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Assortative Learning

By Jan Eeckhout* and Xi Weng†

*UPF Barcelona, ICREA, BSE, CREI †Peking University

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Because of sorting, more skilled workers are more productive in higher-type firms. They also learn at different rates about their productivity and therefore expect different wage paths across firms. We show that under strict supermodularity, there is always positive assortative matching: differential learning is always dominated by the impact of productivity. Surprisingly, this holds even if learning is faster in the low-type firm. The key assumption driving this result is that this is a pure Bayesian learning model. The model provides realistic predictions about wage variance, turnover and the wage distribution that are in line with recent work that estimates the value of learning from co-workers.

Introduction

High-ability workers sort into more productive jobs. Due to complementarities in production, their higher marginal product allows them to command higher wages. The Beckerian model of assortative matching is very well suited to explain those patterns of sorting. Unfortunately, it is mute on the issue of turnover of workers between different jobs. Instead, the Jovanovic (1979) learning model has long been the canonical framework for analysing turnover in the labour market[1] over the lifecycle. Workers and firms learn about match-specific human capital and will tend to stay in a match if learning reveals that the match is good. Experimentation occurs early on, which leads to decreasing turnover over the lifecycle. However, the large literature following Jovanovic (1979) (e.g. Jovanovic 1984; Felli and Harris 1996; Moscarini 2005) is silent on how experimentation affects sorting. Because in Jovanovic (1979) learning is about the match and not about the worker, there is neither worker heterogeneity nor sorting.

In this paper, we consider learning about general human capital and thus offer a unified approach of studying experimentation and sorting. In the labour market, the learning experiences of workers for workers with different general human capital are most likely to differ across firms. Starting in a top law firm or a multinational will induce different paths of information revelation than working in a local family business. The worker now faces a trade-off between different experimentation experiences: take a lower wage at a high productivity firm where information may be revealed at a different rate, or accept a higher wage and learn more slowly. It is intuitive that sorting and learning are connected intimately: the worker may choose to sacrifice instantaneous productive efficiency for dynamic gains from learning.[2]

We introduce a model of a market equilibrium in a continuous time economy with multiple learning opportunities (multi-armed bandit). Our main result states that under supermodularity, positive assortative matching always obtains in equilibrium, even if learning rates differ across firms. Modelling the labour market as a multi-armed bandit problem and solving it is challenging. Most existing learning models and continuous time games are tractable because they are essentially one-armed bandit problems with a fixed outside option that acts as an absorbing

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Jan Eeckhout is ICREA research professor of Economics at UPF Barcelona.
Jan graduated from the LSE in 1998 with a PhD in Economics. Jan also lectured at the LSE from 1996–1997.

state. One-armed bandit problems typically have attractive properties, including reservation strategies. Instead, multi-armed bandits in general do not have reservation strategies when arms are correlated, even if the learning rate is the same across firms.[3] But our labour market is not exactly identical to the canonical bandit problem. First, there is a continuum of experimenters, and as a result of two-sided heterogeneity, deviations and off-equilibrium path beliefs affect equilibrium non-trivially. Second, because of competitive wage determination à la Jovanovic (1979), the payoffs are endogenous. Finally, because workers learn about general human capital instead of match-specific human capital, the arms are positively correlated.

We find that the competitive wage determination (endogenous payoffs) enables us to apply the theoretical results in Eeckhout and Weng (2015), and derive a new condition that we call the *no-deviation condition*. In addition to the well-known conditions of value matching (level) and smooth pasting (first derivative), this condition implies that the second derivatives of the value function must be equal whenever the agent switches action. We further show that supermodularity of the production technology is a sufficient condition for positive assortative matching, and that the equilibrium allocation is unique. Those workers with the highest beliefs about their ability will in equilibrium sort into those firms that are most productive. Moreover, we can solve analytically for the equilibrium allocation in terms of the cut-off belief, and we derive in closed form the stationary distribution of beliefs.

While in most of the analysis we consider common variance across firms, it turns out that the sorting result holds for different learning rates (noise) across firms, even if the rate of learning is slower in the high-type firm. It is conceivable that with supermodularity and a learning rate no smaller in high-type firms, there will be positive sorting. The high-type firm is superior in both the learning rate and productive efficiency. But if high-type firms learn at a sufficiently slower rate (the noise is sufficiently high), then the signal-to-noise ratio in the high-type firm may well be lower. The reason why this nonetheless does not affect the learning is that the value of learning also depends on the degree of convexity of the value function (from Ito's Lemma), in addition to the signal-to-noise ratio. But by the no-deviation condition, at the cut-off belief, the degree of convexity is the same in both firms and therefore the equilibrium value of learning is the same, no matter the difference in signal-to-noise ratios. Key here is that wages are endogenous and determined competitively. That is why this property does not necessarily hold in the canonical multi-armed bandit problem.

In our decentralized equilibrium, there is a source of market incompleteness: wages are spot market prices only and cannot be made contingent on future realizations. In order to study a counterfactual economy with commitment, we consider the same problem but with the modification that the high-type firms can commit to maximizing profits contingent on future realizations, and we analyse this problem in the steady state. We find that this problem is equivalent to the planner's problem under stationarity. We show that the planner's optimal stationary allocation coincides with the decentralized equilibrium allocation without commitment, even if learning rates differ across different firms. Hence the decentralized equilibrium is efficient in the sense that introducing commitment does not change the equilibrium allocation. It turns out that the efficiency result and proof crucially hinge on the martingale property inherent in Bayesian learning. The martingale property implies that no matter how fast workers learn, the expected beliefs about their ability will stay the same. Since under strict supermodularity, the differential in expected output between working in high- and low-productivity firms is monotonically increasing in the likelihood that the worker has high ability, reallocating a group of low-belief workers to a better match will decrease expected outputs no matter how fast they learn.

We extend our analysis of Bayesian learning to allow for *observable* human capital accumulation. This adds realism in the sense that workers learn on the job and increase their productivity with tenure, yet we do not resort to non-Bayesian updating. Now cut-off types that characterize the equilibrium allocation depend on the degree of observable experience, and beliefs continue to follow a martingale process. The properties of our equilibrium extend to this more general human capital accumulation case. We also establish that positive assortative matching can fail if the updating process is not Bayesian (this can be interpreted, for example, as a technology of *unobserved* human capital accumulation in addition to the information extraction).

Literature

The motivation of our analysis and the results is obviously closest related to the labour market learning literature such as Jovanovic (1979, 1984), Harris and Holmström (1982), Moscarini (2005), Papageorgiou (2014), Pastorino (2015), Li and Weng (2017), and Felli and Harris (2018).[4] Yet there is a close relation to both the experimentation literature (Bolton and Harris 1999; Keller *et al.* 2005; Strulovici 2010) and the literature on continuous time games (Sannikov 2007, 2008). Most models of learning have a finite set of players and have an absorbing state. Our model has a continuum of agents and there is learning in all states. Moreover, it is essentially a competitive model with equilibrium prices, therefore payoffs from learning are endogenous.

The idea of analysing a matching model where the current allocation determines the future type is first explored in Anderson and Smith (2010). They find the opposite result to ours: positive assortative matching fails even under supermodularity.[5] They analyse a two-sided matching model of reputations with imperfect information about both matched types.[6] Our setup differs substantially, but the main difference is in the information extraction. Their agents infer the type of each of the matched partners from the realization of a *joint* signal.[7]

I. The Model Economy

Population of firms and workers

The economy is populated by a unit measure of workers and a unit measure of firms. Both firms and workers are *ex ante* heterogeneous. The firm's type $y \in \{H, L\}$ represents its productivity. The type y is observable to all agents in the economy. The fraction of H firms is π, and all firms are infinitely-lived. The worker ability $x \in \{H, L\}$ is not observable, to both firms and workers; that is, information is symmetric.[8] Nonetheless, both hold a common belief about the worker type, denoted by $p \in [0, 1]$. Upon entry, a newly born worker is of type H with probability p_0, and of type L with probability $1 - p_0$. Workers die with exogenous probability δ. New workers are born at the same rate.[9]

Preferences and production

Workers and firms are risk-neutral and discount future payoffs at rate $r > 0$. Utility is perfectly transferable. Output is produced in pairs of one worker and one firm, (x, y). Time is continuous. Positive output produced consists of a divisible consumption good and is denoted by μ_{xy}. We assume that more able workers are more productive in any firm, $\mu_{Hy} \geq \mu_{Ly}$ for all y, and refer to this as worker monotonicity. While it is often useful, we do not in general assume firm monotonicity, which would be $\mu_{xH} \geq \mu_{xL}$ for all x. Strict supermodularity is defined in the usual way as

$$\mu_{HH} - \mu_{LH} > \mu_{HL} - \mu_{LL},$$

with the opposite sign for strict submodularity. Throughout the paper, we will mean strict supermodularity when we mention just supermodularity, and likewise for submodularity.

Information

Because worker ability is not observable to both the worker and the firm, parties face an information extraction problem. They observe a noisy measure of productivity, denoted by X_t. Cumulative output is assumed to be a Brownian motion with drift μ_{xy} and common variance σ^2,

$$X_t = \mu_{xy} t + \sigma Z_t,$$

where Z_t is a standard Wiener process, and as a result, X_t is normally distributed with mean $\mu_{xy} t$ and variance $\sigma^2 t$. By Girsanov's Theorem, the probability measures over the paths of two diffusion processes with the same volatility but different bounded drifts are equivalent, that is, they have the same zero-probability events. Since the volatility of a continuous-time diffusion process is effectively observable, the worker's type could be learned directly from the observed volatility if σ depends on workers' types.[10]

Equilibrium

We consider a stationary competitive equilibrium in this economy. With two types of firms and a continuum of p values in this market, take a competitive wage schedule $w_y(p)$ as given, which specifies the wage for every possible type p worker working in firm y.[11] Denote by V_y the stationary discounted present value of the competitive profits for firm y. The flow profit can be written as rV_y.[12] Now we are ready to define the notion of competitive equilibrium.

Definition 1. A stationary competitive equilibrium consists of a competitive wage schedule $w_y(p) = \mu_y(p) - rV_y$, where $\mu_y(p) = p\mu_{Hy} + (1-p)\mu_{Ly}$ denotes the expected productivity of worker p in firm $y \in \{H, L\}$, and worker p chooses the firm y with the highest discounted present value. The market clears such that the measure of workers in L firms is $1-\pi$, and the measure of workers in H firms is π.

We would like to point out several things about this definition. First, the definition of competitive equilibrium implies that identical types will obtain the same payoff. A firm y earns the same flow profit for every p. Our notion of competitive equilibrium puts restrictions on the off-equilibrium prices, as does the Beckerian definition of a matching equilibrium. Although a type p worker is not employed by firm y on the equilibrium path, the hypothetical wage is still $w_y(p) = \mu_y(p) - rV_y$ to guarantee that the firm cannot make or lose money if the employment suddenly happens. Second, our wage definition concerns a spot market wage and captures the idea that firms cannot commit to future actions or realizations (see also Hörner and Samuelson (2013) for a model of experimentation in the presence of spot market contracts). Together with sequential rationality, this therefore requires that the wage contract is self-enforcing. We believe that this is realistic since it is consistent with the at-will employment doctrine in which parties are free to terminate employment relations with no liability. Our spot market wage assumption is in contrast with Anderson and Smith (2010), who parse the wage into a static wage plus a dynamic human capital effect. Their wage setting process therefore corresponds to the Pareto-efficient allocation. Third, like all price-taking economies, the wage schedule essentially transforms our problem into a decision problem for the workers.

II. Preliminaries

Benchmark: no learning

We assume that workers are characterized by the common beliefs p that they are a high type. We shut down learning so that beliefs are invariant. This can be viewed as a special case of the learning model with variance σ^2 going to infinity. We further assume that there is no birth or death, so essentially we have a static problem. Suppose without loss of generality that p is uniformly distributed on $[0, 1]$. We continue to maintain the assumption that the worker does not know her true type or that she has no private information about it. We define $\mu_y(p)$ as in Definition 1, and let r denote the discount rate.

Under the above notion of competitive equilibrium, it is easy to verify the following claim. (All of the results in this paper are in the sense of 'almost surely' because we allow a zero measure of agents to behave differently.)

Claim 1. Under strict supermodularity, positive assortative matching (PAM) is the unique (stationary) competitive equilibrium allocation: H firms match with workers $p \in [1-\pi, 1]$, and L firms match with workers $p \in [0, 1-\pi)$. The opposite (negative assortative matching, NAM) holds under strict submodularity: H firms match with workers in $[0, \pi)$.

Since there is no learning, essentially this result is identical to the result of Becker (1973), but with uncertainty. Noteworthy about this version of Becker is that even though for PAM there is supermodularity of the *ex post* payoffs ($\mu_{HH} + \mu_{LL} > \mu_{HL} + \mu_{LH}$), there need not be monotonicity in expected payoffs; that is, $\mu_H(1-\pi)$ may be smaller than $\mu_L(1-\pi)$. In fact, that will be reflected in the firm's equilibrium payoffs: $V_H \geq V_L$ if and only if $\mu_H(1-\pi) \geq \mu_L(1-\pi)$.

As in Becker (1973), the equilibrium allocation is unique, but there may be multiple splits of the surplus. In the case of PAM, we require at the cut-off type $\underline{p} = 1-\pi$ only that $w_H(\underline{p}) = w_L(\underline{p})$. There are multiple equilibrium payoffs if the surplus of a match between L and $\underline{p} = 0$ is positive. Instead, if $\mu_L(0) = 0$,[13] then there is a unique equilibrium payoff.

Belief updating

In the presence of learning, we can now derive the beliefs and subsequently the value functions. The posterior belief p_t that the worker has a high productivity is a sufficient statistic for the output history. Now we can use the following well-known result: conditional on the output process $(X_t)_{t\geq 0}$, $(p_t)_{t\geq 0}$ is a martingale diffusion process. Moreover, this process can be represented as a Brownian motion. Based on the framework of our model, denote $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma$, $y \in \{H, L\}$, and $\Sigma_y(p) = \frac{1}{2}p^2(1-p)^2 s_y^2$, and then we obtain the following lemma, which is used widely in the literature (e.g. Moscarini 2005; Daley and Green 2012; Papageorgiou 2014).

Lemma 1 (Belief consistency). Consider any worker who works for firm y between t_0 and t_1. Given a prior $p_{t_0} \in (0, 1)$, the posterior belief $(p_t)_{t_0<t\leq t_1}$ is consistent with the output process $(X_{y,t})_{t_0<t\leq t_1}$ if and only if it satisfies

$$\mathrm{d}p_t = p_t(1-p_t)s_y\,\mathrm{d}\overline{Z}_{y,t},$$

where

$$\mathrm{d}\overline{Z}_{y,t} = \frac{1}{\sigma}\left[\mathrm{d}X_{y,t} - \left(p_t\mu_{Hy} + (1-p_t)\mu_{Ly}\right)\mathrm{d}t\right].$$

The proof of Lemma 1 is in Liptser and Shyryaev (1977), and this lemma establishes that dp depends on three elements: $p(1-p)$, which peaks at $\frac{1}{2}$; the signal-to-noise ratio of output, $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma$; and $\mathrm{d}\overline{Z}_y$, the normalized difference between realized and unconditionally expected flow output, which is a standard Wiener process with respect to the filtration $\{X_{y,t}\}$. Obviously, beliefs move faster the more uncertainty there is about a worker's quality (p close to $\frac{1}{2}$), the less variation there is in the output process (smaller σ), and the larger the productivity difference (higher $\mu_{Hy} - \mu_{Ly}$).

Learning considerations will change the benchmark results. Moreover, supermodularity not only affects the value of the static output created as in the standard Beckerian model, but also has a dynamic effect by changing the speed of learning. For example, under supermodularity ($\mu_{HH} - \mu_{HL} > \mu_{LH} - \mu_{LL}$), the learning speed is faster in the high-type firm, which is especially significant for p close to $\frac{1}{2}$. Intuitively speaking, learning makes it more attractive to match with a high-type firm even though statically it is better to match with a low-type firm without learning.

