purposes, it suffices to recover the hazard function to compute the

First, we exploit the relationship between the model's hazard rate $\Lambda(\hat{k})$ and the density of capital-productivity ratios $g(\hat{k})$, and the data's adjustment frequency $\mathcal{N}$ and investment density $h(\Delta\hat{k})$. Specifically, for any $\hat{k}$, these objects satisfy the following relationship:

$$\underbrace{\Lambda(\hat{k})g(\hat{k})}_{\text{model}} = \underbrace{\mathcal{N}h(\Delta\hat{k})}_{\text{data}}, \quad \text{where} \quad \Delta\hat{k} = \hat{k}^*(\hat{k}) - \hat{k}. \tag{73}$$

Second, we parameterize the investment density, $h(\Delta\hat{k})$, using a Gamma distribution that accounts for asymmetries between positive and negative adjustments. This is achieved by introducing separate frequency (Υ), shape (ϱ^-, ϱ^+), and scale $(\varsigma^-, \varsigma^+)$ parameters for positive and negative adjustments:

$$h(\Delta\hat{k} \mid \Upsilon, \varrho^-, \varrho^+, \varsigma^-, \varsigma^+) = \begin{cases} \frac{\Upsilon}{\Gamma(\varrho^-)(\varsigma^-)^{\varrho^-}}(\Delta\hat{k})^{\varrho^- - 1}\exp\left(-\frac{\Delta\hat{k}}{\varsigma^-}\right) & \text{if } \Delta\hat{k} > 0, \\ \frac{1-\Upsilon}{\Gamma(\varrho^+)(\varsigma^+)^{\varrho^+}}(-\Delta\hat{k})^{\varrho^+ - 1}\exp\left(\frac{\Delta\hat{k}}{\varsigma^+}\right) & \text{if } \Delta\hat{k} < 0. \end{cases} \tag{74}$$

We estimate these five parameters $(\Upsilon, \varrho^-, \varrho^+, \varsigma^-, \varsigma^+)$ using a simulated method of moments. Using the estimated values, we substitute the investment density, $h(\Delta\hat{k})$, into the KFE. We then solve the resulting system of equations for $g(\hat{k})$ at each $\hat{k}$, employing finite differences and incorporating the boundary conditions: $g(\hat{k}^-) = g(\hat{k}^+) = 0$ and $\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k} = 1$, where the boundaries depend on the price wedge ω. By varying the price wedge, we compute the corresponding hazards and distributions.[29]

Finally, with the estimated density $g(\hat{k})$, we recover the hazard function as:

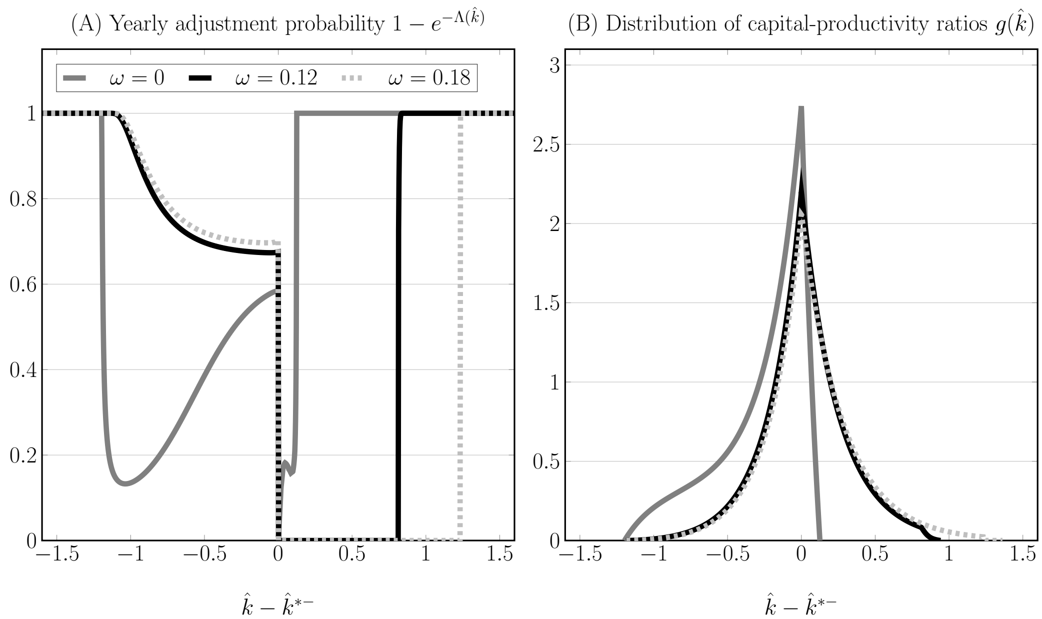
$$\Lambda(\hat{k}) = \begin{cases} \frac{\mathcal{N}h(\hat{k}^{*-} - \hat{k})}{g(\hat{k})} & \text{if } \hat{k} < \hat{k}^{*-}, \\ \frac{\mathcal{N}h(\hat{k}^{*+} - \hat{k})}{g(\hat{k})} & \text{if } \hat{k} > \hat{k}^{*+}. \end{cases} \tag{75}$$

Panel A of Figure VII displays the yearly adjustment probability, $1 - e^{-\Lambda(\hat{k})}$, derived as a transformation of the hazard function. Panel B shows the capital-productivity distribution, $g(\hat{k})$. The x-axes plot capital-productivity ratios relative to the lower reset point $\hat{k} - \hat{k}^{*-}$. We analyze three different wedges, $\omega \in \{0, 0.12, 0.18\}$. Notably, all three specifications align with the empirical investment distribution in Figure VI, despite substantial variations in adjustment probabilities and distributions across price wedges. In essence, given the adjustment frequency $\mathcal{N}$, the generalized hazard approach ensures the existence of an adjustment hazard $\Lambda(\hat{k})$ and a capital-productivity distribution $g(\hat{k})$ such that their product rationalizes the data, $\mathcal{N}h(\Delta\hat{k})$. However, as illustrated in Figure VII, these components can differ significantly, resulting in distinct implications for aggregate fluctuations.

CIR's sufficient statistics.

[29] See Appendix E.5 for additional technical details.

Figure VII: Adjustment Probability and Capital-Productivity Distribution



Notes: Panel A plots the yearly adjustment probability $1 - \exp(-\Lambda(\hat{k}))$ and Panel B plots the capital-productivity distribution $g(\hat{k})$ for price wedges $\omega \in \{0, 0.12, 0.18\}$. In both figures, the x-axis shows capital-productivity ratios relative to the lower reset point $\hat{k} - \hat{k}^{*-}$.

Through the lens of a model without irreversibility ($\omega = 0$, solid black line), the few negative investments arise by limiting $\hat{k}$'s growth and immediately correcting positive capital-productivity ratios through disinvestment. The large mass of small positive investment rates and the convexity of the investment distribution are driven by a decreasing hazard function: firms are more likely to adjust when $\hat{k}$ deviates slightly from the reset point than when the deviation is more considerable. As a result, the adjustment probability is non-monotonic, and the capital-productivity distribution is skewed toward lower ratios.

In contrast, a model with irreversibility ($\omega > 0$, gray lines) introduces an inner inaction region, breaking away from the single reset point. This allows capital-productivity ratios to grow and positively covary with firm age. In this case, firms with high capital-productivity ratios account for the few negative investments observed. The large mass of small positive investment rates and the convexity of the investment distribution is explained by an increasing hazard function, which is the natural outcome under profit optimization: the likelihood of adjustment rises as $\hat{k}$ deviates further from the optimal point.

The adjustment hazard is also higher under irreversibility than without for all $\hat{k}$ in the outer inaction region. Consequently, the density of capital-productivity ratios in these regions is lower, ensuring that the product of the hazard function and the capital-productivity distribution delivers consistent values. This mechanism highlights the distinct dynamics introduced by irreversibility in the model and its implications for micro-level investment behavior.

Table II: CIR's Sufficient Statistics: Data vs. Model

	$\omega = 0.00$				$\omega = 0.12$				$\omega = 0.18$			
	Data		**Model**		**Data**		**Model**		**Data**		**Model**	
CIR	2.54		0.92		2.60		1.93		**2.33**		**2.39**	
Suff. statistics:												
(i) Variance term	1.88	(74)	1.73	(188)	1.68	**(65)**	1.40	**(72)**	1.38	(56)	1.58	(66)
(ii) Covariance term	0.65	(26)	−0.81	(−88)	0.30	**(11)**	0.19	**(10)**	−0.13	(−5)	0.50	(21)
(iii) Irreversibility term	0.00	(0)	0.00	(0)	0.61	**(23)**	0.34	**(18)**	1.20	(49)	0.31	(13)

Notes: CIR computed using Chilean data and calibrated model. The relative importance of each sufficient statistic, expressed in %, is reported in parenthesis. Variance term: $\mathbb{V}ar[\hat{k}]/\sigma^2$, Covariance term: $\nu\mathbb{C}ov[\hat{k}, a]/\sigma^2$, and Irreversibility term: $\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))]\right]/\sigma^2$.

This analysis demonstrates that fully capturing micro-level investment dynamics requires more than just matching the investment distribution. As the figures show, capital-productivity distribution changes with the price wedge. Extra information is necessary to identify the correct parameter configuration. The CIR provides this critical information. Next, we exploit the CIR's sufficient statistics as model discrimination tools to discipline the price wedge.

5.3 Sufficient Statistics for Aggregate Capital Fluctuations

With the adjustment hazard function recovered, we compute the CIR's sufficient statistics to evaluate aggregate capital fluctuations. Table II reports the CIR and its sufficient statistics for Chilean manufacturing plants between 1980 and 2011. We consider three price wedges: $\omega = 0$, $\omega = 0.12$, and $\omega = 0.18$. These wedges illustrate the model's performance across different degrees of irreversibility and provide a basis for selecting the value of ω that best matches the data.

The first case, $\omega = 0$, represents the baseline scenario where irreversibility is absent. While this is a valuable benchmark, it fails to match the CIR's level and decomposition into sufficient statistics. Specifically, the variance term is overstated, the covariance term is negative instead of positive, and the irreversibility term is absent. These discrepancies highlight the necessity of introducing irreversibility to explain the dynamics of aggregate capital fluctuations.[30]

At the other extreme, $\omega = 0.18$ achieves a CIR level (2.39) that matches the data (2.33) remarkably well. However, the decomposition into sufficient statistics diverges significantly from the observed contributions. The variance term dominates excessively, while the covariance and irreversibility terms deviate from their empirical counterparts. Although this wedge captures the overall CIR level, it fails to reflect the actual economic channels driving capital adjustment, making

[30](Baley and Blanco, 2021) show that even with a zero wedge, an extremely high fixed cost for downward adjustments can revert the sign of the covariance to positive. Nevertheless, that parametrization does not generate the observed investment rate distribution.

it unsuitable for accurately representing the underlying dynamics.

The wedge $\omega = 0.12$ represents the intermediate case and provides the best balance. While the CIR level (1.93) is slightly below the observed value, the decomposition into sufficient statistics aligns closely with the data. The variance term accounts for 72%, followed by irreversibility and covariance, mirroring the empirical structure. This wedge correctly captures the mechanisms driving aggregate fluctuations, making it the preferred choice.

5.4 Disciplining the Price Wedge

We select $\omega = 0.12$ as the preferred value for the price wedge because it maximizes consistency between the model and the data regarding sufficient statistics. While it does not perfectly match the CIR level, it accurately captures the decomposition, reflecting the relative importance of variance, covariance, and irreversibility in driving aggregate capital fluctuations. Prioritizing mechanisms over level ensures the model represents the actual economic dynamics.

At $\omega = 0.12$, we recover a CIR of 1.93, meaning a 1% decrease in aggregate productivity leads to a nearly 2% deviation of average capital-productivity ratios from their steady-state value. The first sufficient statistic, $\mathbb{V}ar[\hat{k}]/\sigma^2$, accounts for 72% of the CIR, highlighting the dominant role of variance. The second statistic, $-\nu\mathbb{C}ov[\hat{k}, a]/\sigma^2$, captures the covariance channel, which represents 10% of the CIR. Lastly, the third statistic, $\mathbb{E}\left[\mathbb{E}_s\left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)\right]/\mathrm{d}s\right]/\sigma^2$, measures the contribution of irreversibility which is 18% of the CIR. Together, these statistics provide a clear decomposition of the CIR and validate $\omega = 0.12$ as the most reliable representation of capital adjustment dynamics. Besides the CIR, these sufficient statistics match the distribution of investment rates.

5.5 Price Wedges in the Literature

We compare our preferred price wedge value, $\omega = 0.12$, with estimates from existing literature and discuss alternative approaches. To express values in the same units, the price wedge is calculated as one minus the recovery rate—the liquidation value over replacement cost net of depreciation.[31]

Empirical studies provide direct evidence of price wedges, often based on observed recovery rates. Ramey and Shapiro (2001) analyze capital reallocation from closing aerospace plants in the United States, estimating a price wedge of 0.72, which reflects significant discounts during liquidation. Kermani and Ma (2023) document industry-level wedges of around 0.65 for plant, property, and equipment, consistent with the high levels of asset specificity in these sectors. These estimates are likely upward biased due to selection effects, as they are based on liquidating firms subject to fire-sale dynamics. Surveys offer complementary insights. Dibiasi, Mikosch and Sarferaz (2021) survey Swiss CEOs and CFOs and find an average price wedge of approximately 0.47.

[31] Appendix F summarizes the values reported in the literature.

Structural quantitative models estimate wedges by calibrating them to static features of the investment rate distribution. For instance, Bloom (2009) calibrates a wedge of 0.34, while Fang (2023) and Senga and Varotto (2024) report values between 0.30 and 0.41. These larger wedges typically reflect settings with substantial capital specificity or adjustment frictions. In contrast, models such as Cooper and Haltiwanger (2006), Khan and Thomas (2013), and Lanteri (2018) rely on smaller wedges, ranging from 0.025 to 0.07, indicating less severe frictions or more fluid capital adjustment.

Our estimate of $\omega = 0.12$, which relies on matching the CIR and the micro-dynamic moments, lies in between. This value likely captures heterogeneity across sectors and capital types and the effects of internal capital transfers, such as mergers and acquisitions, that may mitigate irreversibility for some firms (Bhandari, Martellini and McGrattan, 2024).

6 Final Thoughts

We investigate how partially irreversible investment shapes aggregate fluctuations. Our approach innovates by characterizing fluctuations with lumpy adjustments across two reset points, using (i) conditioning on prior resets, (ii) transition probabilities across resets, and (iii) microdata to discipline those transitions. Our flexible methodology can accommodate a finite number of reset points and applies broadly wherever sufficient microdata exist to discipline transitions. Extensions of our framework could study financial frictions by linking reset points to fund availability through firm-level financial data.

We outline four directions for future research. First, while we focus on aggregate productivity as a source of fluctuations, our framework applies to other aggregate shocks, such as changes in profitability, capital prices, or interest rates.[32] This opens avenues for studying corporate tax reforms (Altug, Demers and Demers, 2009; Winberry, 2021; Chen, Jiang, Liu, Suárez-Serrato and Xu, 2023) or monetary policy and their interaction with investment frictions (Fang, 2023; Baley, Blanco and Oviedo, 2024).

Second, our analysis assumes fixed price wedges and interest rates suited to small open economies. However, ample evidence shows that wedges vary across the business cycle and are endogenously determined in secondary markets (Lanteri, 2018; Gavazza and Lanteri, 2021). Incorporating time-variation in price wedges and interest rates are natural extensions in this direction, allowing the assessment of general equilibrium effects (Veracierto, 2002; Gourio and Kashyap, 2007).

Third, while we focus on first-moment shifts in the capital-productivity distribution, our methodology can analyze higher-order moments, such as dispersion or skewness, by adapting the CIR's sufficient statistics framework. We characterize the CIR for any continuous function

[32] Shocks to the price wedge or the stochastic process of capital-productivity ratios would entail changes in the sufficient statics, so we leave them for future study.

$f(\hat{k})$; thus, setting $f(\hat{k}) = \hat{k}^m$ allows characterizations of cross-sectional moments ($m = 2$ for variance, $m = 3$ for skewness), while $f(\hat{k}) = e^{\alpha\hat{k}}$ can characterize aggregate output.[33]

Finally, we focus on small aggregate shocks (δ) and first-order perturbations, potentially ignoring nonlinearities and the response to large shocks. Appendix G studies numerically the non-linearities in the generalized hazard model regarding the sign and magnitude of aggregate productivity shocks. We find tiny non-linearities and asymmetries for productivity shocks below $\delta = 5\%$.

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[33]In this case, the baseline model requires additional features to make the economy stationary (the baseline economy with Brownian idiosyncratic shocks does not feature a stationary distribution), such as stochastic firm exit or monopolistic competition with quality shocks (see Appendix D).

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The Macroeconomics of Irreversibility

Isaac Baley and Andrés Blanco

Online Appendix

Contents

A Proofs

A.1 Auxiliary Theorems

Auxiliary Theorem 1 (Optional Sampling Theorem, OST). *Let $\hat{k}$ be a martingale on the filtered space $(\Omega, \mathbb{P}, \mathcal{F})$ and let τ be a stopping time. If $(\{\hat{k}_t\}_t, \tau)$ is a well-defined stopping process, then*

$$\mathbb{E}[\hat{k}_\tau] \;=\; \mathbb{E}[\hat{k}_0]. \tag{A.1}$$

See Theorem 4.4 in Stokey (2009). This result establishes that, under certain conditions, the expected value of a martingale at a stopping time is equal to its initial expected value. We use this result to derive the mappings between the cross-sectional moments of adjusters and non-adjusters.

Auxiliary Theorem 2 (Occupancy Measure Theorem, OMT). *Let $\hat{k}_t$ be a Brownian motion, τ a stopping time, and $\hat{k}_\tau = \hat{k}^*$ a constant reset state. Let G be the ergodic distribution of $\hat{k}$. Consider a real-valued function $f(\hat{k})$ such that $\int f(\hat{k})\,\mathrm{d}G(\hat{k}) = \lim_{T\to\infty} T^{-1}\int_0^T f(\hat{k}_t)\,\mathrm{d}t$ for all initial $\hat{k}_0$. Then the following relationship holds:*

$$\underbrace{\mathbb{E}\left[\int_0^\tau f(\hat{k}_t)\,\mathrm{d}t \,\middle|\, \hat{k}_0 = \hat{k}^*\right]}_{\text{occupancy measure}} \;=\; \underbrace{\int f(\hat{k})\,\mathrm{d}G(\hat{k})}_{\text{steady-state mass}} \quad \underbrace{\mathbb{E}\left[\tau \,\middle|\, \hat{k}_0 = \hat{k}^*\right]}_{\text{proportionality constant}}. \tag{A.2}$$

See Stokey (2009) and the Green measure in Chapter 9 of Oksendal (2007)). This result establishes the equivalence between the occupancy measure—the average time an agent's state spends at a given value—and the stationary mass of agents at that particular state, with a proportionality constant equal to the expected time between adjustments. E.g., if $f(\hat{k}) = \hat{k}^m$, then $\mathbb{E}\left[\int_0^\tau \hat{k}_t^m\,\mathrm{d}t|\hat{k}_0 = \hat{k}^*\right] = \mathbb{E}[\hat{k}^m]\mathbb{E}[\tau|\hat{k}_0 = \hat{k}^*]$. We use this theorem to convert occupancy measures, scaled by frequency, into steady-state cross-sectional moments. We use $\overline{\mathbb{E}}[\cdot] = \mathbb{E}[\cdot|\hat{k}_0 = \hat{k}^*]$.

Auxiliary Theorem 3 (Equivalence between sequential and recursive formulations). *Let $\hat{k}_s$ be a Brownian motion ($\mathrm{d}\hat{k}_s = -\nu\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$) inside a continuation region $\mathcal{R}$. Let $g(\hat{k})$ be the flow payoff and $\phi(\hat{k}_s)$ be the terminal payoff(s). Define a (non-optimal) value $w(\hat{k})$ using a sequential formulation as follows:*

$$w(\hat{k}) \;\equiv\; \mathbb{E}\left[\int_0^{\tau_\mathcal{R}} e^{-\rho\tau_\mathcal{R}} g(\hat{k}_s)\,\mathrm{d}s \,\middle|\, \hat{k}_0 = \hat{k}\right] \;+\; \mathbb{E}\left[e^{-\rho\tau_\mathcal{R}}\phi(\hat{k}_{\tau_\mathcal{R}}) \,\middle|\, \hat{k}_0 = \hat{k}\right], \quad \forall \hat{k} \in \mathcal{R}, \tag{A.3}$$

where $\tau_\mathcal{R}$ is <u>any stopping time</u>. Under certain regularity conditions over $\mathcal{R}$, $g(\hat{k})$, and $\phi(\hat{k})$, we have that $\forall \hat{k} \in \mathcal{R}$:

$$\text{(HJB)} \qquad \rho w(\hat{k}) \;=\; g(\hat{k}) \;-\; \nu w'(\hat{k}) \;+\; \frac{\sigma^2}{2} w''(\hat{k}), \tag{A.4}$$

$$\text{(Value Matching)} \qquad \lim_{t\uparrow\tau_\mathcal{R}} w(\hat{k}_t) \;=\; \phi(\hat{k}_{\tau_\mathcal{R}}), \quad a.s. \tag{A.5}$$

If $\tau_\mathcal{R}$ is an <u>optimal stopping time</u> then the smooth-pasting condition also holds:

$$\text{(Smooth Pasting)} \qquad \lim_{t\uparrow\tau_\mathcal{R}} w'(\hat{k}_t) \;=\; \phi(\hat{k}_{\tau_\mathcal{R}}), \quad a.s. \tag{A.6}$$

If there exist a function $w_1 \in \mathbb{C}^2(\mathcal{R})$ and w_1 satisfies (A.4) and (A.5) (and (A.6) if optimal), then $w_1 = w$.

See Chapters 9 and 10 in Oksendal (2007). These results allow us to go back and forth between w's sequential formulation—given by the cumulative flow payoff during inaction plus the value at termination—and the recursive formulations—with an HJB in the interior of the inaction region, value-matching conditions when stopping, and smooth pasting conditions if the stopping policy is optimal.

A.2 Proof of Proposition 1

Proposition 1. ***(Optimal policy)*** *Marginal $q(\hat{k})$ and the optimal policy $\{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ is characterized by the following sufficient optimality conditions:*

(i) Inside the inaction region $\mathcal{R}$, $q(\hat{k})$ solves the Hamilton-Jacobini-Bellman (HJB) equation:

$$\mathcal{U}q(\hat{k}) = \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} \; - \; \nu q'(\hat{k}) \; + \; \frac{\sigma^2}{2} q''(\hat{k}), \qquad \forall\, \hat{k} \in (\hat{k}^-, \hat{k}^+). \tag{11}$$

(ii) In the outer inaction regions, $q(\hat{k})$ satisfies the value-matching conditions:

$$\frac{\theta}{p} \;=\; \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}} \left(q(\hat{k}) - 1\right) \mathrm{d}\hat{k}, \qquad \forall\, \hat{k} \in [\hat{k}^-, \hat{k}^{*-}], \tag{12}$$

$$\frac{\theta}{p} \;=\; \int_{\hat{k}^{*+}}^{\hat{k}^{+}} e^{\hat{k}} \left((1-\omega) - q(\hat{k})\right) \mathrm{d}\hat{k} \qquad \forall\, \hat{k} \in [\hat{k}^{*+}, \hat{k}^+]. \tag{13}$$

(iii) At the borders of the inaction region and reset points, $q(\hat{k})$ satisfies the optimality conditions:

$$q(\hat{k}) \;=\; 1, \qquad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \tag{14}$$

$$q(\hat{k}) \;=\; 1-\omega, \qquad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \tag{15}$$

From these conditions, q's stopping-time formulation is given by

$$q(\hat{k}) \;=\; \mathbb{E}\left[\int_0^{\tau} \frac{\alpha e^{-\mathcal{U}s + (\alpha-1)\hat{k}_s}}{p}\, \mathrm{d}s \;+\; e^{-\mathcal{U}\tau} q(\hat{k}_\tau)\right]. \tag{16}$$

Proof's strategy We divide this proof into three steps.

1. In Step 1, we characterize the two-state value function $V(k,u)$ and optimal policies $(k^-(u), k^{*-}(u), k^{*+}(u), k^+(u))$ through the Hamilton-Jacobi-Bellman (HJB) equation, value matching, optimality, and smooth pasting conditions.

2. In Step 2, we guess that $V(k,u) = uv(\hat{k})$, where $v(\hat{k})$ is a function of the log capital-to-productivity ratio $\hat{k} \equiv \log(k/u)$. Using the guess, we exploit homotheticity in the firm's programming problem to express optimality conditions as joint conditions between $v(\hat{k})$ and the firm's policy $(\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+)$. We use the sufficient conditions that characterize the two-state value function and optimal policies from Step 1 to reduce the state space into one dimension. The corresponding policies are $(\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+)$. We verify the guess by showing that the sufficient conditions for $V(\cdot)$ are equivalent to those satisfied by $v(\cdot)$.

3. In Step 3, we note that $q(\hat{k}) = v'(\hat{k})/(pe^{\hat{k}})$ and establish its sufficient optimality conditions reexpressing the HJB and optimality conditions for v in Step 2. An advantage of characterizing the policy with $q(\hat{k})$ is that it shares the infinitesimal generator with $\hat{k}$; this is not the case with $v(\hat{k})$ as its drift equals $-(\nu + \sigma^2)$.

A.2.1 Step 1: Characterize the two-state value function $V(k,u)$

Substitute output $y_s = u_s^{1-\alpha} k_s^\alpha$ from (1) and the adjustment costs $\theta_s = \theta u_s$ from (3) into the firm problem in (5):

$$V(k_0, u_0) \;=\; \max_{\{T_h,\, i_{T_h}\}_{h=1}^{\infty}} \mathbb{E}\left[\int_0^\infty e^{-rs} u_s^{1-\alpha} k_s^\alpha \,\mathrm{d}s \;-\; \sum_{h=1}^{\infty} e^{-rT_h} \left(\theta u_{T_h} \;+\; p\left(\Delta k_{T_h}\right) i_{T_h}\right)\right]. \tag{A.7}$$

Using the principle of optimality, we get a recursive stopping-time problem with initial conditions $(k_0, u_0) = (k, u)$:

$$\text{(A.8)} \qquad V(k,u) \;=\; \max_{\tau,\ \Delta k_\tau} \; \mathbb{E}\left[\int_0^\tau e^{-rs} u_s^{1-\alpha} k_s^\alpha \, \mathrm{d}s \;+\; e^{-r\tau}\left(-\theta u_\tau - \left(p(\Delta k_\tau)\right)\Delta k_\tau + V\left(k_{\tau^-} + \Delta k_\tau, u_\tau\right)\right)\right],$$

where we change the notation to $i_{T_h} = \Delta k_\tau = k_\tau - k_{\tau^-}$.

Let $\mathcal{R}$ be the firms' inaction region that is equal to $\mathcal{R} \equiv \{(k,u) : k^-(u) < k < k^+(u)\}$, where $k^-(u)$ is the lower inaction threshold that triggers a positive investment, and $k^+(u)$ is the upper inaction thresholds that trigger a negative investment. For each level of productivity u, let $\mathcal{R}^- \equiv \{(k,u) : k = k^-(u)\}$ denote the lower border of the inaction set and $k^{*-}(u)$ the reset capital after positive adjustment, where $\Delta k(u) = k^{*-}(u) - k^-(u) > 0$. Analogously, we denote the upper border of inaction set as $\mathcal{R}^+ \equiv \{(k,u) : k = k^+(u)\}$ with a reset capital after negative adjustment as $k^{*+}(u)$, where $\Delta k(u) = k^{*+}(u) - k^+(u) < 0$.

Optimality conditions for $V(k,u)$ The value $V(k,u)$ and the optimal policy $(k^-(u), k^{*-}(u), k^{*+}(u), k^+(u))$ satisfy the system of sufficient conditions in (A.9) to (A.17):

1. Inside the inaction region $\mathcal{R}$, $V(k,u)$ solves the HJB equation:

$$\text{(A.9)} \qquad rV(k,u) \;=\; u\left(\frac{k}{u}\right)^\alpha - \xi k \frac{\partial V(k,u)}{\partial k} + \left(\mu + \frac{\sigma^2}{2}\right) u \frac{\partial V(k,u)}{\partial u} + \frac{\sigma^2 u^2}{2} \frac{\partial^2 V(k,u)}{\partial u^2} \qquad \forall (k,u) \in \mathcal{R}.$$

2. Value matching conditions equalize the value of action and inaction at the borders of the inaction region:

$$\text{(A.10)} \qquad V(k^{*-}(u), u) \;-\; p \quad \Delta k(u) - \theta u \;=\; V(k^-(u), u) \quad \forall (k,u) \in \mathcal{R}^-,$$

$$\text{(A.11)} \qquad V(k^{*+}(u), u) \;-\; p(1-\omega)\Delta k(u) - \theta u \;=\; V(k^+(u), u) \quad \forall (k,u) \in \mathcal{R}^+.$$

3. The two optimality conditions for the reset capitals $\{k^{*-}, k^{*+}\}$ are:

$$\text{(A.12)} \qquad \frac{\partial V(k^{*-}(u), u)}{\partial k} \;=\; p,$$

$$\text{(A.13)} \qquad \frac{\partial V(k^{*+}(u), u)}{\partial k} \;=\; p(1-\omega).$$

4. The four smooth pasting conditions are:

$$\text{(A.14)} \qquad \frac{\partial V(k,u)}{\partial k} \;=\; p \qquad \forall (k,u) \in \mathcal{R}^-,$$

$$\text{(A.15)} \qquad \frac{\partial V(k,u)}{\partial k} \;=\; p(1-\omega) \quad \forall (k,u) \in \mathcal{R}^+,$$

$$\text{(A.16)} \qquad \frac{\partial V(k^{*-}(u), u)}{\partial u} \;=\; \theta + \frac{\partial V(k,u)}{\partial u} \quad \forall (k,u) \in \mathcal{R}^-,$$

$$\text{(A.17)} \qquad \frac{\partial V(k^{*+}(u), u)}{\partial u} \;=\; \theta + \frac{\partial V(k,u)}{\partial u} \quad \forall (k,u) \in \mathcal{R}^+,$$

For additional details on the sufficiency of these conditions, see Baley and Blanco (2019, 2021).

