# C Online Appendix

## C.1 Default risk and credit spreads

In this appendix, we extend our analysis to incorporate heterogeneity in entrepreneurial funding costs. To do so, we incorporate idiosyncratic risk into the entrepreneurial production technology. If the entrepreneur invests $k$ at $t = 0$, then she succeeds with probability $p$ at $t = 1$, in which case she receives output $A \cdot k$; otherwise, she fails with probability $1 - p$ and receives output 0. We assume that entrepreneurs differ both in the probability of success $p$ and in the productivity $A$ in case of success. As before, we assume an entrepreneur can pledge at most a fraction $\lambda$ of her output to outsiders.

Since it is without loss of generality to assume that the entrepreneur does not default upon success, it follows that $R \cdot (1 + \frac{1-p}{p})$ is the interest rate at which an entrepreneur of type $(A, p)$ can raise funds from lenders, where $R \cdot \frac{1-p}{p}$ is the spread over the risk-free rate due to the possibility of default. Let $e(A, p) = p \cdot A$ denote the expected productivity of an entrepreneur of type $(A, p)$. The distributions over $p$ and $A$ induce a distribution $\widetilde{G}(\cdot)$ over the expected productivity $e$, which we will assume has full support on $[0, 1]$. Observe that the model can accommodate either positive or negative correlation between $p$ and $A$.

Entrepreneurial demand for capital can be shown to take the form:

$$k_{(A,p)}(q, R) = k_e(q, R) \begin{cases} = 0 & \text{if } \frac{e}{R} < q \\ \in \left[0, \frac{1}{q - \frac{\lambda \cdot e}{R}} \cdot w\right] & \text{if } \frac{e}{R} = q \\ = \frac{1}{q - \frac{\lambda \cdot e}{R}} \cdot w & \text{if } \frac{\lambda \cdot e}{R} < q < \frac{e}{R} \\ \infty & \text{if } q \leq \frac{\lambda \cdot e}{R} \end{cases}; \tag{56}$$

that is, the demand of entrepreneur $(A, p)$ depends only on her expected productivity $e = p \cdot A$, and not on $p$ and $A$ separately. As a result, $q$ must be such that:

$$K^S(q) = K = K^D(q, R) = \int_0^1 k_e(q, R) \cdot d\widetilde{G}(e). \tag{57}$$

Aggregate output is in turn given by:

$$Y = \int_0^1 e \cdot k_e(q, R) \cdot d\widetilde{G}(e). \tag{58}$$

By comparing Equations (56)-(58) with their counterparts in the baseline model, it follows immediately that Proposition 1 holds in this extended setting as well (we only need to relabel $A$ with $e$). Importantly, note that for a given distribution over expected productivity $\widetilde{G}$, the correlation of $p$ and $A$ is irrelevant for the equilibrium $q$, $K$, and $Y$.

## C.2 Closed economy: endogenous interest rates and savings gluts

Throughout our main analysis, we considered a small open economy that experienced an exogenous fall in the world interest rate. In this Appendix, we show that none of our main insights would change if the economy were closed and the fall in the interest rate were the result of a savings glut, i.e., an increase in the economy's desired savings.

Suppose now that the economy is closed, that the agents preferences are given by:

$$U = E_0\{c_0 + \beta \cdot c_1\} \tag{59}$$

for some $\beta \in (0,1)$, and that the capitalists have an endowment $w^C > 0$ of the consumption good at $t = 0$. Given these adjustments, we next show that the main results from our baseline setting can be obtained by raising the desired savings in this economy.

**Proposition 3** *The effects of a fall in the interest rate, $R$, as described in Proposition 1 are isomorphic to those of an increase in $w^C$ and/or $\beta$.*

In what follows, we illustrate the proof of this result. First, note that the equilibrium interest rate, $R$, must be weakly greater than $\beta^{-1}$. Otherwise, there would be a positive credit demand but no savings, as all agents who do not invest in capital would want to consume; hence, the credit market would not clear.