Value functions

Consider any interval for the posterior belief $p \in [p_1, p_2]$ where the worker accepts the offer from a type y firm. Then the value function is given by[14]

(1) $$r\ W_y(p) = \mu_y(p) - rV_y + \Sigma_y(p)\ W_y''(p) - \delta\ W_y(p),$$

from Ito's Lemma. The term $\mu_y(p) - rV_y$ is equal to the flow wage payoff and corresponds to the deterministic component of the diffusion $X_{y,t}$, and the term $\Sigma_y(p)\ W_y''(p)$ is the second-order term from the transformation W of the diffusion process $X_{y,t}$. First-order and all higher-order terms vanish as the time interval shrinks to zero. The general solution of this differential equation is

(2) $$W_y(p) = \frac{\mu_y(p) - rV_y}{r+\delta} + k_{y1}p^{1-\alpha_y}(1-p)^{\alpha_y} + k_{y2}p^{\alpha_y}(1-p)^{1-\alpha_y},$$

where

$$\alpha_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{s_y^2}} \geq 1.$$

First notice that the boundedness of the value function implies that if 0 is included in the domain, then $k_{y1} = 0$, and if 1 is included in the domain, then $k_{y2} = 0$. Otherwise, with $\alpha_y > 1$, the value of W shoots off to infinity. Second, $\Sigma_y(p)\ W_y''(p)$ is the value of learning, and this is an option value in the sense that the worker has the choice to change his job as he learns his type p. It is easy to verify that this value is zero if the worker never changes his job.[15] From the martingale property of the Brownian motion, at any p the expected value of p in the next time interval is equal to p. There is as much good news as bad news to be expected in the next period. It is the option value of switching to a *more suitable* match that generates the value of learning. Equation (2) implies that this option value can be decomposed into two parts: $k_{y1}p^{1-\alpha_y}(1-p)^{\alpha_y}$ ($k_{y2}p^{\alpha_y}(1-p)^{1-\alpha_y}$) denotes the option value of switching to a *more suitable* match when p goes down (up). The option value $k_{y1}p^{1-\alpha_y}(1-p)^{\alpha_y}$ ($k_{y2}p^{\alpha_y}(1-p)^{1-\alpha_y}$) must be zero if 0 (1) is included, since no switch happens as p goes down (up).

III. Analysis and Results

Characterization of the equilibrium allocation

Now consider any candidate stationary equilibrium where a type p worker switches from firm y to firm y'. Since the worker is essentially facing a two-armed bandit problem given the wage schedule, optimality in stopping time requires the value-matching condition (the worker gets the same value at the cut-off) and the smooth-pasting condition (the marginal of both value functions is identical) (see Dixit 1993). For example, if for $p \in [p_1, p_2)$ the worker works in the low-type firm, and for $p \in [p_2, p_3)$ the worker works in the high-type firm, then we must have[16]

$$W_L(p_2) = W_H(p_2) \quad \text{and} \quad W_L'(p_2) = W_H'(p_2).$$

Notice that workers are price-takers. As a result, there is no strategic interaction between players where equilibrium solves for the fixed point of individual strategies. It is also important to point out that both the value-matching condition and the smooth-pasting condition are on-equilibrium path conditions. They have nothing to do with the off-equilibrium path (i.e. instead of accepting offers from low-type firms, workers with $p \in [p_1, p_2)$ are tempted to accept offers from high-type firms). In the following lemmas we characterize the value functions establishing convexity and monotonicity.

Lemma 2. The equilibrium value functions W_y are strictly convex for $p \in (0, 1)$.

Proof. See the Appendix. □

The intuition for this lemma is as follows. Preferences and output are linear in p, and the option value of learning is strictly positive, hence the value function with the option of learning is convex. To see this, observe that since the measures of both types of firm are strictly positive, market clearing requires that workers with some p values will be employed by high-type firms, while workers with other p values will be employed by low-type firms. This implies that some worker has to change jobs at some point, and the option value of learning, $\Sigma_y(p)\ W_y''(p)$, is strictly positive. Hence we have $W_y''(p) > 0$ for all $p \in (0, 1)$ since $\Sigma_y(p) > 0$. On the other hand, when $p = 0$ or $p = 1$, the posterior belief will always stay at 0 or 1 by Bayes' Rule, such that learning never happens. It is easy to verify that $W_y''(p) = 0$ for $p = 0$ or $p = 1$.

Given the strict convexity of equilibrium value functions and the smooth-pasting condition, we can immediately derive the following result.

Lemma 3. The equilibrium value functions W_y are strictly increasing.

Proof. See the Appendix. □

One important implication is that if we define $\mathcal{W}(p)$ as the envelope of all equilibrium value functions $W_y(p)$, then this envelope function $\mathcal{W}(p)$ is continuous, strictly increasing and strictly convex for $p \in (0, 1)$. Suppose that workers with $p \in [0, \underline{p})$ are employed by a type y firm, and workers with $p \in (\overline{p}, 1]$ are employed by a type $-y$ firm. Then we should have

$$W_y'(0) = \frac{\mu_{Hy} - \mu_{Ly}}{r + \delta} < W_{-y}'(1) = \frac{\mu^{H,-y} - \mu^{L,-y}}{r + \delta}.$$

This gives us another result.

Lemma 4. Under supermodularity, in any equilibrium, $p = 0$ workers match with L firms, and $p = 1$ workers match with H firms. The opposite holds under strict submodularity. Moreover,

$$\frac{\min(\Delta_H, \Delta_L)}{r+\delta} < W'(p) < \frac{\max(\Delta_H, \Delta_L)}{r+\delta},$$

where $\Delta_H = \mu_{HH} - \mu_{LH}$ and $\Delta_L = \mu_{HL} - \mu_{LL}$.

Intuitively, this result is best understood by using the standard sorting argument from Becker (1973). At $p = 0$ and $p = 1$, there is no value of learning. As a result, there the value function can be interpreted as being determined by the no-learning allocation.

The properties derived above are concerned mainly with on-equilibrium path behaviour. We also need to specify what happens in the event of deviations, and consider behaviour off-equilibrium path. We contemplate the equivalence of a one-shot deviation in continuous time because we think of the continuum as an idealization of discrete time. This amounts to a worker playing the deviant action over an interval $[t, t + \mathrm{d}t)$ according to the belief p at time t, and considering the limit as $\mathrm{d}t \to 0$.[17] This is very important because it allows us to derive the value function for deviation. On the contrary, if the deviation takes place at only a single point in time t, then the value function for deviation is essentially the same as the one without deviation because no information will be extracted from just a single time point.

The next lemma establishes that if we consider off-equilibrium path deviations, we actually derive one additional condition, which we call the *no-deviation* condition.

Lemma 5. To deter possible deviations, a necessary condition is

$$W''_H(\underline{p}) = W''_L(\underline{p}) \quad \text{(No-deviation condition)}$$

for any possible cut-off $\underline{p}$.

The proof of this no-deviation condition can be found in Eeckhout and Weng (2015), and hence is omitted. Notice that this condition is quite unique for the two-armed bandit problem, since it is absent in a one-armed bandit problem. Most of the models in the literature on continuous time learning models (Jovanovic 1979; Moscarini 2005) and continuous time games (see, among others, Sannikov 2007) are essentially investigating a one-armed bandit problem. There, we can look directly at equilibria in cut-off strategies. In the one-armed bandit problems, the safe arm essentially is an absorbing state so we need to worry about only the potential deviation from the risky arm to the safe arm.[18] Then the no-deviation condition becomes $W''_H(\underline{p}) \geq W''_L(\underline{p}) = 0$, but this is already implied by the convexity property.[19]

We provide some intuition for the no-deviation condition. By assuming sequential rationality—that is, the equilibrium is robust to a one-shot deviation—we basically impose that the equilibrium wage is self-enforcing. There is no commitment to future realizations of X_t and therefore of future beliefs p. Now we can interpret W' as the marginal value of learning: W' is the marginal change of W with respect to the posterior p, and learning changes p and is therefore quantified by the change in W', which is W'. The condition states that there is no deviation if the marginal value of learning at $\underline{p}$ is the same in both firms.

Now, in our two-armed bandit problem, we first need to answer the question of whether there exist non-cut-off stationary equilibria, that is, a worker with $p \in [p_1, p_2)$ accepts the offer from a high-type firm, with $p \in [p_2, p_3)$ he accepts the offer from a low-type firm, and with $p \in [p_3, p_4)$ he accepts the offer from a high-type firm again. Surprisingly, Lemmas 2–5 imply that all possible stationary competitive equilibria must be in cut-off strategies. The next theorem therefore establishes uniqueness and sorting under supermodularity. It does not shown existence yet, which we do in Theorem 3 below.

Theorem 1. If an equilibrium exists, then PAM is the unique stationary competitive equilibrium allocation under strict supermodularity. Likewise for NAM under strict submodularity.

To prove this theorem, we need only prove the following claim.

Claim 2. Under strict supermodularity, it is impossible to have $p_1 < p_2$ and equilibrium value functions W_H (for $p \in [p_1, p_2]$), W_{L1} (for $p < p_1$), W_{L2} (for $p > p_2$) such that

$$W_H(p_1) = W_{L1}(p_1) \quad \text{and} \quad W_H''(p_1) = W_{L1}''(p_1),$$
$$W_H(p_2) = W_{L2}(p_2) \quad \text{and} \quad W_H''(p_2) = W_{L2}''(p_2),$$

are satisfied simultaneously.

Under strict submodularity, it is impossible to have $p_1 < p_2$ and equilibrium value functions W_L (for $p \in [p_1, p_2]$), W_{H1} (for $p < p_1$), W_{H2} (for $p > p_2$) such that

$$W_L(p_1) = W_{H1}(p_1) \quad \text{and} \quad W_L''(p_1) = W_{H1}''(p_1),$$
$$W_L(p_1) = W_{H2}(p_2) \quad \text{and} \quad W_L''(p_2) = W_{H2}''(p_2),$$

are satisfied simultaneously.

Proof. See the Appendix. □

This result states that it is not beneficial for a worker of type p to learn in the high-type firm H in the middle as long as there there are still types p on both sides who work in the low-type firms. Given the above claim, it is easy to prove the theorem.

Proof of Theorem 1 Under supermodularity, by Lemma 5, workers with sufficiently low p values will accept a low-type firm's wage offer, and workers with sufficiently high p values will accept a high-type firm's offer. But Claim 2 implies that it is impossible to have a worker first accept a low-type firm's offer, then accept a high-type firm's offer, and finally accept a low-type firm's offer again. Hence we must have some cut-off $\underline{p}$ such that $p < \underline{p}$ will accept a low-type firm's offer, and $p > \underline{p}$ will accept a high-type firm's offer. This is exactly a PAM allocation. Use the same logic, NAM is the only possible stationary competitive equilibrium allocation under strict submodularity. □

Before we turn to the equilibrium distribution, we show that the no-deviation condition in Lemma 5 is not just necessary but also sufficient under strict supermodularity.

Lemma 6. Under strict supermodularity, $W_H''(\underline{p}) = W_L''(\underline{p})$ implies that no deviation will happen for the PAM equilibrium allocation.

Proof. See the Appendix. □

The equilibrium distribution

The previous subsection shows that under strict supermodularity (submodularity), PAM (NAM) is the unique candidate stationary competitive equilibrium allocation. Note that this does not necessarily mean that the equilibrium exists. We still need to construct such an equilibrium. To do that, we assume strict supermodularity, and worker and firm monotonicity: $\mu_{HH} > \mu_{HL}$ and $\mu_{LH} > \mu_{LL}$.[20] Now consider a strictly PAM equilibrium such that workers with beliefs less than $\underline{p}$ will choose L firms, and workers with beliefs higher than $\underline{p}$ will choose H firms. From equation (2), we hence have $k_{L1} = 0$ and $k_{L2} > 0$ for $y = L$, and $k_{H2} = 0$ and $k_{H1} > 0$ for $y = H$. Let $k_L = k_{L2}$ and $k_H = k_{H1}$. Then workers' value functions become

$$W_L(p) = \frac{w_L(p)}{r+\delta} + k_L p^{\alpha_L}(1-p)^{1-\alpha_L}$$

and

$$W_H(p) = \frac{w_H(p)}{r+\delta} + k_H p^{1-\alpha_H}(1-p)^{\alpha_H},$$

where

$$\alpha_y = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{s_y^2}} \geq 1.$$

To discuss market clearing conditions, we need to consider the ergodic distribution of p values. From the Fokker–Planck (Kolmogorov forward) equation, the stationary and ergodic density f_y should satisfy the differential equation

$$0 = \frac{\mathrm{d}f_y(p)}{\mathrm{d}t} = \frac{\mathrm{d}^2}{\mathrm{d}p^2}\left[\Sigma_y(p)\, f_y(p)\right] - \delta\, f_y(p).$$

The general solution of this differential equation is (see also Moscarini 2005)[21]

$$f_y(p) = f_{y0}p^{\gamma_{y1}}(1-p)^{\gamma_{y2}} + f_{y1}(1-p)^{\gamma_{y1}}p^{\gamma_{y2}},$$

where

$$\gamma_{y1} = -\frac{3}{2} + \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} > -1$$

and

$$\gamma_{y2} = -\frac{3}{2} - \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} < -2.$$

First, the integrability of f_y requires that $f_{y1} = 0$ if 0 is included in the domain, and $f_{y0} = 0$ if 1 is included in the domain. Second, the Fokker–Planck (Kolmogorov forward) equation is valid only for $p \neq p_0$. Since there is a flow in of new workers, for $p = p_0$ we should have a kink in the density function. This also raises the issue of the relative position between p_0 and $\underline{p}$. We first consider the case where $\underline{p} < p_0$. We then derive in abbreviated format the result when $\underline{p} > p_0$.

Given any $p_0 \in (0,1)$, if $\underline{p} < p_0$, then the density functions are

$$f_H(p) = \left[f_{H0}p^{\gamma_{H1}}(1-p)^{\gamma_{H2}} + f_{H1}(1-p)^{\gamma_{H1}}p^{\gamma_{H2}}\right]\, \mathbb{I}(\underline{p} < p \leq p_0) + f_{H2}(1-p)^{\gamma_{H1}}p^{\gamma_{H2}}\, \mathbb{I}(p > p_0)$$

and

$$f_L(p) = f_{L0} p^{\gamma_{L1}} (1-p)^{\gamma_{L2}},$$

where $\mathbb{I}$ is the indicator function.

The density functions are subject to the boundary conditions discussed below. (The derivations of these boundary conditions are given in the Appendix.)

First, once the posterior belief reaches the equilibrium separation point $\underline{p}$, we should have the cut-off condition

$$\Sigma_H(\underline{p}+)\, f_H(\underline{p}+) = \Sigma_L(\underline{p}-)\, f_L(\underline{p}-).$$

This condition guarantees that the flow speed of agents who cross $\underline{p}$ from below is equal to the flow speed of agents who cross from above. The implication is that since the speed from above Σ_H is larger than Σ_L, the densities are not continuous: $f_H(\underline{p}+) < f_L(\underline{p}-)$. It is worth comparing this condition to the standard condition when there is an absorbing state (Cox and Miller 1965; Dixit 1993; Moscarini 2005). In the case with only one Brownian motion and an absorbing state, what is required is $\Sigma(\underline{p}+)\, f(\underline{p}+) = 0$ because the probability of absorption in a time interval dt must equal the flow-in speed of the Brownian motion, which is proportional to $\sqrt{\mathrm{d}t}$ (see Cox and Miller 1965, p. 220).

Second, total flows in and out of the high-type firms must balance:

$$\Sigma_H(p_0)\ \left[f_H'(p_0-) - f_H'(p_0+)\right] = \delta\pi + \frac{\mathrm{d}}{\mathrm{d}p}\left[\Sigma_H(p)\, f_H(p)\right]\Big|_{\underline{p}+}.$$

The left-hand side of this equation is the total inflow into high-type firms, which are new workers who enter into this economy. The right-hand side of the equation is the total outflows from the high-type firms, which include workers who reach $\underline{p}$ and transfer to low-type firms, and workers who are hit by the death shock. We manage to show that this equation will further imply

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[\Sigma_L(p)\, f_L(p)\right]\Big|_{\underline{p}-} = \frac{\mathrm{d}}{\mathrm{d}p}\left[\Sigma_H(p)\, f_H(p)\right]\Big|_{\underline{p}+}.$$

Third, the density function has to be continuous at p_0:

$$f_H(p_0-) = f_H(p_0+).$$

It is customary to impose this condition as it approximates entry from a non-degenerate distribution instead of entry of identical types p_0.