A.2.2 Step 2: Characterize the one-state value $v(\hat{k}) = V(k,u)/u$

We guess that $V(k,u)$ is separable:

$$\text{(A.18)} \qquad V(k,u) \;=\; u \times v\left(\log\left(\frac{k}{u}\right)\right) \;=\; uv(\hat{k}),$$

with associated policies

$$(k^-(u), k^{*-}(u), k^{*+}(u), k^+(u)) \; = \; u \times (e^{\hat{k}^-}, e^{\hat{k}^{*-}}, e^{\hat{k}^{*+}}, e^{\hat{k}^+}). \tag{A.19}$$

Given the guess (A.18), the derivatives of $V(k,u)$ and the derivatives of $v(\hat{k})$ satisfy the following relationships:

$$\frac{\partial V(k,u)}{\partial k} \; = \; \frac{u}{k} v'\left(\log\left(\frac{k}{u}\right)\right) = \frac{u}{k} v'(\hat{k}), \tag{A.20}$$

$$\frac{\partial V(k,u)}{\partial u} \; = \; v\left(\log\left(\frac{k}{u}\right)\right) - v'\left(\log\left(\frac{k}{u}\right)\right) = v(\hat{k}) - v'(\hat{k}), \tag{A.21}$$

$$\frac{\partial^2 V(k,u)}{\partial u^2} \; = \; -\frac{v'\left(\log\left(\frac{k}{u}\right)\right)}{u} + \frac{v''\left(\log\left(\frac{k}{u}\right)\right)}{u} = -\frac{v'(\hat{k})}{u} + \frac{v''(\hat{k})}{u}. \tag{A.22}$$

2a. HJB Substituting the guess into (A.9):

$$rV(k,u) = u\left(\frac{k}{u}\right)^{\alpha} - \xi k \frac{\partial V(k,u)}{\partial k} + \left(\mu + \frac{\sigma^2}{2}\right) u \frac{\partial V(k,u)}{\partial u} + \frac{\sigma^2 u^2}{2}\frac{\partial^2 V(k,u)}{\partial u^2}, \tag{A.23}$$

$$ruv(\hat{k}) = \; ue^{\alpha\hat{k}} - \xi k \frac{u}{k} v'(\hat{k}) + \left(\mu + \frac{\sigma^2}{2}\right) u(v(\hat{k}) - v'(\hat{k})) + \frac{\sigma^2 u^2}{2}\left(\frac{v''(\hat{k})}{u} - \frac{v'(\hat{k})}{u}\right). \tag{A.24}$$

Joining terms we get:

$$\left(r - \mu - \frac{\sigma^2}{2}\right) uv(\hat{k}) \; = \; ue^{\alpha\hat{k}} - (\mu + \xi + \sigma^2)uv'(\hat{k}) + \frac{\sigma^2}{2} uv''(\hat{k}). \tag{A.25}$$

Defining new parameters $\nu \equiv \mu + \xi$ and $\rho \equiv r - \mu - \sigma^2/2$, and dividing both sides by u, we obtain the HJB:

$$\rho v(\hat{k}) \; = \; e^{\alpha\hat{k}} - (\nu + \sigma^2)v'(\hat{k}) + \frac{\sigma^2}{2} v''(\hat{k}). \tag{A.26}$$

2b. Value matching Substituting the guess into (A.10) and (A.11):

$$u_\tau v(\hat{k}^{*-}) \; - \; p\Delta k \; - \; \theta u_\tau \; = \; u_\tau v(\hat{k}^-) \; , \tag{A.27}$$

$$u_\tau v(\hat{k}^{*+}) \; - \; p(1-\omega)\Delta k \; - \; \theta u_\tau \; = \; u_\tau v(\hat{k}^-) \; . \tag{A.28}$$

Next, we express investment in terms of changes in capital productivity ratios $\hat{k}$. The expression (9), which reads $\Delta\hat{k} = \log(1 + \Delta k / k_{\tau^-})$, implies $\Delta k = e^{\Delta\hat{k}} k_{\tau^-} - k_{\tau^-}$; multiplying and dividing by u_τ and substituting the definition of $\hat{k}$ yields: $\Delta k = u_\tau\left(e^{\Delta\hat{k} + \hat{k}_\tau} - e^{\hat{k}_\tau}\right)$. Then we use $\hat{k}_\tau = \hat{k}^+$ or $\hat{k}_\tau = \hat{k}^-$ accordingly. Using this notation, we rewrite positive investment as $\Delta k = u_\tau(e^{\Delta\hat{k} + \hat{k}^-} - e^{\hat{k}^-})$ and negative investment as $\Delta k = u_\tau(e^{\Delta\hat{k} + \hat{k}^+} - e^{\hat{k}^+})$. Substituting into (A.27) and (A.28) and dividing both sides by u_τ

$$v(\hat{k}^{*-}) - p(e^{\hat{k}^{*-}} - e^{\hat{k}^-}) - \theta \; = \; v(\hat{k}^-), \tag{A.29}$$

$$v(\hat{k}^{*+}) - p(1-\omega)(e^{\hat{k}^{*+}} - e^{\hat{k}^+}) - \theta \; = \; v(\hat{k}^+). \tag{A.30}$$

2c. Optimality Substituting the guess into (A.12) and (A.13):

$$\frac{\partial V(k^{*-}(u), u)}{\partial k} \; = \; p \iff \frac{u}{k^{*-}(u)} v'(\hat{k}^{*-}) \; = \; p \iff v'(\hat{k}^{*-}) \; = \; pe^{\hat{k}^{*-}}, \tag{A.31}$$

$$\frac{\partial V(k^{*+}(u), u)}{\partial k} \; = \; p(1-\omega) \iff \frac{u}{k^{*+}(u)} v'(\hat{k}^{*-}) \; = \; p(1-\omega) \iff v'(\hat{k}^{*-}) \; = \; p(1-\omega)e^{\hat{k}^{*+}}. \tag{A.32}$$

2d. Smooth pasting for capital Substituting the guess into (A.14) and (A.15)

$$\text{(A.33)} \quad \frac{\partial V(k^-(u),u)}{\partial k} = p \iff \frac{u}{k^-(u)} v'(\hat{k}^-) = p \iff v'(\hat{k}^-) = pe^{\hat{k}^-},$$

$$\text{(A.34)} \quad \frac{\partial V(k^+(u),u)}{\partial k} = p(1-\omega) \iff \frac{u}{k^+(u)} v'(\hat{k}^+) = p(1-\omega) \iff v'(\hat{k}^+) = p(1-\omega)e^{\hat{k}^+}.$$

2e. Smooth pasting for idiosyncratic productivity To verify the smooth-pasting for idiosyncratic productivity, we substitute the guess into (A.16) and (A.17) and then substitute $v'(\hat{k}) = pe^{\hat{k}}$ and $v'(\hat{k}) = p(1-\omega)e^{\hat{k}}$ in the outer inaction regions to rewrite $v'(\cdot)$ in terms of prices, which yields

$$\text{(A.35)} \quad v(\hat{k}^{*-}) = \theta + v(\hat{k}^-) + p(e^{\hat{k}^{*-}} - e^{\hat{k}^-})$$

$$\text{(A.36)} \quad v(\hat{k}^{*+}) = \theta + v(\hat{k}^+) + p(1-\omega)(e^{\hat{k}^{*+}} - e^{\hat{k}^+})$$

Summary The value $v(\hat{k})$ and the optimal policy $\{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ satisfy the conditions:

(i) In the inaction region $\mathcal{R}$, $v(\hat{k})$ solves the HJB equation:

$$\text{(A.37)} \quad \rho v(\hat{k}) = e^{\alpha\hat{k}} - (\nu+\sigma^2)v'(\hat{k}) + \frac{\sigma^2}{2}v''(\hat{k}), \ \forall \hat{k} \in (\hat{k}^-, \hat{k}^+).$$

(ii) At the borders of inaction, $v(\hat{k})$ satisfies the value-matching conditions:

$$\text{(A.38)} \quad v(\hat{k}^-) = v(\hat{k}^{*-}) - \theta - p(e^{\hat{k}^{*-}} - e^{\hat{k}^-}),$$

$$\text{(A.39)} \quad v(\hat{k}^+) = v(\hat{k}^{*+}) - \theta + p(1-\omega)(e^{\hat{k}^+} - e^{\hat{k}^{*+}}).$$

(iii) At the borders of inaction and reset states, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$\text{(A.40)} \quad v'(\hat{k}) = pe^{\hat{k}}, \qquad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\},$$

$$\text{(A.41)} \quad v'(\hat{k}) = p(1-\omega)e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}.$$

A.2.3 Step 3: Characterizing $q = v'(\hat{k})/pe^{\hat{k}}$

From the definition $q(\hat{k}) \equiv \frac{\partial V(k,u)}{\partial k}/p$, and the decomposition $V(k,u) = uv(\hat{k})$ from Step 2, we have that $q(\hat{k}) = \frac{v'(\hat{k})}{pe^{\hat{k}}}$. Thus, the following relationships hold:

$$\text{(A.42)} \quad q'(\hat{k}) = \frac{v''(\hat{k})}{pe^{\hat{k}}} - \frac{v'(\hat{k})}{pe^{\hat{k}}} = \frac{v''(\hat{k})}{pe^{\hat{k}}} - q(\hat{k}) \iff \frac{v''(\hat{k})}{pe^{\hat{k}}} = q'(\hat{k}) + q(\hat{k})$$

$$\text{(A.43)} \quad q''(\hat{k}) = \frac{v'''(\hat{k})}{pe^{\hat{k}}} - 2\frac{v''(\hat{k})}{pe^{\hat{k}}} + \frac{v'(\hat{k})}{pe^{\hat{k}}} = \frac{v'''(\hat{k})}{pe^{\hat{k}}} - 2q'(\hat{k}) - q(\hat{k}) \iff \frac{v'''(\hat{k})}{pe^{\hat{k}}} = q''(\hat{k}) + 2q'(\hat{k}) + q(\hat{k}).$$

3a. HJB We take the first derivative of the HBJ equation for v in (A.37) and then divide by $pe^{\hat{k}}$:

$$\text{(A.44)} \quad \rho\frac{v'(\hat{k})}{pe^{\hat{k}}} = \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} - (\nu+\sigma^2)\frac{v''(\hat{k})}{pe^{\hat{k}}} + \frac{\sigma^2}{2}\frac{v'''(\hat{k})}{pe^{\hat{k}}}, \ \forall \hat{k} \in (\hat{k}^-, \hat{k}^+).$$

Substituting q's definition and the second and third derivatives of v in (A.42) and (A.43):

(A.45) $$\rho q(\hat{k}) \;=\; \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} \;-\; (\nu+\sigma^2)\left(q'(\hat{k})+q(\hat{k})\right) \;+\; \frac{\sigma^2}{2}\left(q''(\hat{k})+2q'(\hat{k})+q(\hat{k})\right).$$

Joining common terms:

(A.46) $$\left(\rho+\nu+\frac{\sigma^2}{2}\right)q(\hat{k}) \;=\; \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} \;-\; \nu q'(\hat{k}) \;+\; \frac{\sigma^2}{2}q''(\hat{k}),$$

Since $\rho \equiv r-\mu-\sigma^2/2$, then $\rho+\nu+\frac{\sigma^2}{2} = r+\xi := \mathcal{U}$. Substitute to obtain the final expression for $q's$ HJB:

(A.47) $$\mathcal{U}q(\hat{k}) \;=\; \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} \;-\; \nu q'(\hat{k}) \;+\; \frac{\sigma^2}{2}q''(\hat{k}), \qquad \forall\;\hat{k}\in(\hat{k}^-,\hat{k}^+).$$

3b. Value matching The value matching at $\hat{k}^-$ in (A.38) can be written as a definite integral:

(A.48) $$\theta \;=\; v(\hat{k}^{*-}) \;-\; pe^{\hat{k}^{*-}} \;-\; (v(\hat{k}^-)-pe^{\hat{k}^-}) \;=\; \int_{\hat{k}^-}^{\hat{k}^{*-}}\left(v'(\hat{k}) \;-\; pe^{\hat{k}}\right)\mathrm{d}\hat{k}.$$

Dividing both sides by p, factoring $e^{\hat{k}}$ on the right, and substituting q, we obtain the value matching for q at $\hat{k}^-$:

(A.49) $$\frac{\theta}{p} \;=\; \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left(\frac{v'(\hat{k})}{pe^{\hat{k}}} \;-\; 1\right)\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left(q(\hat{k}) \;-\; 1\right)\mathrm{d}\hat{k}.$$

Similarly, we use (A.39) to obtain the value matching for q at $\hat{k}^+$:

(A.50) $$\frac{\theta}{p} \;=\; \int_{\hat{k}^{*+}}^{\hat{k}^+} e^{\hat{k}}\left((1-\omega)-q(\hat{k})\right)\mathrm{d}\hat{k}.$$

3c. Optimality Substituting q's definition in the optimality conditions for v in (A.40) and (A.41)

(A.51) $$v'(\hat{k}) \;=\; pe^{\hat{k}} \iff q(\hat{k})=1 \qquad \hat{k}\in\left\{\hat{k}^-,\hat{k}^{*-}\right\}$$

(A.52) $$v'(\hat{k}) \;=\; p(1-\omega)e^{\hat{k}} \iff q(\hat{k})=(1-\omega) \quad \hat{k}\in\left\{\hat{k}^{*+},\hat{k}^+\right\}.$$

3d. Stopping-time formulation Given the sufficient conditions, we write the optimal $q(\hat{k})$ using a stopping-time formulation (note that there is no maximization involved):

(A.53) $$q(\hat{k}) \;\equiv\; \mathbb{E}\left[\int_0^\tau \frac{\alpha e^{-\mathcal{U}s+(\alpha-1)\hat{k}_s}}{p}\,\mathrm{d}s \;+\; e^{-\mathcal{U}\tau}Q^*(\hat{k}_\tau)\right],$$

where the reset function takes two values: $Q^*(\hat{k}^{*-})=1$ and $Q^*(\hat{k}^{*+})=1-\omega$.

A.3 Cross-sectional distributions

Consider $\theta_s = \theta u_s$, where $\theta > 0$ is a constant fixed adjustment cost. The density and frequencies solve the KFE

$$\nu g'(\hat{k}) \; + \; \frac{\sigma^2}{2} g''(\hat{k}) \; = \; 0, \quad \text{for all} \quad \hat{k} \in (\hat{k}^-, \hat{k}^+) \setminus \{\hat{k}^{*-}, \hat{k}^{*+}\}; \tag{A.54}$$

three border conditions

$$g(\hat{k}) \; = \; 0, \quad \text{for} \;\; \hat{k} \in \{\hat{k}^{*-}, \hat{k}^{*+}\}, \tag{A.55}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k}) \, \mathrm{d}\hat{k} \; = \; 1; \tag{A.56}$$

two resetting conditions

$$\underbrace{\frac{\sigma^2}{2} \lim_{\hat{k} \downarrow \hat{k}^-} g'(\hat{k})}_{\mathcal{N}^-} \; = \; \frac{\sigma^2}{2} \left[\lim_{\hat{k} \uparrow \hat{k}^{*-}} g'(\hat{k}) - \lim_{\hat{k} \downarrow \hat{k}^{*-}} g'(\hat{k}) \right], \tag{A.57}$$

$$\underbrace{-\frac{\sigma^2}{2} \lim_{\hat{k} \uparrow \hat{k}^+} g'(\hat{k})}_{\mathcal{N}^+} \; = \; \frac{\sigma^2}{2} \left[\lim_{\hat{k} \uparrow \hat{k}^{*+}} g'(\hat{k}) - \lim_{\hat{k} \downarrow \hat{k}^{*+}} g'(\hat{k}) \right], \tag{A.58}$$

and two continuity conditions at the reset points:

$$g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^{*-}, \hat{k}^{*+}\}), \mathbb{C}^2(\{\hat{k}^{*-}, \hat{k}^{*+}\}). \tag{A.59}$$

Condition (A.55) sets the mass of firms at the inaction thresholds equal to zero. Condition (A.56) ensures that g is a density. Conditions (A.57) and (A.58) relate the masses of upward and downward adjustments to the discontinuities in the derivative of g at the reset points. In a small period of time $\mathrm{d}s$, the mass $\mathcal{N}^-$ that "exits" the inaction region by hitting the lower threshold—equal to $\frac{\sigma^2}{2} \lim_{\hat{k} \downarrow \hat{k}^-} g'(\hat{k})$—must coincide with the mass of firms that "enters" at the reset point $\hat{k}^{*-}$—equal to the jump in g'. This argument is analogous for $\mathcal{N}^+$; in fact, it is straightforward to verify that conditions (A.54) to (A.57) jointly imply condition (A.58), and thus it is redundant.

A.4 Distributions of stopping times τ

Conditional on current capital-productivity ratio Given the inaction thresholds $\hat{k}^- < \hat{k}^+$, we derive the densities of stopping times (first passage time) when firms hit the ℓower threshold $\ell(\tau|\hat{k})$, the υpper threshold $\upsilon(\tau|\hat{k})$, or $h$itting either threshold $h(\tau|\hat{k})$, conditional on a current capital-productivity ratio $\hat{k}$. The first passage time is set to zero after a reset. We use the formulas of the exit times densities when barriers are flat (15) and (16) in Kolkiewicz (2002), adjusted for the drift $-\nu$ and the volatility σ.

- The measure of times for hitting the *lower threshold* for current $\hat{k}$ is

$$\ell(\tau|\hat{k}) = \left(\frac{\pi\sigma^2}{(\hat{k}^+ - \hat{k}^-)^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left[\pi n\frac{(\hat{k}^+ - \hat{k})}{(\hat{k}^+ - \hat{k}^-)}\right]\exp\left[-\frac{n^2\pi^2\sigma^2\tau}{2(\hat{k}^+ - \hat{k}^-)^2}\right]\right) \times \exp\left[\frac{-\nu}{2\sigma^2}(2(\hat{k}^+ - \hat{k}) + \nu\tau)\right]. \tag{A.60}$$

- The measure of times for hitting the *upper threshold* for current $\hat{k}$ is

$$\upsilon(\tau|\hat{k}) = \left(\frac{\pi\sigma^2}{(\hat{k}^+ - \hat{k}^-)^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left[\pi n\frac{(\hat{k} - \hat{k}^-)}{(\hat{k}^+ - \hat{k}^-)}\right]\exp\left[-\frac{n^2\pi^2\sigma^2\tau}{2(\hat{k}^+ - \hat{k}^-)^2}\right]\right) \times \exp\left[\frac{-\nu}{2\sigma^2}(2(\hat{k}^- - \hat{k}) + \nu\tau)\right]. \tag{A.61}$$

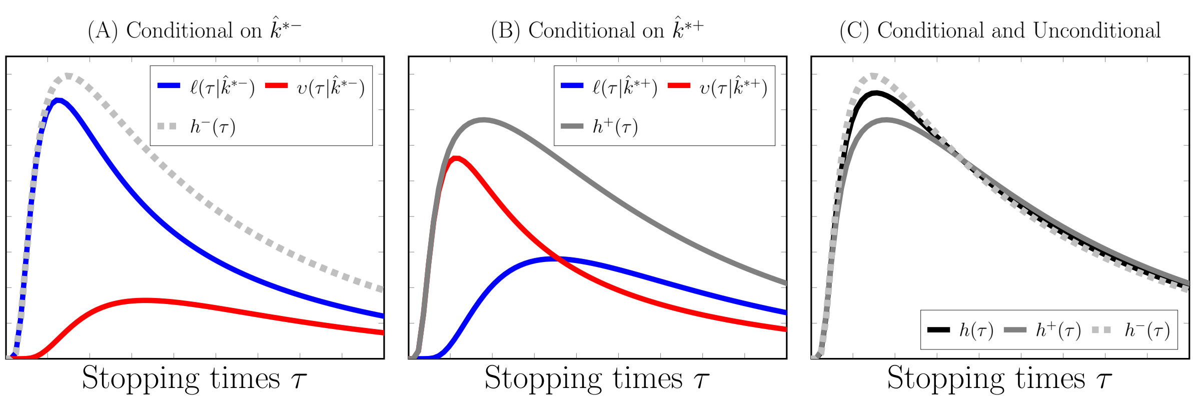
- The density of times for hitting *either threshold* for current $\hat{k}$ is the sum of the two previous measures:

$$h(\tau|\hat{k}) = \ell(\tau|\hat{k}) + \upsilon(\tau|\hat{k}). \tag{A.62}$$

Evaluating (A.60), (A.61), and (A.62) at the reset points, we obtain the duration densities conditional on a previous upsizing $\ell(\tau|\hat{k}^{*-})$, $\upsilon(\tau|\hat{k}^{*-})$ or downsizing $\ell(\tau|\hat{k}^{*+})$, $\upsilon(\tau|\hat{k}^{*+})$. The unconditional duration densities are weighted averages of the conditional densities, averaged using the updating shares:

$$z(\tau) = \frac{\mathcal{N}^-}{\mathcal{N}} z(\tau|\hat{k}^{*-}) + \frac{\mathcal{N}^+}{\mathcal{N}} z(\tau|\hat{k}^{*+}), \quad \text{for} \quad z \in \{\upsilon, \ell, h\}. \tag{A.63}$$

Figure A.1: Distributions of stopping times



Notes: These figures present the conditional densities of stopping times (τ) for hitting the lower and upper thresholds: Panel A depicts the densities following a purchase, and Panel B shows those following a sale. Panel C illustrates the stopping time distribution's conditional and unconditional densities.

A.5 Illustrative example on adjusted shares

Consider an economy where half of the firms adjust their capital every year ($\mathcal{N} = 0.5$), with 80% purchasing ($\mathcal{N}^-/\mathcal{N} = 0.8$) and 20% selling capital ($\mathcal{N}^+/\mathcal{N} = 0.2$). The conditional duration of inaction following a purchase is $\mathbb{E}^-[\tau] = 1.5$ years and following a sale is $\mathbb{E}^+[\tau] = 4$ years. From (24), the economy-wide average duration is computed using shares is $\mathbb{E}[\tau] = (\mathcal{N}^-/\mathcal{N})\mathbb{E}^-[\tau] + (\mathcal{N}^+/\mathcal{N})\mathbb{E}^+[\tau] = 0.8(1.5) + 0.2(4) = 2$ years.[34] The average adjustment $\overline{\mathbb{E}}[\Delta\hat{k}]$ is also computed using these shares.

Now, let us consider the distribution of $\hat{k}$. Assume the average capital-productivity ratio after a purchase is $\mathbb{E}^-[\hat{k}] = -0.2$ (capital is 80% of productivity) and after a sale is $\mathbb{E}^+[\hat{k}] = 0.2$ (capital is 120% of productivity). To compute the economy-wide mean $\mathbb{E}[\hat{k}]$, the naive aggregation using shares is biased as it does not consider the duration of inaction. While only 20% of adjustments are downward, they happen after longer inaction spells with twice the average duration, implying that the capital-productivity ratios generating those adjustments are occupied for more extended periods. According to (25), the renewal weights $r^- = (\mathcal{N}^-/\mathcal{N})(\mathbb{E}^-[\tau]/\mathbb{E}[\tau]) = 0.8(0.75) = 0.6$ and $r^+ = (\mathcal{N}^+/\mathcal{N})(\mathbb{E}^+[\tau]/\mathbb{E}[\tau]) = 0.2(2) = 0.4$ appropriately account for the higher occupancy. Therefore, the average ratio computed with (26) is $\mathbb{E}[\hat{k}] = 0.6(-0.2) + 0.4(0.2) = -0.04$ (capital is 96% of productivity). Using the wrong aggregation delivers a lower mean and biases the estimation of investment frictions.

[34] Note that $\overline{\mathbb{E}}[\tau] = 1/\mathcal{N}$ but $\overline{\mathbb{E}}^{\pm}[\tau] \neq 1/\mathcal{N}^{\pm}$.

B Generalized Hazard Model

In the main text, we specialize investment frictions to a symmetric adjustment cost θ paid indistinctly for positive and negative investments and a price wedge that gives rise to partial irreversibility.

We examine this model mainly for pedagogical reasons, as it simplifies the exposition of the theory. In this section, we expand the scope of the analysis and present an asymmetric generalized hazard model, which follows the contributions by Caballero and Engel (1999, 2007) and examined in contemporary work by Alvarez, Lippi and Oskolkov (2022), which may accommodate other empirically-relevant frictions. All the following proofs in the next sections are shown for the generalized hazard model and thus apply to the parsimonious environments as a special case setting $\Lambda(\hat{k}) = 0$.

B.1 Environment

The generalized hazard function depends mainly on the assumption of the fixed adjustment cost. Therefore, in this section, we assume technology and shocks as in Section 2. Moreover, the firm can control its capital stock through buying and selling investment goods at prices p^{buy} and p^{sell}, with $p^{\text{buy}} > p^{\text{sell}}$.

Adjustment costs The first step generalizes the adjustment cost structure. For every investment $i = \Delta k$, the firm must pay an adjustment cost θ_s proportional to current productivity u_s and measured in consumption units (Caballero and Engel, 1999):

$$\theta_s \;=\; \Theta(i_s, \mathrm{d}N_s^-, \mathrm{d}N_s^+, \vartheta_s^-, \vartheta_s^+)u_s, \tag{B.1}$$

where the function $\Theta(\cdot) > 0$ is described by

$$\Theta(i, \mathrm{d}N^+, \mathrm{d}N^-, \vartheta^-, \vartheta^+) = \begin{cases} 0 & \text{if} \quad i = 0 \\ \bar{\theta}^+(1 - \mathrm{d}N) \;+\; \mathrm{d}N\vartheta^+ & \text{if} \quad i < 0 \\ \bar{\theta}^-(1 - \mathrm{d}N) \;+\; \mathrm{d}N\vartheta^- & \text{if} \quad i > 0. \end{cases} \tag{B.2}$$

Let us describe each element in equation (B.2).

(i) N_s^+ and N_s^- follows Poisson counter with unit increments and arrival rates λ^+ and λ^-;

(ii) $\bar{\theta}^+$ and $\bar{\theta}^-$ are non negative number; and

(iii) ϑ_s^+ and ϑ_s^- are *i.i.d.* random variables with support $\mathbb{S}upp(\vartheta^+) = [0, \bar{\vartheta}^+]$ and $\mathbb{S}upp(\vartheta^-) = [0, \bar{\vartheta}^-]$. We assume that $\vartheta^- \le \bar{\theta}^-$ and $\vartheta^+ \le \bar{\theta}^+$. Define $J^+(x) \equiv \Pr(\vartheta^+ < x)$ and $J^-(x) \equiv \Pr(\vartheta^- < x)$ the cumulative distribution for each random variable.

B.1.1 Relationship to the literature

The stochastic process of fixed cost in (B.2) can derive the majority of lumpy adjustment models used in previous work.

1. Setting $\lambda^+ = \lambda^- = 0$ and $\bar{\theta}^+ = \bar{\theta}^-$ yields the standard fixed cost model of adjustment, originally proposed by Scarf (1959) in an inventory model and Sheshinski and Weiss (1977) in a pricing context.

2. Setting $\lambda^+ = \lambda^- > 0$ and $\mathbb{S}upp(\vartheta^+) = \mathbb{S}upp(\vartheta^-) = \{0\}$, and $\bar{\theta}^+ = \bar{\theta}^- > 0$ yields the CalvoPlus model proposed by Nakamura and Steinsson (2010), which nests the standard fixed cost model and the time-dependent Calvo model.

3. Under this case, if $\bar{\theta}^{+} \neq \bar{\theta}^{-}$, then we have the Bernoulli fixed cost model or asymmetric Bernoulli fixed cost model if $\lambda^{+} \neq \lambda^{-}$, see Baley and Blanco (2021).
4. Finally, setting $\lambda^{+} = \lambda^{-} > 0$ and $\bar{\vartheta}^{+} = \bar{\vartheta}^{-} = \bar{\theta}^{-} = \bar{\theta}^{+}$ yields the generalized hazard model originally proposed by Caballero and Engel (1993).

B.1.2 Value function and optimal policy

Value Let $V(k,u)$ denote the value of a firm with capital stock k and productivity u. Given initial conditions (k_0,u_0), the firm chooses a sequence of adjustment dates $\{T_h\}_{h=1}^{\infty}$ and investments $\{i_{T_h}\}_{h=1}^{\infty}$, where h counts the number of adjustments, to maximize its expected discounted stream of profits. The sequential problem is

$$\text{(B.3)} \qquad V(k_0,u_0) \;=\; \max_{\{T_h, i_{T_h}\}_{h=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} e^{-\rho s}\pi_s \,\mathrm{d}s - \sum_{h=1}^{\infty} e^{-\rho T_h}\left(\theta_{T_h} + p\left(i_{T_h}\right) i_{T_h}\right)\right],$$

subject to the production technology (1), the idiosyncratic productivity shocks (2), the investment price function (4), the law of motion for the capital stock (6), and the stochastic process of adjustment cost in (B.2).

Capital-productivity ratios $\hat{k}$ As in the main text, it is easy to show that $v(k,u) = uv(\hat{k})$ where

$$\text{(B.4)} \qquad v(\hat{k}) = \max_{\tau,\Delta\hat{k}} \mathbb{E}\left[\int_0^{\tau} Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s + e^{-r\tau}\left(-\theta_\tau(\Delta\hat{k}) - p(\Delta\hat{k})(e^{\hat{k}_\tau+\Delta\hat{k}} - e^{\hat{k}_\tau}) + v(\hat{k}_\tau+\Delta\hat{k})\right)\Big|\hat{k}_0 = \hat{k}\right].$$

Here, $\theta_\tau(\Delta\hat{k})$ is a random variable instead of a number, a function of the adjustment direction—similar to the investment price.

Optimal investment policy The optimal investment policy is characterize by four numbers $\mathcal{K} \equiv \{\hat{k}^{-} \leq \hat{k}^{*-} \leq \hat{k}^{*+} \leq k^{+}\}$, and a hazard rate of adjustment $\Lambda(\hat{k})$. The numbers $\hat{k}^{-}$ and k^{+} correspond to the lower and upper borders of the inaction region $\mathcal{R} = \left\{\hat{k} : \hat{k}^{-} < \hat{k} < \hat{k}^{+}\right\}$, and $\hat{k}^{*-} < \hat{k}^{*+}$ to the two reset points following a positive and a negative investment, respectively. $\Lambda(\hat{k}) : \overline{\mathcal{R}} \to \mathbb{R}^{+}$ is a non-negative function corresponding to the arrival rate of a new Poisson counter N^{Λ}. Given $\mathcal{R}$ and N^{Λ}, the optimal adjustment dates are

$$\text{(B.5)} \qquad T_h \;=\; \inf\left\{s \geq T_{h-1} : \hat{k}_s \notin \mathcal{R} \text{ or } \mathrm{d}N_s^{\Lambda}(\hat{k}) = 1\right\} \quad \text{with} \quad T_0 = 0.$$

Following Øksendal and Sulem (2005) and Oksendal (2007), Lemma B.1 establishes the optimality conditions that characterize (B.4).

Lemma B.1. *The value function $v(\hat{k})$ and the policy $\mathcal{K} \equiv \{\hat{k}^{-}, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^{+}\}$ satisfy:*

(i) For all $\hat{k} \in \mathcal{R}$, $v(\hat{k})$ solves the HJB equation:

$$\text{(B.6)} \qquad rv(\hat{k}) \;=\; Ae^{\alpha\hat{k}} \;-\; \nu v'(\hat{k}) \;+\; \frac{\sigma^2}{2}v''(\hat{k})$$
$$+\,\lambda^{-}\int_0^{\bar{\vartheta}^{-}} \max\left\{v^{buy}(\hat{k}) - \vartheta, 0\right\}\mathrm{d}J^{-}(\vartheta) + \lambda^{+}\int_0^{\bar{\vartheta}^{+}} \max\left\{v^{sell}(\hat{k}) - \vartheta, 0\right\}\mathrm{d}J^{+}(\vartheta)$$

where the values v^{buy} and v^{sell} are defined as follows:

$$\text{(B.7)} \qquad v^{buy}(\hat{k}) \;\equiv\; v(\hat{k}^{-}) - v(\hat{k}) - p^{buy}(e^{\hat{k}^{-}} - e^{\hat{k}}),$$

$$\text{(B.8)} \qquad v^{sell}(\hat{k}) \;\equiv\; v(\hat{k}^{+}) - v(\hat{k}) - p^{sell}(e^{\hat{k}^{+}} - e^{\hat{k}}).$$

(ii) At the borders of the inaction region, $v(\hat{k})$ satisfies the value-matching conditions:

$$v^{buy}(\hat{k}^-) = \bar{\theta}^-; \qquad v^{sell}(\hat{k}^+) = \bar{\theta}^+; \tag{B.9}$$

(iii) At the borders of the inaction region and the two reset states, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$\frac{\mathrm{d}v^{buy}(\hat{k})}{\mathrm{d}\hat{k}} = p^{buy}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \tag{B.10}$$

$$\frac{\mathrm{d}v^{sell}(\hat{k})}{\mathrm{d}\hat{k}} = p^{sell}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \tag{B.11}$$

Hazard rate of adjustment $\Lambda(\hat{k})$ We are now ready to define $\Lambda(\hat{k})$, which gives the probability of adjustment $\Lambda(\hat{k})\,\mathrm{d}t$ in a time period $\mathrm{d}t$ a firm with $\hat{k} \in \mathcal{R}$. The hazard rate of adjustment is given by

$$\Lambda(\hat{k}) = \lambda^- J^- \left(v^{\text{buy}}(\hat{k})\right) \mathbb{1}_{\left\{\hat{k}\in(\hat{k}^-,\hat{k}^{*-})\right\}} + \lambda^+ J^+ \left(v^{\text{sell}}(\hat{k})\right) \mathbb{1}_{\left\{\hat{k}\in(\hat{k}^{*+},\hat{k}^+)\right\}}. \tag{B.12}$$

The hazard function $\Lambda(\hat{k})$ satisfies the following properties:

1. $\Lambda(\hat{k}) = 0$ in the inner inaction region, i.e., for all $\hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+})$,
2. $\Lambda(\hat{k})$ is weakly decreasing in $(\hat{k}^-, \hat{k}^{*-})$ and weakly increasing in $(\hat{k}^{*+}, \hat{k}^+)$;
3. If $J^-(0) > 0$ then $\Lambda(\hat{k})$ is bounded below in the domain $(\hat{k}^-, \hat{k}^{*-})$ by $\Lambda(\hat{k}) = \lambda^- J^-(0)$.
4. If $J^+(0) > 0$ then $\Lambda(\hat{k})$ is bounded below in the domain $(\hat{k}^{*+}, \hat{k}^+)$ by $\Lambda(\hat{k}) = \lambda^+ J^+(0)$

B.1.3 Cross-sectional distribution

Without irreversibility

$$\Lambda(\hat{k})g(\hat{k}) = \nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k}), \quad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^*\} \tag{B.13}$$

$$g(\hat{k}^\pm) = 0 \tag{B.14}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k} = 1, \tag{B.15}$$

$$g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^*\}), \mathbb{C}^2(\{\hat{k}^*\}) \tag{B.16}$$

With irreversibility

$$\Lambda(\hat{k})g(\hat{k}) = \nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k}), \qquad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^{*-}, \hat{k}^{*+}\} \tag{B.17}$$

$$g(\hat{k}^\pm) = 0 \tag{B.18}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k} = 1, \tag{B.19}$$

$$g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^{*-}, \hat{k}^{*+}\}), \mathbb{C}^2(\{\hat{k}^{*-}, \hat{k}^{*+}\}) \tag{B.20}$$

C Proofs under Generalized Hazard

Proofs' overview. In Proposition 2, we express the CIR as the integral of a value function $m(\hat{k})$ and $g'(\hat{k})$. In Proposition 3, we characterize the terminal value of the value function. In Proposition 4, we characterize the CIR as a function of steady-state moments. In all propositions, we examine cases without and with irreversibility, in that order.

C.1 Proof of Proposition 2

Proposition 2. ***(CIR)*** *Up to the first order, the CIR equals*

$$\frac{CIR(\delta)}{\delta} \;=\; -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g'(\hat{k})\, \mathrm{d}\hat{k} + o(\delta), \tag{40}$$

where $m(\hat{k})$ is a continuously differentiable function equal to the average cumulative deviations of the capital-productivity ratio $\hat{k}$ from the economy's mean $\mathbb{E}[\hat{k}]$, satisfying the HJB

$$0 \;=\; \hat{k} \;-\; \mathbb{E}[\hat{k}] \;-\; \nu m'(\hat{k}) \;+\; \frac{\sigma^2}{2} m''(\hat{k}) \qquad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+), \tag{41}$$

with two boundary conditions

$$m(\hat{k}^-) \;=\; m(\hat{k}^{*-}), \quad \text{and} \quad m(\hat{k}^+) \;=\; m(\hat{k}^{*+}), \tag{42}$$

and a stationarity condition

$$\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k})\, \mathrm{d}\hat{k} \;=\; 0. \tag{43}$$

C.1.1 Step 1: First-order approximation and exchange order of integration

Let $g(\hat{k})$ be the capital-productivity steady-state distribution and $g_t(\hat{k})$ the distribution t-periods after an aggregate productivity shock of size $\delta > 0$, with $g_0(\hat{k}) = g(\hat{k} - \delta)$. Let $f(\hat{k})$ be a continuous function of $\hat{k}$ (in the main text, we take $f(\hat{k}) = \hat{k}$, the proof here is more general). Define the cumulative impulse response of the function f as:

$$\text{CIR}(f, \delta) \equiv \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} f(\hat{k}) \;\Big(g_s(\hat{k}) - g(\hat{k}) \Big)\, \mathrm{d}\hat{k}\, \mathrm{d}s. \tag{C.21}$$

We show that in a general environment, with or without irreversibility, up to first order, the CIR is equal to

$$\text{CIR}(f, \delta) \;=\; -\delta \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g'(\hat{k})\, \mathrm{d}\hat{k} + o(\delta^2). \tag{C.22}$$

where $m_{\mathcal{T}}(\hat{k}_0)$ is defined as

$$m_{\mathcal{T}}(\hat{k}_0) \equiv \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \Big(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \Big)\, g_s(\hat{k}|\hat{k}_0)\, \mathrm{d}\hat{k}\, \mathrm{d}s. \tag{C.23}$$

Starting from the CIR's definition in (C.21), we do the following steps: Equality (1) operates over the integral; (2) uses the Chapman-Kolmogorov equation to substitute the conditional expectation with respect to $\hat{k}$, with density $g_s(\hat{k})$, with a conditional expectation with respect to the initial condition $\hat{k}_0$, with density $g_s(\hat{k}|\hat{k}_0)g_0(\hat{k}_0)$, where

$g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}$ is the probability of the state $\hat{k}$ at date s with initial condition $\hat{k}_0$; (3) writes the initial density following the shock in terms of the steady-state density $g_0(\hat{k}_0) = g(\hat{k}_0 - \delta)$; (4) applies Fubini's theorem to exchange orders of integration; (5) writes the integral using a limit; (6) defines and substitutes the function $m_{\mathcal{T}}(\hat{k})$ as in (C.23) and changes the variable of integration from $\hat{k}_0$ to $\hat{k}$; and (7) applies a first-order approximation over δ.

$$\begin{aligned}
\mathrm{CIR}(f,\delta) &= \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} f(\hat{k}) \left(g_s(\hat{k}) - g(\hat{k})\right) \mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(1)} \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k})\,\mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(2)} \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) g_0(\hat{k}_0)\,\mathrm{d}\hat{k}_0\right] \mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(3)} \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0 - \delta)\,\mathrm{d}\hat{k}_0\right] \mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(4)} \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s\right] g(\hat{k}_0 - \delta)\,\mathrm{d}\hat{k}_0 \\
&=^{(5)} \int_{\hat{k}^-}^{\hat{k}^+} \left[\lim_{\mathcal{T}\to\infty} \underbrace{\int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s}_{\equiv\, m_{\mathcal{T}}(\hat{k}_0)}\right] g(\hat{k}_0 - \delta)\,\mathrm{d}\hat{k}_0 \\
&=^{(6)} \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g(\hat{k} - \delta)\,\mathrm{d}\hat{k} \\
&=^{(7)} -\delta \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2).
\end{aligned}$$

C.1.2 Step 2: Show that the cross-sectional mean of $m_{\mathcal{T}}$ is zero.

Show that $\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = 0$. Substitute the integral's definition of $m_{\mathcal{T}}(\hat{k})$ from (C.23) into $\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k}$. In the following equalities, (1) uses Fubini's theorem, (2) uses Bayes' theorem, (3) uses the fact that $g(\hat{k})$ is the steady-state distribution, and (4) solves the first and second integrals.

$$\begin{aligned}
\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} &= \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s\right] g(\hat{k}_0)\,\mathrm{d}\hat{k}_0 \\
&=^{(1)} \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}\hat{k}_0\,\mathrm{d}s \\
&=^{(2)} \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) \left[\int_{\hat{k}^-}^{\hat{k}^+} g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0)\,\mathrm{d}\hat{k}_0\right] \mathrm{d}\hat{k}\,\mathrm{d}s \\
&=^{(3)} \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g(\hat{k})\,\mathrm{d}\hat{k}\,\mathrm{d}s =^{(4)} 0.
\end{aligned} \tag{C.24}$$

C.1.3 Step 3: Derive HJB and border conditions for $m_{\mathcal{T}}$.