Second, observe that, given prices $\{q, R\}$, the aggregate savings of the savers (i.e., the capitalists and entrepreneurs with productivity $A < q \cdot R$) are given by:

$$S(q,R) \begin{cases} = w^C + q \cdot K^S(q) - \chi(K^S(q)) + w \cdot G(q \cdot R) & \text{if } R > \beta^{-1}, \\ \in [0,\ w^C + q \cdot K^S(q) - \chi(K^S(q)) + w \cdot G(q \cdot R)] & \text{if } R = \beta^{-1} \end{cases}. \tag{60}$$

Equation (60) states that if $R > \beta^{-1}$, then the savers save all their resources, which are given by their endowments of the consumption good plus the profits of the capitalists. If $R = \beta^{-1}$, then the savers are indifferent between saving and consuming these resources. As a result, the credit market clearing condition is given by:

$$S(q,R) = \int_{q \cdot R}^{1} b_A(q,R) \cdot dG(A), \tag{61}$$

which together with Equations (5), (6), (8), (9) and (10), characterizes the equilibrium.

Lastly, observe that the aggregate credit demand can be expressed as:

$$\int_{q \cdot R}^{1} b_A(q,R) \cdot dG(A) = q \cdot K^S(q) - w \cdot (1 - G(q \cdot R)), \tag{62}$$

since the entrepreneurs who invest in capital use all of their endowment plus borrowing to finance purchases of capital.

Therefore, we can immediately see that there are two possibilities in equilibrium.

*Case 1.* Consider a candidate equilibrium where the interest rate, $R$, is equal to $\beta^{-1}$. For this to be an equilibrium, it must be that:

$$w^C + q \cdot K^S(q) - \chi(K^S(q)) + w \cdot G(q \cdot \beta^{-1}) \geq q \cdot K^S(q) - w \cdot (1 - G(q \cdot R)), \tag{63}$$

which holds if and only if:

$$w^C + w \geq \chi(K^S(q)), \tag{64}$$

where the equilibrium price of capital, $q$, clears the capital market:

$$K^S(q) = \int_{q \cdot R}^{1} k_A(q, \beta^{-1}) \cdot dG(A). \tag{65}$$

It is therefore immediate that in this case the effects of an increase $\beta$ on the aggregate capital and output are isomorphic to those of a fall in $R$ analyzed in Section 3. Moreover, observe that this candidate is an equilibrium if $w^C$ and/or $\beta$ are large enough.

*Case 2.* Consider a candidate equilibrium where the interest rate $R$ is above $\beta^{-1}$. This candidate is an equilibrium if at $R = \beta^{-1}$ the inequality (64) is violated, i.e., if $w^C$ and/or $\beta$ are small. Hence, in this case, the equilibrium prices $\{q, R\}$ are such that:

$$w^C + w = \chi(K^S(q)), \tag{66}$$

and

$$K^S(q) = \int_{q \cdot R}^{1} k_A(q, R) \cdot dG(A). \tag{67}$$

Here, a rise in $w^C$ raises the capital price (as $\chi(K^S(q))$ is increasing in $q$) and lowers the interest rate (to offset the effect of a higher $q$ that depresses capital demand). Hence, the effects of an increase in $w^C$ on the aggregate capital and output are isomorphic to those of a fall in $R$ analyzed in Section 3.

Lastly, note that if the equilibrium is initially in Case 2, then an increase in $w^C$ eventually moves the equilibrium into Case 1.

## C.3 Our mechanism in the Kiyotaki-Moore model

In this Appendix, we show that the capital-reallocation effects induced by falling interest rates that we emphasized through the main text are also present in the class macro-finance model of Kiyotaki and Moore (1997).

Time is now infinite, $t = 0, 1, ....$ Assume, for simplicity, that all entrepreneurs in the modern sector have the same productivity $A \in (0, 1)$, and that the capital stock is fixed at $\bar{K} > 0$. Thus, aggregate output in any period $t$ depends solely on the allocation of capital between the modern and traditional sectors:

$$Y_t = A \cdot K_t + a \cdot f(\bar{K} - K_t), \tag{68}$$

where $K_t$ denotes the aggregate stock of capital employed in the modern sector at time $t$.