Finally, usual market clearing conditions apply:

$$\int_{\underline{p}}^{1} f_H(p)\,\mathrm{d}p = \pi \quad \text{and} \quad \int_0^{\underline{p}} f_L(p)\,\mathrm{d}p = 1-\pi.$$

In summary, when $\underline{p} < p_0$, the equilibrium is characterized by a system of eight equations with nine unknowns $(V_L, V_H, k_L, k_H, \underline{p}, f_{H0}, f_{H1}, f_{H2}, f_{L0})$:[2]

(3) $$W_H(\underline{p}) = W_L(\underline{p})$$ (Value-matching condition),

(4) $$W_H'(\underline{p}) = W_L'(\underline{p})$$ (Smooth-pasting condition),

(5) $$W_H''(\underline{p}) = W_L''(\underline{p}) \qquad \text{(No-deviation condition).}$$

(6) $$\Sigma_H(\underline{p}+)\, f_H(\underline{p}+) = \Sigma_L(\underline{p}-)\, f_L(\underline{p}-) \qquad \text{(Boundary condition),}$$

(7) $$\int_{\underline{p}}^{1} f_H(p)\,\mathrm{d}p = \pi \qquad \text{(Market clearing } H\text{),}$$

(8) $$\int_{0}^{\underline{p}} f_L(p)\,\mathrm{d}p = 1-\pi \qquad \text{(Market clearing } L\text{),}$$

(9) $$\frac{\mathrm{d}}{\mathrm{d}p}\left[\Sigma_L(p)\, f_L(p)\right]\Big|_{\underline{p}-} = \frac{\mathrm{d}}{\mathrm{d}p}\left[\Sigma_H(p)\, f_H(p)\right]\Big|_{\underline{p}+} \qquad \text{(Flow equation at } \underline{p}\text{),}$$

(10) $$f_H(p_0-) = f_H(p_0+) \qquad \text{(Continuous density at } p_0\text{).}$$

Fortunately, equations (6)–(10) can be solved separately from equations (3)–(5). In other words, the procedure for solving this system of equations could be: first we solve for $\underline{p}$ jointly with $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ from equations (6)–(10), and then we plug $\underline{p}$ into equations (3)–(5) to pin down the other unknowns.

Proposition 1. Equations (6)–(10) imply, in equilibrium, $\underline{p} < p_0$ if and only if

(11) $$\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2}\, \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\,\mathrm{d}p}{\int_{0}^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p} < \frac{\pi}{1-\pi}.$$

Moreover, if such a $\underline{p}$ exists, then it must be unique.

Proof. See the Appendix. □

The proof of Proposition 1 is quite straightforward. The idea of the proof is as follows. Since we have five equations with five unknowns, we can first express $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ as functions of $\underline{p}$, and then use the last equation to pin down $\underline{p}$.

The existence and uniqueness of the solution to the system require that $f_{H0}, f_{H1}, f_{H2}, f_{L0}$ change monotonically with $\underline{p}$. Fortunately, this is the case, as shown in the Appendix. The monotonicity guarantees that if a solution exists, it must be unique. Furthermore, it enables us to check only the boundaries when determining whether a solution exists. Equation (11) given in the proposition is thus derived.

The other case to consider is $\underline{p} \geq p_0$. Given $p_0 \in (0,1)$, $\underline{p} \geq p_0$ implies that the density functions are

$$f_L(p) = f_{L0}p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathbb{I}(p<p_0) + [f_{L1}p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} + f_{L2}(1-p)^{\gamma_{L1}}p^{\gamma_{L2}}]\,\mathbb{I}(p_0 \leq p \leq \underline{p})$$

and

$$f_H(p) = f_{H0}(1-p)^{\gamma_{H1}}p^{\gamma_{H2}}.$$

Then we obtain a system of equilibrium equations exactly the same as equations (3)–(9), while equation (10) becomes $f_L(p_0-) = f_L(p_0+)$. Based on the equilibrium equations, we can prove the following proposition, the counterpart to Proposition 1, in a similar fashion.

Proposition 2. In equilibrium, $\underline{p} \geq p_0$ if and only if

$$\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\,\mathrm{d}p}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p} \geq \frac{\pi}{1-\pi}. \tag{12}$$

Moreover, if such a $\underline{p}$ exists, then it must be unique.

Proof. See the Appendix. □

The idea for the proof of Proposition 2 is exactly the same as that for the proof of Proposition 1. Propositions 1 and 2 together provide the following existence and uniqueness result.

Theorem 2. Under strict supermodularity, for any pair $(p_0, \pi) \in (0,1)^2$, there exists a unique PAM cut-off $\underline{p}$. Moreover, $\underline{p} < p_0$ if and only if

$$\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\,\mathrm{d}p}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p} < \frac{\pi}{1-\pi}. \tag{13}$$

One of the nice properties of equation (13) is that the whole equation depends only on p_0, π, δ/s_H^2 and δ/s_L^2. This provides a feasible way to compute $\underline{p}$. Given p_0, π, δ/s_H^2 and δ/s_L^2, we first need to decide the sign of

$$\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{L2}} \frac{\delta/s_H^2}{\delta/s_L^2} \frac{\int_{p_0}^{1} p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\,\mathrm{d}p}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p} - \frac{\pi}{1-\pi}.$$

If the sign is negative, then we know that $\underline{p}$ is smaller than p_0, and we can use the system of equations in the case $\underline{p} < p_0$ to figure out $\underline{p}$. On the contrary, if the sign is not negative, then we know that $\underline{p}$ is larger than p_0, and we can use the system of equations in the case $\underline{p} \geq p_0$ to compute $\underline{p}$. This turns out to be a convenient way to determine the equilibrium cut-off numerically.

Before presenting the numerical results, we have a simple theoretical comparative static result.

Corollary 1. $\underline{p}$ is strictly increasing in p_0, and decreasing in π.

Proof. See the Appendix. □

The intuition for the proof of this corollary is quite straightforward. Decreasing in π means that there are more low-type firms in the economy, hence $\underline{p}$ must become larger so that more workers are matched with low-type firms. Increasing in p_0 means that the overall quality of the workers is becoming better in the economy, and $\underline{p}$ must increase to make sure that low-type firms are also matched with better workers.

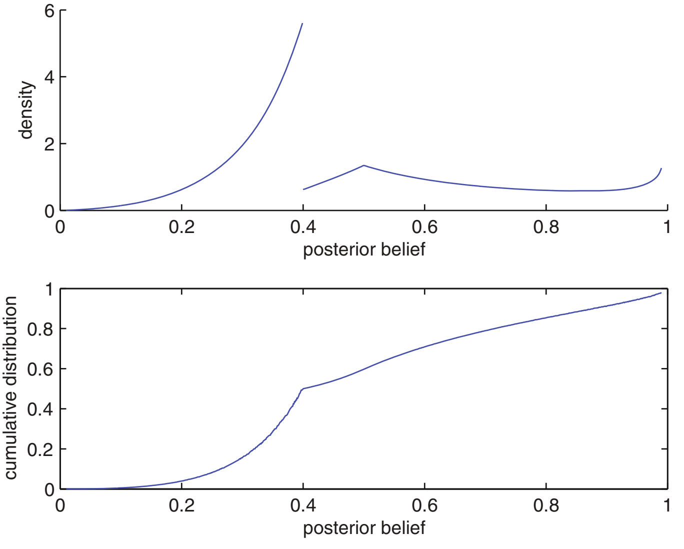

FIGURE 1. Equilibrium distribution of posterior beliefs

Mathematically, it is not easy to derive comparative statics between $\underline{p}$ and δ/s_H^2 or δ/s_L^2. But intuitively speaking, as s_L increases, the degree of supermodularity will be reduced while the speed of learning in low-type firms will increase. Both of these factors make the low-type firms more attractive, hence $\underline{p}$ should increase in s_L. On the other hand, as s_H becomes larger, both the degree of supermodularity and the speed of learning in high-type firms will increase, which will lead to a reduction in $\underline{p}$.

Figure 1 plots the stationary distribution of beliefs p for the case of PAM and with parameter values $s_H = 0.15$, $s_L = 0.05$, $p_0 = 0.5$, $\pi = 0.5$, $\delta = 0.01$.

Equilibrium analysis: value functions

Theorem 2 implies that under strict supermodularity, the PAM cut-off $\underline{p}$ can be determined uniquely. But given this $\underline{p}$, we still have the following conditions (originally equations (3)–(5)) to satisfy:

(14) $$W_H(\underline{p}) = W_L(\underline{p}) \quad \text{(Value-matching condition)},$$

(15) $$W'_H(\underline{p}) = W'_L(\underline{p}) \quad \text{(Smooth-pasting condition)},$$

(16) $$W''_H(\underline{p}) = W''_L(\underline{p}) \quad \text{(No-deviation condition)}.$$

Equations (14)–(16) are three equations for four unknowns. The equilibrium is indeterminate in the sense that although the allocation $\underline{p}$ is unique, there could be multiple ways to divide the surplus.[23] To make the system determinate, we assume firm monotonicity and set $\mu_{LL} = 0$. Then limited liability requires that $w_L(0)$ must be zero and hence $V_L = 0$. Equations (14)–(16) thus could be written as

$$\frac{\mu_L(\underline{p})}{r+\delta} + k_L \underline{p}^{\alpha_L}(1-\underline{p})^{1-\alpha_L} = \frac{\mu_H(\underline{p}) - rV_H}{r+\delta} + k_H \underline{p}^{1-\alpha_H}(1-\underline{p})^{\alpha_H},$$

$$\frac{\mu_{HL}-\mu_{LL}}{r+\delta} + k_L \underline{p}^{\alpha_L}(1-\underline{p})^{1-\alpha_L}\ \frac{\alpha_L - \underline{p}}{\underline{p}(1-\underline{p})} = \frac{\mu_{HH}-\mu_{LH}}{r+\delta} + k_H \underline{p}^{1-\alpha_H}(1-\underline{p})^{\alpha_H}\ \frac{1-\alpha_H-\underline{p}}{\underline{p}(1-\underline{p})},$$

$$k_L \underline{p}^{\alpha_L - 2}(1-\underline{p})^{-1-\alpha_L}\alpha_L(\alpha_L - 1) = k_H \underline{p}^{-1-\alpha_H}(1-\underline{p})^{\alpha_H - 2}\alpha_H(\alpha_H - 1).$$

This system of equations will give us a unique formula for V_H:

$$rV_H = (\mu_{LH} - \mu_{LL}) + \frac{\alpha_H(\alpha_L - 1)(\Delta_H - \Delta_L)\underline{p}}{\alpha_H(\alpha_L - 1) - (1-\underline{p})(\alpha_L - \alpha_H)}. \tag{17}$$

As usual, $\Delta_H = \mu_{HH} - \mu_{LH}$ and $\Delta_L = \mu_{HL} - \mu_{LL}$. Furthermore, it is easy to check that both k_H and k_L are strictly larger than zero so that the option value of learning is strictly positive.

Therefore we finally reach our main result.

Theorem 3. Under strict supermodularity, the stationary competitive equilibrium is unique in the sense that all equilibria are PAM and the allocation is uniquely determined by Theorem 2. Moreover, assuming firm monotonicity and normalizing $V_L = 0$, we obtain a unique formula for V_H, as given by equation (17).

Wage gap at $\underline{p}$

The analysis of the value functions allows us to determine equilibrium wages. We start with an interesting observation:

$$\begin{aligned} w_H(\underline{p}) = \mu_H(\underline{p}) - rV_H &= \Delta_H\underline{p} + \mu_{LL} - \frac{\alpha_H(\alpha_L - 1)(\Delta_H - \Delta_L)\underline{p}}{\alpha_H(\alpha_L - 1) - (1-\underline{p})(\alpha_L - \alpha_H)} \\ &< \Delta_L\underline{p} + \mu_{LL} = w_L(\underline{p}). \end{aligned}$$

This implies that the worker with posterior belief slightly higher than $\underline{p}$ will accept the high firm's offer even though the wage provided is lower than the wage at the low firm. This obviously comes from the fact that the learning speed in the high firm is higher, and this would compensate the loss in the flow wages.

On the other hand, we can see that the difference in expected productivity at $\underline{p}$ is

$$\mu_H(\underline{p}) - \mu_L(\underline{p}) = (\mu_{HL} - \mu_{LL}) + (\Delta_H - \Delta_L)\underline{p} < rV_H.$$

This implies that the high firm can enjoy a strictly positive rent from a higher learning speed, as illustrated by Figure 2. This result actually does not depend on the assumption $V_L = 0$, and it can be generalized for any possible division of surplus,[24] as shown by Lemma 7.

Lemma 7. Under strict supermodularity, we have $w_H(\underline{p}) < w_L(\underline{p})$ and $rV_H - rV_L > \mu_H(\underline{p}) - \mu_L(\underline{p})$.

Proof. See the Appendix. □

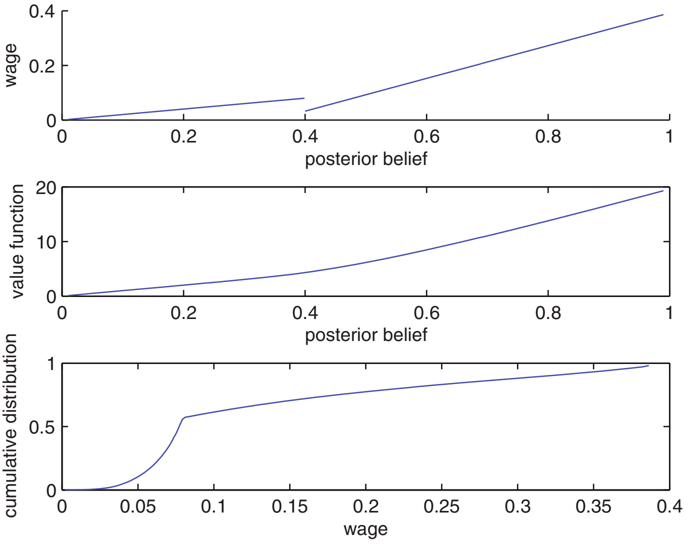

FIGURE 2. Equilibrium wage function and value function in terms of beliefs p; stationary wage distribution

IV. DISCUSSION

Firm-dependent volatility: σ_y

A valid criticism of our approach is that we give the H firms too much of an edge under supermodularity (likewise for the L firms under submodularity). Not only are they superior in the production of output, by assuming that the volatility σ is common to both types of firms, effectively the signal-to-noise ratio is higher in H firms:

$$s_H = \frac{\mu_{HH} - \mu_{LH}}{\sigma} > \frac{\mu_{HL} - \mu_{LL}}{\sigma} = s_L,$$

from supermodularity. With firm-dependent volatility, that need not be the case. In particular, for σ_H sufficiently high, it may well be the case that $s_H < s_L$.

Mere observation of the value function in equation (1), $rW_y(p) = \mu_y(p) - rV_y + \Sigma_y(p)\, W_y''(p) - \delta\, W_y(p)$, reveals that firm-dependent volatility will play a crucial role here. Since $\Sigma_y = \frac{1}{2}p^2(1-p)^2 s_y^2$ for sufficiently high σ_H and therefore low s_H, it appears intuitive that the value of W_H can be smaller than the value of W_L for high p. It turns out that this intuition is wrong. First, in this competitive equilibrium, wages are endogenous, and therefore as the value of learning changes, so does $\mu_y(p) - rV_y$. Second, the no-deviation condition requires that at the marginal type $\underline{p}$, $W_H' = W_L'$. It turns out that as a result these two features, in equilibrium the learning effect is the same in both firms, no matter what the volatility σ_y is.