We start from the stopping time definition of $m_{\mathcal{T}}(\hat{k}_0) \equiv \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s$ in equation (C.23), and use the conditions in Auxiliary Theorem (A.3) to characterize its value.

Without irreversibility

$$0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\mathcal{T}} - \nu\frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}^2} + \Lambda(\hat{k})(m_{\mathcal{T}}(\hat{k}^*) - m_{\mathcal{T}}(\hat{k})) \tag{C.25}$$

$$0 = m_{\mathcal{T}}(\hat{k}^*) - m_{\mathcal{T}}(\hat{k}^{\pm}) \tag{C.26}$$

$$0 = \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} \tag{C.27}$$

With irreversibility Using the property that $\Lambda(\hat{k}) = 0$ for all $\hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+})$, we can write in a simple form the HJB and border conditions satisfied by $m_{\mathcal{T}}(\hat{k})$:

$$0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\mathcal{T}} - \nu\frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{d\hat{k}} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}^2} + \Lambda(\hat{k})(\mathcal{M}_{\mathcal{T}}(\hat{k}) - m_{\mathcal{T}}(\hat{k})) \tag{C.28}$$

$$0 = \mathcal{M}_{\mathcal{T}}(\hat{k}^{*\pm}) - m_{\mathcal{T}}(\hat{k}^{\pm}), \tag{C.29}$$

$$0 = \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k}. \tag{C.30}$$

where $\mathcal{M}_{\mathcal{T}}(\hat{k}) \in \mathbb{C}^2$ is defined as

$$\mathcal{M}_{\mathcal{T}}(\hat{k}) \equiv \begin{cases} m_{\mathcal{T}}(\hat{k}^{*-}) & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ m_{\mathcal{T}}(\hat{k}^{*+}) & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+] \end{cases} \tag{C.31}$$

C.1.4 Step 4: Show pointwise converge of $m_{\mathcal{T}}$ to m

Let $m(\hat{k})$ be defined as:

$$m(\hat{k}) \equiv \mathbb{E}\left[\int_0^\infty (\hat{k}_s - \mathbb{E}[\hat{k}])\,\mathrm{d}s \Big| \hat{k}\right] + \mathbb{C}. \tag{C.32}$$

We show that for each $\hat{k}$, $\lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) = m(\hat{k})$.

Without irreversibility See Baley and Blanco (2021).

With irreversibility Let $\{T_i\}_{i=0}^{N(\mathcal{T})}$ be the adjustment dates between 0 and $\mathcal{T}$, where i denotes the counter of adjustments for all $i = 1, 2, \ldots, N(\mathcal{T})-1$, $N(\mathcal{T})$ is the maximum number of adjustments until $\mathcal{T}$ and $T_0 = 0$. Then, for any $\mathcal{T}$, we rewrite $m_{\mathcal{T}}(\hat{k})$ as a sum between adjustment dates:

$$m_{\mathcal{T}}(\hat{k}) = \mathbb{E}\left[\sum_{i=1}^{N(\mathcal{T})-1} \int_{T_{i-1}}^{T_i} \Big(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})]\Big)\,\mathrm{d}s + \int_{T_{N(\mathcal{T})-1}}^{\mathcal{T}} \Big(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})]\Big)\,\mathrm{d}s \,\Big|\, \hat{k}_0 = \hat{k}\right] \tag{C.33}$$

We take the limit $\mathcal{T} \to \infty$ to show convergence. We conduct the following steps in the next equalities: (1) splits the sum; (2) uses the indicator function to write the finite sum in the first term; (3) uses the fact that $N(\mathcal{T})$ always exceeds i, thus $\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\mathbb{I}(N(\mathcal{T})\geq i)|\,\hat{k}_0=\hat{k}\right]=1,\forall i$; (4) recognizes that the first term is independent of $\mathcal{T}$.

$$
\begin{aligned}
\text{(C.34)}\quad & \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) \\
&=^{(1)} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\sum_{i=1}^{N(\mathcal{T})-1}\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s+\lim_{\mathcal{T}\to\infty}\int_{T_{N(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\right|\hat{k}_0=\hat{k}\Bigg] \\
&=^{(2)} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\sum_{i=1}^{\infty}\mathbb{I}(N(\mathcal{T})\geq i)\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s+\lim_{\mathcal{T}\to\infty}\int_{T_{N(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\right|\hat{k}_0=\hat{k}\Bigg] \\
&=^{(3)} \mathbb{E}\left[\sum_{i=1}^{\infty}\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s+\lim_{\mathcal{T}\to\infty}\int_{T_{N(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\right|\hat{k}_0=\hat{k}\Bigg] \\
&=^{(4)} \text{terms independent of } \mathcal{T}+\underbrace{\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\int_{T_{\mathcal{N}(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\right|\hat{k}_0=\hat{k}\Bigg]}_{\text{tail term } \mathcal{E}}.
\end{aligned}
$$

Tail term Next, we show the "tail" term is independent of the initial condition $\hat{k}$ and the $\mathcal{T}$. To do this, we consider tails conditional on the previous reset, defined as:

$$
\text{(C.35)}\qquad \mathcal{E}(\hat{k}^{*\pm},\mathcal{T})\ \equiv\ \mathbb{E}\left[\int_{T_{\mathcal{N}(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\right|\hat{k}_{T_{\mathcal{N}(\mathcal{T})-1}}=\hat{k}^{*\pm}\Bigg].
$$

Define $\mathbb{P}^+_{\mathcal{T}}(\hat{k}_0)\equiv\mathbb{E}\left[\hat{k}_{T_{\mathcal{N}(\mathcal{T})-1}}\geq\hat{k}^{*+}\middle|\hat{k}_0\right]$ and $\mathbb{P}^-_{\mathcal{T}}(\hat{k}_0)\equiv\mathbb{E}\left[\hat{k}_{T_{\mathcal{N}(\mathcal{T})-1}}\leq\hat{k}^{*-}\middle|\hat{k}_0\right]$ be the probabilities of downsizing or upsizing, given a current $\hat{k}$. In the following equalities, we do the following steps. In Step 1, we use the law of iterated expectations (only two contingencies, upsizing or downsizing) and use conditional expectation to substitute the tails and probabilities, *conditional on the initial condition* $\hat{k}_0$. Step 2 eliminates the dependence of probabilities on the initial condition using the convergence of discrete Markov chains (see chapter 11 of Stokey (1989)) and the proof at the end. In other words, $\lim_{\mathcal{T}\to\infty}\mathbb{P}^+_{\mathcal{T}}(\hat{k}_0)$ is independent of $\mathcal{T}$ and $\hat{k}_0$. Finally, the convergence of $\lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*\pm},\mathcal{T})=\mathcal{E}^{\infty}(\hat{k}^{*\pm})$ is shown in Baley and Blanco (2021) and Alexandrov (2021).

$$
\begin{aligned}
&\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\int_{T_{\mathcal{N}(\mathcal{T})-1}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\right|\hat{k}_0\Bigg] \\
&=^{(1)} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*+},\mathcal{T})\lim_{\mathcal{T}\to\infty}\mathbb{P}^+_{\mathcal{T}}(\hat{k}_0)+\lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*-},\mathcal{T})\lim_{\mathcal{T}\to\infty}\mathbb{P}^-_{\mathcal{T}}(\hat{k}_0)\middle|\hat{k}_0\right] \\
&=^{(2)} \lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*+},\mathcal{T})\mathbb{P}^{+,\infty}+\lim_{\mathcal{T}\to\infty}\mathcal{E}(\hat{k}^{*-},\mathcal{T})\mathbb{P}^{-,\infty} \\
&=^{(3)} \mathcal{E}^{\infty}(\hat{k}^{*+})\mathbb{P}^{+,\infty}+\mathcal{E}^{\infty}(\hat{k}^{*-})\mathbb{P}^{-,\infty}.
\end{aligned}
$$

Extra: Convergence of discrete Markov chains Let $\mathbb{P}_N(\hat{k})\equiv\left[\mathbb{P}^-_N(\hat{k});\mathbb{P}^+_N(\hat{k})\right]\in\mathbb{R}^{2\times1}$, then

$$
\text{(C.36)}\qquad \mathbb{P}_N(\hat{k})=\mathbb{P}^T\mathbb{P}_{N-1}(\hat{k}),
$$

where $\mathbb{P}=[\mathbb{P}^-(\hat{k}^{*-}),1-\mathbb{P}^-(\hat{k}^{*-});1-\mathbb{P}^+(\hat{k}^{*+});\mathbb{P}^+(\hat{k}^{*+})]\in\mathbb{R}^{2\times2}$ is a 2×2 transition probability where the rows are the transition probability and P^T is its transpose. If $\mathbb{P}^-_1(\hat{k}^{*-}),\mathbb{P}^+_1(\hat{k}^{*+})\in(0,1)$, then

$$
\text{(C.37)}\qquad \lim_{N\to\infty}\mathbb{P}_N(\hat{k})=\lim_{N\to\infty}\mathbb{P}^{N-1}\mathbb{P}_1(\hat{k})=[\mathbb{P}^{-\infty};\mathbb{P}^{+\infty}].
$$

where the last equality comes from Theorem 11.1 in Stokey (1989).

C.1.5 Step 5: Show convergence of CIR

We will show that

$$\text{(C.38)}\qquad \text{CIR}(f,\delta) = -\delta \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2) = -\delta \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2)$$

Without irreversibility We also need to show that the HJB and border conditions converge. For this, we take the limit $\mathcal{T}\to\infty$ of conditions (C.25), (C.26) and (C.27) and use point-wise convergence of $m_{\mathcal{T}}(\hat{k})$ from Step 4:

$$\text{(C.39)}\qquad 0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \Lambda(\hat{k})(m(\hat{k}^*) - m(\hat{k})),$$

$$\text{(C.40)}\qquad 0 = m(\hat{k}^*) - m(\hat{k}^\pm),$$

With irreversibility Finally, we take the limit $\mathcal{T}\to\infty$ of conditions (C.28), (C.29) and (C.30) and use point-wise convergence of $m_{\mathcal{T}}(\hat{k})$ from Step 4 to obtain:

$$\text{(C.41)}\qquad 0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \Lambda(\hat{k})(\mathcal{M}(\hat{k}) - \mathbb{C}ov[\hat{k},a] - m(\hat{k})),$$

$$\text{(C.42)}\qquad 0 = \mathcal{M}(\hat{k}^\pm) - \mathbb{C}ov[\hat{k},a] - m(\hat{k}^\pm),$$

(C.43)

where $\mathcal{M}(\hat{k}) \in \mathbb{C}^2$ is defined as

$$\text{(C.44)}\qquad \mathcal{M}(\hat{k}) \equiv \begin{cases} m(\hat{k}^{*-}) + \mathbb{C}ov[\hat{k},a] & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ m(\hat{k}^{*+}) + \mathbb{C}ov[\hat{k},a] & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+] \end{cases}$$

Stability condition To show the stability condition that require the cross-sectional average of $m(\hat{k}) = 0$ in the cases with and without irreversibility, we write $m(\hat{k}) = \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k})$ inside the integral, pull the limit outside the integral, and use the previous result in (C.24) to get:

$$\text{(C.45)}\qquad \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = \lim_{\mathcal{T}\to\infty} \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = 0$$

C.1.6 Step 6: Without general hazard

To obtain the characterization in the baseline model, just set $\Lambda(\hat{k}) = 0$.

C.2 Proof of Proposition 3

Proposition 3. ***(Expected sum of deviations)*** *The expected sum of deviations after upsizing $\mathcal{M}(\hat{k}^{*-}) \equiv m(\hat{k}^{*-}) + \mathbb{C}ov[\hat{k},a]$ and after downsizing $\mathcal{M}(\hat{k}^{*+}) \equiv m(\hat{k}^{*+}) + \mathbb{C}ov[\hat{k},a]$ are equal to*

$$\text{(47)}\qquad \mathcal{M}(\hat{k}^{*-}) = (\mathbb{E}^-[\hat{k}] - \mathbb{E}[\hat{k}])\, \overline{\mathbb{E}}^-[\tau]\, \frac{\mathbb{E}[\mathbb{P}^+(\hat{k})]}{\mathbb{P}^{-+}} < 0$$

$$\text{(48)}\qquad \mathcal{M}(\hat{k}^{*+}) = (\mathbb{E}^+[\hat{k}] - \mathbb{E}[\hat{k}])\, \overline{\mathbb{E}}^+[\tau]\, \frac{\mathbb{E}[\mathbb{P}^-(\hat{k})]}{\mathbb{P}^{+-}} > 0,$$

where the average downsizing and upsizing probabilities are equal to

$$\mathbb{E}[\mathbb{P}^{-}(\hat{k})] = \frac{\overline{\mathbb{E}}\left[\tau' \mathbb{1}_{\{\hat{k}_{\tau'}=\hat{k}^{-}\}}\right]}{\overline{\mathbb{E}}[\tau]}, \qquad \mathbb{E}[\mathbb{P}^{+}(\hat{k})] = \frac{\overline{\mathbb{E}}\left[\tau' \mathbb{1}_{\{\hat{k}_{\tau'}=\hat{k}^{+}\}}\right]}{\overline{\mathbb{E}}[\tau]}. \tag{49}$$

C.2.1 Without irreversibility

For any continous function f, we characterize the terminal value $m(\hat{k}^*) = -\mathbb{C}ov[a, f(\hat{k})]$. From Proposition 2, $m(\hat{k})$ satisfies the following recursive representation

$$m(\hat{k}) \;=\; \mathbb{E}\left[\int_0^{\tau} (f(\hat{k}_s) - \mathbb{E}[f(\hat{k})])\,\mathrm{d}s + m(\hat{k}^*) \middle| \hat{k}_0 = \hat{k}\right]. \tag{C.46}$$

For a given parameter $\varphi \geq 0$, define the auxiliary function $z(\hat{k}|\varphi)$ as follows

$$z(\hat{k}|\varphi) \;\equiv\; \mathbb{E}\left[\int_0^{\tau} e^{\varphi s}(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})])\,\mathrm{d}s + e^{\varphi\tau} m(\hat{k}^*) \middle| \hat{k}_0 = \hat{k}\right], \tag{C.47}$$

Using Auxiliary Theorem A.3, the auxiliary function $z(\hat{k}|\varphi)$ satisfies the following HBJ and border conditions:

$$-\varphi z(\hat{k}|\varphi) \;=\; f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu \frac{\partial z(\hat{k}|\varphi)}{\partial \hat{k}} + \frac{\sigma^2}{2}\frac{\partial^2 z(\hat{k}|\varphi)}{\partial \hat{k}^2} \;+\; \Lambda(\hat{k})\left(m(\hat{k}^*) - z(\hat{k}|\varphi)\right), \tag{C.48}$$

$$z(\hat{k}^{\pm}, \varphi) \;=\; m(\hat{k}^*). \tag{C.49}$$

Taking the derivatives of (C.48) and (C.49) with respect to φ:

$$(\Lambda(\hat{k}) - \varphi)\frac{\partial z(\hat{k}|\varphi)}{\partial \varphi} - z(\hat{k}|\varphi) \;=\; -\nu \frac{\partial^2 z(\hat{k}, \varphi)}{\partial \hat{k}\partial\varphi} + \frac{\sigma^2}{2}\frac{\partial^3 z(\hat{k}|\varphi)}{\partial \hat{k}^2 \partial\varphi} \quad \text{and} \quad \frac{\partial z(\hat{k}^{\pm}|\varphi)}{\partial \varphi} \;=\; 0. \tag{C.50}$$

Using the Schwarz's theorem to exchange partial derivatives, evaluating at $\varphi = 0$, and using $z(\hat{k}|0) = m(\hat{k})$, the two expressions become:

$$\Lambda(\hat{k})\frac{\partial z(\hat{k}|0)}{\partial \varphi} \;=\; m(\hat{k}) - \nu\frac{\partial}{\partial \hat{k}}\left(\frac{\partial z(\hat{k}|0)}{\partial \varphi}\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial \hat{k}^2}\left(\frac{\partial z(\hat{k}|0)}{\partial \varphi}\right) \quad \text{and} \quad \frac{\partial z(\hat{k}^{\pm}|0)}{\partial \varphi} \;=\; 0. \tag{C.51}$$

From Auxiliary Theorem A.3, equations in (C.51) are the HBJ and border conditions of $\frac{\partial z(\hat{k}|0)}{\partial \varphi}$, and therefore:

$$\frac{\partial z(\hat{k}|0)}{\partial \varphi} \;=\; \mathbb{E}\left[\int_0^{\tau} m(\hat{k}_s)\,\mathrm{d}s \middle| k_0 = \hat{k}\right] \tag{C.52}$$

Evaluating at $\hat{k}^*$, using the Auxiliary Theorem OMT in (A.2), providing the equivalence of occupancy measure and steady-state moments, we write the previous equation as:

$$\frac{\partial z(\hat{k}^*|0)}{\partial \varphi} \;=\; \mathbb{E}\left[\int_0^{\tau} m(\hat{k}_s)\,\mathrm{d}s \middle| k_0 = \hat{k}^*\right] \;=\; \mathbb{E}[\tau]\mathbb{E}[m(\hat{k})] \;=\; 0 \tag{C.53}$$

where we used $\mathbb{E}[m(\hat{k})] = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0$ by (43) .

At the same time, taking the derivative of (C.47) with respect to φ and evaluating at $\varphi = 0$ yields

$$\frac{\partial z(\hat{k}^*|0)}{\partial \varphi} \;=\; \mathbb{E}\left[\int_0^{\tau} s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s + \tau m(\hat{k}^*) \middle| \hat{k}_0 = \hat{k}^*\right]. \tag{C.54}$$

Together (C.53) and (C.54) imply:

$$0 = \mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s\middle|\,\hat{k}_0 = \hat{k}^*\right] + \mathbb{E}\left[\tau|\,\hat{k}_0 = \hat{k}^*\right]m(\hat{k}^*). \tag{C.55}$$

Solving for $m(\hat{k}^*)$:

$$m(\hat{k}^*) = -\frac{\mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s\middle|\,\hat{k}_0 = \hat{k}^*\right]}{\mathbb{E}\left[\tau|\,\hat{k}_0 = \hat{k}^*\right]}. \tag{C.56}$$

Note that s captures the time elapsed since the last adjustment, that is, capital age a. Using the OMT in (A.2), we rewrite the occupancy measure as a steady-state moment, which turns out to be equal to minus the covariance of age with the function of capital-productivity ratios $f(\hat{k})$:

$$m(\hat{k}^*) = -\mathbb{E}[a(f(\hat{k}) - \mathbb{E}[f(\hat{k})])] \;=\; -\mathbb{C}ov[a, f(\hat{k})]. \tag{C.57}$$

C.2.2 With irreversibility

Observe that $\mathcal{M}(f, \hat{k})$ satisfies the following recursive representation

$$m(\hat{k}) = \mathbb{E}\left[\int_0^\tau \left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s + m(\hat{k}^*(\hat{k}_\tau))\middle|\,\hat{k}_0 = \hat{k}\right]. \tag{C.58}$$

Define an auxiliary function $z(\hat{k}|\varphi)$ as follows:

$$z(\hat{k}|\varphi) \equiv \mathbb{E}\left[\int_0^\tau e^{\varphi s}\left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})]\right)\mathrm{d}s + e^{\varphi\tau}m(\hat{k}^*(\hat{k}_\tau))\middle|\,\hat{k}_0 = \hat{k}\right]. \tag{C.59}$$

and note the relationship: $z(\hat{k}|0) = m(\hat{k})$, $z(\cdot|\varphi) \in \mathbb{C}^2((\hat{k}^-, \hat{k}^+)) \cap \mathbb{C}$ for all φ, and

$$-\varphi z(\hat{k}|\varphi) + \Lambda(\hat{k})\left(z(\hat{k}|\varphi) - m(\hat{k})\right) = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu z'(\hat{k}|\varphi) + \frac{\sigma^2}{2}z''(\hat{k}|\varphi), \tag{C.60}$$

$$z(\hat{k}^{\pm}|\varphi) = m(\hat{k}^{*\pm}). \tag{C.61}$$

Since $z(\hat{k}|0) = m(\hat{k})$, then we have $\int_{\hat{k}^-}^{\hat{k}^+} z(\hat{k}|0)g(\hat{k})\,\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0$. Taking the derivative with respect to φ in (C.60), we have that

$$(\Lambda(\hat{k}) - \varphi)\frac{\partial z(\hat{k}|\varphi)}{\partial\varphi} - z(\hat{k}|\varphi) = -\nu\frac{\partial^2 z(\hat{k}|\varphi)}{\partial\hat{k}\partial\varphi} + \frac{\sigma^2}{2}\frac{\partial^3 z(\hat{k}|\varphi)}{\partial\hat{k}^2\partial\varphi} \quad \text{and} \quad \frac{\partial z(\hat{k}^{\pm}, \varphi)}{\partial\varphi} = 0. \tag{C.62}$$

Using the Schwarz's theorem to exchange partial derivatives and evaluating at $\varphi = 0$:

$$\Lambda(\hat{k})\left.\frac{\partial z(\hat{k}, \varphi)}{\partial\varphi}\right|_{\varphi=0} - m(\hat{k}) = -\nu\frac{\partial\frac{\partial\left.\frac{\partial z(\hat{k},\varphi)}{\partial\varphi}\right|_{\varphi=0}}{\partial\varphi}}{\partial\hat{k}} + \frac{\sigma^2}{2}\frac{\partial^2\left.\frac{\partial z(\hat{k},\varphi)}{\partial\varphi}\right|_{\varphi=0}}{\partial\hat{k}^2} \quad \text{and} \quad \left.\frac{\partial z(\hat{k}, \varphi)}{\partial\varphi}\right|_{\varphi=0} = 0. \tag{C.63}$$

From the previous equation, using OMT in (A.2) and the renewal distribution, we have that

$$\overline{\mathbb{E}}\left[\frac{\partial z(\hat{k}^*(\Delta\hat{k})|0)}{\partial\varphi}\right] \;=\; \overline{\mathbb{E}}\left[\mathbb{E}\left[\int_0^\tau m(\hat{k}_s)\,\mathrm{d}s|\hat{k}_0 = \hat{k}^*\right]\right]\overline{\mathbb{E}}[\tau] \;=\; \overline{\mathbb{E}}[\tau]\mathbb{E}[m(\hat{k})] \;=\; 0. \tag{C.64}$$

Therefore, $\overline{\mathbb{E}}\left[\frac{\partial z(\hat{k}^*(\Delta\hat{k})|0)}{\partial\varphi}\right] = 0$. Using this result, the renewal distribution, the OST, and the the definition of $\overline{\mathbb{E}}$ with shares, we get:

$$
\begin{aligned}
0 &= \overline{\mathbb{E}}\left[\frac{\partial z(\hat{k}^*(\Delta\hat{k})|0)}{\partial\varphi}\right] \\
&= \overline{\mathbb{E}}\left[\mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s + \tau m(\hat{k}^*(\hat{k}_\tau))\middle|\hat{k}_0 = \hat{k}^*\right]\right] \\
&= \overline{\mathbb{E}}[\tau]\mathbb{E}\left[a\left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right)\right] \;+\; \overline{\mathbb{E}}\left[\mathbb{E}[\tau m(\hat{k}^*(\hat{k}_\tau))|\hat{k}_0 = \hat{k}^*]\right] \\
&= \overline{\mathbb{E}}[\tau]\mathbb{C}ov\left[a, f(\hat{k})\right] \;+\; \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-\left[\tau m(\hat{k}^*(\hat{k}_\tau))\right] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+\left[\tau m(\hat{k}^*(\hat{k}_\tau))\right] \\
&= \overline{\mathbb{E}}[\tau]\mathbb{C}ov\left[a, f(\hat{k})\right] \;+\; \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-\left[\tau\left(m(\hat{k}^{*+})\mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}} + m(\hat{k}^{*-})\left(1 - \mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}}\right)\right)\right] \\
&\qquad + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+\left[\tau\left(m(\hat{k}^{*+})\mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}} + m(\hat{k}^{*-})\left(1 - \mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}}\right)\right)\right] \\
&= \overline{\mathbb{E}}[\tau]\mathbb{C}ov\left[a, f(\hat{k})\right] + m(\hat{k}^{*-}) + (m(\hat{k}^{*+}) - m(\hat{k}^{*-}))\overline{\mathbb{E}}[\tau\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]
\end{aligned}
\tag{C.65}
$$

To characterize the difference in cumulative deviations, $m(\hat{k}^{*+}) - m(\hat{k}^{*-})$, observe that

$$
m(\hat{k}^{*-}) = \left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\overline{\mathbb{E}}^-[\tau] + (1 - \mathbb{P}^{--})m(\hat{k}^{*+}) + \mathbb{P}^{--}m(\hat{k}^{*-})
\tag{C.66}
$$

where $\mathbb{E}^-[f(\hat{k})]$ is the expected $\hat{k}$ conditional of a positive investment. Thus,

$$
-(m(\hat{k}^{*+}) - m(\hat{k}^{*-})) = \frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\overline{\mathbb{E}}^-[\tau]}{1 - \mathbb{P}^{--}}
\tag{C.67}
$$

From (C.65) and (C.67), we have that

$$
m(\hat{k}^{*-}) \;+\; \mathbb{C}ov\left[f(\hat{k}), a\right] \;=\; \frac{\overline{\mathbb{E}}[\tau\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\overline{\mathbb{E}}[\tau]}\frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\overline{\mathbb{E}}^-[\tau]}{1 - \mathbb{P}^{--}}.
\tag{C.68}
$$

With similar steps as before, it is easy to show that

$$
m(\hat{k}^{*+}) \;+\; \mathbb{C}ov\left[f(\hat{k}), a\right] \;=\; \frac{\mathbb{E}[\tau\mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]}\frac{\left(\mathbb{E}^+[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^+[\tau]}{1 - \mathbb{P}^{++}}.
\tag{C.69}
$$

C.2.3 Expected probabilities

Next, we characterize the average adjustment probabilities in terms of stopping times: $\mathbb{E}[\mathbb{P}^-(\hat{k})] = \frac{\mathbb{E}[\tau\mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]}$ and $\mathbb{E}[\mathbb{P}^+(\hat{k})] = \frac{\mathbb{E}[\tau\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\mathbb{E}[\tau]}$. Define the function

$$
\tilde{P}^+(\hat{k}, \varphi) \equiv \mathbb{E}\left[e^{\varphi\tau}\mathbb{I}[\hat{k}_\tau \geq \hat{k}^{*+}]|\hat{k}_0 = \hat{k}\right],
\tag{C.70}
$$

which satisfies the following HBJ conditions and border conditions

$$-\varphi \tilde{P}^{+}(\hat{k},\varphi) + \Lambda(\hat{k})\left(\tilde{P}^{+}(\hat{k},\varphi) - \mathbb{I}[\hat{k} \geq \hat{k}^{*+}]\right) = -\nu \frac{\partial \tilde{P}^{+}(\hat{k},\varphi)}{\partial \hat{k}} + \frac{\sigma^2}{2}\frac{\partial^2 \tilde{P}^{+}(\hat{k},\varphi)}{\partial \hat{k}^2}, \tag{C.71}$$

$$\tilde{P}^{+}(\hat{k}^{+},\varphi) = 1, \tag{C.72}$$

$$\tilde{P}^{+}(\hat{k}^{-},\varphi) = 0. \tag{C.73}$$

Note that $\tilde{P}^{+}(\hat{k},0) = \mathbb{P}^{+}(\hat{k})$. Taking the derivative with φ and evaluating at $\varphi = 0$

$$\Lambda(\hat{k})\frac{\partial \tilde{P}^{+}(\hat{k},0)}{\partial \varphi} = \mathbb{P}^{+}(\hat{k}) - \nu \frac{\partial \frac{\partial \tilde{P}^{+}(\hat{k},0)}{\partial \varphi}}{\partial \hat{k}} + \frac{\sigma^2}{2}\frac{\partial^2 \frac{\partial \tilde{P}^{+}(\hat{k},0)}{\partial \varphi}}{\partial \hat{k}^2}, \tag{C.74}$$

$$\frac{\partial \tilde{P}^{+}(\hat{k}^{+},0)}{\partial \varphi} = 0, \tag{C.75}$$

$$\frac{\partial \tilde{P}^{+}(\hat{k}^{-},0)}{\partial \varphi} = 0. \tag{C.76}$$

From Auxiliary Theorem A.3, we convert the HJB and borders into the first of two formulations:

$$\frac{\partial \tilde{P}^{+}(\hat{k},0)}{\partial \varphi} = \mathbb{E}\left[\int_0^{\tau} \mathbb{P}^{+}(\hat{k}_t)\,\mathrm{d}t \middle| \hat{k}_0 = \hat{k}\right]. \tag{C.77}$$

To obtain the second formulation, note that by definition

$$\frac{\partial \tilde{P}^{+}(\hat{k},\varphi)}{\partial \varphi} = \mathbb{E}\left[\tau e^{\tau\varphi}\mathbb{I}[\hat{k}_{\tau} \geq \hat{k}^{*+}] \middle| \hat{k}_0 = \hat{k}\right], \tag{C.78}$$

evaluating at zero

$$\frac{\partial \tilde{P}^{+}(\hat{k},0)}{\partial \varphi} = \mathbb{E}\left[\tau \mathbb{I}[\hat{k}_{\tau} \geq \hat{k}^{*+}] \middle| \hat{k}_0 = \hat{k}\right]. \tag{C.79}$$

Using relations (C.77) and (C.79), we have that

$$\mathbb{E}\left[\int_0^{\tau} \mathbb{P}^{+}(\hat{k}_t)\,\mathrm{d}t \middle| \hat{k}_0 = \hat{k}\right] = \mathbb{E}\left[\tau \mathbb{I}[\hat{k}_{\tau} \geq \hat{k}^{*+}] \middle| \hat{k}_0 = \hat{k}\right] \tag{C.80}$$

Evaluating in $\hat{k}^{*\pm}$ and operating

$$\frac{r^{-}\mathbb{E}^{-}[\tau]}{\mathbb{E}[\tau]}\mathbb{E}^{-}[\mathbb{P}^{+}(\hat{k})] = r^{-}\frac{\bar{\mathbb{E}}^{-}\left[\tau \mathbb{I}[\hat{k}_{\tau} \geq \hat{k}^{*+}]\right]}{\mathbb{E}[\tau]}, \tag{C.81}$$

$$\frac{r^{+}\mathbb{E}^{+}[\tau]}{\mathbb{E}[\tau]}\mathbb{E}^{+}[\mathbb{P}^{+}(\hat{k})] = r^{+}\frac{\bar{\mathbb{E}}^{+}\left[\tau \mathbb{I}[\hat{k}_{\tau} \geq \hat{k}^{*+}]\right]}{\mathbb{E}[\tau]}. \tag{C.82}$$

Suming the two equations,

$$\mathbb{E}[\mathbb{P}^{+}(\hat{k})] = \frac{\bar{\mathbb{E}}\left[\tau \mathbb{I}[\hat{k}_{\tau} \geq \hat{k}^{*+}]\right]}{\mathbb{E}[\tau]}. \tag{C.83}$$

C.3 Proof of Proposition 4

Proposition 4. ***(Sufficient statistics)*** *Up to the first order, the CIR of average capital-productivity ratios equals the sum of three steady-state cross-sectional moments:*

$$\frac{CIR(\delta)}{\delta} = \underbrace{\frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} + \frac{\nu\mathbb{C}ov[\hat{k},a]}{\sigma^2}}_{\text{up to first adjustment}} + \underbrace{\frac{1}{\sigma^2}\,\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))]\right]}_{\text{subsequent adjustments}} + o(\delta). \tag{51}$$

Proof's strategy We prove the proposition without and with irreversibility. In each case, we construct the master equation that combines the HJB for deviations $m(\hat{k})$ and the KFE describing the distribution of $\hat{k}$. The trick is substituting the hazard $\Lambda(\hat{k})$ from the KFE into the HJB. Then, we multiply by $\hat{k}$ and compute the cross-sectional average. Depending on the case, we get three or four terms T_j that we compute using integration by parts, exploiting the border conditions of m and g. We prove the results for any continuous function $f(\hat{k})$.

C.3.1 Without irreversibility

We will show that

$$\frac{\mathrm{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}),\frac{\hat{k}+\nu a}{\sigma^2}\right] = \frac{\mathbb{C}ov\left[f(\hat{k}),\hat{k}\right]}{\sigma^2} + \frac{\nu\mathbb{C}ov\left[f(\hat{k}),a\right]}{\sigma^2}. \tag{C.84}$$

Rearranging the HJB for $m(\hat{k})$ in (C.39), we get:

$$\Lambda(\hat{k})m(\hat{k}) = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2}m''(\hat{k}) + \Lambda(\hat{k})m(\hat{k}^*)$$

Solve for $\Lambda(k) = \frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}$ from the KFE and substitute it into (B.13) to obtain

$$\left[\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}\right]m(\hat{k}) = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2}m''(\hat{k}) + \left[\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}\right]m(\hat{k}^*). \tag{C.85}$$

Multiplying both sides by $g(\hat{k})\hat{k}$ and taking the definite integral between $\hat{k}^-$ and $\hat{k}^+$ (effectively, we compute the cross-sectional average) we obtain the following expression:

$$0 = \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}[f(\hat{k})]\mathbb{E}[\hat{k}] - \nu T_1 + \frac{\sigma^2}{2}T_2 + m(\hat{k}^*)T_3. \tag{C.86}$$

where we define the following three terms, characterized next:

$$T_1 \equiv \int_{\hat{k}^-}^{\hat{k}^+}\hat{k}\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k} \tag{C.87}$$

$$T_2 \equiv \int_{\hat{k}^-}^{\hat{k}^+}\hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} \tag{C.88}$$

$$T_3 \equiv \int_{\hat{k}^-}^{\hat{k}^+}\hat{k}\left(\nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k})\right)\mathrm{d}\hat{k}. \tag{C.89}$$

We will now rearrange T_1, T_2 and T_3. We will often use the product rule, integration by parts, and the continuity and border conditions of g, namely $g(\hat{k}^+) = g(\hat{k}^-) = 0$, the border conditions of m, namely $m(\hat{k}^+) = m(\hat{k}^-) = m(\hat{k}^*)$, and the continuity of $m(\cdot)$ and $g(\cdot)$ around k^*.