We focus on equilibria in which the traditional sector is active in all periods and, hence, its demand for capital is given by:

$$\frac{a \cdot f'\left(\bar{K} - K_{t+1}\right) + q_{t+1}}{q_t} = R, \tag{69}$$

i.e., the return to capital within the traditional sector must equal the interest rate.

As in the static model, we introduce a financial friction by assuming that – in any period – an entrepreneur can walk away with a fraction $1 - \lambda$ of her resources, which now include her output and the market value of her capital. It thus follows that entrepreneurs face the following borrowing constraint:

$$R \cdot B_t \leq \lambda \cdot (A + q_{t+1}) \cdot K_{t+1}, \tag{70}$$

where $B_t$ and $K_{t+1}$ respectively denote entrepreneurial borrowing and investment in period $t$.[30] Note that, since all entrepreneurs are identical, $B_t$ and $K_{t+1}$ also represent aggregate borrowing and investment in the modern sector.

In any period $t$, the net worth of entrepreneurs equals the sum of their output and the market value of their capital minus repayments to creditors: $A \cdot K_t + q_t \cdot K_t - R \cdot B_{t-1}$. We assume that entrepreneurs consume a fraction $1 - \rho$ of this net worth in every period, where $\rho \cdot R < 1$.[31] This ensures, in the spirit of Kiyotaki and Moore (1997), that the financial constraint holds with equality in all periods. As a result, the modern-sector demand for capital is given by:

$$K_{t+1} = \frac{1}{q_t - \lambda \cdot \frac{A + q_{t+1}}{R}} \cdot \rho \cdot (1 - \lambda) \cdot (A + q_t) \cdot K_t, \tag{71}$$

where we make parametric assumptions to ensure that both sectors are active in a neighborhood of the steady state.[32]

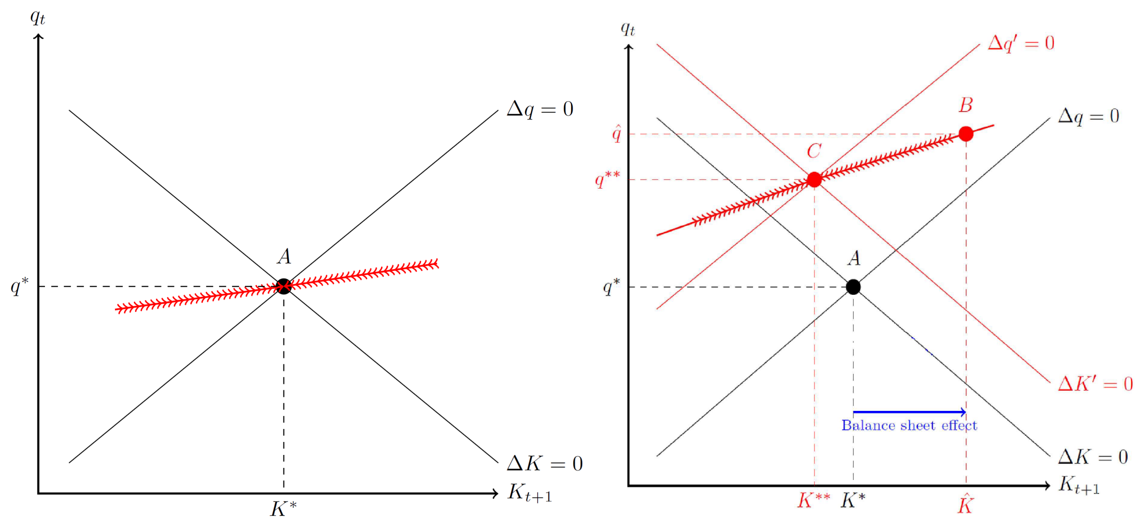
Thus, given an initial value for $K_0 > 0$ and a no bubbles condition on the price of capital, Equations (69) and (71) fully characterize the equilibrium of this economy. Panel (a) of Figure 9 portrays the equilibrium dynamics with the help of a phase diagram in the $(K_{t+1}, q_t)$-space. The $\Delta q = 0$ locus depicts all the combinations of $K_{t+1}$ and $q_t$ for which Equation (69) is satisfied with $q_t = q_{t+1}$. The locus is upward sloping because a higher level of modern-sector investment, $K_{t+1}$, is associated with a higher productivity of capital in the traditional sector and – since capital is priced by this sector – with a higher level of $q_t$. The $\Delta K = 0$ locus depicts instead all the combinations of $K_{t+1}$ and $q_t$ for which Equation (69) is satisfied with $K_t = K_{t+1}$. The locus is downward sloping because a higher level of modern-sector investment,