To make this argument formal, when $\sigma_H \neq \sigma_L$, we generally define $s_y = (\mu_{Hy} - \mu_{Ly})/\sigma_y$, $y \in \{H, L\}$. It is trivial to show that belief updating also satisfies the formula

$$\mathrm{d}p_t = p_t(1-p_t)s_y\, \mathrm{d}\overline{Z}_{y,t}.$$

Furthermore, Lemmas 2–5 still hold because none of these results depends explicitly on σ_y. As shown in the Appendix, the statement in Claim 2 is generalized to *any* combination of (σ_H, σ_L).[25]

With the proof of Claim 2 in hand, the result of Theorem 1 immediately extends: the statement 'PAM (NAM) is the unique stationary competitive equilibrium allocation under strict supermodularity (submodularity)' thus holds for *any* combination of (σ_H, σ_L). Surprisingly, this implies that under strict supermodularity, even if we have an extremely high σ_H such that the learning rate in high-type firms is smaller than that in low-type firms, we still have PAM. It is equivalent to assert that the direct productivity consideration dominates the learning in our model. The reason for this comes from the fact that the equilibrium wage schedules adjust to offset the impact of change in learning rate. The key insight here is the no-deviation condition. At $\underline{p}$, the no-deviation condition requires that the second-order effect on the value function is the same in both firms. This second-order effect W_y' exactly captures the effect of learning through $\Sigma_y(\underline{p})\ W_y'(\underline{p})$, where $\Sigma_y = \frac{1}{2}p^2(1-p)^2 s_y^2$. Because equilibrium wages adjust to satisfy the no-deviation condition at the cut-off, the impact of differential learning rates is completely offset by the change of wage schedule, and the equilibrium allocation is determined solely by the productivity consideration.

The commitment problem

A priori, we might not expect the competitive equilibrium to be efficient as the spot wage contracts cannot condition on future realizations or actions, and these contracts are assumed to be self-enforcing. As a result of this lack of commitment, there is a missing market. In this subsection, we introduce commitment into our model. As we show below, it turns out that the decentralized equilibrium without commitment is efficient in the sense that introducing commitment does not change the equilibrium allocation.

A model with commitment When we allow firms to commit to a contract, a potential problem is that different types of firms may make conflicting commitments, which causes intricate strategic interactions. To avoid this problem, we normalize $V_L = 0$ as in Theorem 3 and hence focus on the problem where the high-type firms can commit to maximizing profits contingent on future realizations. This can happen when the low-type firms are fully competitive.

Under the above assumption, the high-type firms need to commit only to an acceptance region $\Omega_H \subset [0, 1]$ as the wages can be set as $W_H(p) = W_L(p) = \mu_L(p)$ to make the workers indifferent. The optimal acceptance region obviously depends on the distribution of beliefs, hence solving the optimal problem with commitment is a complicated non-stationary problem. As in the non-commitment model, we can focus on the optimal solution under stationarity, that is, in the presence of an ergodic distribution. This can also be viewed as the limit of optimal commitment as the distribution of beliefs becomes stationary.

Formally, denote the ergodic density of beliefs associated with the acceptance region Ω_H as $f_H(p)$. The ergodic density has to satisfy the Kolmogorov forward equation

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2}\left[\Sigma_H(p)\, f_H(p)\right] - \delta\, f_H(p) = \frac{\mathrm{d}f_H(p)}{\mathrm{d}t} = 0$$

and the market clearing condition

$$\int_{\Omega_H} f_H(p)\,\mathrm{d}p = \pi.$$

The optimal Ω_H maximizes the high-type firms' stationary profit

$$\int_{\Omega_H} \left[\mu_H(p) - \mu_L(p)\right] f_H(p)\,\mathrm{d}p$$

subject to the above constraints.

The planner's problem It is not straightforward to compare directly the above model with the competitive equilibrium. As an intermediate step, we introduce a planner's problem under stationarity.[26] The planner chooses an allocation rule, and as a consequence of the Kolmogorov forward equation, the ergodic distribution associated with this allocation rule. The objective is to maximize the aggregate flow of output. Given the stationarity of the problem, the focus on output maximization yields the same outcome as maximization of aggregate values.

Consider any allocation with multiple cut-offs:

$$0 < \underline{p}_N < \cdots < \underline{p}_1 < 1, \quad N \text{ odd.}$$

Without loss of generality, we assume that workers with $p \in (p_1, 1]$ are allocated to the high-type firms while workers with $p \in [0, p_N)$ are allocated to the low-type firms, since for workers with $p = 0$ or $p = 1$, there is no need for learning and it is optimal to allocate them according to instantaneous production efficiency (PAM).[27] This also implies that generically, N is odd.

Denote again by Ω_y the set of p values that match with firms of type y. Similar to the problem with commitment, the planner will choose Ω_y to solve

$$\max_{\Omega_y} S = \int_{\Omega_H} \mu_H(p)\, f_H(p)\, \mathrm{d}p + \int_{\Omega_L} \mu_L(p)\, f_L(p)\, \mathrm{d}p$$

subject to

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2}\left[\Sigma_y(p)\, f_y(p)\right] - \delta\, f_y(p) = \frac{\mathrm{d}f_y(p)}{\mathrm{d}t} = 0 \qquad \text{(Kolmogorov forward equation)},$$

$$\int_{\Omega_H} p\, f_H(p)\, \mathrm{d}p + \int_{\Omega_L} p\, f_L(p)\, \mathrm{d}p = p_0 \qquad \text{(Martingale property)},$$

$$\int_{\Omega_L} f_L(p)\, \mathrm{d}p = 1 - \pi, \quad \int_{\Omega_H} f_H(p)\, \mathrm{d}p = \pi \qquad \text{(Market clearing)}.$$

Notice that the main difference between the problem with commitment and the planner's problem is that only the ergodic density $f_H(p)$ enters into the former problem, while both $f_H(p)$ and $f_L(p)$ enter into the latter problem. As a result, we also need to include the martingale property into the planner's constraint.

Efficiency of the competitive equilibrium We first show that the optimal allocation in the planner's problem also solves the problem with commitment. It turns out that the martingale property plays an important role in establishing this result, hence the following lemma.

Lemma 8. Denote the optimal allocation in the planner's problem as $(\widetilde{\Omega}_H, \widetilde{\Omega}_L)$. Then $\widetilde{\Omega}_H$ is also the optimal acceptance region in the problem with commitment.

Proof. See the Appendix. □

Our next theorem shows that the competitive equilibrium allocation solves the planner's problem, and by Lemma 8, we conclude that introducing commitment does not change the equilibrium allocation.

Theorem 4. The competitive equilibrium decentralizes the planner's stationary solution that maximizes the aggregate flow of output.

Proof. See the Appendix. □

To prove that the competitive equilibrium decentralizes the planner's stationary solution under supermodularity, it suffices to show that the PAM allocation is better than *any* allocation with multiple cut-offs because from Theorem 2, we know that the PAM allocation is unique and will be the same as the competitive equilibrium allocation for any combination of (s_H, s_L). The key technical issue is that the ergodic distribution is determined endogenously by the allocation rule. It is infeasible to compute the ergodic density functions for each possible allocation. Our strategy of proof is therefore to use a variational argument to circumvent this difficulty.

The proof uses the martingale property heavily and works as follows. First, we consider a candidate allocation with three cut-offs. Under this candidate allocation, there will be an interior interval of p values that are matched to L firms associated with some ergodic distribution. We move the bounds of that interval slightly to the left, thus generating a new density in this interval while keeping all other cut-offs and distributions unchanged. The new interval is chosen by imposing market clearing conditions. Lemma 8 then shows that under supermodularity, this experiment strictly increases aggregate output. This holds until cut-offs coincide such that the interior range of p values matched with L firms disappears, thus reducing the number of cut-offs to $N = 1$. We use a similar argument to establish that output increases when moving from N to $N - 2$ cut-offs. The result then follows by induction. We derive the result under supermodularity. The same logic applies under submodularity.

V. On-the-job Human Capital Accumulation

On the job, workers and firms not only learn about their unknown innate skills, they also accumulate human capital. In reality, human capital accumulation is an ongoing, continuous process. The longer the tenure of a worker, the higher her productivity. This monotonically increasing relation between tenure and human capital experience is likely also to be concave. For modelling purposes, here we consider a very simple form that captures this relation. With probability λ, a worker transitions from being inexperienced to being experienced.[28] Once a worker is experienced, her productivity increases to $\mu_{xy} + \xi_x$ and the status of experience is complete information.[29] Now there are the same value functions for experienced workers as before, W_y^e:

$$rW_y^e(p) = \mu_y(p) + \xi(p) - rV_y + \Sigma_y^e(p)\, W_y^{e''}(p) - \delta W_y^e(p), \tag{18}$$

where $\xi(p) = p\xi_H + (1-p)\xi_L$ is the expected experience.[30] For the inexperienced worker, there is now one additional value function. As before, there are inexperienced workers who are matched with L firms, and who continue to match with L firms; and there are those who match with H firms both when inexperienced as well as when experienced. We denote those values by W_{LL}^u, W_{HH}^u. There are now also some types p who match with an L firm when inexperienced and who switch to an H firm when they become experienced, the value of which is denoted by W_{LH}^u. This requires that the reservation type of an experienced worker ($\underline{p}^e$) is lower than that of the inexperienced worker ($\underline{p}^u$). We start from this premise and later verify that this is indeed the case. The value functions are then

$$(19) \qquad rW^u_{yy}(p) = \mu_y(p) - rV_y + \Sigma^u_y(p)\, W^{u''}_{yy}(p) + \lambda W^e_y(p) - (\delta + \lambda)\, W^u_{yy}(p),$$

$$(20) \qquad rW^u_{LH}(p) = \mu_L(p) - rV_L + \Sigma^u_L(p)\, W^{u''}_{LH}(p) + \lambda W^e_H(p) - (\delta + \lambda)\, W^u_{LH}(p).$$

Observe that even though experience is completely observable, it does affect the inference from learning in the sense that the signal-to-noise ratio changes to $\left[(\mu_{Hy} + \xi_H - \mu_{Ly} - \xi_L)\right]/\sigma^2$. As a result, Σ_y depends on experience, u, e.

There are now two cut-offs, $\underline{p}^u, \underline{p}^e$. Since we just want to compare $\underline{p}^u$ and $\underline{p}^e$, we can consider the following thought experiment. First, we assume that $\underline{p}^u = \underline{p}^e = \underline{p}$. Then we can obtain two systems of equations: one system is the set of value-matching, smooth-pasting and no-deviation conditions for the inexperienced workers, and the other is for the experienced workers. Second, we can solve $\Delta V = V_H - V_L$ the way we did previously, but now we can obtain two possible values for ΔV. Denote them as ΔV^e and ΔV^u. Notice that ΔV^e and ΔV^u are both increasing in the cut-off $\underline{p}$. Finally, we compare ΔV^e and ΔV^u under the assumption that $\underline{p}^u = \underline{p}^e = \underline{p}$. If $\Delta V^e > \Delta V^u$, then we should decrease $\underline{p}^e$ or increase $\underline{p}^u$, hence $\underline{p}^u > \underline{p}^e$; on the contrary, if $\Delta V^e < \Delta V^u$, then we should decrease $\underline{p}^u$ or increase $\underline{p}^e$, hence $\underline{p}^u < \underline{p}^e$. We derive this in the Appendix and can show it to hold when human capital accumulation is not too different for H and L types.

Proposition 3. Assume supermodularity and $\xi_H \simeq \xi_L$. Then $\underline{p}^e < \underline{p}^u$.

Proof. See the Appendix. □

With human capital accumulation, we can now characterize the entire equilibrium, including wage schedules and the ergodic distribution of types. Even though there are types who gradually learn that they are of low productivity, wages need not decrease over the lifecycle as they accumulate human capital.

Turnover and tenure

We express the expected future duration of a match by tenure $\tau_y(p)$. Tenure relates inversely to turnover. For $p < \underline{p}^e$ and $p > \underline{p}^u$, $\tau_y(p)$ satisfies the differential equation (see also Moscarini 2005)

$$\Sigma_y(p)\, \tau''_y(p) - \delta\, \tau_y(p) = -1.$$

If $p \in (\underline{p}^e, \underline{p}^u)$, then the only difference is that

$$\Sigma_y(p)\, \tau^{u''}_L(p) - (\delta + \lambda)\, \tau^u_L(p) = -1,$$

since inexperienced workers will switch jobs once they become experienced. An immediate implication of Proposition 3 is the following.

Corollary 2 (Tenure). Assume supermodularity and $\xi_H \simeq \xi_L$. Then $\tau^u_L(p) > \tau^e_L(p)$ for $p < \underline{p}^e$, and $\tau^u_H(p) < \tau^e_H(p)$ for $p > \underline{p}^u$. For $p \in (\underline{p}^e, \underline{p}^u)$, there is a cut-off such that $\tau^u_L(p) < \tau^e_H(p)$ for p larger than this cut-off, and $\tau^u_L(p) > \tau^e_H(p)$ for p smaller than this cut-off.

For the lowest types p, tenure for the inexperienced worker is longer as the experienced workers are more likely to be hired by an H firm given positive information revelation. The opposite is true for the highest p: the inexperienced types face a higher cut-off type and will therefore, upon bad information, be more likely to switch to an L firm. In the intermediate range, tenure depends on how close p is to either of the cut-offs.

Non-Bayesian updating

So far, we have assumed that the belief updating follows a martingale process, a property satisfied by Bayesian learning. We will now show, as an example, that under some non-Bayesian learning process, the competitive equilibrium can be non-PAM even if there is supermodularity.

In this example, we abandon the assumption of experienced and inexperienced workers, and just consider the benchmark model where the belief updating is not a martingale. Suppose that the belief updating process in firm y is given by $\mathrm{d}p = \lambda_y p\,\mathrm{d}t$ for $p < 1$, with λ_y a constant, and once p reaches 1, $\mathrm{d}p = 0$. We may think of p as a particular form of human capital, with 1 as an upper bound on the accumulation. The value function of a worker is given by[31]

$$(r+\delta)\; W_y(p) = w_y(p) + \lambda_y p W_y'(p). \tag{21}$$

Suppose that PAM is the equilibrium allocation (i.e. there exists a cut-off $\underline{p}$ such that workers with $p \in [\underline{p}, 1]$ are matched with the high-type firms, and workers with $p \in [0, \underline{p})$ are matched with the low-type firms). Then the value functions should satisfy the following.

(i) Continuity at $p = 1$:

$$\lim_{p\to 1} W_H(p) = W_H(1) = \frac{\Delta_H}{r+\delta} + \frac{\mu_{LH} - rV_H}{r+\delta}.$$

(ii) Boundary conditions at $\underline{p}$:

$$W_H(\underline{p}) = W_L(\underline{p}) \qquad \text{(Value-matching condition)},$$
$$W_H'(\underline{p}) = W_L'(\underline{p}) \qquad \text{(Smooth-pasting \& No-deviation conditions)}.$$

(iii) The desirability of the low-type firm for $p = 0$ workers:

$$\frac{\mu_{LL} - rV_L}{r+\delta} > \frac{\mu_{LH} - rV_H}{r+\delta}.$$

Notice that for this belief-updating process, the no-deviation condition coincides with the smooth-pasting condition. This is consistent with the finding in Eeckhout and Weng (2015), and we also derive the no-deviation condition in the Appendix.

It turns out that when $\lambda_L > \lambda_H$, $r + \delta > \lambda_H$ and the degree of supermodularity is sufficiently small, the above conditions cannot be satisfied simultaneously. As a result, we reach the following proposition.

Proposition 4. Under the non-Bayesian learning model, PAM cannot be an equilibrium allocation when $\lambda_L > \lambda_H$, $r + \delta > \lambda_H$ and the degree of supermodularity is sufficiently small.

Proof. See the Appendix. □

Proposition 4 implies that the martingale property of Bayesian learning plays an important role in delivering our main results. Under Bayesian learning, the beliefs can go either up or down, and on average there is no drift in the belief updating process. Therefore productive efficiency always dominates learning speed in the sense that PAM is the unique equilibrium allocation under strict supermodularity, no matter what the learning speed is. In the above non-Bayesian learning case, the drift in the belief updating process introduces a trade-off between instantaneous productive efficiency and the dynamic gain from learning when $\lambda_L > \lambda_H$. If the workers are patient enough ($r + \delta > \lambda_H$) and the degree of supermodularity is sufficiently small, then workers will sacrifice instantaneous productive efficiency for the dynamic gain from learning. By looking at the equilibrium conditions above, we conclude immediately that the equilibrium allocation in this case should be that workers with belief close to 0 or 1 are matched with the high-type firms, while workers with intermediate beliefs are matched with the low-type firms. This is because the gain in learning speed is close to 0 at $p = 0$ ($(\lambda_L - \lambda_H)p \simeq 0$) or $p = 1$ (upper bound of belief updating).