(i) We re-write T_1 by the following steps: in step (1) we split the integral, in step (2) the product rule – $m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k}) = \frac{\mathrm{d}[m(\hat{k})g(\hat{k})]}{\mathrm{d}\hat{k}}$, in step (3) we use integration by parts, in step (4) we rely on the border conditions and continuity of $m(\cdot)$ and $g(\cdot)$ around k^*, in step (5) we join the integral, and in step (6) we use that the cross-sectional mean of $m(\cdot)$ is zero:

$$
\begin{aligned}
T_1 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right] \mathrm{d}\hat{k} \\
&=^{(1)} \int_{\hat{k}^-}^{\hat{k}^*} \hat{k}\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right] \mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right] \mathrm{d}\hat{k} \\
&=^{(2)} \int_{\hat{k}^-}^{\hat{k}^*} \hat{k}(m(\hat{k})g(\hat{k}))' \,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}(m(\hat{k})g(\hat{k}))' \,\mathrm{d}\hat{k} \\
&=^{(3)} \hat{k}m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}m(\hat{k})g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} - \left[\int_{\hat{k}^-}^{\hat{k}^*} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(4)} 0 + 0 - \left[\int_{\hat{k}^-}^{\hat{k}^*} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}\right] =^{(5)} -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} =^{(6)} 0
\end{aligned}
\tag{C.90}
$$

(ii) To characterize T_2, we do the following steps. In step (1), we split the integral; in step (2), we use the equality $m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k}) = (m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k}))'$ and integration by parts; in step (3), we use continuity of $m'(\hat{k})$ and $g(\hat{k})$ around $\hat{k}^*$ and the border condition $g(\hat{k}^+) = g(\hat{k}^-) = 0$ for $\hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+} = 0$; in step (4), we use the border conditions of m; in step (5) we apply integration by parts to t$\int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}$; and step (6) groups common terms and relies on the continuity and border conditions of $m(\cdot)$ and $g(\cdot)$:

(C.91)

$$
\begin{aligned}
T_2 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right] \mathrm{d}\hat{k} \\
&= ^{(1)} \int_{\hat{k}^-}^{\hat{k}^*} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right] \mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right] \mathrm{d}\hat{k} \\
&= ^{(2)} \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^*}^{\hat{k}^+} \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^*} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right] \mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right] \mathrm{d}\hat{k}\right] \\
&= ^{(3)} \underbrace{\hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0} - m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^*} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right] \mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right] \mathrm{d}\hat{k}\right] \\
&= ^{(4)} - m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] - \int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\
&= ^{(5)} - m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] - \left[\underbrace{m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right] + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\
&= ^{(6)} - m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}.
\end{aligned}
$$

(iii) To characterize T_3 we perform the following steps: In step (1) we split the integral, in step (2) we use integration by parts, in step (3) we use the border conditions of $g(\cdot)$, the definition of a density function and solve the integral $\int_{\hat{k}^-}^{\hat{k}^+} g'(\hat{k})\,\mathrm{d}\hat{k}$, in step (4) we use the border conditions of $g(\cdot)$:

$$
\text{(C.92)}\quad \begin{aligned}
T_3 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left(\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})\right)\mathrm{d}\hat{k} \\
&=^{(1)} \nu\left[\int_{\hat{k}^-}^{\hat{k}^*} \hat{k}g'(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}g'(\hat{k})\,\mathrm{d}\hat{k}\right] + \frac{\sigma^2}{2}\left[\int_{\hat{k}^-}^{\hat{k}^*} \hat{k}g''(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}g''(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(2)} \nu\left[\underbrace{\hat{k}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{=0} - \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k}}_{=1}\right] + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} - \int_{\hat{k}^-}^{\hat{k}^+} g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(3)} -\nu + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} - \underbrace{g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0}\right] \\
&=^{(4)} -\nu + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right]
\end{aligned}
$$

Substituting back the expressions for T_1, T_2, and T_3 from equations (C.90) to (C.92) into (C.86)

(C.93)

$$
\begin{aligned}
0 &= \mathbb{E}\Big[f(\hat{k})\hat{k}\Big] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu T_1 + \frac{\sigma^2}{2}T_2 + m(\hat{k}^*)T_3 \\
&= \mathbb{E}\Big[f(\hat{k})\hat{k}\Big] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu 0 + \frac{\sigma^2}{2}\left[-m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&\cdots + m(\hat{k}^*)\left[-\nu + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right]\right] \\
&= \mathbb{E}\Big[f(\hat{k})\hat{k}\Big] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} + \nu\mathbb{E}\left[a\left(f(\hat{k}) - \mathbb{E}\left[f(\hat{k})\right]\right)\right] \\
&= \mathbb{C}ov\left[f(\hat{k}), \hat{k} + \nu a\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}. \iff -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k}) = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}+\nu a}{\sigma^2}\right]
\end{aligned}
$$

Combining this result with the CIR's first-order approximation in Proposition 2 yields:

$$
\text{(C.94)}\qquad \frac{\text{CIR}(f,\delta)}{\delta} = -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}+\nu a}{\sigma^2}\right]
$$

C.3.2 With irreversibility

From Propositions 2 and 3, we know that

$$0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \Lambda(\hat{k}) \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a] - m(\hat{k}) \right), \tag{C.95}$$

$$0 = \mathcal{M}(\hat{k}^{\pm}) - \mathbb{C}ov[f(\hat{k}), a] - m(\hat{k}^{\pm}), \tag{C.96}$$

$$0 = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k}) \, \mathrm{d}\hat{k}. \tag{C.97}$$

where $\mathcal{M}(\hat{k})$ is defined in (C.98)

$$\mathcal{M}(\hat{k}) \equiv \begin{cases} m(\hat{k}^{*-}) + \mathbb{C}ov(\hat{k}, a) < 0 & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ m(\hat{k}^{*+}) + \mathbb{C}ov(\hat{k}, a) > 0 & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+]. \end{cases} \tag{C.98}$$

From the KFE in (B.17) we solve for the adjustment hazard $\Lambda(k) = \frac{\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})}{g(\hat{k})}$ and using equation (C.41)

$$\frac{\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})}{g(\hat{k})} m(\hat{k}) = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \frac{\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})}{g(\hat{k})} \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a] \right) \tag{C.99}$$

Multiplying by $g(\hat{k})\hat{k}$ and taking the integral between $\hat{k}^-$ and $\hat{k}^+$:

$$0 = \mathbb{E}\left[f(\hat{k}) \hat{k} \right] - \mathbb{E}\left[\hat{k} \right] \mathbb{E}\left[f(\hat{k}) \right] - \nu T_1 + \frac{\sigma^2}{2} T_2 + \nu T_3 + \frac{\sigma^2}{2} T_4 \tag{C.100}$$

where we define the following four terms, characterized next:

$$T_1 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k} \left[m'(\hat{k}) g(\hat{k}) + m(\hat{k}) g'(\hat{k}) \right] \mathrm{d}\hat{k} \tag{C.101}$$

$$T_2 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k} \left[m''(\hat{k}) g(\hat{k}) - m(\hat{k}) g''(\hat{k}) \right] \mathrm{d}\hat{k} \tag{C.102}$$

$$T_3 = \int_{\hat{k}^-}^{\hat{k}^+} (\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]) \hat{k} g'(\hat{k}) \, \mathrm{d}\hat{k} \tag{C.103}$$

$$T_4 = \int_{\hat{k}^-}^{\hat{k}^+} (\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]) \hat{k} g''(\hat{k}) \, \mathrm{d}\hat{k}. \tag{C.104}$$

(i) To characterize T_1, we use the following: (1) the product rule – $m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k}) = \frac{\mathrm{d}[m(\hat{k})g(\hat{k})]}{\mathrm{d}\hat{k}}$, (2) integration by parts, border and continuity conditions of $m(\cdot)$ and $g(\cdot)$, and the zero expectation of $m(\cdot)$:

$$\begin{aligned} T_1 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k} \left[m'(\hat{k}) g(\hat{k}) + m(\hat{k}) g'(\hat{k}) \right] \mathrm{d}\hat{k} \\ &=^{(1)} \int_{\hat{k}^-}^{\hat{k}^+} \hat{k} \left(m(\hat{k}) g(\hat{k}) \right)' \mathrm{d}\hat{k} \\ &=^{(2)} \underbrace{\hat{k} m(\hat{k}) g(\hat{k}) \Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \hat{k} m(\hat{k}) g(\hat{k}) \Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k} m(\hat{k}) g(\hat{k}) \Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k}) \, \mathrm{d}\hat{k} = 0 \end{aligned} \tag{C.105}$$

(ii) To rewrite T_2 we carry out the following steps: (1) substitutes $m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})$ with $\frac{\mathrm{d}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}$

using the product rule and splits the integral; (2) applies integration by parts; (3) rearranges terms; (4) uses the continuity and border conditions of $m(\cdot)$ and $g(\cdot)$; (5) uses integration by parts; (6) uses the continuity and border conditions of $m(\cdot)$ and $g(\cdot)$ and joins common terms; (7) uses $m(\hat{k}) = \mathcal{M}(\hat{k}) + \mathbb{C}ov[f(\hat{k}), a]$:

(C.106)

$$
\begin{aligned}
T_2 \;&=\; \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} \\
&=^{(1)}\; \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\frac{\mathrm{d}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\frac{d\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}d\hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\frac{\mathrm{d}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}} \\
&=^{(2)}\; \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^{*+}}^{\hat{k}^+} \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^{*-}}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\;\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\;\hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\;\hat{k}\right] \\
&=^{(3)}\; \underbrace{\hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+} + \hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^-}\right] \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\;\hat{k} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(4)}\; -\left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] - \int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\;\hat{k} + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\;\hat{k} \\
&=^{(5)}\; -\left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\
&- \left[\underbrace{m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\
&=^{(6)}\; -\left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\;\hat{k}. \\
&=^{(7)}\; -\left[\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}}\right] \\
&- \left[\Big(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}), a]\Big)\,\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\;\hat{k}
\end{aligned}
$$

(iii) For T_3, step (1) divides the integration domain into the discontinuity points; step (2) uses continuity of $\mathcal{M}(\hat{k})$ and $g(\hat{k})$, together with the boundaries conditions of $g(\hat{k}^{\pm}) = 0$; step (3) re-writes the integral:

(C.107)

$$
\begin{aligned}
T_3 \;&=\; \int_{\hat{k}^-}^{\hat{k}^+} \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g'(\hat{k})\,\mathrm{d}\hat{k} \\
&=\; ^{(1)} \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) g'(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) g'(\hat{k})\,\mathrm{d}\hat{k} \\
&+ \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) g'(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(2)} \underbrace{\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} \\
&- \int_{\hat{k}^-}^{\hat{k}^+} \left[\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a] + \hat{k}\mathcal{M}'(\hat{k})\right] g(\hat{k})\,\mathrm{d}\hat{k} \\
&=\; ^{(3)}\mathbb{C}ov[f(\hat{k}),a] - \mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right]
\end{aligned}
$$

(iv) For T_4, step (1) breaks the integral, step (2) uses integration by parts, step (3) uses integration by parts, step (4) uses the border conditions for $g(\cdot)$:

(C.108)

$$
\begin{aligned}
T_4 \;&=\; \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) g''(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(1)} \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) g''(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) g''(\hat{k})\,\mathrm{d}\hat{k} \\
&+ \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) g''(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(2)} \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} \\
&+ \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+} - \int_{\hat{k}^-}^{\hat{k}^+} \left[\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a] + \hat{k}\mathcal{M}'(\hat{k})\right] g'(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(3)} \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+} \\
&\cdots - \underbrace{\left[\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a] + \hat{k}\mathcal{M}'(\hat{k})\right] g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0} + \int_{\hat{k}^-}^{\hat{k}^+} \left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right] g(\hat{k})\,\mathrm{d}\hat{k} \\
&=\; \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right) \hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+} \\
&+ \mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right]
\end{aligned}
$$

From equations (C.104) to (C.109)

(C.109)

$$0 = \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu T_1 + \frac{\sigma^2}{2}T_2 + \nu T_3 + \frac{\sigma^2}{2}T_4$$

(C.110)

$$= \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu 0 + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} + \nu\left[\mathbb{C}ov[f(\hat{k}),a] - \mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right]\right]$$

(C.111)

$$\pm\left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right)\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right)\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \left(\mathcal{M}(\hat{k}) - \mathbb{C}ov[f(\hat{k}),a]\right)\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}$$

$$+\frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right]$$

$$= \mathbb{C}ov\left[f(\hat{k}),\hat{k}\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} + \nu\mathbb{C}ov[f(\hat{k}),a] - \nu\mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right]$$

Recalling that $\frac{CIR(f,\delta)}{\delta} = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}$ and rearranging the terms we obtain:

$$\frac{\mathrm{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}),\frac{\hat{k}+\nu a}{\sigma^2}\right] - \frac{\nu}{\sigma^2}\mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + \frac{1}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right] + o(\delta) \tag{C.112}$$

Finally, if we apply Ito's lemma to $\hat{k}\mathcal{M}(\hat{k})$, we have that

$$\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))|\hat{k}_s = \hat{k}] = \left[-\nu\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k}_s)\right]\right]\mathrm{d}s \tag{C.113}$$

such that:

$$\frac{\mathrm{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}),\frac{\hat{k}+\nu a}{\sigma^2}\right] + \frac{1}{\sigma^2}\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)]\right] \tag{C.114}$$

This concludes the proof.

C.4 Proof of Proposition 5

Proposition 5. ***(Extreme cases)*** *Up to the first order, the CIR's sufficient statistics as a function of investment frictions are as follows.*

(i) No drift and only fixed cost: If $\nu = \omega = 0$ and $\theta > 0$, then

$$\frac{CIR(\delta)}{\delta} = \frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} = \left(\frac{12\tilde{\theta}}{(1-\alpha)\sigma^6}\right)^{1/4}, \quad \text{where} \quad \tilde{\theta} = \frac{\theta}{\alpha}\left(\frac{p\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}. \tag{53}$$

(ii) No drift and only partial irreversibility: If $\nu = \theta = 0$ and $\omega > 0$, then

$$\frac{CIR(\delta)}{\delta} = 2 \times \frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} = \left(\frac{12\tilde{\omega}}{(1-\alpha)\sigma^4}\right)^{1/3}, \quad \text{where} \quad \tilde{w} = \frac{\omega/2}{\mathcal{U}(1-\omega/2)}. \tag{54}$$

(iii) Large drift: If $\sigma^2 > 0$ and $\nu \to \infty$, then the price wedge is irrelevant and

$$\mathbb{E}\left[\frac{\mathbb{E}_s[\mathrm{d}(\hat{k}_s \mathcal{M}(\hat{k}_s))]}{\mathrm{d}s}\right] = 0, \qquad \nu\mathbb{C}ov[\hat{k}, a] = -\mathbb{V}ar[\hat{k}], \qquad \frac{CIR(\delta)}{\delta} = 0. \tag{55}$$

Proof's strategy We prove Proposition 5 in a sequence lemmas by departing from Proposition 1. First, Lemma C.2 shows that the investment policy can be separated into a static frictionless component and a dynamic frictional component, where we characterize the latter, introducing the notion of effective investment frictions. From a firm's perspective, what matters for investment decisions is the fixed adjustment cost relative to frictionless profits and the price wedge relative to the frictionless profits-capital ratio, respectively. The first lemma uses the dynamic frictional component to characterize CIR's sufficient statistics. Then, we divide the proof into two major cases: $\nu \to 0$ and $\nu \to \infty$. Within the first case, we consider $\theta = 0$ and $\omega = 0$.

C.4.1 Static and dynamic investment policies

Lemma C.2. *Let $\mathcal{K} \equiv \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ denote the firm's optimal investment policy. Fix any investment price $\hat{p}$. The optimal investment policy can be decomposed as the sum of a static and a dynamic component $\mathcal{K} = \hat{k}^{ss} + \mathcal{X}$, where $\hat{k}^{ss}$ is the static log capital-productivity ratio that firms would set in the absence of frictions under the investment price $\hat{p}$*

$$\hat{k}^{ss} = \frac{1}{1-\alpha}\log\left(\frac{\alpha}{\hat{p}\mathcal{U}}\right) \tag{C.115}$$

and $\mathcal{X} \equiv \{x^-, x^{*-}, x^{*+}, x^+\}$ is the dynamic component that solves the following stopping-time problem for the normalized capital-productivity ratio $x := \hat{k} - \hat{k}^{ss}$

$$\tilde{q}(x) = \mathbb{E}\left[\int_0^\tau e^{-\mathcal{U}s}\left(e^{(\alpha-1)x_s} - 1\right)\mathrm{d}s + e^{-\mathcal{U}s}\left(\tilde{q}(x_\tau + \Delta x) - \tilde{p}(\Delta x)\right)\middle| x_0 = x\right], \tag{C.116}$$

$$\mathrm{d}x_t = -\nu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{C.117}$$

with the additional restriction

$$\tilde{\theta} = \int_{x^-}^{x^{*-}} e^x \left(\tilde{q}(x) - \tilde{p}^{buy}\right) \mathrm{d}x, \tag{C.118}$$

$$\tilde{\theta} = \int_{x^{*+}}^{x^+} e^x \left(\tilde{p}^{sell} - \tilde{q}(x)\right) \mathrm{d}x. \tag{C.119}$$

The effective fixed cost $\tilde{\theta}$ and the effective price wedge $\tilde{\omega}$ are define as

$$\tilde{\theta} = \frac{\theta}{\alpha e^{\alpha \hat{k}^{ss}}} = \frac{\theta}{\alpha}\left(\frac{\hat{p}\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} \tag{C.120}$$

$$\tilde{p}^{buy} = \frac{p - \hat{p}}{\alpha e^{(\alpha-1)\hat{k}^{ss}}} = \frac{p - \hat{p}}{\mathcal{U}\hat{p}} \tag{C.121}$$

$$\tilde{p}^{sell} = \frac{p(1-\omega) - \hat{p}}{\alpha e^{(\alpha-1)\hat{k}^{ss}}} = \frac{p(1-\omega) - \hat{p}}{\mathcal{U}\hat{p}} \tag{C.122}$$

The static optimal policy $\hat{k}^{ss}$ in (C.115) sets the capital-productivity ratio to a constant, and its value reflects profitability α, the average user cost of capital $\mathcal{U}$, and the investment price p. By definition, investment frictions do not affect the static choice $\hat{k}^{ss}$. In contrast, the dynamic policy $\mathcal{X}$ characterized by (C.116) and (C.117) takes into account the fixed cost and the price wedge, scaled by static profits or the profit-capital ratio, respectively. The flow payoff in the dynamic problem $e^{(\alpha-1)x_s} - 1$ only depends on the curvature of the profit function α, and thus is invariant to frictions. Finally, any price can be used to construct $\hat{k}^{ss}$, because $\mathcal{X}$ moves accordingly so that $\mathcal{K}$ is invariant to the price. We use this property below to obtain symmetry in the problem.

Proof. The equilibrium conditions for the Tobin's q are given by:

$$\mathcal{U}q(\hat{k}) = \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} - \nu q'(\hat{k}) + \frac{\sigma^2}{2} q''(\hat{k}), \quad \forall\ \hat{k} \in (\hat{k}^-, \hat{k}^+). \tag{C.123}$$

$$\frac{\theta}{p} = \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}} \left(q(\hat{k}) - 1\right) \mathrm{d}\hat{k}, \tag{C.124}$$

$$\frac{\theta}{p} = \int_{\hat{k}^{*+}}^{\hat{k}^+} e^{\hat{k}} \left((1-\omega) - q(\hat{k})\right) \mathrm{d}\hat{k}, \tag{C.125}$$

$$q(\hat{k}) = 1, \qquad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\} \tag{C.126}$$

$$q(\hat{k}) = (1-\omega), \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \tag{C.127}$$

Normalized $q^\star$ Define the normalized Tobins' q as

$$q^\star(x) \equiv \frac{q(x + \hat{k}^{ss}) - \hat{p}/p}{\alpha e^{(\alpha-1)\hat{k}^{ss}}}, \tag{C.128}$$

which satisfies the following properties:

$$q(\hat{k}) = q^\star(\hat{k} - \hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}} + \hat{p}/p \tag{C.129}$$

$$q'(\hat{k}) = q^{\star\prime}(\hat{k} - \hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}} \tag{C.130}$$

$$q''(\hat{k}) = q^{\star\prime\prime}(\hat{k} - \hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}}. \tag{C.131}$$

From the HJB, we have that for all $\hat{k} \in (\hat{k}^-, \hat{k}^+)$

$$
\begin{aligned}
\text{(C.132)} \qquad \mathcal{U}q(\hat{k}) &= \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} - \nu q'(\hat{k}) + \frac{\sigma^2}{2} q''(\hat{k}), \iff \\
\mathcal{U}(q^\star(\hat{k}-\hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}} + \hat{p}/p) &= \frac{\alpha e^{(\alpha-1)\hat{k}}}{p} - \nu {q^\star}'(\hat{k}-\hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}} + \frac{\sigma^2}{2} {q^\star}''(\hat{k}-\hat{k}^{ss})\alpha e^{(\alpha-1)\hat{k}^{ss}}, \iff \\
\mathcal{U}q^\star(x) &= \frac{e^{(\alpha-1)x}-1}{p} - \nu {q^\star}'(x) + \frac{\sigma^2}{2} {q^\star}''(x),
\end{aligned}
$$

where the last equation holds for all $x \in (x^-, x^+)$. From the optimality condition

$$
\text{(C.133)} \qquad q(\hat{k}) = 1, \qquad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\} \iff q^\star(x) = \frac{p-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}, \qquad x \in \left\{x^-, x^{*-}\right\}
$$

$$
\text{(C.134)} \qquad q(\hat{k}) = (1-\omega), \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\} \iff q^\star(x) = \frac{p(1-\omega)-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}, \qquad x \in \left\{x^+, x^{*+}\right\}
$$

From the value-matching condition with $\hat{k}^-$, we have that

$$
\begin{aligned}
\text{(C.135)} \qquad \frac{\theta}{p} &= \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}} \left(q(\hat{k}) - 1\right) \mathrm{d}\hat{k} \iff \\
\frac{\theta}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}} &= \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}} \left(q^\star(\hat{k}-\hat{k}^{ss}) - \frac{p-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}\right) \mathrm{d}\hat{k} \iff \\
\frac{\theta}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}} &= \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}-\hat{k}^{ss}+\hat{k}^{ss}} \left(q^\star(\hat{k}-\hat{k}^{ss}) - \frac{p-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}\right) \mathrm{d}\hat{k} \iff \\
\frac{\theta}{p\alpha e^{\alpha\hat{k}^{ss}}} &= \int_{x^-}^{x^{*-}} e^{x} \left(q^\star(x) - \frac{p-\hat{p}}{p\alpha e^{(\alpha-1)\hat{k}^{ss}}}\right) \mathrm{d}x \iff \\
\frac{\theta}{\alpha e^{\alpha\hat{k}^{ss}}} &= \int_{x^-}^{x^{*-}} e^{x} \left(\tilde{q}(x) - \frac{p-\hat{p}}{\alpha e^{(\alpha-1)\hat{k}^{ss}}}\right) \mathrm{d}x.
\end{aligned}
$$

where we define $\tilde{q}(x) \equiv pq^\star(x)$,. Similar steps apply to the value matching condition for $\hat{k}^+$.

□

C.4.2 Proof for $\nu = 0$

Lemma C.3. *Let $\nu = 0$ and set $\hat{p} = p(1-\omega/2)$. Consider a first-order approximation of the flow profits*

$$
\text{(C.136)} \qquad e^{(\alpha-1)x} - 1 \approx -(1-\alpha)x
$$

and assume the unweighted boundary conditions are a good approximation of the weighted conditions:

$$
\text{(C.137)} \qquad \int_{x^-}^{x^{*-}} e^x \left(\tilde{q}(x) - \tilde{p}^{buy}\right) \mathrm{d}x \approx \int_{x^-}^{x^{*-}} \left(\tilde{q}(x) - \tilde{p}^{buy}\right) \mathrm{d}x
$$

$$
\text{(C.138)} \qquad \int_{x^{*+}}^{x^+} e^x \left(\tilde{p}^{sell} - \tilde{q}(x)\right) \mathrm{d}x \approx \int_{x^{*+}}^{x^+} \left(\tilde{p}^{sell} - \tilde{q}(x)\right) \mathrm{d}x.
$$

Then $\tilde{q}(x)$ is anti-symmetric, the policies are $x^- = -x^+$ and $x^- = -x^+$, and satisfy:

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x \; + \; \frac{\sigma^2}{2}\tilde{q}''(x), \qquad \forall \;\; x \in (0, x^+), \tag{C.139}$$

$$\tilde{q}(x) = -\tilde{\omega}, \;\; x \in \left\{x^+, x^{*+}\right\}, \qquad \tilde{q}(0) = 0 \tag{C.140}$$

$$-\tilde{\theta} = \int_{x^{*+}}^{x^+} (\tilde{q}(x) + \tilde{\omega})\,\mathrm{d}x. \tag{C.141}$$

Moreover, x^+ and x^{*+} satisfy the non-linear system of equations

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + (1-\alpha)\frac{(x^+)^2-(x^{*+})^2}{2} - \mathcal{U}\tilde{\omega}(x^+ - x^{*+})}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}} (e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}) - (1-\alpha)x^{*+} = -\omega\mathcal{U} \tag{C.142}$$

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + (1-\alpha)\frac{(x^+)^2-(x^{*+})^2}{2} - \mathcal{U}\tilde{\omega}(x^+ - x^{*+})}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}} (e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{+}}) - (1-\alpha)x^{+} = -\omega\mathcal{U} \tag{C.143}$$

- *If $\tilde{\omega} = 0$, then*

$$x^{*+} = 0; \qquad x^+ = \left(\frac{12\tilde{\theta}\sigma^2}{1-\alpha}\right)^{1/4}. \tag{C.144}$$

- *If $\tilde{\theta} = 0$, then*

$$x^{*+} = x^+ = \left(\frac{3\tilde{\omega}\sigma^2}{2(1-\alpha)}\right)^{1/3}. \tag{C.145}$$

Proof. We show that $q(x) = -q(-x)$, $x^- = x^+$, and $x^{*-} = -x^{*+}$ using a guess and verify strategy in the equilibrium conditions.

Step 1: q is antisymmetric. Observe that if $q(x) = -q(-x)$, then $q''(x) = -q''(-x)$. Assume that for all $x \in (x^-, 0]$

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x + \nu q'(x) \; + \; \frac{\sigma^2}{2}q''(x), \qquad \forall \;\; \hat{k} \in (x^-, x^+) \tag{C.146}$$

Multiplying by -1 both sizes of the equality

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x \; + \; \frac{\sigma^2}{2}q''(x), \qquad \forall \;\; x \in (x^-, 0] \tag{C.147}$$

$$\mathcal{U}(-\tilde{q}(x)) = -(1-\alpha)(-x) \; + \; \frac{\sigma^2}{2}(-q''(x)), \qquad \forall \;\; x \in (x^-, 0] \tag{C.148}$$

$$\mathcal{U}(\tilde{q}(-x)) = -(1-\alpha)(-x) \; + \; \frac{\sigma^2}{2}(q''(-x)), \qquad \forall \;\; x \in (x^-, 0] \tag{C.149}$$

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x \; + \; \frac{\sigma^2}{2}q''(x), \qquad \forall \;\; x \in [0, x^+). \tag{C.150}$$

Observe that under $\hat{p} = p(1-\omega)$

$$\tilde{p}^{buy} = \frac{p - p(1-\omega/2)}{\mathcal{U}p(1-\omega/2)} = \frac{\omega/2}{\mathcal{U}(1-\omega/2)} =: \tilde{\omega} \tag{C.151}$$

$$\tilde{p}^{sell} = \frac{p(1-\omega) - \hat{p}}{\mathcal{U}\hat{p}} = \frac{\omega/2}{\mathcal{U}(1-\omega/2)} =: -\tilde{\omega} \tag{C.152}$$

Then

$$\tilde{q}(x) = \tilde{\omega},\ x \in \left\{x^-, x^{*-}\right\} \iff -\tilde{q}(x) = -\tilde{\omega},\ x \in \left\{x^-, x^{*-}\right\} \iff \tilde{q}(x) = -\tilde{\omega},\ x \in \left\{x^+, x^{*+}\right\}. \tag{C.153}$$

Finally, using the unweighted boundary condition, using a change of variables $s = -x$ and $\mathrm{d}x = -\,\mathrm{d}s$

$$\tilde{\theta} = \int_{x^-}^{x^{*-}} (\tilde{q}(x) - \tilde{\omega})\,\mathrm{d}x \quad \iff -\tilde{\theta} = \int_{x^-}^{x^{*-}} (-\tilde{q}(x) + \tilde{\omega})\,\mathrm{d}x \quad \iff -\tilde{\theta} = \int_{x^-}^{x^{*-}} (\tilde{q}(-x) + \tilde{\omega})\,\mathrm{d}x \quad \iff \tag{C.154}$$

$$-\tilde{\theta} = -\int_{-x^-}^{-x^{*-}} (\tilde{q}(s) + \tilde{\omega})\,\mathrm{d}s \quad \iff -\tilde{\theta} = \int_{-x^{*+}}^{x^+} (\tilde{q}(s) + \tilde{\omega})\,\mathrm{d}s. \tag{C.155}$$

Thus, we have shown that $\tilde{q}(x)$ is anti-symmetric.

Step 2: q equilibrium conditions. Since $\tilde{q}(x) = -\tilde{q}(-x)$, we have that $\tilde{q}(0) = -\tilde{q}(0)$ if and only if $\tilde{q}(0)$. Thus, the equilibrium conditions for $\tilde{q}(x)$ and $\{x^{*+}, x^+\}$ are given by

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x \ + \ \frac{\sigma^2}{2}\tilde{q}''(x), \qquad \forall \ \ x \in (0, x^+), \tag{C.156}$$

$$\tilde{q}(x) = -\tilde{\omega},\ x \in \left\{x^+, x^{*+}\right\}, \qquad \tilde{q}(0) = 0 \tag{C.157}$$

$$-\tilde{\theta} = \int_{x^{*+}}^{x^+} (\tilde{q}(x) + \tilde{\omega})\,\mathrm{d}x. \tag{C.158}$$

The solution to the HJB in (C.156) is given by

$$\tilde{q}(x) = \frac{Ae^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} + Be^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} - (1-\alpha)x}{\mathcal{U}} \tag{C.159}$$

Since $\tilde{q}(0) = 0$, we find that $A = -B$ and thus

$$\tilde{q}(x) = \frac{A(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x}) - (1-\alpha)x}{\mathcal{U}} \tag{C.160}$$

To find A, we use the border condition (C.158)

$$\begin{aligned}
-\tilde{\theta} &= \int_{x^{*+}}^{x^+} \left(\frac{A}{\mathcal{U}}(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x}) - \frac{(1-\alpha)x}{\mathcal{U}} + \tilde{\omega}\right)\mathrm{d}x \\
&= \frac{A}{\mathcal{U}}\left(\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\right)^{-1}\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x}\right)\Bigg|_{x=x^{*+}}^{x^+} - (1-\alpha)\frac{(x^+)^2 - (x^{*+})^2}{2\mathcal{U}} + \tilde{\omega}(x^+ - x^{*+}) \\
&= \frac{A}{\mathcal{U}}\left(\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\right)^{-1}\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}\right) - (1-\alpha)\frac{(x^+)^2 - (x^{*+})^2}{2\mathcal{U}} + \tilde{\omega}(x^+ - x^{*+})
\end{aligned}$$

Solving for A we get:

$$A = \sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + \frac{1-\alpha}{2}\left((x^+)^2 - (x^{*+})^2\right) - \tilde{\omega}\mathcal{U}(x^+ - x^{*+})}{\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}.\right)} \tag{C.161}$$

The equilibrium policy satisfies the following system of equations

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + (1-\alpha)\frac{(x^+)^2-(x^{*+})^2}{2} - \mathcal{U}\tilde{\omega}(x^+ - x^{*+})}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}}(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}) - (1-\alpha)x^{*+} = -\omega\mathcal{U} \tag{C.162}$$

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{-\tilde{\theta}\mathcal{U} + (1-\alpha)\frac{(x^+)^2-(x^{*+})^2}{2} - \mathcal{U}\tilde{\omega}(x^+ - x^{*+})}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{*+}}}(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{+}} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^{+}}) - (1-\alpha)x^{+} = -\omega\mathcal{U} \tag{C.163}$$

C.4.3 Proof for $\omega = 0$

If $\tilde{\omega} = 0$, then $x^{*+} = 0$ we have that

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \left((1-\alpha)\frac{(x^+)^2}{2} - \tilde{\theta}\mathcal{U}\right) \frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2} - (1-\alpha)x^+ = 0 \tag{C.164}$$

We can operate over the previous equation, and we have

$$(1-\alpha)x^+ \underbrace{\left(-1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}} \frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2}\right)}_{=(2)} = \sqrt{\frac{2\mathcal{U}}{\sigma^2}}\tilde{\theta}\mathcal{U} \underbrace{\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2}}_{=(1)} \tag{C.165}$$

First, we approximate the term (1), $e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} / \left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2\right)$ for low value of x^+. Observe that when $x^+ \downarrow 0$, $e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} \downarrow 0$ and $\left(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2\right) \downarrow 0$. Thus, we use a Taylor approximation to approximate the ratio—keeping the lowest order to determine the sign of the denominator and numerator. For the denominator and numerator, we have that

$$e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} = \underbrace{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0}}_{=0} + \underbrace{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0})(x^+ - 0)}_{=\sqrt{\frac{2\mathcal{U}}{\sigma^2}}2x^+}$$

$$e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2 = \underbrace{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} - 2}_{=0} + \underbrace{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0})(x^+ - 0)}_{=0x^+}$$
$$+ \frac{1}{2}\underbrace{\left(\frac{2\mathcal{U}}{\sigma^2}\right)(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}0})(x^+ - 0)^2}_{=\frac{2\mathcal{U}}{\sigma^2}(x^+)^2}$$

Using this approximation

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\tilde{\theta}\mathcal{U} \frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2} \approx \sqrt{\frac{2\mathcal{U}}{\sigma^2}}\tilde{\theta}\mathcal{U}\frac{2}{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} = \frac{2\tilde{\theta}\mathcal{U}}{x^+} \tag{C.166}$$

Now, we approximate the term (2),

$$(2) = -1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2} \tag{C.167}$$

for low value of x^+. Doing a third-order Taylor approximation over x near 0 in the numerator and a second in the denominator the denominator

$$(2) = -1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2}, \tag{C.168}$$

$$= -1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\frac{2\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+ + \frac{2}{3!}\left(\frac{2\mathcal{U}}{\sigma^2}\right)^{3/2}(x^+)^3}{\frac{2\mathcal{U}}{\sigma^2}(x^+)^2}, \tag{C.169}$$

$$= -1 + 1 + \left(\frac{2\mathcal{U}}{\sigma^2}\right)\frac{(x^+)^2}{3!}. \tag{C.170}$$

Using this approximation

$$(1-\alpha)x^+\left(-1 + \frac{x^+}{2}\sqrt{\frac{2\mathcal{U}}{\sigma^2}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} - 2}\right) = (1-\alpha)\left(\frac{2\mathcal{U}}{\sigma^2}\right)\frac{(x^+)^3}{3!}. \tag{C.171}$$

Thus,

$$\frac{2\tilde{\theta}\mathcal{U}}{x^+} = (1-\alpha)\left(\frac{2\mathcal{U}}{\sigma^2}\right)\frac{(x^+)^3}{3!} \iff x^+ = \left(\frac{12\tilde{\theta}\sigma^2}{1-\alpha}\right)^{1/4}. \tag{C.172}$$

Since $\omega = 0$, we have that $\tilde{\theta} = \frac{\theta}{\alpha}\left(\frac{\hat{p}\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} = \frac{\theta}{\alpha}\left(\frac{p(1-\omega/2)\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} = \frac{\theta}{\alpha}\left(\frac{p\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}$ and

$$x^+ = \left(\frac{12\tilde{\theta}\sigma^2}{1-\alpha}\right)^{1/4}, \text{ with } \tilde{\theta} = \frac{\theta}{\alpha}\left(\frac{p\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} \tag{C.173}$$

C.4.4 Proof for $\theta = 0$

Since $q(x) = -\tilde{\omega}$ has two roots for $x \geq 0$ with $q(0) = 0$, it is easy to see that when $\tilde{\theta} \downarrow 0$, then $x^{*+} \to x^+$ with $q'(x^+) = 0$. Thus, we replace $-\tilde{\theta} = \int_{x^{*+}}^{x^+}(\tilde{q}(x) + \tilde{\omega})\,\mathrm{d}x$, by the reflecting barrier condition $q'(x^+) = 0$.

$$\mathcal{U}\tilde{q}(x) = -(1-\alpha)x + \frac{\sigma^2}{2}\tilde{q}''(x), \quad \forall\ x \in (0, x^+), \tag{C.174}$$

$$\tilde{q}(x^+) = -\tilde{\omega}, \qquad \tilde{q}(0) = 0, \qquad \tilde{q}'(x^+) = 0. \tag{C.175}$$

Given the solution

$$\tilde{q}(x) = \frac{A(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x} - e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x}) - (1-\alpha)x}{\mathcal{U}} \tag{C.176}$$

The border condition $\tilde{q}'(x^+) = 0$ implies

$$\sqrt{\frac{2\mathcal{U}}{\sigma^2}}A(e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}) - (1-\alpha) = 0 \iff A = \sqrt{\frac{\sigma^2}{2\mathcal{U}}}\frac{1-\alpha}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+} + e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}} \tag{C.177}$$

Thus, x^+ satisfies

$$(1-\alpha)\left(\sqrt{\frac{\sigma^2}{2\mathcal{U}}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}-e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}+e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}-x^+\right)=-\tilde{w}\mathcal{U} \tag{C.178}$$

Applying similar steps as before

$$\sqrt{\frac{\sigma^2}{2\mathcal{U}}}\frac{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}-e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}{e^{\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}+e^{-\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+}}-x^+ \tag{C.179}$$

$$\approx\sqrt{\frac{\sigma^2}{2\mathcal{U}}}\frac{2\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^++\frac{2}{3!}\left(\sqrt{\frac{2\mathcal{U}}{\sigma^2}}x^+\right)^3}{2+\frac{2\mathcal{U}}{\sigma^2}(x^+)^2}-x^+ \tag{C.180}$$

$$=\frac{2x^++\frac{1}{3}\left(\frac{2\mathcal{U}}{\sigma^2}\right)(x^+)^3-2x^+-\frac{2\mathcal{U}}{\sigma^2}(x^+)^3}{2+\frac{2\mathcal{U}}{\sigma^2}(x^+)^2} \tag{C.181}$$

$$=\frac{2\mathcal{U}}{2\sigma^2}(x^+)^3\frac{1/3-1}{1+\frac{\mathcal{U}}{\sigma^2}(x^+)^2} \tag{C.182}$$

$$=-\frac{\mathcal{U}}{\sigma^2}(x^+)^3\frac{2}{3} \tag{C.183}$$

Therefore, we get the result:

$$x^+=\left(\frac{3\tilde{\omega}\sigma^2}{2(1-\alpha)}\right)^{1/3},\ \text{with } \tilde{w}=\frac{\omega/2}{\mathcal{U}(1-\omega/2)} \tag{C.184}$$

□

C.4.5 Sufficient statistics

We finish the proof with the investment statistics and the CIR′(0) components. To simplify the exposition, we work in the space of normalized capital-productivity ratios $x\equiv\hat{k}-\hat{k}^{ss}$

Lemma C.4. *The variance is given by*

$$\mathbb{V}ar[x]=\frac{(\bar{x}^+)^2+(x^{*+})^2}{6}. \tag{C.185}$$

The covariance is given by

$$\mathbb{C}ov[a,x]=0. \tag{C.186}$$

The irreversibility term is given by

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(x_s)x_s)]\right]=\frac{x^{*+}x^+}{3}. \tag{C.187}$$

The CIR is given by

$$\frac{CIR(\delta)}{\delta}=\frac{(\bar{x}^+)^2+(x^{*+})^2}{6\sigma^2}+\frac{x^{*+}x^+}{3\sigma^2}+o(\delta). \tag{C.188}$$

If $\theta = 0$, *then*

$$\frac{CIR(\delta)}{\delta} = 2\frac{\mathbb{V}ar[x]}{\sigma^2} = \left(\frac{12\tilde{\omega}}{(1-\alpha)\sigma^4}\right)^{1/3} \tag{C.189}$$

If $\omega = 0$, *then*

$$\frac{CIR(\delta)}{\delta} = \frac{\mathbb{V}ar[x]}{\sigma^2} = \left(\frac{12\tilde{\theta}}{\sigma^6(1-\alpha)}\right)^{1/4} \tag{C.190}$$

Proof. We exploit symmetry in x to compute all the inputs for the CIR.