[30] In Kiyotaki and Moore (1997), the output of investment is not pledgeable but the resale value of capital is fully pledgeable. Although our results would also go through under that specification, we have chosen the current specification in order to preserve symmetry with the baseline model of Section 2.

[31] E.g., it is sufficient to assume that entrepreneurs have log-preferences, i.e., $U^E = \sum_{t=0}^{\infty} \rho^t \cdot log(c_t)$. Note that the preferences of other agents (i.e., capitalists and traditional investors) are irrelevant for the evolution of $q_t$, $K_t$ and $Y_t$.

[32] In particular, if $K_0$ is close to steady state, this requires that:

$$\frac{a \cdot f'(0)}{R - 1} > \frac{R \cdot \rho \cdot (1 - \lambda) + \lambda}{R - R \cdot \rho \cdot (1 - \lambda) - \lambda} \cdot A > \frac{a \cdot f'(\bar{K})}{R - 1}.$$




(a) Convergence to steady-state

(b) Effects of unexpected fall in $R$

Figure 9: **Equilibrium dynamics and balance sheet effects.** The figure illustrates a phase diagram for the joint evolution of the price of capital and the stock of capital in the modern sector. The saddle path of the system is depicted by a red curve with arrows pointing to the steady state: the left panel depicts the dynamics before the unexpected decline in the interest rate, whereas the right panel depicts the dynamics after it.

$K_{t+1}$, is only affordable to constrained entrepreneurs if the equilibrium price of capital, $q_t$, is lower. As the figure shows, the system displays saddle-path dynamics. From an initial condition $K_0 < K^*$, both $K$ and $q$ increase monotonically as the economy transitions to the steady state and modern-sector entrepreneurs accumulate net worth. The opposite dynamics follow from an initial condition $K_0 > K^*$.

The right-hand panel of Figure 9 portrays the response to a permanent and unanticipated decline in $R$ in a given period $t_0$. In response to a lower $R$, both loci shift upwards. The $\Delta q = 0$ locus shifts up because the traditional sector's willingness to pay for capital increases alongside the net present value of dividends; the $\Delta K = 0$ also shifts up because entrepreneurs' ability to pay for capital increases as lower interest rates relax their borrowing constraint. The presence of financial frictions, however, mitigates the shift in the $\Delta K = 0$ locus. Thus, as the figure shows, a decline in $R$ triggers an increase in the steady-state price of capital to $q^{**}$, and a reduction in the capital employed in the modern sector to $K^{**}$. Hence, a reduction in the interest rate leads to a fall in the steady-state level of output despite the presence of dynamics.