VI. Concluding Remarks

In this paper, we have proposed a competitive equilibrium model of the labour market that unifies frictionless sorting and a learning-based theory of turnover. In equilibrium under supermodularity, workers with better posteriors about their ability tend to sort into more productive jobs. We find that as a result of sequential rationality in the presence of competitively determined payoffs, the second derivative of the workers' value function must equate. In addition to the standard conditions of value-matching (zeroth derivative) and smooth-pasting (first derivative), we now also have the no-deviation condition (second derivative).

What is possibly most surprising is that the result of positive sorting under supermodularity is not determined by the speed of learning. In the trade-off between learning speed and instantaneous productive efficiency, productive efficiency always takes the upper hand. As such, the equilibrium allocation does not depend on the signal-to-noise ratio (the ratio of the average payoff gain, which measures the efficiency, to the noise term). This seems to indicate that in this competitive environment, the sorting aspect dominates the learning. Quite surprisingly, this sorting result does not hinge on the particular information structure and is robust to general Bayesian learning processes.

Our analysis has certain limitations, and several issues remain unanswered.

First, like most experimentation models, payoffs are linear and agents are risk-neutral. Non-linearity is desirable for the economic interpretation. However, it renders the solution of the differential equation of the value function much harder to find.

Second, ideally we would like to extend the analysis to general distributions of worker and firm types. As in much of the experimentation literature, the realized type is either high or low on a risky arm. Here, in addition we have *two* risky arms that are correlated, since there is learning in both types of firms. The focus on the two-firm-type case (two arms) keeps down the dimensionality of the continuous time problem. With more than two firm types, analysing the Brownian motion process is mathematically substantially more demanding. Finally, our result that PAM obtains under supermodularity, and that the planner's problem can be decentralized, is established for a stationary equilibrium. While a solution of a general non-stationary equilibrium is too complex, one can easily construct a two-period counterexample in which PAM will not necessarily obtain in a non-stationary environment.

APPENDIX

Proof of Lemma 2

The worker $p \in (0,1)$ always has the choice to stay in one firm y forever. Then

$$p = \frac{\mu_y(p) - rV_y}{r+\delta}.$$

But obviously, this is not an optimal choice. (Suppose otherwise; then all of the workers will stay in one type of firm, and the market is not cleared.) So we have that the equilibrium value function $W_y(p)$ must satisfy

$$W_y(p) > \frac{\mu_y(p) - rV_y}{r+\delta}.$$

This immediately implies that

$$\Sigma_y(p)\; W_y''(p) = (r+\delta)\; W_y(p) - (\mu_y(p) - rV_y) > 0.$$

So the equilibrium value function $W_y(p)$ is convex for $p \in (0,1)$.

Proof of Lemma 3

Suppose that workers with $p \in [0, \underline{p})$ are employed by a type y firm. This implies that

$$W_y(p) = \frac{\mu_y(p) - rV_y}{r+\delta} + k_{y2} p^{\alpha_y}(1-p)^{1-\alpha_y},$$

since 0 is included in the domain. It is easy to see that

$$W_y'(0) = \frac{\mu_{Hy} - \mu_{Ly}}{r+\delta} > 0,$$

and since W_y is strictly convex, $W_y'(p) > 0$ for all $p \in [0, \underline{p})$. At $\underline{p}$, workers will transfer to a type $-y$ firm, but the smooth-pasting condition implies that $W_{-y}'(\underline{p}) = W_y'(\underline{p}) > 0$. Strict convexity implies $W_{y'}'(p) > 0$, and so on. Therefore we must have that the equilibrium value functions W_y are strictly increasing.

Proof of Claim 2

We will actually prove a more general claim, that the result holds for any combination (s_H, s_L), including $s_H < s_L$. This makes the proof also applicable to the case $\sigma_H \neq \sigma_L$.

Under strict supermodularity, for any combination (s_H, s_L), it is impossible to have $p_1 < p_2$ and equilibrium value functions W_H (for $p \in [p_1, p_2]$), W_{L1} (for $p < p_1$) and W_{L2} (for $p > p_2$) such that

$$W_H(p_1) = W_{L1}(p_1) \quad \text{and} \quad W_H''(p_1) = W_{L1}''(p_1),$$
$$W_H(p_2) = W_{L2}(p_2) \quad \text{and} \quad W_H''(p_2) = W_{L2}''(p_2),$$

are satisfied simultaneously.

Suppose, on the contrary, that the equations described above hold simultaneously. Then from equation (1), we should get

$$w_H(p_1) + \Sigma_H(p_1)\; W_H''(p_1) = w_L(p_1) + \Sigma_L(p_1)\; W_{L1}''(p_1)$$

and

$$w_H(p_2) + \Sigma_H(p_2)\ W_H''(p_2) = w_L(p_2) + \Sigma_L(p_2)\ W_{L2}''(p_2),$$

since

$$W_H(p_2) = W_{L2}(p_2) \quad \text{and} \quad W_H(p_1) = W_{L1}(p_1).$$

Notice that

$$W_H''(p_2) = W_{L2}''(p_2) \quad \text{and} \quad W_H''(p_1) = W_{L1}''(p_1),$$

by Lemma 5, and hence

$$\text{(A1)} \qquad \frac{\Sigma_H(p_1) - \Sigma_L(p_1)}{\Sigma_H(p_1)}\ (r+\delta)\ W_H(p_1) = w_L(p_1) - \frac{\Sigma_L(p_1)}{\Sigma_H(p_1)}\ w_H(p_1)$$

and

$$\text{(A2)} \qquad \frac{\Sigma_H(p_2) - \Sigma_L(p_2)}{\Sigma_H(p_2)}\ (r+\delta)\ W_H(p_2) = w_L(p_2) - \frac{\Sigma_L(p_2)}{\Sigma_H(p_2)}\ w_H(p_2).$$

By definition,

$$\frac{\Sigma_H(p_1) - \Sigma_L(p_1)}{\Sigma_H(p_1)} = \frac{\Sigma_H(p_2) - \Sigma_L(p_2)}{\Sigma_H(p_2)} = \frac{s_H^2 - s_L^2}{s_H^2}.$$

First, if $s_H^2 = s_L^2$, then equations (A1) and (A2) imply that $w_H(p_1) - w_L(p_1) = w_H(p_2) - w_L(p_2) = 0$, which cannot hold for $p_1 \neq p_2$ since $w_H(\cdot)$ and $w_L(\cdot)$ are linear functions with different slopes Δ_H and Δ_L.

Second, if $s_H^2 > s_L^2$, then equations (A1) and (A2) could be simplified as

$$\text{(A3)} \qquad \frac{s_H^2 - s_L^2}{s_H^2}\ (r+\delta)\left(W_H(p_2) - W_H(p_1)\right) = w_L(p_2) - w_L(p_1) - \frac{\Sigma_L(p_2)}{\Sigma_H(p_2)}\ \left(w_H(p_2) - w_H(p_1)\right).$$

Under strict supermodularity, the left-hand side of this equation is strictly larger than

$$\frac{s_H^2 - s_L^2}{s_H^2}\ (r+\delta)\ W_H'(p_1)\ (p_2 - p_1)$$

by the convexity of the value function. And

$$\frac{s_H^2 - s_L^2}{s_H^2}\ (r+\delta)\ W_H'(p_1)\ (p_2 - p_1) \quad \text{is larger than} \quad \frac{s_H^2 - s_L^2}{s_H^2}\ \Delta_L(p_2 - p_1)$$

by Lemma 4. Meanwhile, the right-hand side of equation (A3) is strictly smaller than

$$\Delta_L(p_2 - p_1) - \frac{\Sigma_L(p_2)}{\Sigma_H(p_2)}\ \Delta_H(p_2 - p_1)) = \frac{s_H^2 - s_L^2}{s_H^2}\ \Delta_L(p_2 - p_1),$$

which contradicts the fact that the two sides of equation (A3) should be equal. The impossibility in the $s_H^2 < s_L^2$ case could be proved similarly, and is thus omitted. By contradiction, we immediately know that the claim at the beginning of the proof is correct.

For the strict submodularity case, it suffices to re-label 'H' by 'L', and 'L' by 'H'. The claim is obviously correct given that we have already proved the strict supermodularity result.

Proof of Lemma 6

We will actually prove a more general lemma, that the result holds for any combination (s_H, s_L), including $s_H < s_L$. This makes the proof also applicable to the case $\sigma_H \neq \sigma_L$.

First, we want to show that all of the one-shot deviations are ruled out by our no-deviation condition as $\mathrm{d}t \to 0$.

Under strict supermodularity, PAM is the only candidate equilibrium allocation by Theorem 1. The value functions are thus given by

$$W_L(p) = \frac{w_L(p)}{r+\delta} + k_L p^{\alpha_L}(1-p)^{1-\alpha_L}$$

and

$$W_H(p) = \frac{w_H(p)}{r+\delta} + k_H p^{1-\alpha_H}(1-p)^{\alpha_H}.$$

Let

$$\mathcal{G}_L(p) = k_L p^{\alpha_L}(1-p)^{1-\alpha_L}\ \frac{\alpha_L - p}{p(1-p)} > 0$$

and

$$\mathcal{G}_H(p) = k_H p^{1-\alpha_H}(1-p)^{\alpha_H}\ \frac{1-\alpha_H - p}{p(1-p)} < 0$$

be the first derivatives for the non-linear parts of the value functions. Smooth pasting at $\underline{p}$ implies

$$\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p}) = \frac{\Delta_H}{r+\delta} + \mathcal{G}_H(\underline{p}).$$

From Eeckhout and Weng (2015), it suffices to show that the inequality

$$w_H(p) - w_L(p) + \left[\Sigma_H(p) - \Sigma_L(p)\right]\ W_L''(p) < 0$$

holds for any $p < \underline{p}$, and the inequality

$$w_L(p) - w_H(p) + \left[\Sigma_L(p) - \Sigma_H(p)\right]\ W_H''(p) < 0$$

holds for $p > \underline{p}$.

For $p < \underline{p}$, define

$$Z_L(p) = w_H(p) - w_L(p) + \frac{s_H^2 - s_L^2}{s_L^2}\ \left((r+\delta)\ W_L(p) - w_L(p)\right).$$

Obviously, we have $\lim_{p \nearrow \underline{p}} Z_L(p) = 0$ from Lemma 5. If we can show that $Z_L(p)$ is increasing in p as p increases from 0 to $\underline{p}$, then we are done since $Z_L(p) < Z_L(\underline{p}) = 0$. Notice that

$$Z_L'(p) = \Delta_H - \frac{s_H^2}{s_L^2}\ \Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}\ (r+\delta)\ W_L'(p),$$

and $W_L'(p)$ lies between $\Delta_L/(r+\delta)$ and $(\Delta_L/(r+\delta)) + \mathcal{G}_L(\underline{p})$ for $p \in [0, \underline{p}]$.[32]

If $s_H^2 \geq s_L^2$, then

$$Z_L'(p) \geq \Delta_H - \frac{s_H^2}{s_L^2}\ \Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}\ (r+\delta)\ \frac{\Delta_L}{r+\delta} = \Delta_H - \Delta_L > 0,$$

and if $s_H^2 < s_L^2$, then

$$\begin{aligned} Z_L'(p) &\geq \Delta_H - \frac{s_H^2}{s_L^2}\,\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}\,(r+\delta)\left[\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p})\right] \\ &= \Delta_H - \frac{s_H^2}{s_L^2}\,\Delta_L + \frac{s_H^2 - s_L^2}{s_L^2}\,(r+\delta)\left[\frac{\Delta_H}{r+\delta} + \mathcal{G}_H(\underline{p})\right] \\ &= \frac{s_H^2}{s_L^2}\,(\Delta_H - \Delta_L) + \frac{s_H^2 - s_L^2}{s_L^2}\,(r+\delta)\,\mathcal{G}_H(\underline{p}) \\ &> 0. \end{aligned}$$

Therefore we conclude that $Z_L'(p) > 0$ for both $s_H \geq s_L$ and $s_H < s_L$, which implies that $Z_L(p) < 0$ for all $p < \underline{p}$, and hence there is no profitable one-shot deviation as dt is sufficiently small.

For $p > \underline{p}$, similarly define

$$Z_H(p) = w_L(p) - w_H(p) + \left[\Sigma_L(p) - \Sigma_H(p)\right]\, W_H''(p).$$

Under PAM equilibrium, we have $Z_H(\underline{p}+) = 0$ from Lemma 5. Notice that

$$\begin{aligned} Z_H(p) &= w_L(p) - w_H(p) + \left[\Sigma_L(p) - \Sigma_H(p)\right]\, W_H''(p) \\ &= w_L(p) - w_H(p) + \frac{s_L^2 - s_H^2}{s_H^2}\,\left((r+\delta)\, W_H(p) - w_H(p)\right), \end{aligned}$$

and $W_H'(p)$ lies between $(\Delta_H/(r+\delta)) + \mathcal{G}_H(\underline{p})$ and $\Delta_H/(r+\delta)$ for $p \in [\underline{p}, 1]$. Similar to the proof for the $p < \underline{p}$ case, if $s_L^2 > s_H^2$, then

$$Z_H'(p) \leq \Delta_L - \Delta_H < 0,$$

and if $s_L^2 \leq s_H^2$, then

$$Z_H'(p) \leq \Delta_L - \frac{s_L^2}{s_H^2}\,\Delta_H + \frac{s_L^2 - s_H^2}{s_H^2}\,(r+\delta)\left(\frac{\Delta_L}{r+\delta} + \mathcal{G}_L(\underline{p})\right) < 0.$$

Therefore $Z_H'(p) < 0$ for both $s_H \geq s_L$ and $s_H < s_L$, and hence $Z_H(p) < 0$ for all $p > \underline{p}$.

Second, since there is no one-shot deviation for any p, obviously there will be no deviation for any p. Consider any deviation starting at p. Then the above result says that it is better not to deviate for at least time dt. Suppose that after dt, we achieve a new p'. Similarly, there should be no profitable deviation for at least time dt'. Keep using the same logic, and we can conclude that any deviation is not profitable.

Derivation of the boundary conditions

Here, we just investigate the boundary conditions (6)–(10) for the case $\underline{p} < p_0$. The derivation is similar for the case $\underline{p} \geq p_0$.