Cross-sectional distribution. The stationary density $g(x)$ solves the KFE with border, continuity, and reinjection (exit mass equals entry mass) conditions:

$$0 = \frac{\sigma^2}{2} g''(x), \tag{C.191}$$

$$g(\overline{x}) = g(-\overline{x}) = 0, \tag{C.192}$$

$$\int_{-\overline{x}}^{\overline{x}} g(x)\,\mathrm{d}x = 1, \tag{C.193}$$

$$\lim_{x \downarrow -x^*} g(x) = \lim_{x \uparrow -x^*} g(x), \qquad \lim_{x \downarrow x^*} g(x) = \lim_{x \uparrow x^*} g(x), \tag{C.194}$$

$$\lim_{x \downarrow -\overline{x}} g'(x) = \lim_{x \uparrow -x^*} g'(x) - \lim_{x \downarrow -x^*} g'(x), \qquad \lim_{x \uparrow \overline{x}} g'(x) = \lim_{x \downarrow x^*} g'(x) - \lim_{x \uparrow x^*} g'(x). \tag{C.195}$$

Solving for $g(x)$, we obtain a linear function:

$$g''(x) = 0, \quad g'(x) = A, \quad g(x) = Ax + B. \tag{C.196}$$

We split the state-space into three segments $[-\overline{x}, -x^*] \cup [-x^*, x^*] \cup [x^*, \overline{x}]$ and consider three different functions $g_k(x) = A_k x + B_k$ for $j = 1, 2, 3$, one for each segment. Evaluating at the border conditions, we obtain relationships for (A_1, B_1) and (A_3, B_3):

$$\left.\begin{aligned} -A_1\overline{x} + B_1 &= 0 \\ A_3\overline{x} + B_3 &= 0 \end{aligned}\right\} \implies \overline{x} = B_1/A_1 = -B_3/A_3. \tag{C.197}$$

Evaluating at the reinjection conditions, we obtain A_2:

$$\left.\begin{aligned} A_1 &= A_1 - A_2 \\ A_3 &= A_3 - A_2 \end{aligned}\right\} \implies A_2 = 0. \tag{C.198}$$

Evaluating at the continuity conditions, using $A_2 = 0$ we obtain for (A_1, B_1) and (A_3, B_3):

$$\left.\begin{aligned} B_2 &= -A_1 x^* + B_1 \\ B_2 &= \ \ A_3 x^* + B_3 \end{aligned}\right\} \implies x^* = \frac{B_1 - B_2}{A_1} = \frac{B_2 - B_3}{A_3}. \tag{C.199}$$

Finally, we use the fact that the density integrates to one:

$$\text{(C.200)}\qquad \begin{aligned} 1 &= \int_{-\overline{x}}^{-x^*} (A_1 x + B_1)\,\mathrm{d}x + \int_{-x^*}^{x^*} B_2\,\mathrm{d}x + \int_{x^*}^{\overline{x}} (A_3 x + B_3)\,\mathrm{d}x \\ &= \left(A_1\frac{x^2}{2} + B_1 x\right)\Big|_{-\overline{x}}^{-x^*} + B_2 x\Big|_{-x^*}^{x^*} + \left(A_3\frac{x^2}{2} + B_3 x\right)\Big|_{x^*}^{\overline{x}} \\ &= A_1\left(\frac{x^{*2} - \overline{x}^2}{2}\right) + B_1(\overline{x} - x^*) + 2B_2 x^* + A_3\left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + B_3(\overline{x} - x^*) \\ &= (A_3 - A_1)\left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + 2B_2 x^* + (B_1 + B_3)(\overline{x} - x^*). \end{aligned}$$

Substituting $B_1 = \overline{x}A_1$ and $B_3 = -\overline{x}A_3$ from (C.197) into the previous expression:

$$\text{(C.201)}\qquad \begin{aligned} 1 &= (A_3 - A_1)\left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + 2B_2 x^* - \overline{x}\,(A_3 - A_1)\,(\overline{x} - x^*) \\ &= (A_3 - A_1)\,(\overline{x} - x^*)\left[\frac{\overline{x} + x^*}{2} - \overline{x}\right] + 2B_2 x^* \\ &= (A_3 - A_1)\,\frac{(\overline{x} - x^*)^2}{2} + 2B_2 x^*. \end{aligned}$$

Therefore, the cross-sectional density is equal to:

$$\text{(C.202)}\qquad g(x) = \frac{1}{\overline{x}^2 - x^{*2}} \begin{cases} \overline{x} + x & \text{for} \quad x \in [-\overline{x}, -x^*] \\ \overline{x} - x^* & \text{for} \quad x \in [-x^*, x^*] \\ \overline{x} - x & \text{for} \quad x \in [x^*, \overline{x}]. \end{cases}$$

Renewal probabilities and relative shares. The renewal probabilities (the mass of adjusters from each reset point) are equal to:

$$\text{(C.203)}\qquad \mathcal{N}^- = \frac{\sigma^2}{2}\lim_{x\downarrow -\overline{x}} g'(x) = \frac{\sigma^2}{2}A_1 = \frac{\sigma^2}{2}\frac{1}{(\overline{x}^2 - x^{*2})}$$

$$\text{(C.204)}\qquad \mathcal{N}^+ = -\frac{\sigma^2}{2}\lim_{x\uparrow \overline{x}} g'(x) = -\frac{\sigma^2}{2}A_3 = \frac{\sigma^2}{2}\frac{1}{(\overline{x}^2 - x^{*2})}.$$

The shares of total, upward, and downward adjustment are:

$$\text{(C.205)}\qquad \mathcal{N} = \mathcal{N}^- + \mathcal{N}^+ = \frac{\sigma^2}{(\overline{x}^2 - x^{*2})}$$

$$\text{(C.206)}\qquad \frac{\mathcal{N}^-}{\mathcal{N}} = \frac{1}{2}; \qquad \frac{\mathcal{N}^+}{\mathcal{N}} = \frac{1}{2}.$$

Probability of negative adjustment. Let $\mathbb{P}^+(x) \equiv \Pr[\Delta x < 0|x]$ denote the probability of doing a negative adjustment (after hitting the upper bound) conditional on the state x. It solves the HJB with border conditions:

$$\text{(C.207)}\qquad 0 = {\mathbb{P}^+}''(x); \quad \mathbb{P}^+(\overline{x}) = 1; \quad \mathbb{P}^+(-\overline{x}) = 0$$

Solving for $\mathbb{P}^+(x) = Ax + B$ and evaluating at the border conditions:

$$\text{(C.208)}\qquad \left.\begin{aligned} A\overline{x} + B = 1 \\ -A\overline{x} + B = 0 \end{aligned}\right\} \implies \left.\begin{aligned} A = 1/2\overline{x} \\ B = 1/2 \end{aligned}\right\} \implies \mathbb{P}^+(x) = \frac{\overline{x} + x}{2\overline{x}} = \frac{1}{2} + \frac{x}{2\overline{x}}.$$

The unconditional probability of a negative adjustment is:

$$\mathbb{E}[\mathbb{P}^{+}] \;=\; \frac{1}{2} + \frac{1}{2\overline{x}}\mathbb{E}[x] \;=\; \frac{1}{2}. \tag{C.209}$$

The probability of a negative adjustment conditional on the last adjusting being positive (a switch in adjustment sign) equals:

$$\mathbb{P}^{+}(-x^*) \equiv \Pr[\Delta x < 0| - x^*] \;=\; \frac{\overline{x} - x^*}{2\overline{x}}. \tag{C.210}$$

Probability of positive adjustment. Let $\mathbb{P}^{-}(x) \equiv \Pr[\Delta x > 0|x]$ denote the probability of doing a positive adjustment (after hitting the lower bound) conditional on the state x. It solves the HJB with border conditions:

$$0 = \mathbb{P}^{-\prime\prime}(x); \quad \mathbb{P}^{-}(-\overline{x}) = 1; \quad \mathbb{P}^{-}(\overline{x}) = 0. \tag{C.211}$$

Solving for $\mathbb{P}^{-}(x) = Ax + B$ and evaluating at the border conditions:

$$\left.\begin{aligned} -A\overline{x} + B = 1 \\ A\overline{x} + B = 0 \end{aligned}\right\} \quad \left.\begin{aligned} A = -1/2\overline{x} \\ B = 1/2 \end{aligned}\right\} \quad \mathbb{P}^{-}(x) = \frac{\overline{x} - x}{2\overline{x}} \;=\; \frac{1}{2} - \frac{x}{2\overline{x}}. \tag{C.212}$$

The unconditional probability of a positive adjustment is:

$$\mathbb{E}[\mathbb{P}^{-}] \;=\; \frac{1}{2} - \frac{1}{2\overline{x}}\mathbb{E}[x] \;=\; \frac{1}{2}. \tag{C.213}$$

The probability of a positive adjustment conditional on the last adjusting being negative (a switch in adjustment sign) equals:

$$\mathbb{P}^{-}(x^*) \equiv \Pr[\Delta x > 0|x^*] \;=\; \frac{\overline{x} - x^*}{2\overline{x}}. \tag{C.214}$$

Expected duration of inaction. Let $T(x) \equiv \mathbb{E}[\tau|x]$. It solves the HJB with border conditions:

$$0 \;=\; 1 + \frac{\sigma^2}{2}T''(x), \qquad T(\overline{x}) \;=\; T(-\overline{x}) \;=\; 0. \tag{C.215}$$

Solving for $T(x)$:

$$T''(x) \;=\; -\frac{2}{\sigma^2}, \quad T'(x) \;=\; -\frac{2}{\sigma^2}x + A, \quad T(x) \;=\; -\frac{x^2}{\sigma^2} + Ax + B. \tag{C.216}$$

Evaluating at the border conditions, we obtain values for A and B:

$$\left.\begin{aligned} -\frac{\overline{x}^2}{\sigma^2} + A\overline{x} + B = 0 \\ -\frac{\overline{x}^2}{\sigma^2} - A\overline{x} + B = 0 \end{aligned}\right\} \implies \left.\begin{aligned} 2A\overline{x} = 0 \\ -\frac{2\overline{x}^2}{\sigma^2} + 2B = 0 \end{aligned}\right\} \implies \left.\begin{aligned} A = 0 \\ B = \frac{\overline{x}^2}{\sigma^2} \end{aligned}\right\} \implies T(x) = \frac{\overline{x}^2 - x^2}{\sigma^2}. \tag{C.217}$$

The expected duration of inaction given the current state $\mathbb{E}[\tau|x]$, the expected duration of a complete inaction spell conditional on the last reset point $(\overline{\mathbb{E}}^{+}[\tau], \overline{\mathbb{E}}^{-}[\tau])$, and the unconditional expected duration of inaction $\overline{\mathbb{E}}[\tau]$

are given by:

$$\mathbb{E}[\tau|x] = \frac{\overline{x}^2 - x^2}{\sigma^2}, \tag{C.218}$$

$$\overline{\mathbb{E}}^+[\tau] = \overline{\mathbb{E}}^-[\tau] = \frac{\overline{x}^2 - x^{*2}}{\sigma^2}, \tag{C.219}$$

$$\overline{\mathbb{E}}[\tau] = \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau] + \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau] = \frac{\overline{x}^2 - x^{*2}}{\sigma^2}, \tag{C.220}$$

where the shares of upward and downward adjustment are identical: $\mathcal{N}^+/\mathcal{N} = \mathcal{N}^-/\mathcal{N} = 1/2$.

Cross-sectional means Let $m(x) \equiv \mathbb{E}\left[\int_0^\tau x_s \, \mathrm{d}s | x_0 = x\right]$. It solves the HJB with border conditions:

$$0 = x + \frac{\sigma^2}{2} m''(x), \quad m(\overline{x}) = m(-\overline{x}) = 0. \tag{C.221}$$

Solving for $m(x)$:

$$m''(x) = -\frac{2}{\sigma^2} x, \qquad m'(x) = -\frac{x^2}{\sigma^2} + A, \qquad m(x) = -\frac{x^3}{3\sigma^2} + Ax + B. \tag{C.222}$$

Evaluating at the border conditions, we obtain values for A and B:

$$\left.\begin{array}{r} -\frac{\overline{x}^3}{3\sigma^2} + A\overline{x} + B = 0 \\ \frac{\overline{x}^3}{3\sigma^2} - A\overline{x} + B = 0 \end{array}\right\} \implies \left.\begin{array}{r} A = \frac{\overline{x}^2}{3\sigma^2} \\ B = 0 \end{array}\right\} \implies m(x) = \frac{\overline{x}^2 x - x^3}{3\sigma^2} = \frac{x}{3}\frac{\overline{x}^2 - x^2}{\sigma^2} = \frac{x}{3}\mathbb{E}[\tau|x] \tag{C.223}$$

Unconditional means. Using the occupancy measure, we obtain the means conditional on the last rest point:

$$\overline{\mathbb{E}}^-[x] = \frac{m(-x^*)}{\overline{\mathbb{E}}^-[\tau]} = -\frac{x^*}{3}; \qquad \overline{\mathbb{E}}^+[x] = \frac{m(x^*)}{\overline{\mathbb{E}}^+[\tau]} = \frac{x^*}{3}, \tag{C.224}$$

where $\overline{\mathbb{E}}^-[\tau] = \mathbb{E}[\tau| - x^*]$ and $\overline{\mathbb{E}}^+[\tau] = \mathbb{E}[\tau|x^*]$.

Conditional mean. By symmetry, $\mathbb{E}[x] = 0$. To show this formally, we use the conditional means and the renewal distribution:

$$\mathbb{E}[x] = \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[x] + \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[x] = \frac{1}{2}\left(\frac{x^*}{3}\right) + \frac{1}{2}\left(\frac{-x^*}{3}\right) = 0. \tag{C.225}$$

Unconditional variance. Since $\mathbb{E}[x] = 0$, then $\mathbb{V}ar[x] = \mathbb{E}[x^2]$. Using the cross-sectional distribution, the second moment equals:

$$
\text{(C.226)} \qquad \mathbb{V}ar[x] = \int_{-\overline{x}}^{\overline{x}} x^2 g(x)\,\mathrm{d}x
$$

$$
= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\int_{-\overline{x}}^{-x^*} x^2(\overline{x}+x)\,\mathrm{d}x \;+\; (\overline{x}-x^*)\int_{-x^*}^{x^*} x^2\,\mathrm{d}x \;+\; \int_{x^*}^{\overline{x}} x^2(\overline{x}-x)\,\mathrm{d}x\right]
$$

$$
= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\left(\frac{x^3\overline{x}}{3}+\frac{x^4}{4}\right)\Big|_{-\overline{x}}^{-x^*} \;+\; (\overline{x}-x^*)\frac{x^3}{3}\Big|_{-x^*}^{x^*} \;+\; \left(\frac{x^3\overline{x}}{3}-\frac{x^4}{4}\right)\Big|_{x^*}^{\overline{x}}\right]
$$

$$
= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\frac{-x^{*3}\overline{x}}{3}+\frac{x^{*4}}{4}+\frac{\overline{x}^{*4}}{3}-\frac{\overline{x}^4}{4}+(\overline{x}-x^*)\frac{x^{*3}+x^{*3}}{3}+\frac{\overline{x}^4}{3}-\frac{\overline{x}^4}{4}-\frac{x^{*3}\overline{x}}{3}+\frac{x^{*4}}{4}\right]
$$

$$
= \frac{1}{(\overline{x}^2 - x^{*2})}\left(\frac{\overline{x}^4 - x^{*4}}{6}\right) = \frac{1}{(\overline{x}^2 - x^{*2})}\left(\frac{(\overline{x}^2 - x^{*2})(\overline{x}^2 + x^{*2})}{6}\right)
$$

$$
= \frac{\overline{x}^2 + x^{*2}}{6}
$$

CIR. From (51) and (62), the CIR without drift equals:

$$
\text{(C.227)} \qquad \frac{\text{CIR}(\delta)}{\delta} = \frac{\mathbb{V}ar[x]}{\sigma^2} - \frac{\overline{\mathbb{C}ov}[\Delta\hat{k}, \mathcal{M}(\Delta\hat{k})]}{\sigma^2\overline{\mathbb{E}}[\tau]} + o(\delta).
$$

Cumulative deviations. Recall the values for the unconditional probabilities of a negative and a positive adjustment $\mathbb{E}[\mathbb{P}^+] = \mathbb{E}[\mathbb{P}^-] = 1/2$ in (C.209) and (C.213), and the conditional probabilities of switching adjustment sign $\mathbb{P}^+(-x^*) = \mathbb{P}^-(x^*) = (\overline{x} - x^*)/2\overline{x}$ in (C.210) and (C.214). Substituting these probabilities, the conditional means $\mathbb{E}^-[x] = -x^*/3$ and $\mathbb{E}^-[x] = x^*/3$ in (C.224), and the conditional durations $\mathbb{E}^-[\tau] = \mathbb{E}^+[\tau] = (\overline{x}^2 - x^{*2})/\sigma^2$ in (C.219) into the definition of cumulative deviations $m(\hat{k}^{*-})$ and $m(\hat{k}^{*+})$ yields:

$$
\text{(C.228)} \quad m(\hat{k}^{*-}) = \mathbb{E}[\mathbb{P}^-]\frac{1}{\mathbb{P}^+(-x^*)}(\mathbb{E}^-[x] - \mathbb{E}[x])\mathbb{E}^-[\tau] = \frac{1}{2}\left(\frac{2\overline{x}}{\overline{x}-x^*}\right)\left(-\frac{x^*}{3}-0\right)\frac{\overline{x}^2 - x^{*2}}{\sigma^2} = -\frac{x^*\overline{x}(\overline{x}+x^*)}{3\sigma^2},
$$

$$
\text{(C.229)} \quad m(\hat{k}^{*+}) = \mathbb{E}[\mathbb{P}^+]\frac{1}{\mathbb{P}^-(x^*)}(\mathbb{E}^+[x] - \mathbb{E}[x])\mathbb{E}^+[\tau] = \frac{1}{2}\left(\frac{2\overline{x}}{\overline{x}-x^*}\right)\left(\frac{x^*}{3}-0\right)\frac{\overline{x}^2 - x^{*2}}{\sigma^2} = \frac{x^*\overline{x}(\overline{x}+x^*)}{3\sigma^2}.
$$

Irreversibility term. The irreversibility term for the CIR equals the covariance of the adjustment size and the auxiliary capital-deviation deviation function $\mathcal{M}(\Delta x)$ defined in (C.98). Recall $x^* = \overline{x} - \Delta x$ for $\Delta x < 0$ and $-x^* = -\overline{x} - \Delta x$ for $\Delta x > 0$ and by symmetry $\overline{\mathbb{E}}[\Delta x] = 0$. The numerator of the irreversibility term equals:

$$
\begin{aligned}
\overline{\mathbb{C}ov}[\Delta x, \mathcal{M}(\Delta x)] &= \overline{\mathbb{E}}[\Delta x\mathcal{M}(\Delta x)] - \overline{\mathbb{E}}[\Delta x]\,\overline{\mathbb{E}}[\mathcal{M}(\Delta x)] \\
&= \frac{1}{2}\left[\overline{\mathbb{E}}^-[\Delta x\mathcal{M}(\Delta x)] + \overline{\mathbb{E}}^+[\Delta x\mathcal{M}(\Delta x)]\right] \\
&= \frac{1}{2}\left[(\overline{x}-x^*)m(\hat{k}^{*-}) + (x^*-\overline{x})m(\hat{k}^{*+})\right] = (\overline{x}-x^*)m(\hat{k}^{*-}), \\
&= -(\overline{x}-x^*)\frac{x^*\overline{x}(\overline{x}+x^*)}{3\sigma^2} = -\frac{x^*\overline{x}}{3}\left(\frac{\overline{x}^2 - x^{*2}}{\sigma^2}\right) \\
&= -\frac{x^*\overline{x}}{3}\overline{\mathbb{E}}[\tau].
\end{aligned}
$$

Therefore, the irreversibility term of the CIR equals:

$$
\text{(C.230)} \qquad -\frac{\overline{\mathbb{C}ov}[\Delta x, \mathcal{M}(\Delta x)]}{\overline{\mathbb{E}}[\tau]} = \frac{x^*\overline{x}}{3} > 0.
$$

Finally, substituting the expression for the cross-sectional variance in (C.226) and the irreversibility term in (C.230) into the CIR yields:

$$\frac{\text{CIR}(\delta)}{\delta} = \frac{\overline{x}^2 + x^{*2}}{6\sigma^2} + \frac{x^*\overline{x}}{3\sigma^2} + o(\delta). \tag{C.231}$$

In the benchmark cases:

$$\frac{\text{CIR}(\delta)}{\delta} = \begin{cases} \frac{\overline{x}^2}{6\sigma^2} + o(\delta) & \text{if} \quad \tilde{p} = 0, \\ \\ \frac{2\overline{x}^{*2}}{3\sigma^2} + o(\delta) & \text{if} \quad \tilde{\theta} = 0, \end{cases} \tag{C.232}$$

□

C.4.6 Proof for $\nu \to \infty$

Lemma C.5. *Let $\nu > 0$ and $\sigma^2 \to 0$ such that $\nu/\sigma^2 \to \infty$. The mean and variance of $\hat{k} - \hat{k}^{ss}$ satisfy the joint system*

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} = -\frac{(\mathcal{U} - \nu)\tilde{\theta}}{\sqrt{12}(1-\alpha)} \quad ; \quad \mathbb{V}ar[x] = 2\left(\frac{\nu}{\mathcal{U}}\right)^2 \frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{1 - \left(1 - \frac{\nu}{\mathcal{U}}(1-\alpha)\right)^2 e^{(\alpha-1)\mathbb{E}[x]}} \tag{C.233}$$

The covariance is given by

$$\mathbb{C}ov[a, x] = -\nu\mathbb{V}ar[x]. \tag{C.234}$$

The irreversibility term is given by

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(x_s)x_s)]\right] = 0. \tag{C.235}$$

The CIR is given by

$$\frac{CIR(\delta)}{\delta} = 0 + o(\delta). \tag{C.236}$$

Proof. We depart from the equilibrium condition for $\tilde{q}(x)$ to characterize the policy in the case with $\sigma = 0$. Let $\hat{p} = p$. Then $\tilde{p}^{buy} = 0$ and since $\sigma = 0$, the domain in $x > x^{*-}$ is not operating. $\tilde{q}(x)$ satisfies

$$\mathcal{U}\tilde{q}(x) = e^{(\alpha-1)x} - 1 - \nu\tilde{q}(x), \tag{C.237}$$

$$\tilde{q}(x^-) = \tilde{q}(x^{*-}) = 0, \tag{C.238}$$

$$\tilde{\theta} = \int_{x^-}^{x^{*-}} e^x \tilde{q}(x)\,\mathrm{d}x. \tag{C.239}$$

Next, we obtain a system of two equations to characterize x^- and x^{*-}. Multiplying the HJB equation by $e^{\frac{\mathcal{U}}{\nu}x}$, we have that

$$e^{\frac{\mathcal{U}}{\nu}x}(\mathcal{U}\tilde{q}(x) + \nu\tilde{q}'(x)) = e^{\frac{\mathcal{U}}{\nu}x}(e^{(\alpha-1)x} - 1), \tag{C.240}$$

which is equivalent to

$$\nu \frac{\mathrm{d} e^{\frac{\mathcal{U}}{\nu} x} \tilde{q}(x)}{\mathrm{d} x} = e^{\frac{\mathcal{U}}{\nu} x} (e^{(\alpha-1)x} - 1). \tag{C.241}$$

Integrating from x^- to x

$$\nu (e^{\frac{\mathcal{U}}{\nu} x} \tilde{q}(x) - e^{\frac{\mathcal{U}}{\nu} x_-} \tilde{q}(x_-)) = \int_{x^-}^{x} e^{\frac{\mathcal{U}}{\nu} s} (e^{(\alpha-1)s} - 1) \, \mathrm{d}s \tag{C.242}$$

Using the optimality condition $q(x^{*-}) = 0$, we have that

$$\tilde{q}(x) = \frac{\int_{x^-}^{x} e^{\frac{\mathcal{U}}{\nu}(s-x)} (e^{(\alpha-1)s} - 1) \, \mathrm{d}s}{\nu}. \tag{C.243}$$

Evaluation at x^{*+}, we have the first equilibrium condition

$$0 = \int_{x^-}^{x^{*+}} e^{\frac{\mathcal{U}}{\nu}(s - x^{*+})} (e^{(\alpha-1)s} - 1) \, \mathrm{d}s \iff 0 = \int_{x^-}^{x^{*+}} e^{\frac{\mathcal{U}}{\nu} s} (e^{(\alpha-1)s} - 1) \, \mathrm{d}s. \tag{C.244}$$

To obtain the second optimality condition, we multiply the HJB equation by e^x, we have that

$$\mathcal{U} e^x \tilde{q}(x) = e^x (e^{(\alpha-1)x} - 1) - \nu e^x \tilde{q}'(x). \tag{C.245}$$

Integrating between x^- to x^{*-},

$$\mathcal{U} \int_{x^-}^{x^{*-}} e^x \tilde{q}(x) \, \mathrm{d}x = \int_{x^-}^{x^{*-}} e^x (e^{(\alpha-1)x} - 1) \, \mathrm{d}x - \nu \int_{x^-}^{x^{*-}} e^x \tilde{q}'(x) \, \mathrm{d}x. \tag{C.246}$$

Doing integration by part and using the boundary condition for $\tilde{q}(x)$,

$$\int_{x^-}^{x^{*-}} e^x \tilde{q}'(x) \, \mathrm{d}x = \underbrace{e^x \tilde{q}(x) |_{x^-}^{x^{*-}}}_{=0} - \int_{x^-}^{x^{*-}} e^x \tilde{q}(x) \, \mathrm{d}x. \tag{C.247}$$

Rearranging

$$(\mathcal{U} - \nu) \int_{x^-}^{x^{*-}} e^x \tilde{q}(x) \, \mathrm{d}x = \int_{x^-}^{x^{*-}} e^x (e^{(\alpha-1)x} - 1) \, \mathrm{d}x. \tag{C.248}$$

Finally, since $\tilde{\theta} = \int_{x^-}^{x^{*-}} e^x \tilde{q}(x) \, \mathrm{d}x$,

$$(\mathcal{U} + \nu) \tilde{\theta} = \int_{x^-}^{x^{*-}} (e^{\alpha x} - e^x) \, \mathrm{d}x. \tag{C.249}$$

In conclusion, the optimality conditions are given by

$$0 = \int_{x^-}^{x^{*+}} e^{\frac{\mathcal{U}}{\nu} x} (e^{(\alpha-1)x} - 1) \, \mathrm{d}x \tag{C.250}$$

$$(\mathcal{U} - \nu) \tilde{\theta} = \int_{x^-}^{x^{*-}} e^x (e^{(\alpha-1)x} - 1) \, \mathrm{d}x \tag{C.251}$$

Observe that the equilibrium conditions are similar to Sheshinski and Weiss (1977) with the objective function

$F(x) = \frac{x^\alpha}{\alpha} - x$ and discounting $\rho = \mathcal{U} - \nu$. Moreover, for this problem to be well-defined, $\mathcal{U} > \nu$ (if not, firms value is infinite). Now, we describe steady-moments as a function of the investment friction. Using a first-order Taylor approximation around zero for $e^x(e^{(\alpha-1)x} - 1)$, we have

$$e^x(e^{(\alpha-1)x} - 1) \approx e^0(e^{(\alpha-1)0} - 1) + (\alpha e^0 - e^0)(x - 0) = -(1-\alpha)x \tag{C.252}$$

Applying this approximation to the first optimality condition in (C.251) yields:

$$x^{*-2} - x^{-2} \;=\; -\frac{2(\mathcal{U}-\nu)\tilde{\theta}}{1-\alpha}. \tag{C.253}$$

Since the cross-sectional distribution is uniform in the range $[x^-, x^{*-}]$, it has the following moments:

$$\mathbb{V}ar[x] \;=\; \frac{(x^{*-} - x^-)^2}{12}; \qquad \mathbb{E}[x] \;=\; \frac{x^{*-} + x^-}{2}. \tag{C.254}$$

Thus, we can write the first optimality condition in (C.253) as:

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} \;=\; -\frac{(\mathcal{U}-\nu)\tilde{\theta}}{\sqrt{12}(1-\alpha)}. \tag{C.255}$$

It is easy to see that $\mathbb{E}[x] < 0$. Firms compensate, but not undo, capital depresiation and productivity growth. Define $j(x) = e^{\frac{\mathcal{U}}{\nu}x}(e^{(\alpha-1)x} - 1)$. Observe that if we divide and multiply by $x^{*-} - x^-$, we can re-express as

$$0 = \int_{x^-}^{x^{*+}} e^{\frac{\mathcal{U}}{\nu}x}(e^{(\alpha-1)x} - 1)\,\mathrm{d}x = (x^{*-} - x^-)\int_{x^-}^{x^{*+}} j(x)\frac{1}{x^{*-} - x^-}\,\mathrm{d}x \iff 0 = \mathbb{E}[j(x)] \tag{C.256}$$

Thus, the expected discounted marginal product of capital relative to its cost is equal to zero. Doing a second-order Taylor approximation over $j(x)$ around the mean

$$0 = \mathbb{E}[j(x)] \approx \mathbb{E}[j(\mathbb{E}[x]) + j'(\mathbb{E}[x])(x - \mathbb{E}[x]) + \frac{j''(\mathbb{E}[x])}{2}(x - \mathbb{E}[x])^2] = j(\mathbb{E}[x]) + \frac{j''(\mathbb{E}[x])}{2}\mathbb{V}ar[x]. \tag{C.257}$$

Re-expressing the previous equation

$$\mathbb{V}ar[x] = 2\frac{j(\mathbb{E}[x])}{-j''(\mathbb{E}[x])} \tag{C.258}$$

Since $\mathbb{E}[x] < 0$, $j(\mathbb{E}[x]) > 0$ and since $j(x)$ is concave (because $\mathcal{U}/\nu > 1$ and $\alpha - 1 < 0$), $-j''(\mathbb{E}[x]) > 0$. Thus,

$$\mathbb{V}ar[x] = -2\frac{e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}}{\left(\frac{\mathcal{U}}{\nu}+\alpha-1\right)^2 e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - \left(\frac{\mathcal{U}}{\nu}\right)^2 e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}} \tag{C.259}$$

Thus, the equilibrium conditions are given by

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} \;=\; -\frac{(\mathcal{U}-\nu)\tilde{\theta}}{\sqrt{12}(1-\alpha)} \quad ; \quad \mathbb{V}ar[x] = -2\frac{e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}}{\left(\frac{\mathcal{U}}{\nu}+\alpha-1\right)^2 e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - \left(\frac{\mathcal{U}}{\nu}\right)^2 e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}} \tag{C.260}$$

Rearranging the second equation

$$\mathbb{V}ar[x] = -2\frac{e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}}{\left(\frac{\mathcal{U}}{\nu}+\alpha-1\right)^2 e^{(\frac{\mathcal{U}}{\nu}+\alpha-1)\mathbb{E}[x]} - \left(\frac{\mathcal{U}}{\nu}\right)^2 e^{\frac{\mathcal{U}}{\nu}\mathbb{E}[x]}} \tag{C.261}$$

$$= -2\frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{\left(\frac{\mathcal{U}}{\nu}+\alpha-1\right)^2 e^{(\alpha-1)\mathbb{E}[x]} - \left(\frac{\mathcal{U}}{\nu}\right)^2} \tag{C.262}$$

$$= 2\frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{\left(\frac{\mathcal{U}}{\nu}\right)^2 - \left(\frac{\mathcal{U}}{\nu}+\alpha-1\right)^2 e^{(\alpha-1)\mathbb{E}[x]}} \tag{C.263}$$

$$= 2\left(\frac{\nu}{\mathcal{U}}\right)^2 \frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{1-\left(1-\frac{\nu}{\mathcal{U}}(1-\alpha)\right)^2 e^{(\alpha-1)\mathbb{E}[x]}} \tag{C.264}$$

Thus, the equilibrium mean and variance of capital-to-productivity ratios is given by

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} \;=\; -\frac{(\mathcal{U}-\nu)\tilde{\theta}}{\sqrt{12}(1-\alpha)} \quad ; \quad \mathbb{V}ar[x] = 2\left(\frac{\nu}{\mathcal{U}}\right)^2 \frac{e^{(\alpha-1)\mathbb{E}[x]} - 1}{1-\left(1-\frac{\nu}{\mathcal{U}}(1-\alpha)\right)^2 e^{(\alpha-1)\mathbb{E}[x]}} \tag{C.265}$$

The proof for the fact that $\mathbb{C}ov[x,a] = -\nu\mathbb{V}ar[x]$ is in Baley and Blanco (2021). □

C.5 Preliminaries for Proofs of Propositions 6, 7, 8 and 9

We review some properties of Markov chains in this model. Following the main text notation, let $\mathbb{P}^{++}$ and $\mathbb{P}^{--}$ be the transition probabilities. In a steady state, the probability of a current upsizing (downsizing) equals the ergodic probability of a subsequent upsizing (downsizing):

$$\frac{\mathcal{N}^-}{\mathcal{N}} \;=\; \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{--} \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{+-} \tag{C.266}$$

$$\frac{\mathcal{N}^+}{\mathcal{N}} \;=\; \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{++} \;+\; \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{-+} \tag{C.267}$$

Ergodicity of stopping times We show that

$$\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau'] \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau'] \;=\; \overline{\mathbb{E}}[\tau]. \tag{C.268}$$

This relationship follows directly from the law of iterated expectations.