This does not mean, however, that balance sheet effects do not play a role. Indeed, on impact, in response to a decline in the interest rate, the value of capital increases from $q^* \cdot K^*$ to $q_{t_0} \cdot K^*$ while entrepreneurial debt payments - which are pre-determined - remain unaffected and equal to $R \cdot B^*$.[33] Therefore:

$$K_{t+1} = \begin{cases} \frac{1}{q_t - \lambda \cdot \frac{A + q_{t+1}}{R}} \cdot \rho \cdot ((1-\lambda) \cdot (A + q^*) + q_t - q^*) \cdot K^* & \text{if } t = t_0 \\ \frac{1}{q_t - \lambda \cdot \frac{A + q_{t+1}}{R}} \cdot \rho \cdot (1-\lambda) \cdot (A + q_t) \cdot K_t & \text{if } t > t_0 \end{cases}. \tag{72}$$

[33] As in Kiyotaki and Moore (1997), these balance sheet effects require that entrepreneurs' debt payments are not indexed to the price of capital.

The evolution of $q_t$ is still given by Equation (69). This means that the adjustment of $K$ to the new steady-state is not monotonic. As the right-hand panel of Figure 9 shows, $K_{t+1}$ rises to $\widehat{K}$ on impact: this, as stated in the figure, is the balance sheet effect. The expansion of the modern sector is short-lived, though, since from that period onwards the economy evolves along the saddle-path towards its new steady state, which features a higher price of capital but a lower capital stock in the modern sector and thus a lower level of output. This decline from $\widehat{K}$ to $K^{**}$ is, as stated in the figure, due to the reallocation effect: the higher demand of capital by the traditional sector keeps capital prices high, and these slowly erode the net worth of modern-sector entrepreneurs. As a result, the dynamic behavior of aggregate output in this economy resembles closely that of the dynamic economy in Section 4, illustrated in Figure 4.

The key takeaway is that the same reallocation forces that we analyzed in our baseline model of Section 2 are also at work in a dynamic environment. Moreover, these forces are persistent in response to a permanent decline in the interest rate, while the balance-sheet effects that are often highlighted in the literature are transitory. To be sure, an unexpected decline in the interest rate does have an initial balance-sheet effect that benefits productive entrepreneurs and reallocates capital towards them, raising average productivity and output. But this effect is by nature temporary: the reason is that it represents a one-time shock to the level of entrepreneurial net worth, but it does not affect the dynamic evolution of net worth thereafter.

## C.4 Closed-form solution to the steady state of the dynamic model

In this Appendix, we derive the steady state of the dynamic model in closed form, for the case in which productivity is i.i.d. over time and uniformly distributed on the unit interval.

As Equation (35) in the text shows, in steady state:

$$W_A = \begin{cases} \frac{\Theta}{\Theta+\rho-\frac{1}{1-\lambda}\cdot\left(\frac{A}{q}-\lambda\cdot r\right)} \cdot g(A) \cdot W & \text{if } A \geq r \cdot q \\ \frac{\Theta}{\Theta+\rho-r} \cdot g(A) \cdot W & \text{otherwise} \end{cases}, \tag{73}$$

where $g(A) = 1$ in this case because of the uniform distribution. Let $x \equiv A/(r \cdot q)$, then:

$$\frac{W_{r\cdot q\cdot x}}{W} = \begin{cases} \frac{\Theta}{\Theta+\rho-\frac{x-\lambda}{1-\lambda}\cdot r} & \text{if } x \in \left[1, \frac{1}{r\cdot q}\right] \\ \frac{\Theta}{\Theta+\rho-r} & \text{if } x \in [0,1] \end{cases}, \tag{74}$$

with:

$$\int_0^{\frac{1}{r\cdot q}} \frac{W_{r\cdot q\cdot x}}{W} dx = 1 \,. \tag{75}$$

From substituting (74) into (75), it follows that:

$$\frac{1}{r \cdot q} = \lambda + \frac{1-\lambda}{r} \cdot \left[\Theta + \rho - \frac{\Theta+\rho-r}{\exp\left\{\frac{1}{\Theta}\cdot\frac{r}{1-\lambda}\cdot\frac{\rho-r}{\Theta+\rho-r}\right\}}\right]. \tag{76}$$

Equation (76) allows us to express price $q$ as a function of model parameters.