In a stationary equilibrium, both the total measure $\int_0^1 f_y(p,t)\,\mathrm{d}p$ and the expectations $\int_0^1 p\, f_y(p,t)\,\mathrm{d}p$ are constant over time. Hence it must be the case that

$$\int_0^1 \frac{\partial f_y(p,t)}{\partial t}\,\mathrm{d}p = 0 \quad \text{and} \quad \int_0^1 p\,\frac{\partial f_y(p,t)}{\partial t}\,\mathrm{d}p = 0.$$

From

$$\frac{\partial f_y(p,t)}{\partial t} = \frac{\mathrm{d}^2}{\mathrm{d}p^2}\left[\Sigma_y(p)\, f_y(p,t)\right] - \delta\, f_y(p,t)$$

we should have

$$\text{(A4)} \qquad \int_0^{\underline{p}} \left(\frac{\mathrm{d}^2}{\mathrm{d}p^2} \left[\Sigma_L(p)\, f_L(p) \right] - \delta\, f_L(p) \right) \mathrm{d}p = 0$$

and

$$\text{(A5)} \qquad \int_{\underline{p}}^{p_0} \left(\frac{\mathrm{d}^2}{\mathrm{d}p^2} \left[\Sigma_H(p)\, f_H(p) \right] - \delta\, f_H(p) \right) \mathrm{d}p + \int_{p_0}^{1} \left(\frac{\mathrm{d}^2}{\mathrm{d}p^2} \left[\Sigma_H(p)\, f_H(p) \right] - \delta\, f_H(p) \right) \mathrm{d}p = 0.$$

Equations (A4) and (A5) give us

$$\frac{\mathrm{d}}{\mathrm{d}p} \left[\Sigma_L(p)\, f_L(p) \right] \Big|_{\underline{p}-} = \delta(1-\pi)$$

and

$$\Sigma_H(p_0)\, \left[f_H'(p_0-) - f_H'(p_0+) \right] = \frac{\mathrm{d}}{\mathrm{d}p} \left[\Sigma_H(p)\, f_H(p) \right] \Big|_{\underline{p}+} + \delta\pi,$$

since the market clearing conditions imply

$$\int_0^{\underline{p}} f_L(p)\, \mathrm{d}p = 1-\pi, \qquad \int_{\underline{p}}^{1} f_H(p)\, \mathrm{d}p = \pi,$$

and there is continuity at p_0:

$$f_H(p_0-) = f_H(p_0+).$$

Meanwhile, notice that inflow at p_0 must be the same as δ, which implies that $\Sigma_H(p_0) \left[f_H'(p_0-) - f_H'(p_0+) \right] = \delta$. This immediately gives us the flow equation at $\underline{p}$:

$$\frac{\mathrm{d}}{\mathrm{d}p} \left[\Sigma_L(p)\, f_L(p) \right] \Big|_{\underline{p}-} = \frac{\mathrm{d}}{\mathrm{d}p} \left[\Sigma_H(p)\, f_H(p) \right] \Big|_{\underline{p}+}.$$

Now applying similar logic gives

$$\int_0^{\underline{p}} \left(p\, \frac{\mathrm{d}^2}{\mathrm{d}p^2} \left[\Sigma_L(p)\, f_L(p) \right] - p\delta\, f_L(p) \right) \mathrm{d}p + \int_{\underline{p}}^{1} \left(p\, \frac{\mathrm{d}^2}{\mathrm{d}p^2} \left[\Sigma_H(p)\, f_H(p) \right] - p\delta\, f_H(p) \right) \mathrm{d}p = 0.$$

Notice that

$$\int_0^{\underline{p}} p\delta\, f_L(p)\, \mathrm{d}p + \int_{\underline{p}}^{1} p\delta\, f_H(p)\, \mathrm{d}p = \delta p_0,$$

by the martingale property. Meanwhile, we still have $\Sigma_H(p_0) \left[f_H'(p_0-) - f_H'(p_0+) \right] = \delta$. Hence, after some tedious algebra, we can get

$$\left(p\, \frac{\mathrm{d}}{\mathrm{d}p} \left[\Sigma_L(p)\, f_L(p) \right] + \Sigma_L(p)\, f_L(p) \right) \Big|_{\underline{p}-} = \left(p\, \frac{\mathrm{d}}{\mathrm{d}p} \left[\Sigma_H(p)\, f_H(p) \right] + \Sigma_H(p)\, f_H(p) \right) \Big|_{\underline{p}+},$$

which gives us the boundary condition at $\underline{p}$:

$$\Sigma_H(\underline{p}+)\, f_H(\underline{p}+) = \Sigma_L(\underline{p}-)\, f_L(\underline{p}-).$$

Proof of Proposition 1

First, we can express f_{H0},f_{H1},f_{H2},f_{L0} as functions of $\underline{p}$. Equations (8) and (10) imply

$$f_{L0} = \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p}$$

and

$$f_{H2} = f_{H0}\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{H2}} + f_{H1}.$$

From equations (6) and (9), f_{H0} and f_{H1} could be written as

$$f_{H0} = \frac{\eta_H+\eta_L}{2\eta_H}\;\frac{s_L^2}{s_H^2}\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L-\eta_H} f_{L0}$$

and

$$f_{H1} = -\frac{\eta_L-\eta_H}{2\eta_H}\;\frac{s_L^2}{s_H^2}\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L+\eta_H} f_{L0}.$$

Here,

$$\eta_L = \sqrt{\frac{1}{4}+\frac{2\delta}{s_L^2}} > \eta_H = \sqrt{\frac{1}{4}+\frac{2\delta}{s_H^2}} > \frac{1}{2}.$$

Next, we want to show that both f_{H0} and f_{H1} are decreasing in $\underline{p}$.
Rewrite f_{H0} as

$$f_{H0} = \frac{\eta_H+\eta_L}{2\eta_H}\;\frac{s_L^2}{s_H^2}\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L-\eta_H}\frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p},$$

and it suffices to show that

$$\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L-\eta_H}\frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p}$$

is decreasing in $\underline{p}$. Notice that

$$\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L-\eta_H} = \int_0^{\underline{p}}\left[\left(\frac{p}{1-p}\right)^{\eta_L-\eta_H}\right]'\mathrm{d}p = \int_0^{\underline{p}}(\eta_L-\eta_H)\left(\frac{p}{1-p}\right)^{\eta_L-\eta_H-1}\left(\frac{1}{1-p}\right)^2\mathrm{d}p.$$

Let

$$G_1(p) = p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} \quad\text{and}\quad G_2(p) = \left(\frac{p}{1-p}\right)^{\eta_L-\eta_H-1}\left(\frac{1}{1-p}\right)^2$$

such that

$$\frac{G_1(p)}{G_2(p)} = p^{-\frac{1}{2}+\eta_H}(1-p)^{-\frac{1}{2}-\eta_H}$$

is increasing in p. Therefore we could derive that

$$\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L-\eta_H}\frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p}$$

is decreasing in $\underline{p}$,[33] and hence f_{H0} is decreasing in $\underline{p}$ as well.

Similarly, we can rewrite f_{H1} as

$$f_{H1} = -\frac{\eta_L - \eta_H}{2\eta_H}\ \frac{s_L^2}{s_H^2}\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L+\eta_H} \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p},$$

and we have

$$\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L+\eta_H} = \int_0^{\underline{p}} (\eta_L+\eta_H)\left(\frac{p}{1-p}\right)^{\eta_L+\eta_H-1}\left(\frac{1}{1-p}\right)^2 \mathrm{d}p.$$

Let

$$G_3(p) = \left(\frac{p}{1-p}\right)^{\eta_L+\eta_H-1}\left(\frac{1}{1-p}\right)^2,$$

and we have that

$$\frac{G_1(p)}{G_3(p)} = p^{-\frac{1}{2}-\eta_H}(1-p)^{-\frac{1}{2}+\eta_H}$$

is decreasing in p. Therefore it must be the case that

$$-\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L+\eta_H} \frac{1-\pi}{\int_0^{\underline{p}} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p}$$

is decreasing in $\underline{p}$, and hence f_{H1} is also decreasing in $\underline{p}$.

Finally, it is immediate that

$$f_{H2} = f_{H0}\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{H2}} + f_{H1}$$

is also decreasing in $\underline{p}$. Therefore we can express f_{H0}, f_{H1} and f_{H2} as $\xi_0(\underline{p})$, $\xi_1(\underline{p})$ and $\xi_2(\underline{p})$, respectively, such that $\xi_0' < 0$, $\xi_1' < 0$ and $\xi_2' < 0$.

Hence the market clearing condition (7) implies that

$$H(\underline{p}) = \int_{\underline{p}}^{p_0}\left[\xi_0(\underline{p})\,p^{\gamma_{H1}}(1-p)^{\gamma_{H2}} + \xi_1(\underline{p})\,p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\right]\mathrm{d}p + \int_{p_0}^1 \xi_2(\underline{p})\,p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\,\mathrm{d}p = \pi.$$

It is easy to check that $H' < 0$ since $\xi_0' < 0$, $\xi_1' < 0$ and $\xi_2' < 0$. There exists $\underline{p} \in (0, p_0)$ such that $H(\underline{p}) = \pi$ if and only if $\lim_{p\to 0} H(p) > \pi$ and $\lim_{p\to p_0} H(p) < \pi$.

As $\underline{p} \to 0$, $f_{H0} = \xi_0(\underline{p}) \to \infty$ and $f_{H1} = \xi_1(\underline{p}) \to 0$, which imply

$$\lim_{p\to 0} H(p) \to \infty > \pi.$$

Meanwhile, when $\underline{p} \to p_0$, it is obvious that

$$H(\underline{p}) \to \int_{p_0}^1 f_{H2}\,p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\,\mathrm{d}p.$$

Notice that

$$f_{H2} = f_{H0}\left(\frac{p_0}{1-p_0}\right)^{\gamma_{H1}-\gamma_{H2}} + f_{H1} \to \frac{s_L^2}{s_H^2}\left(\frac{p_0}{1-p_0}\right)^{\eta_L+\eta_H} \frac{1-\pi}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p}$$

as $\underline{p} \to p_0$.

As a result, $\lim_{p \to p_0} H(p) < \pi$ if and only if

$$\frac{s_L^2}{s_H^2}\left(\frac{p_0}{1-p_0}\right)^{\eta_L+\eta_H} \frac{1-\pi}{\int_0^{p_0} p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p} \int_{p_0}^1 p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\,\mathrm{d}p < \pi,$$

which establishes equation (11) in the proposition. Moreover, since $H(\cdot)$ is strictly decreasing, the solution of $H(p) = \pi$ must be at most 1. This completes our proof of Proposition 1.

Proof of Proposition 2

The equilibrium equations are as follows:

(A6) $$\Sigma_H(\underline{p}+)\, f_H(\underline{p}+) = \Sigma_L(\underline{p}-)\, f_L(\underline{p}-) \qquad \text{(Boundary condition)},$$

(A7) $$\int_{\underline{p}}^1 f_H(p)\,\mathrm{d}p = \pi \qquad \text{(Market clearing } H\text{)},$$

(A8) $$\int_0^{\underline{p}} f_L(p)\,\mathrm{d}p = 1-\pi \qquad \text{(Market clearing } L\text{)},$$

(A9) $$\frac{\mathrm{d}}{\mathrm{d}p}\left[\Sigma_L(p)\, f_L(p)\right]\Big|_{\underline{p}-} = \frac{\mathrm{d}}{\mathrm{d}p}\left[\Sigma_H(p)\, f_H(p)\right]\Big|_{\underline{p}+} \qquad \text{(Flow equation at } \underline{p}\text{)},$$

(A10) $$f_L(p_0-) = f_L(p_0+) \qquad \text{(Continuous density at } p_0\text{)}.$$

First, from equation (A7), we have

$$f_{H0} = \frac{\pi}{\int_{\underline{p}}^1 p^{\gamma_{H2}}(1-p)^{\gamma_{H1}}\,\mathrm{d}p}.$$

Second, equations (A6) and (A9) imply that

$$f_{L1} = \frac{\eta_L - \eta_H}{2\eta_L}\,\frac{s_H^2}{s_L^2}\left(\frac{\underline{p}}{1-\underline{p}}\right)^{-\eta_L-\eta_H} f_{H0}$$

and

$$f_{L2} = \frac{\eta_L + \eta_H}{2\eta_L}\,\frac{s_H^2}{s_L^2}\left(\frac{\underline{p}}{1-\underline{p}}\right)^{\eta_L-\eta_H} f_{H0}.$$

Here,

$$\eta_L = \sqrt{\frac{1}{4} + \frac{2\delta}{s_L^2}} > \eta_H = \sqrt{\frac{1}{4} + \frac{2\delta}{s_H^2}} > \frac{1}{2}.$$

It is easy to verify that f_{H0}, f_{L1}, f_{L2} are increasing in $\underline{p}$, and hence

$$f_{L0} = f_{L1} + f_{L2}\left(\frac{p_0}{1-p_0}\right)^{-2\eta_L}$$

is also increasing in $\underline{p}$ by equation (A10).

Hence we can express f_{L0}, f_{L1}, f_{L2} as $\xi_0(\underline{p})$, $\xi_1(\underline{p})$ and $\xi_2(\underline{p})$, respectively, such that $\xi_0' > 0$, $\xi_1' > 0$ and $\xi_2' > 0$.

Finally, the market clearing condition (A8) implies that

$$H(\underline{p}) = \int_0^{p_0} \xi_0(\underline{p})\, p^{\gamma_{L1}}(1-p)^{\gamma_{L2}}\,\mathrm{d}p + \int_{p_0}^{\underline{p}} \left[\xi_1(\underline{p})\, p^{\gamma_{L1}}(1-p)^{\gamma_{L2}} + \xi_2(\underline{p})\, p^{\gamma_{L2}}(1-p)^{\gamma_{L1}}\right]\mathrm{d}p$$
$$= 1-\pi.$$

Obviously, $H(\cdot)$ is strictly increasing, which guarantees that the solution is unique if it exists, and $\lim_{p\to p_0} H(p) \le 1-\pi$ will give us equation (12) in Proposition 2.

Proof of Corollary 1

To make the proof, we have to redefine the $H(\cdot)$ function in the proof of Proposition 1 as $H(p;\pi,p_0)$, with equilibrium cut-off $\underline{p}$ satisfying $H(\underline{p};\pi,p_0)=\pi$. It is obvious to verify that H is linear in $(1-\pi)$. So as π increases, $\pi/(1-\pi)$ increases, and we have to decrease $\underline{p}$ to balance the equation. On the other hand,

$$\frac{\partial H}{\partial p_0} = \xi_0(\underline{p})\, p_0^{\gamma_1^H}(1-p_0)^{\gamma_2^H} + \xi_1(\underline{p})\, p_0^{\gamma_2^H}(1-p_0)^{\gamma_1^H} - \xi_2(\underline{p})\, p_0^{\gamma_2^H}(1-p_0)^{\gamma_1^H}$$
$$+ \int_{p_0}^{1} \frac{\partial \xi_2(\underline{p})}{\partial p_0}\, p^{\gamma_2^H}(1-p)^{\gamma_1^H}\,\mathrm{d}p.$$

It is easy to verify that the first line on the right-hand side of this equation is zero, while the second line is strictly positive. Hence $H(\underline{p};\pi,p_0)$ is increasing in p_0, and we have to increase $\underline{p}$ to keep the equation as p_0 increases.

The proof for the comparative statics for the $\underline{p} > p_0$ case is similar and hence is omitted.

Proof of Lemma 7

Generally, the value-matching and no-deviation conditions imply that

$$(r+\delta)\, W_H(\underline{p}) = w_H(\underline{p}) + \Sigma_H(\underline{p})\, W_H''(\underline{p}) = (r+\delta)\, W_L(\underline{p}) = w_L(\underline{p}) + \Sigma_L(\underline{p})\, W_L''(\underline{p})$$

and

$$W_H''(\underline{p}) = W_L''(\underline{p}).$$

These immediately mean that $w_H(\underline{p}) < w_L(\underline{p})$ and $rV_H - rV_L > \mu_H(\underline{p}) - \mu_L(\underline{p})$, as $\Sigma_H(\underline{p}) > \Sigma_L(\underline{p})$.

Proof of Lemma 8

Let Ω_H be the optimal acceptance region in the problem with commitment. Given Ω_H, we can allocate any worker with $p \notin \Omega_H$ to the low-type firms. This allocation (allocation 1) leads to ergodic density functions $f_H(p), f_L(p)$ satisfying the Kolmogorov forward equation, the martingale property and market clearing. Similarly, the allocation $(\widetilde{\Omega}_H, \widetilde{\Omega}_L)$ (allocation 2) leads to ergodic density functions $\tilde{f}_H(p)$, $\tilde{f}_L(p)$ satisfying the same properties.