Ergodicity of reset points We show that the average reset capital conditional on an investment $\overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right] = \mathcal{H}^-\hat{k}^{*-} + \mathcal{H}^+\hat{k}^{*+}$ is equal to the average reset capital in the next investment conditional on current investment $\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k}')\right]\Big|\Delta k\right] = \overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right]$.

$$\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}^{*\prime}] \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}^{*\prime}] \;=\; \overline{\mathbb{E}}[\hat{k}^*]. \tag{C.269}$$

The first term:

$$\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}^{*\prime}] = \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{--}\hat{k}^{*-} \;+\; \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{-+}\hat{k}^{*+} \tag{C.270}$$

The second term:

$$\frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}^{*'}] = \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{++}\hat{k}^{*+} \ + \ \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{+-}\hat{k}^{*-} \tag{C.271}$$

Summing the two terms up and using the relationships in (C.266) and (C.267) we get:

$$\begin{aligned}
\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}^{*'}] \ + \ \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}^{*'}] \quad &= \quad \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{--}\hat{k}^{*-} \ + \ \frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{-+}\hat{k}^{*+} \ + \ \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{++}\hat{k}^{*+} \ + \ \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{+-}\hat{k}^{*-} \\
&= \quad \left(\frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{--} \ + \ \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{+-}\right)\hat{k}^{*-} \ + \ \left(\frac{\mathcal{N}^-}{\mathcal{N}}\mathbb{P}^{-+} \ + \ \frac{\mathcal{N}^+}{\mathcal{N}}\mathbb{P}^{++}\right)\hat{k}^{*+} \\
&= \quad \frac{\mathcal{N}^-}{\mathcal{N}}\hat{k}^{*-} \ + \ \frac{\mathcal{N}^+}{\mathcal{N}}\hat{k}^{*+} = \overline{\mathbb{E}}[\hat{k}^*]
\end{aligned}$$

C.6 Proof of Proposition 6

Proposition 6. ***(Recovering parameters)*** *The drift ν and volatility σ^2 of capital-productivity ratios implied by investment microdata are recovered through the following mappings:*

$$\nu \ = \ \frac{\overline{\mathbb{E}}[\Delta\hat{k}]}{\overline{\mathbb{E}}[\tau]}, \tag{56}$$

$$\sigma^2 \ = \ \frac{\overline{\mathbb{E}}[(\hat{k}_{\tau'} + \nu\tau')^2 - (\hat{k}^*)^2]}{\overline{\mathbb{E}}[\tau]}. \tag{57}$$

C.6.1 Drift

Conditional on the previous reset point $\hat{k}^*$, the law of motion of capital-productivity ratios implies

$$\hat{k}_s \ + \ \nu s \ - \ \hat{k}^* \ = \ \sigma W_s. \tag{C.272}$$

Evaluate at a future stopping time $s = \tau'$ to get

$$\hat{k}_{\tau'} + \nu\tau' - \hat{k}^* \ = \ \sigma W_{\tau'}. \tag{C.273}$$

Since the expectation of the future stopped capital depends on the previous reset point, we take expectations conditional on the last adjustment:

$$\overline{\mathbb{E}}^{\pm}[\hat{k}_{\tau'}] \ + \ \nu\overline{\mathbb{E}}^{\pm}[\tau'] \ - \ \hat{k}^{*\pm} \ = \ 0. \tag{C.274}$$

Note that from these expressions, we can find mappings for the drift using conditional means:

$$\nu \ = \ \frac{\hat{k}^{*\pm} - \overline{\mathbb{E}}^{\pm}[\hat{k}_{\tau'}]}{\overline{\mathbb{E}}^{\pm}[\tau']}. \tag{C.275}$$

To derive mappings using the unconditional mean, we average the conditional expectations in (C.274) with the shares of upward and downward adjustments:

$$\frac{\mathcal{N}^-}{\mathcal{N}}\left(\overline{\mathbb{E}}^-[\hat{k}_{\tau'}] + \nu\overline{\mathbb{E}}^-[\tau'] - \hat{k}^{*-}\right) + \frac{\mathcal{N}^+}{\mathcal{N}}\left(\overline{\mathbb{E}}^+[\hat{k}_{\tau'}] + \nu\overline{\mathbb{E}}^+[\tau'] - \hat{k}^{*+}\right) \ = \ 0 \tag{C.276}$$

Join similar terms and use the ergodic property of stopping times in (C.268)

$$\text{(C.277)} \qquad \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}_{\tau'}] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}_{\tau'}] + \nu \underbrace{\left(\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau'] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau']\right)}_{\overline{\mathbb{E}}[\tau]} - \frac{\mathcal{N}^+}{\mathcal{N}}\hat{k}^{*-} - \frac{\mathcal{N}^+}{\mathcal{N}}\hat{k}^{*+} = 0.$$

Substitute the relationship $\hat{k}_{\tau'} = \hat{k}^{*'} - \Delta\hat{k}'$ and use the ergodic property of reset points in (C.269)

$$\text{(C.278)} \qquad \underbrace{\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\hat{k}^{*'}] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\hat{k}^{*'}]}_{\overline{\mathbb{E}}[\hat{k}^*]} \underbrace{-\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\Delta\hat{k}'] - \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\Delta\hat{k}']}_{-\overline{\mathbb{E}}[\Delta\hat{k}']} + \nu\overline{\mathbb{E}}[\tau'] \underbrace{-\frac{\mathcal{N}^-}{\mathcal{N}}\hat{k}^{*-} - \frac{\mathcal{N}^+}{\mathcal{N}}\hat{k}^{*+}}_{-\overline{\mathbb{E}}[\hat{k}^*]} \;=\; 0$$

Cancel terms and rearrange

$$\text{(C.279)} \qquad \cancel{\overline{\mathbb{E}}[\hat{k}^*]} \;-\; \overline{\mathbb{E}}[\Delta\hat{k}'] \;+\; \nu\overline{\mathbb{E}}[\tau'] \;-\; \cancel{\overline{\mathbb{E}}[\hat{k}^*]} \;=\; 0$$

to obtain the result:

$$\text{(C.280)} \qquad \nu \;=\; \frac{\overline{\mathbb{E}}[\Delta\hat{k}']}{\overline{\mathbb{E}}[\tau']}.$$

C.6.2 Idiosyncratic volatility

Let $Y_s \equiv (\hat{k}_s + \nu s)^2$. Applying Itō's lemma to Y_s,

$$\text{(C.281)} \qquad \mathrm{d}Y_s \;=\; 2(\hat{k}_s + \nu s)(\mathrm{d}\hat{k}_s + \nu\,\mathrm{d}s) \;+\; (\mathrm{d}\hat{k}_s)^2 \;=\; 2(\hat{k}_s + \nu s)\sigma\,\mathrm{d}W_s + \sigma^2\,\mathrm{d}s.$$

We integrate both sides from 0 to τ' and take expectations conditional on the previous reset point:

$$\text{(C.282)} \qquad \overline{\mathbb{E}}^{\pm}[Y_{\tau'}] - Y_0 \;=\; 2\sigma\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}(\hat{k}_s + \nu s)\,\mathrm{d}W_s\right] + \sigma^2\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'} 1\,\mathrm{d}s\right]$$

We use the OST (Auxiliary Theorem in A.1) to set the martingale to zero, $\overline{\mathbb{E}}^{\pm}[\int_0^{\tau'}(\hat{k}_s + \nu s)\,\mathrm{d}W_s] = 0$, and obtain:

$$\text{(C.283)} \qquad \overline{\mathbb{E}}^{\pm}[Y_{\tau'}] - Y_0 \;=\; \sigma^2\overline{\mathbb{E}}^{\pm}[\tau'].$$

Substituting $Y_{\tau'} \equiv (\hat{k}_{\tau'} + \nu\tau')^2$ and $Y_0 \equiv (\hat{k}^{*\pm})^2$

$$\text{(C.284)} \qquad \overline{\mathbb{E}}^{\pm}\left[(\hat{k}_{\tau'} + \nu\tau')^2\right] - (\hat{k}^{\pm})^2 \;=\; \sigma^2\overline{\mathbb{E}}^{\pm}[\tau']$$

We average the conditional expectations with the shares of upward and downward adjustments

$$\text{(C.285)} \qquad \underbrace{\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-\left[(\hat{k}_{\tau'} + \nu\tau')^2\right] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+\left[(\hat{k}_{\tau'} + \nu\tau')^2\right]}_{\overline{\mathbb{E}}\left[(\hat{k}_{\tau'}+\nu\tau')^2\right]} - \underbrace{\left(\frac{\mathcal{N}^-}{\mathcal{N}}(\hat{k}^{*-})^2 + \frac{\mathcal{N}^+}{\mathcal{N}}(\hat{k}^{*+})^2\right)}_{\overline{\mathbb{E}}[(\hat{k}^*)^2]} = \sigma^2\underbrace{\left(\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau'] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau']\right)}_{\overline{\mathbb{E}}[\tau] \text{ by (C.268)}}$$

Rearranging, we obtain the mapping from data to σ^2:

$$\text{(C.286)} \qquad \sigma^2 \;=\; \frac{\overline{\mathbb{E}}[(\hat{k}_{\tau'} + \nu\tau')^2] - \overline{\mathbb{E}}[(\hat{k}^*)^2]}{\overline{\mathbb{E}}[\tau]}.$$

C.7 Proof of Proposition 7

Proposition 7. ***(Recovering means)*** *Let $r^{\pm}$ be the adjusted shares in (25). The unconditional mean $\mathbb{E}[\hat{k}]$ and means conditional on the previous reset $\mathbb{E}^{\pm}[\hat{k}]$ are recovered as:*

$$\mathbb{E}[\hat{k}] \;=\; r^{-}\mathbb{E}^{-}[\hat{k}] \;+\; r^{+}\mathbb{E}^{+}[\hat{k}], \tag{58}$$

$$\mathbb{E}^{\pm}[\hat{k}] \;=\; \overline{\mathbb{E}}^{\pm}\left[\left(\frac{\hat{k}^{*\pm}+\hat{k}_{\tau'}}{2}\right)\left(\frac{\hat{k}^{*\pm}-\hat{k}_{\tau'}}{\overline{\mathbb{E}}^{\pm}[\hat{k}^{*\pm}-\hat{k}_{\tau'}]}\right)\right] \;+\; \frac{\sigma^2}{2\nu}. \tag{59}$$

Proof. This proposition expresses the cross-sectional moments of $\hat{k}$ as functions of the data. We derive mappings for any moment n of $\hat{k}$. We apply Itō's lemma to $\hat{k}_s^n$ for $n \geq 2$:

$$\mathrm{d}\hat{k}_s^{n+1} \;=\; -\nu(n+1)\hat{k}_s^n\,\mathrm{d}s \;+\; \sigma(n+1)\hat{k}_s^n\,\mathrm{d}W s \;+\; \frac{\sigma^2 n(n+1)}{2}\hat{k}_s^{n-1}\,\mathrm{d}s. \tag{C.287}$$

We integrate this expression from 0 to τ' and take expectations conditional on the initial condition given by the previous reset point $\hat{k}_0 = \hat{k}^{*\pm}$

$$\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\mathrm{d}\hat{k}_s^{n+1}\right] \;=\; -\nu(n+1)\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^n\,\mathrm{d}s\right] \;+\; \sigma(n+1)\overline{\mathbb{E}}^{\pm}\left[\hat{k}_s^n\,\mathrm{d}W s\right] \;+\; \frac{\sigma^2 n(n+1)}{2}\overline{\mathbb{E}}^{\pm}\left[\hat{k}_s^{n-1}\,\mathrm{d}s\right]. \tag{C.288}$$

The term on the LHS is the definite integral of a derivative. On the RHS, we use the OST in (A.1) to set the martingale in the second term to zero, $\overline{\mathbb{E}}^{\pm}\left[\hat{k}_s^n\,\mathrm{d}W s\right] = 0$.

$$\overline{\mathbb{E}}^{\pm}\left[\hat{k}_{\tau'}^{n+1}\right] - (\hat{k}^{*\pm})^{n+1} \;=\; -\nu(n+1)\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^n\,\mathrm{d}s\right] \;+\; \frac{\sigma^2 n(n+1)}{2}\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^{n-1}\,\mathrm{d}s\right]. \tag{C.289}$$

Divide both sides by $\overline{\mathbb{E}}^{\pm}[\tau']$

$$\frac{\overline{\mathbb{E}}^{\pm}\left[\hat{k}_{\tau'}^{n+1}\right] - (\hat{k}^{*\pm})^{n+1}}{\overline{\mathbb{E}}^{\pm}[\tau']} \;=\; -\nu(n+1)\frac{\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^n\,\mathrm{d}s\right]}{\overline{\mathbb{E}}^{\pm}[\tau']} \;+\; \frac{\sigma^2 n(n+1)}{2}\frac{\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^{n-1}\,\mathrm{d}s\right]}{\overline{\mathbb{E}}^{\pm}[\tau']} \tag{C.290}$$

Use OMT in (A.2) to recover steady-state moments in the RHS using the occupancy measure (e.g. $\overline{\mathbb{E}}^{\pm}[\hat{k}^n] = \mathbb{E}^{\pm}\left[\int_0^{\tau'}\hat{k}_s^n\,\mathrm{d}s\right]/\overline{\mathbb{E}}^{\pm}[\tau']$)

$$\frac{\overline{\mathbb{E}}^{\pm}\left[\hat{k}_{\tau'}^{n+1}\right] - (\hat{k}^{*\pm})^{n+1}}{\overline{\mathbb{E}}^{\pm}[\tau']} \;=\; -\nu(n+1)\mathbb{E}^{\pm}[\hat{k}^n] \;+\; \frac{\sigma^2 n(n+1)}{2}\mathbb{E}^{\pm}[\hat{k}^{n-1}] \tag{C.291}$$

Solving for $\mathbb{E}^{\pm}[\hat{k}^n]$ and rearranging, we obtain a mapping for the conditional n-th moment:

$$\mathbb{E}^{\pm}[\hat{k}^n] \;=\; \frac{(\hat{k}^{*\pm})^{n+1} - \overline{\mathbb{E}}^{\pm}\left[\hat{k}_{\tau'}^{n+1}\right]}{\nu(n+1)\overline{\mathbb{E}}^{\pm}[\tau']} \;+\; \frac{\sigma^2 n}{2\nu}\mathbb{E}^{\pm}[\hat{k}^{n-1}]. \tag{C.292}$$

Applying similar steps as before, we compute the conditional moments centered at the economy-wide mean $\mathbb{E}[\hat{k}]$:

$$\mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^n] \;=\; \frac{(\hat{k}^{*\pm}-\mathbb{E}[\hat{k}])^{n+1} - \overline{\mathbb{E}}^{\pm}\left[(\hat{k}_{\tau'}-\mathbb{E}[\hat{k}])^{n+1}\right]}{\nu(n+1)\overline{\mathbb{E}}^{\pm}[\tau']} \;+\; \frac{\sigma^2 n}{2\nu}\mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^{n-1}]. \tag{C.293}$$

Conditional means To obtain the conditional means, we first evaluate (C.292) at $n = 1$,

$$\mathbb{E}^{\pm}[\hat{k}] \;=\; \frac{(\hat{k}^{*\pm})^2 - \overline{\mathbb{E}}^{\pm}\left[\hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^{\pm}[\tau']} \;+\; \frac{\sigma^2}{2\nu}. \tag{C.294}$$

Unconditional mean To compute the unconditional mean, we average the conditional means using the adjusted shares $r^{\pm} = \frac{\mathcal{N}^{\pm}}{\mathcal{N}}\frac{\overline{\mathbb{E}}^{\pm}[\tau']}{\overline{\mathbb{E}}[\tau']}$, where $r^- + r^+ = 1$.

$$\begin{aligned}
\mathbb{E}[\hat{k}] \;&=\; r^-\mathbb{E}^-[\hat{k}] + r^+\mathbb{E}^+[\hat{k}] && \text{(C.295)}\\
&=\; r^-\frac{\overline{\mathbb{E}}^-\left[(\hat{k}^{*-})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^-[\tau']} \;+\; r^+\frac{\overline{\mathbb{E}}^+\left[(\hat{k}^{*+})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^+[\tau']} \;+\; (r^- + r^+)\frac{\sigma^2}{2\nu} && \text{(C.296)}\\
&=\; \frac{\mathcal{N}^-}{\mathcal{N}}\frac{\overline{\mathbb{E}}^-[\tau']}{\overline{\mathbb{E}}[\tau']}\frac{\overline{\mathbb{E}}^-\left[(\hat{k}^{*-})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^-[\tau']} \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\frac{\overline{\mathbb{E}}^+[\tau']}{\overline{\mathbb{E}}[\tau']}\frac{\overline{\mathbb{E}}^+\left[(\hat{k}^{*+})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}^+[\tau']} \;+\; \frac{\sigma^2}{2\nu} && \text{(C.297)}\\
&=\; \frac{\frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-\left[(\hat{k}^{*-})^2 - \hat{k}^2_{\tau'}\right] \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+\left[(\hat{k}^{*+})^2 - \hat{k}^2_{\tau'}\right]}{2\nu\overline{\mathbb{E}}[\tau']} \;+\; \frac{\sigma^2}{2\nu} && \text{(C.298)}\\
&=\; \frac{\overline{\mathbb{E}}[(\hat{k}^*)^2] - \overline{\mathbb{E}}[\hat{k}^2_{\tau'}]}{2\overline{\mathbb{E}}[\Delta\hat{k}]} \;+\; \frac{\sigma^2}{2\nu}. && \text{(C.299)}
\end{aligned}$$

In the last step, we substitute $\nu = \overline{\mathbb{E}}[\Delta\hat{k}']/\overline{\mathbb{E}}[\tau']$ and use the relationship $\overline{\mathbb{E}}[\cdot] = \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\cdot] + \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\cdot]$.

□

C.8 Proof of Proposition 8

Proposition 8. ***(Recovering the variance and covariance)*** *The variance $\mathbb{V}ar[\hat{k}]$ and the covariance $\mathbb{C}ov[\hat{k}, a]$ are recovered from the microdata as:*

$$\mathbb{V}ar[\hat{k}] \;=\; \frac{1}{3}\frac{\mathbb{E}\left[(\hat{k}^* - \mathbb{E}[\hat{k}])^3\right] - \overline{\mathbb{E}}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^3\right]}{\hat{k}^* - \overline{\mathbb{E}}[\hat{k}_{\tau'}]}. \tag{60}$$

$$\mathbb{C}ov[\hat{k}, a] \;=\; \frac{1}{2\nu}\left(\mathbb{V}ar[\hat{k}] \;+\; \sigma^2\mathbb{E}[a] \;-\; \frac{\overline{\mathbb{E}}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^2\tau'\right]}{\overline{\mathbb{E}}[\tau]}\right). \tag{61}$$

Proof. **Variance** To obtain the variance, we evaluate (C.293) at $n = 2$:

$$\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}^{\pm}[\hat{k}])^2] \;=\; \frac{\overline{\mathbb{E}}^{\pm}\left[(\hat{k}^{*\pm} - \mathbb{E}^{\pm}[\hat{k}])^3 - (\hat{k}_{\tau'} - \mathbb{E}^{\pm}[\hat{k}])^3\right]}{3\nu\overline{\mathbb{E}}^{\pm}[\tau']} \;+\; \frac{\sigma^2}{\nu}\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}^{\pm}[\hat{k}])]. \tag{C.300}$$

Substituting the definition of variance on the LHS and setting the second term on the RHS to zero, we obtain

$$\mathbb{V}ar^{\pm}[\hat{k}] \;=\; \frac{1}{3}\frac{(\hat{k}^{*\pm} - \mathbb{E}^{\pm}[\hat{k}])^3 - \overline{\mathbb{E}}^{\pm}\left[(\hat{k}_{\tau'} - \mathbb{E}^{\pm}[\hat{k}])^3\right]}{\nu\overline{\mathbb{E}}^{\pm}[\tau']}. \tag{C.301}$$

To obtain the unconditional average, we average the conditional variances using relative adjusting shares to get the unconditional values:

$$\mathbb{V}ar[\hat{k}] \;=\; \frac{1}{3}\frac{(\hat{k}^* - \mathbb{E}[\hat{k}])^3 - \overline{\mathbb{E}}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^3\right]}{\nu\overline{\mathbb{E}}[\tau']}. \tag{C.302}$$

Finally, substitute the denominator for (C.275).

Joint moments of $\hat{k}$ and a We prove this proposition of any joint moment of $(\hat{k}, a)$. Consider the function $Y_s = (\hat{k}_s - \mathbb{E}[\hat{k}])^{n+1}s$ and apply Itō's lemma to obtain:

$$\begin{aligned} \mathrm{d}Y_s \;=\;& (\hat{k}_s - \mathbb{E}[\hat{k}])^{n+1}\,\mathrm{d}s \;-\; \nu(n+1)(\hat{k}_s - \mathbb{E}[\hat{k}])^n s\,\mathrm{d}s \;+\; \sigma(n+1)(\hat{k}_s - \mathbb{E}[\hat{k}])^n s\,\mathrm{d}W_s \\ &+\; \frac{\sigma^2}{2}n(n+1)(\hat{k}_s - \mathbb{E}[\hat{k}])^{n-1}s\,\mathrm{d}s. \end{aligned} \tag{C.303}$$

We integrate this expression from 0 to τ', take expectations conditional on the previous reset point, use OST in (A.1) to set martingales to zero, and divide both sides by $\overline{\mathbb{E}}^{\pm}[\tau']$:

$$\frac{\overline{\mathbb{E}}^{\pm}\left[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^{n+1}\tau'\right]}{\overline{\mathbb{E}}^{\pm}[\tau']} \;=\; \mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^{n+1}] - \nu(n+1)\mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^n a] + \frac{\sigma^2}{2}n(n+1)\mathbb{E}^{\pm}[(\hat{k}-\mathbb{E}[\hat{k}])^{n-1}a]. \tag{C.304}$$

Rearranging:

$$\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}[\hat{k}])^n a] \;=\; \frac{1}{\nu(n+1)}\left[\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}[\hat{k}])^{n+1}] \;+\; \frac{\sigma^2}{2}n(n+1)\mathbb{E}^{\pm}[(\hat{k} - \mathbb{E}[\hat{k}])^{n-1}a] \;-\; \frac{\overline{\mathbb{E}}^{\pm}[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^{n+1}\tau']}{\overline{\mathbb{E}}^{\pm}[\tau']}\right]. \tag{C.305}$$

To obtain the unconditional average, we average the conditional joint moments of $\hat{k}$ and a using relative adjusting shares to get the unconditional joint moments:

$$\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^n a] \;=\; \frac{1}{\nu(n+1)}\left[\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^{n+1}] \;+\; \frac{\sigma^2}{2}n(n+1)\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^{n-1}a] \;-\; \frac{\overline{\mathbb{E}}[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^{n+1}\tau']}{\overline{\mathbb{E}}[\tau']}\right]. \tag{C.306}$$

Covariance Finally, to compute the covariance between $(\hat{k}, a)$, we evaluate expression (C.306) at $n = 1$ to obtain

$$\mathbb{C}ov[\hat{k}, a] = \frac{1}{2\nu}\left(\mathbb{V}ar[\hat{k}] + \sigma^2\mathbb{E}[a] - \frac{\overline{\mathbb{E}}[(\hat{k}_{\tau'} - \mathbb{E}[\hat{k}])^2\tau']}{\overline{\mathbb{E}}[\tau']}\right) \tag{C.307}$$

□

C.9 Proof of Proposition 9

Proposition 9. ***(Recovering the irreversibility term)*** *The CIR's irreversibility term is recovered from the microdata as*

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))\right]\right] \;=\; \frac{\overline{\mathbb{E}}[\hat{k}_{\tau'}\mathcal{M}(\hat{k}_{\tau'})] - \overline{\mathbb{E}}[\hat{k}^*\mathcal{M}(\hat{k}^*)]}{\overline{\mathbb{E}}[\tau]}, \tag{62}$$

where departing deviations $\mathcal{M}(\hat{k}^{*\pm})$ and ending deviations $\mathcal{M}(\hat{k}_{\tau'})$ are recovered in Proposition 3.

Proof. **C.9.1 Local drift**

Apply Ito's lemma to the product $\hat{k}_s\mathcal{M}(\hat{k}_s)$

$$\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))] = \left[-\nu\left[\mathcal{M}(\hat{k}_s) + \hat{k}_s\mathcal{M}'(\hat{k}_s)\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}_s) + \hat{k}_s\mathcal{M}''(\hat{k}_s)\right]\right]\mathrm{d}s \tag{C.308}$$

Taking the integral between 0 and τ', using the OST in (A.1) to set martingales to zero, and the OMT in (A.2) to convert occupancy measures into cross-sectional moments:

$$\frac{\overline{\mathbb{E}}\left[\mathbb{E}\left[\hat{k}_{\tau'}\mathcal{M}(\hat{k}_{\tau'})|\Delta\hat{k}\right]\right] - \overline{\mathbb{E}}\left[\mathbb{E}\left[\hat{k}^*\mathcal{M}(\hat{k}^*)|\Delta\hat{k}\right]\right]}{\overline{\mathbb{E}}[\tau]} = \mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)\right]\right] \tag{C.309}$$

□

C.10 Proof of proposition 10

Proposition 10. ***(Recovering reset points)*** *Let* $\Phi \equiv \log\big(\alpha/(\mathcal{U} - (1-\alpha)\nu - (1-\alpha)^2\sigma^2/2)\big)$. *The two reset points* $\{\hat{k}^{*-}, \hat{k}^{*+}\}$ *are recovered from the microdata as:*

$$\hat{k}^{*-} = \frac{1}{1-\alpha}\left(\Phi - \log p + \log\frac{1-\overline{\mathbb{E}}^-\left[e^{-\mathcal{U}\tau^* + (1-\alpha)(\hat{k}^{*-}-\hat{k}_{\tau'})}\right]}{1-\overline{\mathbb{E}}^-\left[\frac{p(\Delta\hat{k}')}{p}e^{-\mathcal{U}\tau^*}\right]}\right), \tag{65}$$

$$\hat{k}^{*+} = \frac{1}{1-\alpha}\left(\Phi - \log p(1-\omega) + \log\frac{1-\overline{\mathbb{E}}^+\left[e^{-\mathcal{U}\tau^* + (1-\alpha)(\hat{k}^{*+}-\hat{k}_{\tau'})}\right]}{1-\overline{\mathbb{E}}^+\left[\frac{p(\Delta\hat{k}')}{p(1-\omega)}e^{-\mathcal{U}\tau^*}\right]}\right). \tag{66}$$

C.10.1 Without irreversibility

We begin showing how to recover the unique reset point $\hat{k}^*$ without irreversibility. Recall the HJB for Tobin's q in sequential form in equation (11), evaluated at $q^* = 1$:

$$q(\hat{k}) = \mathbb{E}\left[\int_0^\tau \frac{\alpha e^{-\mathcal{U}s+(\alpha-1)\hat{k}_s}}{p}\mathrm{d}s + e^{-\mathcal{U}\tau}1\right] \tag{C.310}$$

Evaluating at the optimum $\hat{k}^*$

$$1 = q(\hat{k}^*) = \overline{\mathbb{E}}\left[\int_0^\tau \frac{\alpha e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}}{p}\mathrm{d}s + e^{-\mathcal{U}\tau}1\right] \tag{C.311}$$

Next, we characterize in terms of observables. Define $Y_s = e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}$ and apply Ito's lemma:

$$\mathrm{d}Y_s = Y_s[-\mathcal{U}\,\mathrm{d}s - (1-\alpha)\,\mathrm{d}\hat{k}_s + \frac{(1-\alpha)^2}{2}\mathrm{d}\hat{k}_s^2] \tag{C.312}$$

$$= Y_s\left[-\mathcal{U}\,\mathrm{d}s - (1-\alpha)(-\nu\,\mathrm{d}s + \sigma\,\mathrm{d}W_s) + \frac{(1-\alpha)^2\sigma^2}{2}\mathrm{d}s\right] \tag{C.313}$$

$$= -\underbrace{\left[\mathcal{U} - (1-\alpha)\nu - \frac{(1-\alpha)^2\sigma^2}{2}\right]}_{\phi} Y_s\,\mathrm{d}s - (1-\alpha)\sigma Y_s\,\mathrm{d}W_s \tag{C.314}$$

where we define $\phi \equiv \mathcal{U} - (1-\alpha)\nu - \frac{(1-\alpha)^2\sigma^2}{2}$. Integrating both sides from 0 to τ^*, taking expectations conditional on the initial condition $k_0 = k^*$, and using the OST in (A.1) to set the expectation of martingales to zero, we obtain:

$$\underbrace{\overline{\mathbb{E}}\left[\int_0^{\tau^*} \mathrm{d}Y_s\,\mathrm{d}s\right]}_{\overline{\mathbb{E}}[Y_\tau - Y_0\ ]} = -\phi\overline{\mathbb{E}}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right] - (1-\alpha)\sigma\underbrace{\overline{\mathbb{E}}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}W_s\right]}_{=0}. \tag{C.315}$$

or simply

$$\frac{\overline{\mathbb{E}}\left[Y_0 - Y_{\tau^*}\right]}{\phi} = \overline{\mathbb{E}}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right] \tag{C.316}$$

Since $Y_{\tau^*} = e^{-\mathcal{U}\tau^* - (1-\alpha)(\hat{k}^* - \Delta\hat{k})}$ and $Y_0 = e^{-(1-\alpha)\hat{k}^*}$, then $Y_0 - Y_{\tau^*} = e^{-(1-\alpha)\hat{k}^*}[1 - e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}]$. Substituting back and rearranging, we find an expression for the first term of (C.311)

$$\overline{\mathbb{E}}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right] = \frac{e^{-(1-\alpha)\hat{k}^*}\left(1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]\right)}{\phi} \tag{C.317}$$

Using this term, we solve for $\hat{k}^*$ from (C.311) to get:

$$p = \alpha\frac{e^{-(1-\alpha)\hat{k}^*}\overline{\mathbb{E}}\left[1 - e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]}{\phi} + p\overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^*}\right] \tag{C.318}$$

$$e^{(1-\alpha)\hat{k}^*} = \alpha\frac{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]}{\phi p\left(1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^*}\right]\right)} \tag{C.319}$$

$$\hat{k}^* = \frac{1}{1-\alpha}\log\left(\frac{\alpha}{\phi p}\frac{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]}{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^*}\right]}\right) \tag{C.320}$$

$$\hat{k}^* = \frac{1}{1-\alpha}\left(\Phi - \log p + \log\frac{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^* + (1-\alpha)\Delta\hat{k}}\right]}{1 - \overline{\mathbb{E}}\left[e^{-\mathcal{U}\tau^*}\right]}\right) \tag{C.321}$$

where $\Phi \equiv \log(\alpha/\phi) = \log(\alpha) - \log\left(\mathcal{U} - (1-\alpha)\nu - (1-\alpha)^2\sigma^2/2\right)$.

C.10.2 With irreversibility

With irreversibility we characterize the two reset states as a function of the data. The investment price is now a function of the investment sign. Moreover, since future reset points are unknown and to avoid confusion, we distinguish between past and future investments by denoting with primes future variables, such as $\hat{k}'_\tau$ and $\Delta\hat{k}'$.

Using the sequential formulation of the Tobin's q

$$q(\hat{k}) = \mathbb{E}\left[\int_0^{\tau} \frac{\alpha e^{-\mathcal{U}s + (\alpha-1)\hat{k}_s}}{p}\,\mathrm{d}s + e^{-\mathcal{U}\tau}Q^*\left(\Delta\hat{k}\right)\right] \tag{C.322}$$

Because there are two reset points, at this step, we must condition on the appropriate initial condition to evaluate the previous condition. If the last reset point was $\hat{k}_0 = \hat{k}^{*-}$ (there was a capital purchase), then the optimality condition is $q(\hat{k}^{*-}) = 1$:

$$\text{(C.323)} \qquad 1 \;=\; \frac{\alpha}{p}\overline{\mathbb{E}}^{-}\left[\int_0^{\tau^*} e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}\,\mathrm{d}s \;+\; Q(\Delta\hat{k}')e^{-\mathcal{U}\tau^*}\right].$$

If the last reset point is $\hat{k}_0 = \hat{k}^{*+}$ (there was a capital sale), then the optimality condition is $q(\hat{k}^{*+}) = 1-\omega$:

$$\text{(C.324)} \qquad 1-\omega \;=\; \frac{\alpha}{p}\overline{\mathbb{E}}^{+}\left[\int_0^{\tau^*} e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}\,\mathrm{d}s \;+\; Q(\Delta\hat{k}')\;e^{-\mathcal{U}\tau^*}\right].$$

where $\overline{\mathbb{E}}^{+} = \overline{\mathbb{E}}[\cdot|k^{*+}, u_0]$ denotes expectations conditional on a negative investment.