Aggregate output is given by:

$$Y = \frac{1}{\int_1^{\frac{1}{r\cdot q}} W_{r\cdot q\cdot x}\cdot dx}\cdot\left[\int_1^{\frac{1}{r\cdot q}} r\cdot q\cdot x\cdot W_{r\cdot q\cdot x}\cdot dx\right]\cdot\bar{K}\ . \tag{77}$$

Thus, we have that:

$$Y = r\cdot q\cdot\frac{\Theta+\rho-r}{\rho-r}\cdot\Theta\cdot\frac{\left[\left(\Theta+\rho+\frac{\lambda}{1-\lambda}\cdot r\right)\ln\left(\Theta+\rho+\frac{\lambda}{1-\lambda}\cdot r-\frac{r}{1-\lambda}\cdot x\right)+\frac{r}{1-\lambda}\cdot x\right]\Big|_1^{\frac{1}{r\cdot q}}}{\left(\frac{r}{1-\lambda}\right)^2}\cdot\bar{K}\ , \tag{78}$$

which together with Equation (76) allow us to express $Y$ as a function of parameters.

## C.5 Sensitivity analysis with respect to the path of the interest rate

In this Appendix, we display the quantitative effects from (i) the permanent decline in the interest rate with the instantaneous transition (Figure 10); and (ii) the temporary decline (Figure 11).

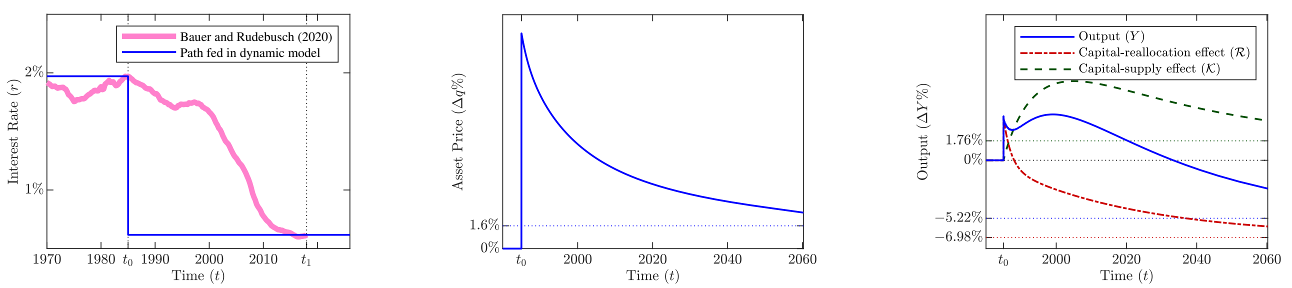



Figure 10: **Permanent decline in the interest rate with the instantaneous transition.**

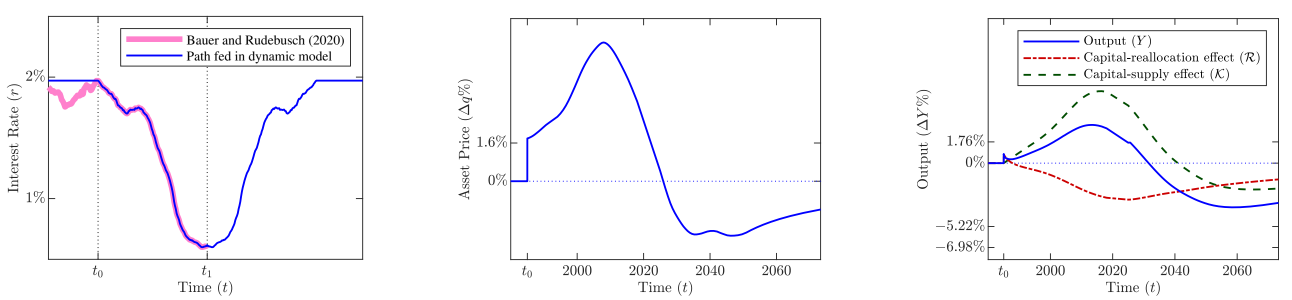



Figure 11: **Temporary decline in the interest rate.**