On the one hand, since allocation 1 is the optimal solution to the problem with commitment, it generates higher profit than allocation 2 for the high-type firms:

$$\int_{\Omega_H} \left[\mu_H(p) - \mu_L(p)\right] f_H(p)\,\mathrm{d}p \ge \int_{\widetilde{\Omega}_H} \left[\mu_H(p) - \mu_L(p)\right] \tilde{f}_H(p)\,\mathrm{d}p.$$

By substituting for $\mu_H(p)$ and $\mu_L(p)$ using the fact that $\mu_H(p) - \mu_L(p) = p$, this inequality implies that

$$\int_{\Omega_H} p\, f_H(p)\,\mathrm{d}p \ge \int_{\widetilde{\Omega}_H} p\, \tilde{f}_H(p)\,\mathrm{d}p.$$

On the other hand, the total expected surplus for allocation 1 could be written as

$$S_1 = \int_{\Omega_H} (\Delta_H p + \mu_{LH}) f_H(p)\,\mathrm{d}p + \int_{\Omega_L} (\Delta_L p + \mu_{LL}) f_L(p)\,\mathrm{d}p.$$

From the market clearing and martingale property conditions, we can furthermore rewrite S_1 as

$$S_1 = (\Delta_H - \Delta_L)\int_{\Omega_H} p\, f_H(p)\,\mathrm{d}p + \Delta_L p_0 + \pi\mu_{LH} + (1-\pi)\mu_{LL},$$

and similarly,

$$S_2 = (\Delta_H - \Delta_L)\int_{\widetilde{\Omega}_H} p\, f_H(p)\,\mathrm{d}p + \Delta_L p_0 + \pi\mu_{LH} + (1-\pi)\mu_{LL}.$$

Therefore $S_1 \leq S_2$ if and only if

$$\int_{\Omega_H} p\, f_H(p)\,\mathrm{d}p \leq \int_{\widetilde{\Omega}_H} p\, \tilde{f}_H(p)\,\mathrm{d}p.$$

So we conclude that

$$\int_{\Omega_H} p\, f_H(p)\,\mathrm{d}p = \int_{\widetilde{\Omega}_H} p\, \tilde{f}_H(p)\,\mathrm{d}p,$$

and hence $\widetilde{\Omega}_H$ also solves the problem with commitment.

Proof of Theorem 4

We establish the proof of Theorem 4 under supermodularity. The same logic goes through for submodularity.

The proof is constructed in the following three steps. (1) For $N = 3$, we show that the planner can increase output when changing the cut-offs. (2) For $N = 3$, no allocation dominates PAM. (3) For any N, the allocation with $N - 2$ cut-offs dominates that with N cut-offs.

(1) For $N = 3$, output increases from changing the cut-offs. Consider any allocation with three cut-offs $0 < \underline{p}_3 < \underline{p}_2 < \underline{p}_1 < 1$ such that workers with $p \in (\underline{p}_1, 1]$ and $p \in (\underline{p}_3, \underline{p}_2)$ are allocated to the high-type firms, while workers with $p \in [0, \underline{p}_3)$ and $p \in (\underline{p}_2, \underline{p}_1)$ are allocated to the low-type firms. Furthermore, denote the ergodic density function for this allocation to be f_y, and for p close to 0, let the density function be $f_L(p) = \tilde{f}_{L0} p^{\gamma_L}(1-p)^{1-\gamma_L}$, while for p close to 1 it is $f_H(p) = \tilde{f}_{H0} p^{1-\gamma_H}(1-p)^{\gamma_H}$, where $\tilde{f}_{L0}$ and $\tilde{f}_{H0}$ are constants. Correspondingly, denote the ergodic density under the PAM allocation to be f_y^*, with the unique cut-off $\underline{p}$.

- Suppose that the planner changes the allocation by moving the interval to the left: $(\underline{p}_2, \underline{p}_1) \to (\underline{p}'_2, \underline{p}'_1)$, where $(\underline{p}'_2, \underline{p}'_1) = (\underline{p}_2 - \varepsilon_2, \underline{p}_1 - \varepsilon_1)$. Choose $\varepsilon_1, \varepsilon_2$ such that market clearing is satisfied:

$$\int_{\underline{p}'_1}^{\underline{p}_1} f_H(p)\,\mathrm{d}p = \int_{\underline{p}'_2}^{\underline{p}_2} f_H(p)\,\mathrm{d}p.$$

- Given the new cut-offs, the Kolmogorov forward equation will pin down a new density $\widehat{f_L}$ in the interval $(\underline{p}'_2, \underline{p}'_1)$. Globally, we need to satisfy market clearing and the martingale property conditions. The market clearing condition for the H types is satisfied by the construction. For the L type firms, we require that

$$\int_{\underline{p}'_2}^{\underline{p}'_1} \widehat{f_L}(p)\,\mathrm{d}p = \int_{\underline{p}_2}^{\underline{p}_1} f_L(p)\,\mathrm{d}p. \tag{A11}$$

The martingale property condition requires that $\mathbb{E}_{\Omega'_H} p + \mathbb{E}_{\Omega'_L} p = p_0$, or

$$\text{(A12)} \qquad \int_0^{p_3} p\, f_L(p)\,\mathrm{d}p + \int_{p_3}^{p_2'} p\, f_H(p)\,\mathrm{d}p + \int_{p_2'}^{p_1'} p\, \widehat{f}_L(p)\,\mathrm{d}p + \int_{p_1'}^{1} p\, f_H(p)\,\mathrm{d}p = p_0.$$

Equations (A11) and (A12) constitute a system of two linear equations in the distributional parameters for $\widehat{f}_L$, and $\widehat{f}_L$, which can thus be solved.[34]

- Then comparing the original allocation to the new one, we get

$$\mathbb{E}_{\Omega'_H} p - \mathbb{E}_{\Omega_H} p = \int_{\underline{p}_1'}^{\underline{p}_1} p\, f_H(p)\,\mathrm{d}p - \int_{\underline{p}_2'}^{\underline{p}_2} p\, f_H(p)\,\mathrm{d}p > 0,$$

since by construction,

$$\int_{\underline{p}_1'}^{\underline{p}_1} f_H(p)\,\mathrm{d}p = \int_{\underline{p}_2'}^{\underline{p}_2} f_H(p)\,\mathrm{d}p,$$

and the interval $[\underline{p}_2', \underline{p}_1']$ is strictly to the left of $[\underline{p}_2, \underline{p}_1]$. From Lemma 8, $\mathbb{E}_{\Omega'_H} p > \mathbb{E}_{\Omega_H} p$ implies that the planner prefers allocation Ω' over Ω.

- Similarly, we can consider another transform, which is to move the interval to the right: $(\underline{p}_3, \underline{p}_2) \to (\underline{p}_3', \underline{p}_2')$, where $(\underline{p}_3', \underline{p}_2') = (\underline{p}_3 + \varepsilon_2, \underline{p}_2 + \varepsilon_1)$. This can also lead to output increases. If we keep on doing such transformations, then eventually we can have both the distance and the measure between $\underline{p}_3'$ and $\underline{p}_1'$ arbitrarily small, while the new $(\underline{p}_1', \underline{p}_2', \underline{p}_3')$ allocation strictly dominates the original $(\underline{p}_1, \underline{p}_2, \underline{p}_3)$ allocation.

(2) For $N = 3$, no allocation dominates PAM.

- We now show by contradiction that no allocation dominates PAM for $N = 3$. Suppose, on the contrary, that there exists an allocation with cut-offs $\tilde{p}_1, \tilde{p}_2$ and $\tilde{p}_3$ that dominates the PAM allocation. Then by Lemma 8, we should have

$$\text{(A13)} \qquad \int_{\tilde{p}_1}^{1} p\, f_H(p)\,\mathrm{d}p + \int_{\tilde{p}_3}^{\tilde{p}_2} p\, f_H(p)\,\mathrm{d}p > \int_{\underline{p}}^{1} p f_H^*(p)\,\mathrm{d}p$$

and

$$\text{(A14)} \qquad \int_{\tilde{p}_2}^{\tilde{p}_1} p\, f_L(p)\,\mathrm{d}p + \int_0^{\tilde{p}_3} p\, f_L(p)\,\mathrm{d}p < \int_0^{\underline{p}} p f_L^*(p)\,\mathrm{d}p.$$

From step (1), we can first fix $\tilde{p}_3$ and make $\tilde{p}_2'$ move towards $\tilde{p}_3$, which is efficiency-improving. $\tilde{p}_1$ could be extended to the left until it reaches $\widehat{p}_1$: $\int_{\widehat{p}_1}^1 f_H(p)\,\mathrm{d}p = \pi$. Since $\int_{\tilde{p}_1'}^1 f_H(p)\,\mathrm{d}p < \pi$, it must be the case that $\widehat{p}_1 < \tilde{p}_1'$. If $\tilde{p}_2'$ is sufficiently close to $\tilde{p}_3$, then we will have $\tilde{p}_2' < \widehat{p}_1$. By hypothesis,

$$\int_{\widehat{p}_1}^{1} p\, f_H(p)\,\mathrm{d}p > \int_{\tilde{p}_1'}^{1} p\, f_H(p)\,\mathrm{d}p + \int_{\tilde{p}_3}^{\tilde{p}_2'} p\, f_H(p)\,\mathrm{d}p > \int_{\underline{p}}^{1} p f_H^*(p)\,\mathrm{d}p.$$

On the other hand, it is also efficiency-improving to fix $\tilde{p}_1$ and make $\tilde{p}_2'$ move towards $\tilde{p}_1$. Similarly, define $\widehat{p}_3$ as $\int_0^{\widehat{p}_3} f_L(p)\,\mathrm{d}p = (1-\pi)$ such that $\widehat{p}_3 > \tilde{p}_3'$. By hypothesis,

$$\int_0^{\widehat{p}_3} p\, f_L(p)\,\mathrm{d}p < \int_0^{\underline{p}} p f_L^*(p)\,\mathrm{d}p,$$

since we can make $\tilde{p}_2'$ sufficiently close to $\tilde{p}_1$.

- The next step of the proof requires Lemma 9, given in the next subsection. The lemma implies that we should have $\tilde{p}_3' < \widehat{p}_3 < \underline{p} < \widehat{p}_1 < \tilde{p}_1'$ to guarantee that

$$\int_{\widehat{p}_1}^{1} p\, f_H(p)\,\mathrm{d}p > \int_{\underline{p}}^{1} p f_H^*(p)\,\mathrm{d}p \quad \text{and} \quad \int_0^{\widehat{p}_3} p\, f_L(p)\,\mathrm{d}p < \int_0^{\underline{p}} p f_L^*(p)\,\mathrm{d}p.$$

Therefore inequalities (A13) and (A14) hold only when $\tilde{p}_1' - \tilde{p}_3' > \widehat{p}_1 - \widehat{p}_3 > 0$, which contradicts the fact that we can make the distance between $\tilde{p}_1'$ and $\tilde{p}_3'$ arbitrarily small while still keeping the inequalities (A13) and (A14). Hence no allocation with $N = 3$ cut-offs could be better than the PAM allocation in terms of aggregate surplus.

(3) For N *cut-offs, the allocation is dominated by any allocation with $N-2$ cut-offs.* Consider three adjacent cut-offs $\underline{p}_{n-1} > \underline{p}_n > \underline{p}_{n+1}$ such that workers with $p \in (\underline{p}_{n-1}, \underline{p}_{n-2})$ and $p \in (\underline{p}_{n+1}, \underline{p}_n)$ are allocated to high-type firms, and workers with $p \in (\underline{p}_n, \underline{p}_{n-1})$ and $p \in (\underline{p}_{n+2}, \underline{p}_{n+1})$ are allocated to low-type firms. Suppose that the density functions are such that the market clears and the expectation of the p values is p_0. Then we just need to choose κ such that

$$\int_{\underline{p}_{n-1}-\kappa}^{\underline{p}_{n-1}} f_H(p)\,\mathrm{d}p = \int_{\underline{p}_{n+1}}^{\underline{p}_n} f_H(p)\,\mathrm{d}p.$$

Now $\underline{p}_{n-1}$, $\underline{p}_n$ and $\underline{p}_{n+1}$ converge to $\underline{p}_{n-1} - \kappa$, but p_{n+2} is kept the same. The market clearing condition requires that

$$\int_{\underline{p}_{n+2}}^{\underline{p}_{n-1}-\kappa} \tilde{f}_L(p)\,\mathrm{d}p = \int_{\underline{p}_n}^{\underline{p}_{n-1}} f_L(p)\,\mathrm{d}p + \int_{\underline{p}_{n+2}}^{\underline{p}_{n+1}} f_L(p)\,\mathrm{d}p.$$

Meanwhile, the martingale property condition requires that

$$\int_{\underline{p}_1}^{1} p\, f_H(p)\,\mathrm{d}p + \cdots + \int_{\underline{p}_{n-1}-\kappa}^{\underline{p}_{n-2}} p\, f_H(p)\,\mathrm{d}p + \int_{\underline{p}_{n+2}}^{\underline{p}_{n-1}-\kappa} p\, \tilde{f}_L(p)\,\mathrm{d}p + \cdots + \int_0^{\underline{p}_N} p\, f_L(p)\,\mathrm{d}p = p_0.$$

Similar to step (1), we have a system of two linear equations for two distributional coefficients, and density $\tilde{f}_L$ can be found. As before,

$$\mathbb{E}_{\Omega_H} p = \int_{\Omega_H} p\, f_H(p)\,\mathrm{d}p$$

must become higher, and this allocation with $N-2$ cut-offs will generate a higher aggregate payoff.

Finally, by the standard induction argument, we can conclude that the PAM allocation with one cut-off dominates any allocation with $N \geq 3$ cut-offs in aggregate surplus.

Lemma for use in proof of Theorem 4 above

Lemma 9. Let $\widehat{p}_1$ be such that $\int_{\widehat{p}_1}^1 f_H(p)\,\mathrm{d}p = \pi$, where $f_H(p)$ satisfies the Kolmogorov forward equation. Then $\int_{\widehat{p}_1}^1 p\, f_H(p)\,\mathrm{d}p$ is increasing in $\widehat{p}_1$. Let $\widehat{p}_3$ be such that $\int_0^{\widehat{p}_3} f_L(p)\,\mathrm{d}p = (1-\pi)$, where $f_L(p)$ satisfies the Kolmogorov forward equation. Then $\int_0^{\widehat{p}_3} p\, f_L(p)\,\mathrm{d}p$ is also increasing in $\widehat{p}_3$.

Proof. We just prove the case that $\widehat{p}_1 > p_0$. The other cases are similar.
Let $f_H(p) = C_H(1-p)^{\gamma_{H1}} p^{\gamma_{H2}}$, where

$$\gamma_{H1} = -\frac{3}{2} + \eta_H \quad \text{and} \quad \gamma_{H2} = -\frac{3}{2} - \eta_H.$$

From the Kolmogorov forward equation,

$$\int_{\widehat{p}_1}^{1} f_H(p)\,\mathrm{d}p = \frac{1}{\delta}\int_{\widehat{p}_1}^{1} \frac{\mathrm{d}^2}{\mathrm{d}p^2}\left[\Sigma_H(p)\, f_H(p)\right] = \pi$$

or

$$\frac{\eta_H + \widehat{p}_1 - \frac{1}{2}}{\widehat{p}_1(1-\widehat{p}_1)}\;\Sigma_H(\widehat{p}_1)\, f_H(\widehat{p}_1) = \delta\pi.$$