To express these moments in terms of microdata, we follow similar steps as in the proof without irreversibility but taking into account that investments happen at two different reset points. Consider the optimality condition of a firm that has bought capital at a price p^{buy} to reset its capital-productivity ratio to $\hat{k}^{*-}$ in (C.323) (the proof for a capital purchase is analogous). As before, we define $Y_s = e^{-\mathcal{U}s-(1-\alpha)\hat{k}_s}$ and apply Ito's lemma: $\mathrm{d}Y_s = -\phi Y_s\,\mathrm{d}s - (1-\alpha)\sigma Y_s\,\mathrm{d}W_s$. Integrating both sides from 0 to τ^*, taking expectations conditional on a positive investment, i.e., with respect to the initial condition $k_0 = k^{*-}$, and using the OST to set the expectation of martingales to zero, we obtain:

$$\text{(C.325)} \qquad \frac{\overline{\mathbb{E}}^{-}\left[Y_0 - Y_{\tau^*}\right]}{\phi} \;=\; \overline{\mathbb{E}}^{-}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right].$$

Since $Y_{\tau^*} = e^{-\mathcal{U}\tau^*-(1-\alpha)\hat{k}'_\tau}$ and $Y_0 = e^{-(1-\alpha)\hat{k}^{*-}}$, then $Y_0 - Y_{\tau^*} = e^{-(1-\alpha)\hat{k}^{*-}} - e^{-\mathcal{U}\tau^*-(1-\alpha)\hat{k}'_\tau}$. We find a common factor but remain alert about the difference between current and future investments:

$$\text{(C.326)} \qquad Y_0 - Y_{\tau^*} \;=\; e^{-(1-\alpha)\hat{k}^{*-}}\left[1 - e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau - \hat{k}^{*-})}\right].$$

Substituting back these difference into the LHS of (C.325),

$$\text{(C.327)} \qquad \frac{e^{-(1-\alpha)\hat{k}^{*-}}\overline{\mathbb{E}}^{-}\left[1 - e^{-\mathcal{U}\tau^*-\mathcal{U}(\hat{k}'_\tau - \hat{k}^{*-})}\right]}{\phi} \;=\; \overline{\mathbb{E}}^{-}\left[\int_0^{\tau^*} Y_s\,\mathrm{d}s\right].$$

Now, we substitute this expression into the first term of (C.323) and multiply and divide the second term by p^{buy} to get:

$$\text{(C.328)} \qquad 1 \;=\; \frac{\alpha}{p^{\text{buy}}}\frac{e^{-(1-\alpha)\hat{k}^{*-}}\overline{\mathbb{E}}^{-}\left[1 - e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau - \hat{k}^{*-})}\right]}{\phi} \;+\; \overline{\mathbb{E}}^{-}\left[Q(\Delta\hat{k}')e^{-\mathcal{U}\tau^*}\right]$$

$$\text{(C.329)} \qquad p^{\text{buy}} \;=\; \alpha\frac{e^{-(1-\alpha)\hat{k}^{*-}}\overline{\mathbb{E}}^{-}\left[1 - e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau - \hat{k}^{*-})}\right]}{\phi} \;+\; p^{\text{buy}}\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p^{\text{buy}}}e^{-\mathcal{U}\tau^*}\right]$$

$$\text{(C.330)} \qquad e^{(1-\alpha)\hat{k}^{*-}} \;=\; \frac{\alpha}{\phi p^{\text{buy}}}\frac{1 - \overline{\mathbb{E}}^{-}\left[e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau - \hat{k}^{*-})}\right]}{1 - \overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p^{\text{buy}}}e^{-\mathcal{U}\tau^*}\right]}$$

$$\text{(C.331)} \qquad \hat{k}^{*-} \;=\; \frac{1}{1-\alpha}\left(\Phi \;-\; \log p^{\text{buy}} \;+\; \log\frac{1 - \overline{\mathbb{E}}^{-}\left[e^{-\mathcal{U}\tau^*-(1-\alpha)(\hat{k}'_\tau - \hat{k}^{*-})}\right]}{1 - \overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p^{\text{buy}}}e^{-\mathcal{U}\tau^*}\right]}\right)$$

The previous expression characterizes the reset point after a positive investment. As a final step, noting that

$\hat{k}'_\tau = \hat{k}^*(\Delta\hat{k}') - \Delta\hat{k}'$, we can rewrite (C.331) as:

$$(C.332)\qquad \hat{k}^{*-} = \frac{1}{1-\alpha}\left(\Phi - \log p^{\text{buy}} + \log \frac{1-\overline{\mathbb{E}}^{-}\left[e^{-(r+\xi)\tau^*-(1-\alpha)(\hat{k}^*(\Delta\hat{k}')-\hat{k}^{*-}-\Delta\hat{k}')}\right]}{1-\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k}')}{p^{\text{buy}}}e^{-(r+\xi)\tau^*}\right]}\right)$$

With one reset point, $\hat{k}^*(\Delta\hat{k}') = \hat{k}^*$ and the expression collapses to that in (C.321). Here, because reset points might be different, $\hat{k}^{*-}$ appears on both sides of (C.331); thus, it is only characterizes it implicitly. We propose an iterative method to compute this value from the microdata.

With analogous steps, we obtain the reset point for negative investments:

$$(C.333)\qquad \hat{k}^{*+} = \frac{1}{1-\alpha}\left(\Phi - \log p^{\text{sell}} + \log \frac{1-\overline{\mathbb{E}}^{+}\left[e^{-(r+\xi)\tau^*-(1-\alpha)(\hat{k}^*(\Delta\hat{k}')-\hat{k}^{*+}-\Delta\hat{k}')}\right]}{1-\overline{\mathbb{E}}^{+}\left[\frac{p(\Delta\hat{k}')}{p^{\text{sell}}}e^{-(r+\xi)\tau^*}\right]}\right)$$

Given the two reset points, the distance between them (the length of the inner inaction region) equals:

$$\hat{k}^{*+} - \hat{k}^{*-} = \frac{1}{1-\alpha}\left(\log\frac{1}{1-\omega} + \log \frac{1-\overline{\mathbb{E}}^{+}\left[e^{-(r+\xi)\tau^*-(1-\alpha)(\hat{k}^*(\Delta\hat{k}')-\hat{k}^{*+}-\Delta\hat{k}')}\right]}{1-\overline{\mathbb{E}}^{-}\left[e^{-(r+\xi)\tau^*-(1-\alpha)(\hat{k}^*(\Delta\hat{k}')-\hat{k}^{*-}-\Delta\hat{k}')}\right]} - \log \frac{1-\overline{\mathbb{E}}^{+}\left[\frac{p(\Delta\hat{k})}{p(1-\omega)}e^{-(r+\xi)\tau^*}\right]}{1-\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k})}{p}e^{-(r+\xi)\tau^*}\right]}\right)$$

D A General Equilibrium Framework

This section provides a general equilibrium model that microfounds the parsimonious investment model presented, allowing for examining macroeconomic fluctuations. Its core components are a small open economy and "capital quality shocks."

D.1 Economic environment

Time is continuous, and it extends forever. Four types of agents live in the economy: (i) A representative household, (ii) a capital–goods producer, (iii) a final–good producer, and (iv) a unit mass of intermediate–good firms indexed by $f \in [0, 1]$ who are subject to capital adjustment frictions.

(i) Representative household. The household chooses the stochastic processes for consumption C_s, risk-free bonds B_s, and equity for each firm E_{fs}, subject to the law of motion for nominal wealth:

$$W_s = \int_0^1 P_{ft} E_{ft} \, \mathrm{d}f + B_t, \tag{D.1}$$

$$\mathrm{d}B_s + \int_0^1 P_{fs} \, \mathrm{d}E_{fs} \, \mathrm{d}f = (\mathcal{Y}_s - C_s) \, \mathrm{d}s, \tag{D.2}$$

where P_{fs} is the price of equity for firm f and $\mathcal{Y}_s$ is the after-tax available income, given by:

$$\mathcal{Y}_s = \left(\int_0^1 D_{fs} E_{fs} \, \mathrm{d}f + \tilde{\rho}_s B_s \right). \tag{D.3}$$

Here, D_{fs} represents firm f's dividend payments, and $\tilde{\rho}_s$ is the world interest rate. We omit the profits of the final-good producer and the capital-good producer, as they exhibit constant returns to scale and do not generate profits for the household. Thus, we exclude those sectors' profits from the household budget constraint. Taking the prices of equity $\{P_{fs}\}_{fs}$ and the real interest rate $\tilde{\rho}_s$ as given, the household's problem is to maximize its expected utility (discounted at rate χ):

$$\max_{\{C_s, B_s, \{E_{ft}\}_f\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\chi t} \log C_s \, \mathrm{d}s \right], \tag{D.4}$$

subject to the budget constraints in (D.2) and (D.3), and the initial conditions B_0 and $\{E_{f0}\}_f$.

(ii) Capital–good producer. The capital–good producer manufactures firm-specific investment goods $\{i_{ft}\}_{f \in [0,1]}$ in a competitive market according to a linear technology:

$$\int_0^1 \left(\frac{\varphi(i_{ft}) i_{fs}}{u_{fs}} \right) \mathrm{d}f = i_s, \tag{D.5}$$

where

$$\varphi(i_{ft}) = \begin{cases} \varphi^- & \text{if } i_{ft} > 0 \\ \varphi^+ & \text{if } i_{ft} \le 0 \end{cases}. \tag{D.6}$$

We refer to u_{fs} as capital quality shocks. The parameters φ^- and φ^+ measure the level of partial irreversibility, with $\varphi^- > \varphi^+$. Taking the prices of firm-specific investment goods p^k_{js} as given, the capital–good firm maximizes its profits:

$$\max_{\{i_{ft}, i_t\}_{t=0}^{\infty}} \left(\int_0^1 p^k_{fs} i_{fs} \, \mathrm{d}f - i_s \right), \tag{D.7}$$

subject to the technology described in (D.5). Here, i_s is the aggregate investment to produce capital. Note that i_{ft} may be positive or negative, as its sign has no technological constraint.

(iii) Final–good producer. The final–good producer assembles output Y_s using intermediate inputs $\{\hat{y}_{fs}\}_{f\in[0,1]}$ according to a linear aggregator:

$$Y_s = \int_0^1 \left(\frac{\hat{y}_{fs}}{u_{fs}} \right) \mathrm{d}f, \tag{D.8}$$

where capital quality u_{fs} decreases the marginal product of the intermediate good f. Taking the prices of intermediate inputs p_{ft} as given, the producer's problem entails choosing final–good supply Y_s and input demands $\hat{y}_{fs}$ to maximize profits:

$$\max_{Y_s, \hat{y}_{fs}} \left(Y_t - \int_0^1 p_{ft} \hat{y}_{ft} \, \mathrm{d}f \right), \tag{D.9}$$

subject to the aggregator in (D.8).

(iv) Intermediate-good firms. These are the most important economic agents for our question as they make investment choices subject to adjustment costs. Intermediate–good firm $f \in [0,1]$ produces output y_{fs} using capital k_{fs} according to a production function with decreasing returns to scale:

$$y_{fs} = u_s^{1-\alpha} k^{\alpha}_{fs}, \quad \alpha < 1. \tag{D.10}$$

An idiosyncratic component drives the firm's total productivity:

$$\mathrm{d}\log(u_{fs}) = \mu \, \mathrm{d}s + \sigma \, \mathrm{d}W_{fs}, \qquad W_{fs} \sim \text{Wiener}, \tag{D.11}$$

where the processes W_{ft} are independent across intermediate–good firms. The profit rate is given by:

$$\pi_{fs} = p_{fs} y_{fs}. \tag{D.12}$$

Taking the prices of the intermediate goods p_{ft}, the marginal investor discount factor Q_t, and firm-specific capital goods $p^k_{ft}(\hat{i})$ as given, together with the adjustment friction θ_{ft}, each firm f chooses a sequence of capital adjustment dates $\{T_{fh}\}_{h=1}^{\infty}$ and investments $\{i_{f,T_{fh}}\}_{h=1}^{\infty}$ to maximize its expected discounted stream of profits:

$$\max_{\{T_{fh}, \hat{i}_{f,T_{fh}}\}_{h=1}^{\infty}} \mathbb{E} \left[\int_0^{\infty} Q_t \pi_{ft} \;\, \mathrm{d}s - \sum_{h=1}^{\infty} Q_{T_{fh}} p^k_{f,T_{fh}} \left(\theta_{fT_{fh}} + p^k_{fT_{fh}}(\hat{i}_{fT_{fh}}) \hat{i}_{fT_{fh}} \right) \right], \tag{D.13}$$

subject to the profit function in (D.12) and the law of motion for its capital stock:

$$\log(k_{ft}) = \log(k_{f0}) - \zeta t + \sum_{h: T_{fh} \leq t} \log \left(1 + \frac{\hat{i}_{f,T_{fh}}}{k_{T^-_{fh}}} \right). \tag{D.14}$$

Market structure. There are three types of goods (respectively, markets) in the economy: (i) final goods, (ii) intermediate goods, and (iii) firm-specific investment goods. There are two assets: (i) risk-free bonds and (ii) equity. All good and asset markets are competitive. We assume equity can only be held by the representative household. Thus, we have segmented the equity market, and the bond market freely trades across countries. The market clearing conditions, respectively, are as follows:

$$E_{fs} = 1 \quad \text{for all } t \text{ and } f, \tag{D.15}$$

$$\hat{y}_{fs} = y_{fs} \quad \text{for all } s \text{ and } f, \tag{D.16}$$

$$\hat{i}_{fs} = i_{fs} \quad \text{for all } s \text{ and } f. \tag{D.17}$$

Equilibrium. *Given a stochastic processes for capital quality* $\{u_{fs}\}_{fs}$*, and adjustment costs* θ_{ft}*, an equilibrium is a set of stochastic processes for prices* $\{\tilde{\rho}_s, \{p_{fs}, p^k_{fs}(i), P_{fs}\}_{f\in[0,1]}\}_{s=0}^{\infty}$*, the household's policy* $\{C_s, B_s, \{E_{fs}\}_{f\in[0,1]}\}_{t=0}^{\infty}$*, the final–good producer's policy* $\left\{Y_s, \{\hat{y}_{fs}\}_{f\in[0,1]}\right\}_{t=0}^{\infty}$*, the capital–good producer's policy* $\left\{\{i_{fs}\}_{f\in[0,1]}, i_s\right\}_{t=0}^{\infty}$*, and the intermediate–good firms' policy* $\left\{\{T_{fh}, i_{f,T_{hf}}\}_{h=1}^{\infty}\right\}$ *such that:*

(i) Given prices $\{\tilde{\rho}_s, P_{fs}\}$*, the household solves (D.4).*

(ii) Given prices $\{p^k_{fs}\}$*, the capital–good producer solves (D.7).*

(iii) Given prices $\{p_{fs}\}$*, the final–good producer solves (D.9).*

(iv) Given prices $\{Q_s, p_{fs}, p^k_{fs}\}$*, intermediate–good firms solve (D.13).*

(v) Market clears in (D.15) to (D.17).

D.2 Equilibrium characterization

We now describe the equilibrium determination of prices and quantities in that order. From now on, we assume that the world interest rate is constant: $\tilde{\rho}_s = \tilde{\rho}$. We derive the aggregate macroeconomic outcomes from their individual counterparts.

Equilibrium determination of prices. The household's optimality conditions over bonds and equity are:

$$\begin{aligned} \tilde{\rho}\,\mathrm{d}s &= \chi\,\mathrm{d}s - \frac{\mathrm{d}(1/C_s)}{1/C_s} \quad \forall s \\ \frac{\mathbb{E}[dP^i_{fs}] + D^i_{fs}\,\mathrm{d}s}{P^i_{fs}} &= \chi\,\mathrm{d}s - \frac{\mathrm{d}(1/C_s)}{1/C_s} \quad \forall s, f \end{aligned} \tag{D.18}$$

The differential equations in (D.18) jointly imply a unique equilibrium for the price of equity. Under the equilibrium condition of unit supply of equity in (D.15), we find:

$$V_0 = P_0 = \mathbb{E}_0\left[\int_0^{\infty} e^{-\tilde{\rho}s} D_s\,\mathrm{d}s\right].$$

Finally, the zero-profit conditions for the final- and capital-good producers imply the following relationships for the input and output prices of the respective goods:

$$p_{ft} = \frac{1}{u_{ft}} \quad ; \quad p^k_{ft}(i) = p_{ft}\varphi(i),$$

where $\varphi(i) = \varphi^+ \mathbf{1}_{i<0} + \varphi^- \mathbf{1}_{i>0}$, and $p^k_{ft}(i)$ represents the relative price of capital.

Equilibrium policy of intermediate good firms. With these facts about equilibrium prices established, we turn to the problem facing an individual intermediate–good firm. Let $V(k,u)$ be the value of a firm with capital k and productivity u. The sufficient optimality conditions satisfied by a firm's policy are (i) the HJB equation valid during periods of inactivity, (ii) the value matching conditions, and (iii) the smooth pasting conditions. The firm policy consists of an inaction region $\mathcal{R} \equiv \{(k,u) : k^-(u) \le k \le k^+(u)\}$, where $k^-(u)$ and $k^+(u)$ are the lower and upper inaction thresholds, together with reset capitals $k^{*-}(u)$ and $k^{*+}(u)$ for positive and negative investments upon adjustment.

Let $r \equiv \tilde{\rho} - \mu$ (without subtracting $\sigma^2/2$, in contrast to the main text) be the adjusted discount factor and let $v(\hat{k}) : \mathbb{R} \to \mathbb{R}$ be a function of the log capital-productivity ratio equal to

$$v(\hat{k}) = \max_{\tau, \Delta\hat{k}} \mathbb{E}\left[\int_0^\tau e^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s + e^{-r\tau}\left(-\theta - p(\Delta\hat{k})(e^{\hat{k}_\tau+\Delta\hat{k}} - e^{\hat{k}_\tau}) + v(\hat{k}_\tau + \Delta\hat{k})\right)\Big|\hat{k}_0 = \hat{k}\right] \tag{D.19}$$

where the price function with taxes is given by:

$$p(i_s) = \left(\varphi^- \mathbf{1}_{i_s>0} + \varphi^+ \mathbf{1}_{i_s<0}\right). \tag{D.20}$$

Then the firm value equals $V_0 = v(\hat{k}_0)$.

A few remarks about the firms' investment policy are in order. The formulations of the capital quality shocks and the adjustment costs allow us to collapse the state-space of the firms (k,u) into the capital-to-productivity ratio $\hat{k} = k/u$. Note that the value of the firm $v(\hat{k}_0)$ is not scaled by the level of productivity to recover the time-0 value V_0. The prices of intermediate goods p_{ft} and capital goods p^k_{ft}, as well as the adjustment costs θ_{ft}, are proportional to capital quality u_{ft}, making profits and investment scaled by total productivity the relevant variables for the firm.

Equilibrium determination of macroeconomic outcomes. With equilibrium prices and firms' policies, we can determine equilibrium aggregate quantities. Lemma D.1 characterizes the equilibrium detrended aggregate quantities: It shows that all aggregates are functions of the distribution of capital-to-productivity ratios.

Lemma D.1. *Let $g(\hat{k})$ be the density of capital-to-productivity ratios and define the following expectations: $\mathbb{E}\left[\exp(\hat{k})\right] \equiv \int_{\hat{k}^-}^{\hat{k}^+} \exp(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k}$. Then, the equilibrium aggregate output is*

$$\hat{Y}_t \quad \equiv \quad \int_0^1 p_{ft} y_{ft}\,\mathrm{d}f \;=\; \mathbb{E}\left[\exp(\alpha\hat{k}_t)\right], \tag{D.21}$$

We cannot sum firms' capital since they are different goods. Still, as in the main text, the capital-productivity ratio is the only input to determine output in this economy. Thus, we define aggregate capital as

$$\hat{K}_t = \mathbb{E}\left[\exp(\hat{k}_t)\right]. \tag{D.22}$$

Equations (D.21) and (D.22) show an important property in this economy: Without fixed costs of adjustment and partial irreversibility, the supply side of this model collapses to a neoclassical firm with technology $\hat{Y} = \hat{K}^\alpha$. Doing a first-order approximation on equation (D.22), it follows the CIR(δ) definition in 32.

With these facts over aggregate quantities, we now describe misallocation. Misallocation is defined as the dispersion of the log of productivity-weighted marginal revenue given by

$$\mathbb{V}\left[\log\left(u_f \frac{\mathrm{d}p_f y_f}{\mathrm{d}k_f}\right)\right] = \mathbb{V}\left[\log\left(\frac{u_f}{k_f}\left(\frac{k_f}{u_f}\right)^\alpha\right)\right] = (1-\alpha)\mathbb{V}\left[\hat{k}\right]. \tag{D.23}$$

The argument for why we need to weigh according to capital quality comes from the technology to produce investment. If the transformation rate from consumption goods to firm-specific investment goods is one, then there is no need to weigh it with capital. This is not the case in our economy since the transformation rate from consumption goods to firm-specific investment goods is given by u_f. This is why we need to weigh the idiosyncratic capital quality shocks to obtain misallocation or productivity in this economy.

D.3 Remarks on the economic framework

General equilibrium structure Capital quality u_{ft} was first used by Baley and Blanco (2021) in the investment context. In the pricing literature, an analogous formulation was first employed by Woodford (2009) to maintain the tractability of their model. It is also used by Midrigan (2011), Alvarez and Lippi (2014), Baley and Blanco (2019), and Blanco (2020), among others. This formulation implies that aggregate feasibility depends only on firms' capital-to-productivity ratios rather than capital and productivity separately. As a result, capital quality shocks reduce the dimensionality of the aggregate state space from the joint distribution of capital and productivity to the distribution of their ratio.

Partial irreversibility The price wedge is a technological constraint for the capital–good producer and, therefore, is exogenous. This formulation follows Veracierto (2002) and Khan and Thomas (2008). Alternatively, partial irreversibility could be the outcome of distortionary taxation. For example, Chen *et al.* (2023) uses China's 2009 VAT reform to study changes in the level of partial irreversibility. It would be straightforward to extend our framework to micro-found partial irreversibility as an outcome of a tax system, as in Chen *et al.* (2023). See Lanteri (2018) for a model that endogenizes partial irreversibility.

Financial markets We assume that the representative household can trade in the bond market, but the economy is closed to the equity market. Only the households in the small open economy own firms, providing the firms' discount factor. While these are extreme assumptions, they are a reasonable approximation for small, open economies. Empirically, it is well known that central banks, firms, and households in emerging economies tend to save in dollar-denominated risk-free assets (e.g., T-bills). Moreover, despite the globalization of finance and financial institutions, market participants commonly allocate most of their wealth to domestic assets. This *home bias* may be due to regulatory constraints, information, and transaction costs, though some attribute it to preferences. While the current version of the model represents an extreme form of the *home bias* phenomenon, it provides a useful starting point for analyzing business cycle fluctuations in an investment model.

Tractability Given the novelty of the general equilibrium framework, further discussion of the assumptions and economic adjustment mechanisms is warranted. The tractability of our framework arises from three main features. First, all aggregate variables are expressed regarding the distribution of capital-to-productivity ratios. This result follows from the introduction of capital quality shocks and the structure of capital adjustment costs. Second, the model produces a constant real interest rate due to the small open economy assumption. Third, the closed equity market assumption allows us to determine the firms' discount factor as a function of the world interest rate.

The theory developed in the main text assumes that the cross-sectional distribution of capital-to-productivity ratios is the relevant aggregate state and that the interest rate is exogenous. The general equilibrium framework presented here provides a microfoundation for the model in the main text.

E Establishment-level investment data

This section describes the sources, the construction of variables, and the filters we apply to clean the data to construct the investment series at the firm level in order

E.1 Source, description and data cleaning

Data come from the *Encuesta Nacional Industrial Anual* (ENIA). The sample period covers 31 years, from 1980 to 2011, with an average of 543 manufacturing plants per year. We have a total number of plant-year observations of 154,591.

1. First, we drop the 3,984 permanently small firms (i.e. with less than 10 workers throughout the sample period, 4% of the sample). This filter is motivated by the lack of good quality data with respect to these firms since ENIA is directed to plants with more than 10 workers.
2. Second, we drop 5,343 observations with a non-positive total value of book capital, wage bill or sales.
3. Third, we drop 12,161 observations that had a frequency of non-zero investment lower than 10% of the sample period.
4. Finally, we drop 5,556 plants with less than 3 years of coverage (6% of the sample). Note that we consider new plants (and give a new ID) that disappear from the sample more than three years and reappear in the sample after that.
5. In total, we drop about 18% of the year-plant observations and keep 127,631 observations. Within this remaining sample, a balanced panel would maintain 101,160 plant-years.

Table E.1 summarizes the cleaning process and shows the number of observations dropped at each step.

Table E.1: Data cleaning

Description	**Chile**
Start year	1980
End year	2011
Avg. number of plants per year	543
Plant-year observations	154,591
Cleaning	**Removed observations**
Less than 10 employees	3,984
Non-positive wage bill, capital, or sales	5,343
Frequency of non-zero investment less than 10	12,161
Less than 3 years of coverage	5,556
Remaining observations	127,631
% of total	82.6
With more than 10 years of data	101,160
% of remaining observations	79.3

Sources: Authors' calculations using establishment-level survey data from Chile. Less than 10 employees refers to plants with less than 10 employees for all the years in the sample.

E.2 Perpetual Inventory Method

To deal with reporting and measurement errors in the surveys, we construct capital series using the standard perpetual inventory method (PIM) with the addition of an investment price wedge.

Capital stocks. Let firm i's stock of capital on year t be given by:

$$k_{i,t} \; = \; (1-\xi^k)k_{i,t-1} \; + \; \frac{I_{i,t}}{p(I_t)D_t} \quad \text{for } k_{i,t_0} \text{ given.} \tag{E.1}$$

We consider the following elements to construct the capital series:

- Capital types considered are $j \in \{\text{structures}, \text{machinery and equipment}, \text{vehicles}\}$.
- Gross investment: $I_{i,j,t}$ is the gross nominal investment into the capital of type j at time t, and it is based on the information on purchases, reforms and improvements, and sales of fixed assets reported by each plant in the ENIA and EAM data sets.

$$I_{i,j,t} \; = \; puchases_{i,j,t} \; + \; reforms_{i,j,t} \; + \; improvements_{i,j,t} \; - \; sales_{i,j,t} \tag{E.2}$$

- Depreciation rate: $\xi^k = 0.09$ is a the depreciation rate.
- Price deflators: $D_{j,t}$ are gross fixed capital formation deflators by capital type from Penn World Tables (PWT).
- Investment prices and wedge.
- Initial capital: K_{i,j,t_0} is given by:

$$K_{i,j,t_0} \; = \; \frac{\tilde{K}_{i,j,t_0}}{D_{t_0}} \quad \text{if } \tilde{K}_{i,j,t_0} \geq 0, \tag{E.3}$$

  where $\tilde{K}_{i,j,t_0}$ is firm i's self-reported nominal stock of capital of type j at current prices on the starting year $t_0 = t_{0,i,j}$, which is the first year in which firm i reports a non-negative capital stock of type j.

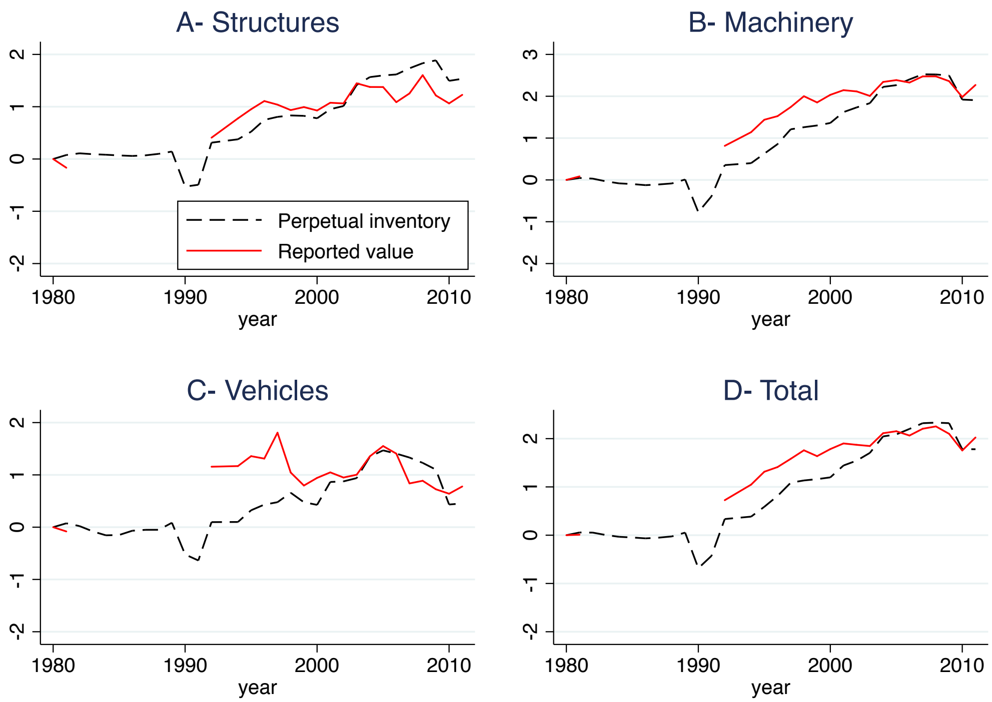
Investment rates. Once we construct the investment and capital stock series, we generate the investment rate $i_{i,j,t}$ by dividing investment by initial capital:

$$i_{i,j,t} \; = \; \frac{I_{i,j,t}}{K_{i,j,t-1}}, \tag{E.4}$$

Outliers. Once we generate the series of investment rates, we eliminate investment rates below the 2nd percentile and above the 98th percentile of the investment rate distribution.

Figure E.1 plots the aggregate capital stock computed with the perpetual inventory method and compares it to the reported book value. In the aggregate, we observe that the reported book value is consistent with the PIM series for each capital type and the total stock. This shows the sound quality of the micro-data. Moreover, given the similarity in the series, we validate our choice of using the initial book value reported by the plant as the initial condition for the PIM construction.

Figure E.1: **Chile:** Reported Book Value vs Perpetual Inventory



Notes: Aggregate capital stock in Chile's manufacturing sector reported by plants and computed through the PIM. All the variables are in logs and real terms, normalized to zero in 1980.

E.3 Comparison with National Accounts

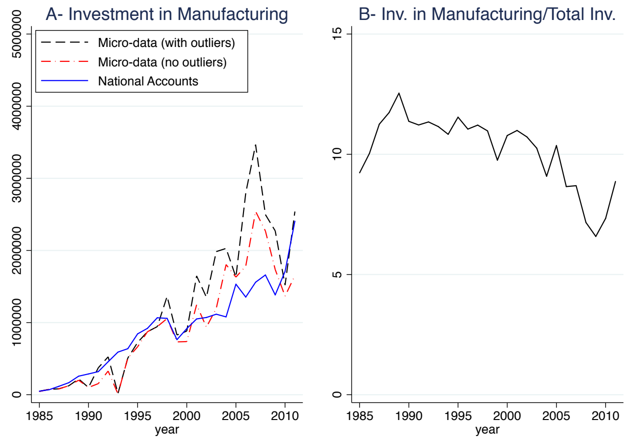
This section verifies that the information in the survey data is consistent with aggregated information from National Accounts.

The national account office in Chile uses the ENIA survey to compute several indices, such as variations in inventories or value added by type of industry. Nevertheless, National Accounts does not use ENIA to compute total investment or investment in the manufacturing sector; for that purpose, it uses sources related to the supply of capital goods (i.e., balance of payments, National Statistical Institute, Corporacion de Desarrollo Tecnologico de Bienes de Capital). Therefore, National Accounts serve as an orthogonal source to verify that the micro-data from the survey is consistent with the total investment in the manufacturing sector.

Panel A of Figure E.2 describes the total nominal investment constructed from the ENIA (dashed black line) and the total nominal investment in the manufacturing sector constructed using National Accounts (solid blue line), in current millions of pesos. As we can see, the two series are very close, with a correlation of 0.62. Total investment from the micro-data for the period 2005-2009 seems to grow much faster than National Accounts, but we found that a few outliers mainly explain this. For example, suppose we drop observations with investment rates larger than 5% of aggregate investment (dashed-dotted red line). In that case, the fit between the aggregate investment from the micro-data and the national account is much better, both in levels and cyclicality. Finally, Panel B of E.2 describes the proportion of total investment done in the manufacturing sector, which represents on average 7% in the sample period but has been declining.

For 2003-2009, the National Accounts calculates the investment distribution by capital types at the sector level. Table E.2 describes the composition of capital across different types from the ENIA and National Accounts.

Figure E.2: **Chile**: Micro-data vs. National Accounts



Notes: Panels A describes investments in the manufacturing sector. Black dashed line plots aggregate nominal investment constructed from ENIA, the red dashed-dotted line plots the same variable but dropping outliers (i.e., investment larger than 5% of aggregate investment), and the blue solid line plots the total investment from National Accounts. Panel B describes investment in the manufacturing sector over total investment. Nominal investment from the national account uses the concatenated investments from the base year 2015.

The proportions invested in structures are similar between National Accounts and ENIA, but the decomposition between machinery and vehicles differs across datasets.

Table E.2: Chile: Distribution of Investment Across types of Capital

	Structures	Machinery	Vehicles
National Accounts	35.4	51.4	13.1
ENIA	29.1	68.6	2.1

Notes: Proportion of investment across different types of capital in the ENIA and national accounts.

E.4 Mappings from microdata to macro outcomes

This section describes the application of the theory with producer-level investment data. Let I_{ft} be the nominal investment in period t for firm f. Below, we show the steps to compute all the macroeconomic outcomes described in Table II.

(I) **Construction of** $\left\{\Delta\hat{k}_{fh}, \tau_{fh}\right\}_{fh}$

- We follow subsection E.1 to drop observations that do not satisfy a set of criteria (e.g., positive sales, wage bill, minimum number of year, etc.).
- We apply the perpetual inventory method to construct the capital stock k_{ft} of firm f in period t

$$k_{ft} \; = \; (1-\xi)k_{ft-1} \; + \; I_{ft}/(p(I_{ft})D_t), \tag{E.1}$$

  where ξ is the physical depreciation rate; I_{ft} is the nominal value of the investment; $p(I_{ft})$ is the investment pricing function, which considers different prices for capital purchases and sales; D_t is the gross fixed capital formation deflator, and k_{f0} is a plant's self-reported nominal capital stock at current prices for the first year enter in the data or its firms investment level.
- We construct the change in the capital-productivity ratio upon action $\Delta\hat{k}_h$ as

$$\Delta\hat{k}_{fh} \; = \; \begin{cases} \log\left(1 + \frac{I_{ft}/(p(I_{ft})D_t)}{k_{ft-1}}\right) & \text{if} \quad |\iota_h| > \underline{\iota}, \\ 0 & \text{if} \quad |\iota_h| < \underline{\iota}. \end{cases} \tag{E.2}$$

- With $\Delta\hat{k}_{fh}$, we construct τ_{fh} as the number of periods between non-zero investments.

(II) **Construction of weights** ω_f Unobserved heterogeneity in the frequency of non-zero investment can generate a bias in estimating moments relevant to the theory. Let N_f be the adjustment frequency of firm f. We construct weights ω_f as the inverse frequency of non-zero investments:

$$\omega_f \propto \frac{1}{N_f}. \tag{E.3}$$

We normalized ω_f s.t.

$$\sum_f \sum_{h=1}^{N_f} \omega_f = 1 \tag{E.4}$$

(III) **Estimation of** $(\hat{k}^{*-}, \hat{k}^{*+}, \nu, \sigma)$**:** First, we estimate the drift as:

$$\nu = \frac{\sum_{fh} \Delta\hat{k}_{fh}\omega_f}{\sum_{fh} \tau_{fh}\omega_f}, \tag{E.5}$$

With the estimate of the drift, we design an iterative method to estimate $(\hat{k}^{*-}, \hat{k}^{*+}, \sigma^2)$. The method constructs a sequence $(\hat{k}^{*-}_j, \hat{k}^{*+}_j, \sigma^2_j)_{j=0}^{\infty}$ that converges to the solution of the implicit equations (57), (65), and (66) from Section 4.

0. Fix a convergence parameter $\psi > 0$ and a dampening parameter $\Gamma \in (0, 1)$.

1. **Construct $(\hat{k}_0^{*-}, \hat{k}_0^{*+}, \sigma_0)$ assuming no irreversibility to compute σ_0 and the right-hand side of (65) and (66):** Without irreversibility, we can estimate σ_0^2 as

$$\sigma_0^2 = \frac{\sum_{fh} \Delta\hat{k}_{fh}^2 \omega_f}{\sum_{fh} \tau_{fh}\omega_f} - 2\nu \left(\frac{\sum_{fh} \Delta\hat{k}_{fh}\omega_f}{2}(1 - \mathbb{CV}^2[\tau]) + \frac{\mathbb{C}ov[\Delta hk, \tau]}{\sum_{fh} \tau_{fh}\omega_f} \right) \tag{E.6}$$

$$\mathbb{CV}^2[\tau] := \frac{\sum_{fh}(\tau_{fh} - \overline{\mathbb{E}}[\tau])^2\omega_f}{\left(\sum_{fh} \tau_{fh}\omega_f\right)^2}, \quad \text{where} \quad \overline{\mathbb{E}}[\tau] = \sum_{fh} \tau_{fh}\omega_f, \tag{E.7}$$

$$\mathbb{C}ov[\Delta\hat{k}, \tau] := \sum_{fh}(\tau_{fh} - \overline{\mathbb{E}}[\tau])(\Delta\hat{k}_{fh} - \overline{\mathbb{E}}[\Delta\hat{k}])\omega_f, \quad \text{where} \quad \overline{\mathbb{E}}[\Delta\hat{k}] = \sum_{fh} \Delta\hat{k}_{fh}\omega_f. \tag{E.8}$$

With σ_0^2, we compute

$$\Phi(\nu, \sigma_0^2) = \log\left(\frac{\alpha}{\mathcal{U} - (1-\alpha)\nu - \frac{\sigma_0^2(1-\alpha)^2}{2}} \right) \tag{E.9}$$

$$\hat{k}_0^{*-} = \frac{1}{1-\alpha}\left(\Phi(\nu, \sigma_0^2) - \log(p^{\text{buy}}) + \log\left(\frac{1 - \mathcal{N}um_0^-}{1 - \mathcal{D}en_0^-} \right) \right) \tag{E.10}$$

$$\hat{k}_0^{*+} = \frac{1}{1-\alpha}\left(\Phi(\nu, \sigma_0^2) - \log(p^{\text{sell}}) + \log\left(\frac{1 - \mathcal{N}um_0^+}{1 - \mathcal{D}en_0^+} \right) \right) \tag{E.11}$$

where the numerators and denominators in the last terms are computed as:

$$\mathcal{N}um_0^+ = \frac{\sum_{fh} \exp(-\mathcal{U}\tau'_{fh} + (1-\alpha)\Delta\hat{k}'_{fh}) I(\Delta\hat{k}_{fh} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} < 0)\omega_f} \tag{E.12}$$

$$\mathcal{D}en_0^+ = \frac{\sum_{fh} \exp(-\mathcal{U}\tau'_{fh}) I(\Delta\hat{k}_{fh} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} < 0)\omega_f} \tag{E.13}$$

$$\mathcal{N}um_0^- = \frac{\sum_{fh} \exp(-\mathcal{U}\tau'_{fh} + (1-\alpha)\Delta\hat{k}'_{fh}) I(\Delta\hat{k}_{fh} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} > 0)\omega_f} \tag{E.14}$$

$$\mathcal{D}en_0^- = \frac{\sum_{fh} \exp(-\mathcal{U}\tau'_{fh}) I(\Delta\hat{k}_{fh} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} > 0)\omega_f} \tag{E.15}$$

Define $\hat{k}_0^*(\Delta\hat{k})$ and $\hat{k}_{\tau,0}(\Delta\hat{k})$ as:

$$\hat{k}_0^*(\Delta\hat{k}) = \hat{k}_j^{*-}\mathbb{1}_{\{\Delta\hat{k}>0\}} + \hat{k}_j^{*+}\mathbb{1}_{\{\Delta\hat{k}<0\}}, \tag{E.16}$$

$$\hat{k}_{h,0}(\Delta\hat{k}) = \hat{k}_0^*(\Delta\hat{k}) - \Delta\hat{k}, \tag{E.17}$$

(E.18)

2. For $j = 1, 2, 3, \dots$, compute $(\hat{k}_j^{*-}, \hat{k}_j^{*+}, \tilde{\sigma}_j^2)$ as

$$\tilde{\sigma}_j^2 = \frac{\sum_{fh}(\hat{k}^*_{\tau,j-1}(\Delta\hat{k}'_{fh}) + \nu\tau'_{fh})\omega_f - \sum_{fh}\hat{k}^*_{j-1}(\Delta\hat{k}_{fh})\omega_f}{\sum_{fh}\tau_{fh}\omega_f} \tag{E.19}$$

$$\Phi(\nu, \sigma_{j-1}^2) = \log\left(\frac{\alpha}{\mathcal{U} - (1-\alpha)\nu - \frac{\sigma_{j-1}^2(1-\alpha)^2}{2}}\right) \tag{E.20}$$

$$\hat{k}_j^{*-} = \frac{1}{1-\alpha}\left(\Phi - \log(p^{\text{buy}}) + \log\left(\frac{1-\mathcal{N}um_j^-}{1-\mathcal{D}en_j^-}\right)\right) \tag{E.21}$$

$$\hat{k}_j^{*+} = \frac{1}{1-\alpha}\left(\Phi - \log(p^{\text{sell}}) + \log\left(\frac{1-\mathcal{N}um_j^+}{1-\mathcal{D}en_j^+}\right)\right) \tag{E.22}$$

where the numerators and denominators in the last terms are computed as:

$$\mathcal{N}um_j^+ = \frac{\sum_{fh}\exp\left(-\mathcal{U}\tau'_{fh} + (1-\alpha)\left(\hat{k}^{*+} - \hat{k}^*_{\tau,j-1}(\Delta\hat{k}'_{fh})\right)\right) I(\Delta\hat{k}_{fh} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} < 0)\omega_f} \tag{E.23}$$

$$\mathcal{D}en_j^+ = \frac{\sum_{fh}\frac{p(\Delta\hat{k}'_{fh})}{p(\Delta\hat{k}_{fh})}\exp\left(-\mathcal{U}\tau'_{fh}\right) I(\Delta\hat{k}_{fh} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} < 0)\omega_f} \tag{E.24}$$

$$\mathcal{N}um_j^- = \frac{\sum_{fh}\exp\left(-\mathcal{U}\tau'_{fh} + (1-\alpha)\left(\hat{k}^{*-}_{j-1} - \hat{k}_{\tau,j-1}(\Delta\hat{k}'_{fh})\right)\right) I(\Delta\hat{k}_{fh} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} > 0)\omega_f} \tag{E.25}$$

$$\mathcal{D}en_j^- = \frac{\sum_{fh}\frac{p(\Delta\hat{k}'_{fh})}{p(\Delta\hat{k}_{fh})}\exp\left(-\mathcal{U}\tau'_{fh}\right) I(\Delta\hat{k}_{fh} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh} > 0)\omega_f} \tag{E.26}$$

Update $(\hat{k}_j^{*-}, \hat{k}_j^{*+}, \sigma_j^2)$ with the dampening parameter Γ

$$\sigma_j^2 = (1-\Gamma)\tilde{\sigma}_j^2 + \Gamma\sigma_{j-1}^2 \tag{E.27}$$

$$\hat{k}_j^{*-} = (1-\Gamma)\tilde{\hat{k}}_j^{*-} + \Gamma\hat{k}_{j-1}^{*+} \tag{E.28}$$

$$\hat{k}_j^{*+} = (1-\Gamma)\tilde{\hat{k}}_j^{*+} + \Gamma\hat{k}_{j-1}^{*+} \tag{E.29}$$

3. Repeat step 2 until there is a j, such that, $||\left(\sigma_j^2 - \sigma_{j-1}^2, \hat{k}_j^{*-} - \hat{k}_{j-1}^{*-}, \hat{k}_j^{*+} - \hat{k}_{j-1}^{*+}\right)|| < \psi$.

(IV) We construct the reset points $\hat{k}^*(\Delta\hat{k})$ and stopped capitals $\hat{k}_\tau(\Delta\hat{k})$ as:

$$\hat{k}^*(\Delta\hat{k}) = \hat{k}^{*-}\mathbb{1}_{\{\Delta\hat{k}>0\}} + \hat{k}^{*+}\mathbb{1}_{\{\Delta\hat{k}<0\}} \tag{E.30}$$

$$\hat{k}_\tau(\Delta\hat{k}) = \hat{k}^*(\Delta\hat{k}) - \Delta\hat{k} \tag{E.31}$$

(V) Cross-sectional mean and variances: We estimate the unconditional mean and variances of capital–productivity ratios as

$$\mathbb{E}[\hat{k}] = \frac{\sum_{fh}\hat{k}^*(\Delta\hat{k}_{fh})^2\omega_f - \sum_{fh}\hat{k}_\tau(\Delta\hat{k}'_{fh})^2\omega_f}{2\sum_{fh}\Delta\hat{k}_{fh}\omega_f} + \frac{\sigma^2}{2\nu} \tag{E.32}$$

$$\mathbb{V}[\hat{k}] = \frac{\sum_{fh}(\hat{k}^*(\Delta\hat{k}_{fh}) - \mathbb{E}[\hat{k}])^3\omega_f - \sum_{fh}(\hat{k}_\tau(\Delta\hat{k}'_{fh}) - \mathbb{E}[\hat{k}])^3\omega_f}{3\sum_{fh}\Delta\hat{k}_{fh}\omega_f} \tag{E.33}$$

We can also compute conditional means

$$\mathbb{E}^{\pm}[\hat{k}] = \frac{(\hat{k}^{*\pm})^2 - \sum_{fh}\hat{k}_\tau(\Delta\hat{k}'_{fh})^2 I(\pm\Delta\hat{k}_{fh} < 0)\omega_f}{2(\hat{k}^{*\pm} - \sum_{fh}\hat{k}_\tau(\Delta\hat{k}'_{fh}) I(\pm\Delta\hat{k}_{fh} < 0)\omega_f)} + \frac{\sigma^2}{2\nu} \tag{E.34}$$

(E.35)

(VI) **Estimation of covariance:** We estimate the covariance using the sample moment of (61) given by

$$\mathbb{C}ov[\hat{k}, a] = -\frac{\sum_{fh}(\hat{k}_{\tau'}(\Delta\hat{k}'_{fh}) - \mathbb{E}[\hat{k}])^2\tau'_{fh}\omega_f}{2\nu\mathbb{E}[\tau]} + \frac{\mathbb{V}[\hat{k}]}{2\nu} + \frac{\sigma^2}{2\nu}\frac{\mathbb{E}[\tau]}{2}(1 + \mathbb{CV}(\tau)), \tag{E.36}$$

where $\mathbb{CV}(\tau)$ and $\bar{\tau}$ are estimated using (E.7), $\mathbb{V}[\hat{k}]$ is estimated using (E.33), and $\mathbb{E}[\hat{k}]$ is estimated using (E.34).

(VII) **Estimation of irreversibility term:** We estimate the CIR's irreversibility term following its sample counterpart:

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)\right]\right] = \frac{\sum_{fh}\left(\hat{k}_\tau(\Delta\hat{k}')\mathcal{M}\left(\Delta\hat{k}'_{fh}\right) - \hat{k}^*(\Delta\hat{k})\mathcal{M}\left(\Delta\hat{k}_{fh}\right)\right)\omega_f}{\overline{\mathbb{E}}[\tau]}. \tag{E.37}$$

The objects are given by:

- $\mathcal{M}(\Delta\hat{k}) := \mathcal{M}(\hat{k}^{*-})\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}} + \mathcal{M}(\hat{k}^{*+})\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}$ with

$$\mathcal{M}(\hat{k}^{*-}) = (\mathbb{E}^-[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^-[\tau]\,\frac{\mathbb{E}[\mathbb{P}^+]}{\mathbb{P}^{-+}} \tag{E.38}$$

$$\mathcal{M}(\hat{k}^{*+}) = (\mathbb{E}^+[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^+[\tau]\,\frac{\mathbb{E}[\mathbb{P}^-]}{\mathbb{P}^{+-}}. \tag{E.39}$$

- Conditional durations of inaction as:

$$\mathbb{E}^-[\tau] = \frac{\sum_{fh}\tau'_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}}\omega_f}{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}}\omega_f}, \tag{E.40}$$

$$\mathbb{E}^+[\tau] = \frac{\sum_{fh}\tau'_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}\omega_f}{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}\omega_f}, \tag{E.41}$$

- Transition probabilities as:

$$\mathbb{P}^{-+} = \frac{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}'_{fh}<0\}}\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}}\omega_f}{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}>0\}}\omega_f}, \tag{E.42}$$

$$\mathbb{P}^{+-} = \frac{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}'_{fh}>0\}}\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}\omega_f}{\sum_{fh}\mathbb{1}_{\{\Delta\hat{k}_{fh}<0\}}\omega_f} \tag{E.43}$$

- Expected probabilities

$$\mathbb{E}[\mathbb{P}^+] = \frac{\sum_{fh}\tau'_{fh}\mathbb{1}_{\{\Delta\hat{k}'_{fh}<0\}}\omega_f}{\overline{\mathbb{E}}[\tau]}, \tag{E.44}$$

$$\mathbb{E}[\mathbb{P}^-] = \frac{\sum_{fh}\tau'_{fh}\mathbb{1}_{\{\Delta\hat{k}'_{fh}>0\}}\omega_f}{\overline{\mathbb{E}}[\tau]}. \tag{E.45}$$

E.5 Calculating $(\lambda^-, \lambda^+, J^-, J^+)$

Let $\left\{\Delta\hat{k}_{fh}, \tau_{fh}\right\}_{fh}$ and weights ω_f be the sample of investment rates and durations with firms' weights ω_f computed in step (i) in Section E.4. We follow step (ii) in Section E.4 to compute $\left(\nu, \sigma^2, \hat{k}^{*+}, \hat{k}^{*-}\right)$. We now the steps to compute the $(\lambda^-, \lambda^+, J^-, J^+)$. From now on, we assume that the sample is i.i.d. Observe that this assumption is incorrect due to partial irreversibility (positive investment begets future positive investment). We find similar results whenever we estimate

(I) **Estimate $h(\Delta k)$ using a parametric specification:** From now on, we assume that the sample is i.i.d.[35] We parameterize $h(\Delta\hat{k})$ with two Gamma distributions where their parameters can differ between positive and negative investment rates.

$$h(\Delta\hat{k}) = \begin{cases} \frac{\Upsilon}{\Gamma(\varrho_-)\varsigma_-^{\varrho_-}} \left(\Delta\hat{k}\right)^{\varrho_- - 1} e^{-\frac{\Delta\hat{k}}{\varsigma_-}} & \text{if } \Delta\hat{k} > 0 \\ \frac{1-\Upsilon}{\Gamma(\varrho_+)\varsigma_+^{\varrho_+}} \left((-\Delta\hat{k})\right)^{\varrho_+ - 1} e^{-\frac{(-\Delta\hat{k})}{\varsigma_+}} & \text{if } \Delta\hat{k} < 0 \end{cases}. \tag{E.46}$$

Here, $\Upsilon \in (0,1)$ is the fraction of positive investment rates, $\varrho_\pm > 0$ are the shape parameters of Gamma distributions and $\varsigma_\pm > 0$ are the scale parameters. We have two methods to estimate the parameters $(\Upsilon, \varrho_-, \varrho_+, \varsigma_-, \varsigma_+)$: Maximum likelihood estimator and method of moments.

(I.a) **Method of Moments:** Using the method of moments, we have that

$$\Upsilon = \frac{\mathcal{N}^-}{\mathcal{N}^- + \mathcal{N}^+}, \tag{E.47}$$

$$\varrho_- = \frac{\bar{\mathbb{E}}\left[\Delta\hat{k}|\Delta\hat{k} > 0\right]^2}{\bar{\mathbb{V}}\left[\Delta\hat{k}|\Delta\hat{k} > 0\right]}, \tag{E.48}$$

$$\varsigma_- = \frac{\bar{\mathbb{V}}[\Delta\hat{k}|\Delta\hat{k} > 0]}{\bar{\mathbb{E}}[\Delta\hat{k}|\Delta\hat{k} > 0]} \tag{E.49}$$

$$\varrho_+ = \frac{\bar{\mathbb{E}}\left[\Delta\hat{k}|\Delta\hat{k} < 0\right]^2}{\bar{\mathbb{V}}\left[\Delta\hat{k}|\Delta\hat{k} < 0\right]}, \tag{E.50}$$

$$\varsigma_+ = \frac{\bar{\mathbb{V}}[\Delta\hat{k}|\Delta\hat{k} < 0]}{\bar{\mathbb{E}}[\Delta\hat{k}|\Delta\hat{k} < 0]}. \tag{E.51}$$

By replacing equations (E.47)-(E.51) with their sample counter-part with weights w_f, we have the estimates of $(\Upsilon, \varrho_-, \varrho_+, \varsigma_-, \varsigma_+)$.

(I.b) **Maximum Likelihood Estimation:** To simplify notation, let $i = 1, 2, \ldots, N$ denotes the index of the sample with weights w_i. To write the likelihood, let us define $\mathcal{A}_-$ the set of positive investment rates with cardinality $\#\mathcal{A}_- = N_-$ (i.e., the total number of positive investment rates in the sample) and $\mathcal{A}_+$ be the set of negative investment rate with cardinality $\#\mathcal{A}_+ = N_+$ (i.e., the total number of negative investment rates in the sample). By construction, $N_- + N_+ = \sum_f N_f$, i.e., the sample size. Under these assumptions, we can

[35]Observe that this assumption is not correct due to partial irreversibility (positive investment begets future positive investment). We find similar results whenever we estimate $h(\Delta k_h)$ conditional on a positive or negative investment rate and then Baye's rule.

Table E.1: Estimation of $h(\Delta\hat{k})$ with Method of Moments and Maximum Likelihood

	Method of Moments		**Maximum Likelihood**	
Parameters				
Share positive investment (Υ)	0.947		0.947	
	$\Delta\hat{k} > 0$	$\Delta\hat{k} < 0$	$\Delta\hat{k} > 0$	$\Delta\hat{k} < 0$
Shape parameter-Gamma distribution (ϱ)	0.758	1.413	0.744	0.915
Scale parameter-Gamma distribution (ς)	0.273	0.031	0.279	0.048

write the likelihood as

(E.52)

$$L(\Upsilon, \varrho_-, \varsigma_-, \varrho_+, \varsigma_+, \alpha|\{\Delta\hat{k}\}) = \sum_{i\in\mathcal{A}_-} w_i \log\left(\frac{\Upsilon}{\Gamma(\varrho_-)\varsigma_-^{\varrho_-}}\left(\Delta\hat{k}_i\right)^{k_- -1} e^{-\frac{\Delta\hat{k}_i}{\varsigma_-}}\right) + \sum_{i\in\mathcal{A}^+} w_i \log\left(\frac{(1-\Upsilon)}{\Gamma(\varrho_+)\varsigma_+^{\varrho_+}}\left(\Delta\hat{k}_i\right)^{k_+ -1} e^{-\frac{\Delta\hat{k}_i}{\varsigma_+}}\right)$$

(E.53)
$$= (k_- - 1)\sum_{i\in\mathcal{A}_-} w_i \log\left(\Delta\hat{k}_i\right) - \sum_{i\in\mathcal{A}_-}\frac{w_i\Delta\hat{k}_i}{\varsigma_-} - N_-\sum_{i\in\mathcal{A}_-} w_i \sim (\varrho_-\log(\varsigma_-) - \log(\Gamma(\varrho_-)))\dots$$

(E.54)
$$\dots + (k_+ - 1)\sum_{i\in\mathcal{A}_+} w_i \log\left(\Delta\hat{k}_i\right) - \sum_{i\in\mathcal{A}_+}\frac{w_i\Delta\hat{k}_i}{\varsigma_+} - N_+\sum_{i\in\mathcal{A}_+} w_i\left(\varrho_+\log(\varsigma_+) - \log(\Gamma(\varrho_+))\right)$$

(E.55)
$$\dots + N_-\sum_{i\in\mathcal{A}_-} w_i \log(\Upsilon) + N_+\sum_{i\in\mathcal{A}_+} w_i \log(1-\Upsilon)$$

From the likelihood optimization, we have that

(E.56)
$$\log(\varrho_-) - \frac{\Gamma'(\varrho_-)}{\Gamma(\varrho_-)} = \log\left(\sum_{i\in\mathcal{A}_-}\frac{w_i\Delta\hat{k}_i}{N_-\sum_{i\in\mathcal{A}_-} w_i}\right) - \frac{\sum_{i\in\mathcal{A}_-} w_i\log\left(\Delta\hat{k}_i\right)}{N_-\sum_{i\in\mathcal{A}_-} w_i}$$

(E.57)
$$\log(\varrho_+) - \frac{\Gamma'(\varrho_+)}{\Gamma(\varrho_+)} = \log\left(\sum_{i\in\mathcal{A}_+}\frac{w_i\Delta\hat{k}_i}{N_+\sum_{i\in\mathcal{A}_+} w_i}\right) - \frac{\sum_{i\in\mathcal{A}_+} w_i\log\left(\Delta\hat{k}_i\right)}{N_+\sum_{i\in\mathcal{A}_+} w_i}$$

(E.58)
$$\varsigma_- = \frac{\sum_{i\in\mathcal{A}_-} w_i\Delta\hat{k}_i}{N_-\sum_{i\in\mathcal{A}_-} w_i\varrho_-},$$

(E.59)
$$\varsigma_+ = \frac{\sum_{i\in\mathcal{A}_+} w_i\Delta\hat{k}_i}{N_+\sum_{i\in\mathcal{A}_+} w_i\varrho_+},$$

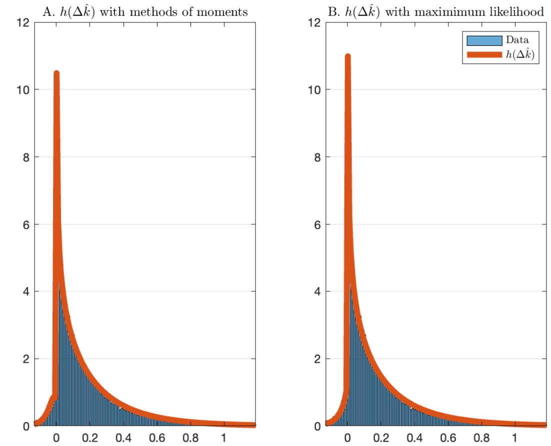
(E.60)
$$\Upsilon = \frac{N_-\sum_{i\in\mathcal{A}_-} w_i\sum}{N_-\sum_{i\in\mathcal{A}_-} w_i + N_+\sum_{i\in\mathcal{A}_+} w_i}.$$

Figure E.1 shows the estimated distributions (E.46) under methods of moments and maximum likelihood. As the figure shows, we have an almost perfect histogram approximation under both methods. Table XX shows the estimates under the two methods.

(II) **Compute $g(\hat{k})$ and $\Lambda(\hat{k})$ using $h(\Delta\hat{k})$:** We use the finite difference to solve the KFE and back up the distribution $g(\hat{k})$. The hazard rate of adjustment is given by

(E.61)
$$\Lambda(\hat{k}) = \begin{cases} \frac{\mathcal{N}h(\hat{k}^{*+}-\hat{k})}{g(\hat{k})} & \text{if } \hat{k} > \hat{k}^{*+}, \\ \frac{\mathcal{N}h(\hat{k}^{*-}-\hat{k})}{g(\hat{k})} & \text{if } \hat{k} < \hat{k}^{*+} \end{cases}.$$

Figure E.1: Estimated $h(\Delta\hat{k})$ under Maximum Likelihood and Method of Moments



Notes: Panel A shows the histogram of investment rates (blue bars) and the estimated distribution (E.46) using the method of moments. Panel B uses the maximum likelihood to show the histogram of investment rates (blue bars) and the estimated distribution (E.46).

E.6 Results for Chile

Next, we present yearly averages of cross-sectional statistics. Inaction frequency is the fraction of observations with investment below 1% in absolute value; positive spikes are investments above 20% and negative spikes below −20%.

Table E.1 presents the yearly average of cross-sectional statistics by capital category for a balanced panel within Chile's ENIA establishment-level survey data. Note that the column total considers the statistics for the total capital stock, which is not the average of the statistics by capital type. For comparison, we include information for the US in Cooper and Haltiwanger (2006) and Zwick and Mahon (2017). Following these papers, investment rates reported in this table are computed as Investment divided by Initial Capital. We use perpetual inventories to compute capital stock.

Table E.1: Investment Rate Statistics (Chile: by capital type)

	Structures	**Machinery**	**Vehicles**	**Total**	**US I**	**US II**
Average Investment	7.3	17.0	17.1	15.8	12.2	10.4
Positive Fraction ($i > 1\%$)	22.2	54.9	25.5	56.8	81.5	—
Negative Fraction ($i < -1\%$)	0.5	1.5	5.3	3.1	10.4	—
Inaction rate ($\|i\| <= 1\%$)	77.3	43.7	69.2	40.1	8.1	23.7
Spike rate ($\|i\| > 20\%$)	8.9	23.2	21.2	22.8	20.4	14.4
Positive spikes ($i > 20\%$)	8.9	23.2	18.7	22.7	18.6	—
Negative spikes ($i < -20\%$)	0.0	0.0	2.5	0.1	1.8	—
Serial correlation	0.0	0.0	0.0	0.0	0.1	0.4

Notes: Authors calculations using establishment-level survey data for Chile (balanced panel). US I shows data from Cooper and Haltiwanger (2006) and US II shows data reported in Zwick and Mahon (2017) for the balanced panel. Following these papers, investment rates reported in this table are computed as investment divided by initial capital. We use the perpetual inventory method to compute capital stocks. We eliminate investment rates below the 1st percentile and above the 99th percentile of the investment rate distribution.

E.7 Comparative Statics

This section conducts a comparative statics exercise concerning the returns to scale α. Other parameters as in the main calibration.

Table E.2: Aggregate Capital Behavior: Comparative Statics

		$\omega = 0.15$		$\alpha = 0.5$	
	Benchmark	$\alpha = 0.4$	$\alpha = 0.6$	$\omega = 0.05$	$\omega = 0.25$
Productivity process					
ν	0.11	0.11	0.11	0.12	0.11
σ	0.23	0.23	0.24	0.23	0.24
Investment Policy					
Difference in reset capitals ($\hat{k}^{*+} - \hat{k}^{*-}$)	0.568	0.472	0.697	0.221	0.914
Exogenous price wedge	0.325	0.271	0.406	0.102	0.575
Endogenous response	0.243	0.201	0.291	0.118	0.339
Capital Allocation					
Variance	0.098	0.097	0.099	0.096	0.098
Capital Valuation					
Tobin's q	1.06	1.07	1.05	1.07	1.05
Productivity	1.09	1.10	1.08	1.08	1.10
Irreversibility	-0.03	-0.03	-0.03	-0.01	-0.05
Capital Fluctuations					
CIR	3.07	3.71	2.50	3.40	2.62
Responsiveness	2.29	2.37	2.17	2.51	1.93
Irreversibility	0.77	1.33	0.33	0.89	0.69

Notes: Objects recovered from theory mappings applied to establishment-level data from Chile. Comparative statics with respect to the wedge ω and the returns to scale α. Other parameters are described in the main text.

F Price wedges in the literature

Asplund (2000) and Ramey and Shapiro (2001) were the first to provide direct empirical evidence on the degree of partial irreversibility of capital investments using data from particular industries. Using data on equipment sales of three aerospace plants, Ramey and Shapiro find an average return on replacement costs of 28 cents per dollar, i.e., an average price wedge of 0.72. Nonetheless, they find that the wedge varies depending on the sale type, private liquidation, or auction, and they find an insider premium on the buyer. Meanwhile, Asplund examined prices for used metalworking machinery in Swedish manufacturing industries. He estimates wedges between 50 and 80 percent for "new" machines once installed. More recently, Kermani and Ma (2023) rely on estimated asset liquidation values of non-financial firms that filed for Chapter 11 bankruptcy. They find a liquidation recovery rate of 35 percent or a 0.65 irreversibility wedge for the average industry.

Contributions to the microeconomic implications of irreversibilities motivated the study of their macroeconomic consequences. Veracierto (2002) proposes a macro model with microeconomic (S,s) policy rules from optimal decision rules of profit-maximizing establishments. The author simulates the model economy with different values of the irreversibility wedge, ranging from 0 (fully flexible) to 1 (entirely irreversible investment), and finds aggregate fluctuations to behave similarly. Bloom (2009) predicts that an uncertainty shock generates a rapid drop and rebound in aggregate output and employment driven by a sudden halt in investment and hiring following the shock; adjustment costs and irreversibilities explain the investment halt. Bloom sets a 34 percent irreversibility wedge based on a simulated method of moments approach which includes joint (cross-sectional and dynamic) moments of the investment, employment, and sales growth series and second-and fourth-order correlations of the investment, employment growth, and sales series. Bloom's estimates are used by Senga and Varotto (2024) in their study of cyclical capital misallocation with partial irreversibilities.

Similarly, Lanteri, Medina and Tan (2023) investigate capital reallocation following an import competition shock. The authors use the method of moments to calibrate their model and set an irreversibility wedge of 0.409 while allowing for an additional wedge for liquidating firms. The frequency of negative investments and the slope of exit thresholds (average slope of survival iso-probability lines) are informative moments for these wedges. Fang (2023) asks about monetary policy effectiveness in a setting with investment frictions. He targets the covariance of the capital gap with the time elapsed since the last adjustment to calibrate an irreversibility wedge of 0.3.

Smaller wedges are used by Khan and Thomas (2013) and by Lanteri (2018). Khan and Thomas base their 0.046 on several steady-state moments reported in Cooper and Haltiwanger (2006). Despite this small wedge, the authors find that negative shocks to borrowing conditions have strong and persistent effects through their capital distribution impact. Lanteri found a 0.067 average wedge in simulations of a GE model, with heterogeneous firms and imperfect substitutability between new and used capital. Despite this, his model features an endogenous resell to purchasing price, which he finds to be pro-cyclical; thus, it takes more work to reverse past investment decisions during recessions.

Table F.3: Price wedges in the literature

Direct evidence	Source	Wedge ω
1. Asplund (2000)	Metalwork machinery in Swedish manufacturing	0.5–0.8
2. Ramey and Shapiro (2001)	Equipment in US aerospace manufacturing	0.72
3. Kermani and Ma (2023)	US firms filling for Chapter 11	0.65

Quantitative models	Calibration	Wedge ω
4. Gilchrist, Sim and Zakrajšek (2014)	Book-value of leverage	0.5
5. Lanteri, Medina and Tan (2023)	Frequency of negative investments	0.41
6. Senga and Varotto (2024)	Investment, employment and sales moments	0.40
7. Bloom (2009); Bloom *et al.* (2018)	Investment, employment and sales moments	0.34
8. Fang (2023)	Covariance of the capital gap with age	0.30
9. Lanteri (2018)	Capital reallocation to expenditures	0.07
10. Khan and Thomas (2013)	Std dev, autocorrelation of investment rates, and spikes	0.05
11. Cooper and Haltiwanger (2006)	Investment spikes and serial correlation	0.025

Surveys	Notes	Wedge ω
12. Dibiasi, Mikosch and Sarferaz (2021)	Survey of Swiss firms	0.47
13. Dibiasi (2022)	Firm surveys and car resale prices	0.21–0.42

Notes: Price wedges are presented in descending order. See Appendix F for further details on these values.

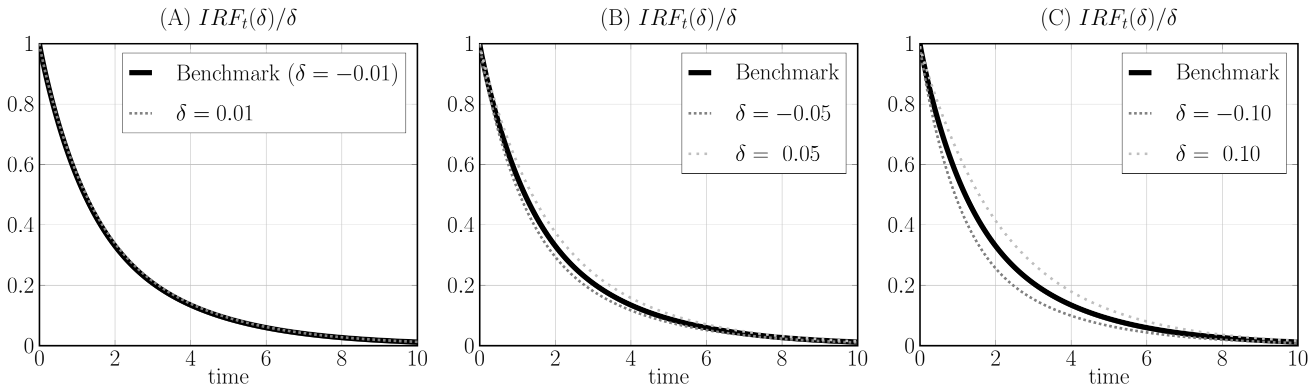
G Asymmetries and Non-Linearities

While the main text focuses on small productivity shocks, this section of the Online Appendix explores non-linearities and asymmetries by calculating impulse-response functions and the CIR(δ) for various aggregate shocks δ, both in terms of sign and magnitude. Our primary finding is that non-linearities and asymmetries are quantitatively insignificant in the generalized hazard model computed in Section 5 with a price wedge of 12% (i.e., $\omega = 0.12$). This result aligns with the distribution of investment rates and the CIR components (to first order).

Figure G.1 displays the impulse-response function normalized by the size of the shocks, i.e., $IRF_t(\delta)/\delta$. Panel A shows results for $\delta = -0.01$ (represented by the black solid line, as analyzed in the main text) and $\delta = 0.01$ (represented by the gray dotted line). A notable property that emerges is symmetry: positive and negative shocks produce symmetric effects on the dynamics of the average capital-productivity ratio. This outcome exemplifies the certainty equivalent principle, where first-order perturbations to aggregate shocks are independent of the volatility of those shocks. The same property can be observed in CIR(δ)/δ. As shown in Table G.1, the numerical computation of CIR(δ)/δ for a small negative shock of -1% is 1.94, while for a small positive shock of 1% it is 1.98. Both values are close to the first-order approximation of CIR$'(0)$, which is 1.93, since the CIR(δ) is differentiable at zero.

Panels B and C of Figure G.1 display impulse-response functions for positive and negative shocks of 5% and 10%. It is important to note that empirical estimates of the standard deviation of productivity shocks are generally less than 1%—see Galí (1999) and Justiano, Primiceri and Tambalotti (2010). Therefore, and under the assumption of a normal distribution, the probability of having a shock larger than 3% is around 0.0027. The figure illustrates that the responses show only minor differences compared to the reaction for $\delta = -0.01$. The main text shows the rationale for the minimal presence of non-linearities in Figure VII. To match the sufficient statistics for the CIR's implying substantial size and dispersion of investment rates, the model incorporates a relatively flat hazard function. This results in a low proportion of firms near the inaction region's boundary, leading to minimal asymmetries and non-linearities.

Figure G.1: Impuse-Response Function for Different Productivity Shocks



Notes: This figures shows the impulse-response function for the general hazard model computed in Section 5. Panel A shows the impulse-response function for $\delta = -0.01$ (black solid line) and for $\delta = 0.01$ (gray dotted line). Panel B and C show the impulse-response functions for $\delta = \pm 0.05$ and for $\delta = 0.10$, respectively.

Table G.1: CIR(δ) for Generalized Hazard Model and $\omega = 12$

	δ					
	-0.10	-0.05	-0.01	0.01	0.05	0.10
$CIR(\delta)/\delta$	1.78	1.87	1.94	1.98	2.05	2.14

Notes: The Table computes the normalized CIR(δ) for different aggregate shocks.