Applying the Kolmogorov forward equation again yields

$$\text{(A15)}\qquad \begin{aligned}\int_{\widehat{p}_1}^{1} p\, f_H(p)\,\mathrm{d}p &= \frac{1}{\delta}\int_{\widehat{p}_1}^{1} p\;\frac{\mathrm{d}^2}{\mathrm{d}p^2}\left[\Sigma_H(p)\, f_H(p)\right]\mathrm{d}p\\ &= -\frac{1}{\delta}\,\widehat{p}_1\;\frac{\mathrm{d}\left[\Sigma_H(p)\, f_H(p)\right]}{\mathrm{d}p}\Big|_{p=\widehat{p}_1} + \Sigma_H(\widehat{p}_1)\, f_H(\widehat{p}_1),\end{aligned}$$

by integration by parts. As

$$-\frac{\mathrm{d}\left[\Sigma_H(p)\, f_H(p)\right]}{\mathrm{d}p}\Big|_{p=\widehat{p}_1} = \delta\pi \quad \text{and} \quad \Sigma_H(\widehat{p}_1)\, f_H(\widehat{p}_1) = \frac{\delta\pi\widehat{p}_1(1-\widehat{p}_1)}{\eta_H + \widehat{p}_1 - \frac{1}{2}},$$

equation (A15) could be simplified as

$$\int_{\widehat{p}_1}^{1} p\, f_H(p)\,\mathrm{d}p = \pi\widehat{p}_1 + \frac{\pi\widehat{p}_1(1-\widehat{p}_1)}{\eta_H + \widehat{p}_1 - \frac{1}{2}} = \frac{\pi\widehat{p}_1\left(\eta_H + \frac{1}{2}\right)}{\eta_H + \widehat{p}_1 - \frac{1}{2}},$$

which is increasing in $\widehat{p}_1$ since

$$\eta_H = \sqrt{\frac{1}{4} + \frac{2\delta}{s_y^2}} > \frac{1}{2}.$$

□

On-the-job human capital accumulation

The solutions of equations (18)–(20) are

$$\begin{aligned}W_{yy}^u(p) = {} & \frac{\mu_y(p) - rV_y}{r+\delta+\lambda} + k_{y1}^u p^{1-\alpha_y^u}(1-p)^{\alpha_y^u} + k_{y2}^u p^{\alpha_y^u}(1-p)^{1-\alpha_y^u}\\ & + \frac{\lambda}{(r+\delta)(r+\delta+\lambda)}\;\left[\mu_y(p) + \xi(p) - rV_y\right]\\ & + \frac{\lambda}{(\lambda+\delta+r) - (s_y^u)^2(r+\delta)/(s_y^e)^2}\;\left[k_{y1}^e p^{1-\alpha_y^e}(1-p)^{\alpha_y^e} + k_{y2}^e p^{\alpha_y^e}(1-p)^{1-\alpha_y^e}\right],\end{aligned}$$

$$\begin{aligned}W_{LH}^u(p) = {} & \frac{\mu_L(p) - rV_L}{r+\delta+\lambda} + k_{L1}^u p^{1-\alpha_L^u}(1-p)^{\alpha_L^u} + k_{L2}^u p^{\alpha_L^u}(1-p)^{1-\alpha_L^u}\\ & + \frac{\lambda}{(r+\delta)(r+\delta+\lambda)}\;\left[\mu_H(p) + \xi(p) - rV_H\right]\\ & + \frac{\lambda}{(\lambda+\delta+r) - (s_L^u)^2(r+\delta)/(s_H^e)^2}\;\left[k_{H1}^e p^{1-\alpha_H^e}(1-p)^{\alpha_H^e} + k_{H2}^e p^{\alpha_H^e}(1-p)^{1-\alpha_H^e}\right]\end{aligned}$$

and

$$W_y^e(p) = \frac{\mu_y(p) + \xi(p) - rV_y}{r+\delta} + k_{y1}^e p^{1-\alpha_y^e}(1-p)^{\alpha_y^e} + k_{y2}^e p^{\alpha_y^e}(1-p)^{1-\alpha_y^e},$$

where

$$\alpha_y^u = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta+\lambda)}{(s_y^u)^2}} \geq 1, \quad \alpha_y^e = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{(s_y^e)^2}} \geq 1.$$

Under the assumption $\underline{p}^u = \underline{p}^e = \underline{p}$, the value functions can be written as

$$\begin{aligned} W_y^u(p) =& \frac{\mu_y(p) - rV_y}{r+\delta+\lambda} + k_{y1}^u p^{1-\alpha_y^u}(1-p)^{\alpha_y^u} + k_{y2}^u p^{\alpha_y^u}(1-p)^{1-\alpha_y^u} \\ &- \frac{\lambda (s_y^u)^2/(s_y^e)^2}{(r+\delta+\lambda)\left[(\lambda+\delta+r) - (s_y^u)^2(r+\delta)/(s_y^e)^2\right]} \left[\mu_y(p) + \xi(p) - rV_y\right] \\ &+ \frac{\lambda}{(\lambda+\delta+r) - (s_y^u)^2(r+\delta)/(s_y^e)^2} W_y^e(p) \end{aligned}$$

and

$$W_y^e(p) = \frac{\mu_y(p) + \xi(p) - rV_y}{r+\delta} + k_{y1}^e p^{1-\alpha_y^e}(1-p)^{\alpha_y^e} + k_{y2}^e p^{\alpha_y^e}(1-p)^{1-\alpha_y^e},$$

where

$$\alpha_y^u = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta+\lambda)}{(s_y^u)^2}} \geq 1, \quad \alpha_y^e = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(r+\delta)}{(s_y^e)^2}} \geq 1.$$

Boundary conditions

$$W_L^e(\underline{p}) = W_H^e(\underline{p}), \quad W_L^{e\prime}(\underline{p}) = W_H^{e\prime}(\underline{p}), \quad W_L^{e''}(\underline{p}) = W_H^{e''}(\underline{p})$$

would imply (by normalizing $V_L = 0$ as usual)

$$r\widetilde{V}_H^e = (\mu_{LH} - \mu_{LL}) + \frac{\alpha_H^e(\alpha_L^e - 1)(\Delta_H - \Delta_L)\underline{p}}{\alpha_H^e(\alpha_L^e - 1) - (1-\underline{p})(\alpha_L^e - \alpha_H^e)}.$$

And from

$$W_L^u(\underline{p}) = W_H^u(\underline{p}), \quad W_L^{u\prime}(\underline{p}) = W_H^{u\prime}(\underline{p}), \quad W_L^{u''}(\underline{p}) = W_H^{u''}(\underline{p}),$$

another equilibrium payoff $\widetilde{V}_H^u$ could be derived as

$$\begin{aligned} r\widetilde{V}_H^u =& \left(\mu_{LH} - \frac{A_L}{B_L}\frac{B_H}{A_H}\mu_{LL}\right) - \frac{B_H}{A_H}\frac{\lambda\xi_L}{r+\delta+\lambda}\left(\frac{1-A_H}{B_H} - \frac{1-A_L}{B_L}\right) \\ &+ \frac{B_H}{A_H}\frac{\alpha_H^u(\alpha_L^u - 1)(D_H - D_L)\underline{p}}{\alpha_H^u(\alpha_L^u - 1) - (1-\underline{p})(\alpha_L^u - \alpha_H^u)}, \end{aligned}$$

where

$$\begin{aligned} D_H &= \frac{A_H}{B_H}\Delta_H - \frac{1-A_H}{B_H}\frac{\lambda\Delta_\xi}{r+\delta+\lambda}, \\ D_L &= \frac{A_L}{B_L}\Delta_L - \frac{1-A_L}{B_L}\frac{\lambda\Delta_\xi}{r+\delta+\lambda}, \\ A_H &= 1 - \frac{(s_H^u)^2}{(s_H^e)^2}, \quad B_H = (\lambda+\delta+r) - \frac{(s_H^u)^2}{(s_H^e)^2}(r+\delta), \\ A_L &= 1 - \frac{(s_L^u)^2}{(s_L^e)^2}, \quad B_L = (\lambda+\delta+r) - \frac{(s_L^u)^2}{(s_L^e)^2}(r+\delta). \end{aligned}$$

Proof of Proposition 3

Supermodularity is equivalent to $\Delta_H > \Delta_L$, and $\xi_H \simeq \xi_L$ is equivalent to $\Delta_\xi = \xi_H - \xi_L \to 0$. The proof can be divided into three parts: as a sufficient condition, the three inequalities

$$\text{(A16)} \qquad \left(\mu_{LH} - \frac{A_L}{B_L}\frac{B_H}{A_H}\mu_{LL}\right) - \frac{B_H}{A_H}\,\frac{\lambda\xi_L}{r+\delta+\lambda}\left(\frac{1-A_H}{B_H} - \frac{1-A_L}{B_L}\right) < \mu_{LH} - \mu_{LL},$$

$$\text{(A17)} \qquad \frac{B_H}{A_H}\,(D_H - D_L) < \Delta_H - \Delta_L,$$

$$\text{(A18)} \qquad \frac{\alpha_H^u(\alpha_L^u - 1)\underline{p}}{\alpha_H^u(\alpha_L^u-1) - (1-\underline{p})(\alpha_L^u - \alpha_H^u)} < \frac{\alpha_H^e(\alpha_L^e - 1)\underline{p}}{\alpha_H^e(\alpha_L^e-1) - (1-\underline{p})(\alpha_L^e - \alpha_H^e)},$$

should be satisfied simultaneously.

First, notice that $(s_H^u)^2/(s_H^e)^2 > (s_L^u)^2/(s_L^e)^2$ since $\Delta_H > \Delta_L$. As a result,

$$\frac{A_H}{B_H} < \frac{A_L}{B_L} \quad \text{and} \quad \frac{1-A_H}{B_H} > \frac{1-A_L}{B_L}.$$

Inequality (A16) holds since

$$\mu_{LH} - \frac{A_L}{B_L}\,\frac{B_H}{A_H}\,\mu_{LL} < \mu_{LH} - \mu_{LL}$$

and

$$\frac{A_L}{B_L}\,\frac{B_H}{A_H}\,\mu_{LL} - \frac{B_H}{A_H}\,\frac{\lambda\xi_L}{r+\delta+\lambda}\left(\frac{1-A_H}{B_H} - \frac{1-A_L}{B_L}\right) > 0.$$

Inequality (A17) could be proved similarly.

For inequality (A18), we just need to compare

$$\alpha_H^u(\alpha_L^u - 1)\left[\alpha_H^e(\alpha_L^e - 1) - (1-\underline{p})(\alpha_L^e - \alpha_H^e)\right]$$

and

$$\alpha_H^e(\alpha_L^e - 1)\left[\alpha_H^u(\alpha_L^u - 1) - (1-\underline{p})(\alpha_L^u - \alpha_H^u)\right].$$

So it suffices to show that

$$\alpha_H^u(\alpha_L^u - 1)(\alpha_L^e - \alpha_H^e) > \alpha_H^e(\alpha_L^e - 1)(\alpha_L^u - \alpha_H^u).$$

The direct proof is not easy. But notice from the expressions of the α terms that

$$(\alpha_L^e - \alpha_H^e)(\alpha_L^e + \alpha_H^e - 1) = 2(r+\delta)\left[\frac{\sigma^2}{(\Delta_L + \Delta_\xi)^2} - \frac{\sigma^2}{(\Delta_H + \Delta_\xi)^2}\right]$$

and

$$(\alpha_L^u - \alpha_H^u)(\alpha_L^u + \alpha_H^u - 1) = 2(r+\delta+\lambda)\left[\frac{\sigma^2}{\Delta_L^2} - \frac{\sigma^2}{\Delta_H^2}\right].$$

Hence when $\Delta_\xi = 0$,

$$\frac{\alpha_L^e - \alpha_H^e}{\alpha_L^u - \alpha_H^u} = \frac{r+\delta}{r+\delta\lambda}\,\frac{\alpha_L^u + \alpha_H^u - 1}{\alpha_L^e + \alpha_H^e - 1}.$$

The original inequality (A18) is transformed to compare

$$(r+\delta)\alpha_H^u(\alpha_L^u - 1)(\alpha_L^u + \alpha_H^u - 1)$$

and

$$(r+\delta+\lambda)\alpha_H^e(\alpha_L^e-1)(\alpha_L^e+\alpha_H^e-1).$$

Meanwhile, we have

$$\begin{aligned}(r+\delta)\alpha_H^u(\alpha_L^u-1)\alpha_L^u &= (r+\delta)\alpha_H^u\ \frac{2(r+\delta+\lambda)}{\Delta_L^2}\\ &> (r+\delta+\lambda)\alpha_H^e(\alpha_L^e-1)\alpha_L^e\\ &= (r+\delta+\lambda)\alpha_H^e\frac{2(r+\delta)}{\Delta_L^2}\end{aligned}$$

and

$$\begin{aligned}(r+\delta)\alpha_H^u(\alpha_L^u-1)(\alpha_H^u-1) &= (r+\delta)(\alpha_L^u-1)\ \frac{2(r+\delta+\lambda)}{\Delta_H^2}\\ &> (r+\delta+\lambda)\alpha_H^e(\alpha_L^e-1)(\alpha_H^e-1)\\ &= (r+\delta+\lambda)(\alpha_L^e-1)\frac{2(r+\delta)}{\Delta_H^2},\end{aligned}$$

since $\alpha_y^u > \alpha_y^e$. This implies that

$$\alpha_H^u(\alpha_L^u-1)(\alpha_L^e-\alpha_H^e) > \alpha_H^e(\alpha_L^e-1)(\alpha_L^u-\alpha_H^u),$$

and therefore

$$\frac{\alpha_H^u(\alpha_L^u-1)\underline{p}}{\alpha_H^u(\alpha_L^u-1)-(1-\underline{p})(\alpha_L^u-\alpha_H^u)} < \frac{\alpha_H^e(\alpha_L^e-1)\underline{p}}{\alpha_H^e(\alpha_L^e-1)-(1-\underline{p})(\alpha_L^e-\alpha_H^e)}.$$

Notice from the above proof that inequality (A18) holds only when Δ_ξ is small, and will not hold as Δ_ξ becomes sufficiently large.

Finally, we can conclude that $\widetilde{V}_H^u < \widetilde{V}_H^e$ when $\xi_H \simeq \xi_L$, and as a result, $\underline{p}^e < \underline{p}^u$.

Proof of Proposition 4

The general solution of equation (21) is

$$W_y(p) = C_y p^{(r+\delta)/\lambda_y} + \frac{\Delta_y}{r+\delta-\lambda_y}\ p + \frac{\mu_{Ly}-rV_y}{r+\delta}. \tag{A19}$$

Suppose that PAM is the equilibrium allocation. Then $\lim_{p\to 1} W_H(p) = W_H(1)$ implies that

$$C_H = -\frac{\lambda_H\Delta_H}{(r+\delta)(r+\delta-\lambda_H)}.$$

The boundary conditions at the cut-off $\underline{p}$ imply a system of equations in C_L and $\underline{p}$, where C_L is the coefficient of value function $W_L(p)$ in equation (A19). Substituting for C_L, we have

$$\begin{aligned}&\frac{\Delta_L}{r+\delta}\ \underline{p} + \frac{\mu_{LL}-rV_L}{r+\delta}\\ &= \frac{\lambda_L-\lambda_H}{r+\delta}\ \frac{\Delta_H}{r+\delta-\lambda_H}\ \underline{p}^{(r+\delta)/\lambda_H} + \left(1-\frac{\lambda_L}{r+\delta}\right)\frac{\Delta_H}{r+\delta-\lambda_H}\ \underline{p} + \frac{\mu_{LH}-rV_H}{r+\delta}\end{aligned}$$

or

$$\text{(A20)} \qquad \frac{\Delta_L - \Delta_H}{r+\delta}\,\underline{p} + \frac{\mu_{LL} - rV_L}{r+\delta} = \frac{\lambda_H - \lambda_L}{r+\delta}\,\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - \underline{p}^{(r+\delta)/\lambda_H}] + \frac{\mu_{LH} - rV_H}{r+\delta}.$$

The inequality

$$\frac{\mu_{LL} - rV_L}{r+\delta} > \frac{\mu_{LH} - rV_H}{r+\delta}$$

implies that if we can show that

$$\text{(A21)} \qquad \frac{\Delta_H - \Delta_L}{r+\delta}\,\underline{p} < \frac{\lambda_L - \lambda_H}{r+\delta}\,\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - \underline{p}^{(r+\delta)/\lambda_H}],$$

then equation (A20) cannot hold as an equality, which is the result that we are looking for. First notice that the left-hand side of inequality (A21) goes to zero as $\Delta_H - \Delta_L$ decreases to zero. Meanwhile, the belief updating process implies that the ergodic distribution depends only on λ terms and will not depend on Δ terms. From previous sections, if PAM is indeed the equilibrium allocation, then $\underline{p}$ should not depend on Δ terms. Therefore if we fix any $\lambda_L > \lambda_H$ and $r+\delta > \lambda_H$, we can derive some corresponding $\underline{p} \in (0,1)$. Then let $\Delta_H - \Delta_L$ decrease to zero, and it is immediate to see that eventually we will have

$$\frac{\Delta_H - \Delta_L}{r+\delta}\,\underline{p} < \frac{\lambda_L - \lambda_H}{r+\delta}\,\frac{\Delta_H}{r+\delta-\lambda_H}[\underline{p} - \underline{p}^{(r+\delta)/\lambda_H}].$$

This implies that PAM cannot be an equilibrium allocation if $\lambda_L > \lambda_H$, $r+\delta > \lambda_H$ and the degree of supermodularity is sufficiently small.

No-deviation condition for the non-Bayesian learning example

Under the non-Bayesian learning case, suppose that it is optimal for a p worker to choose firm y. Then the value function for this worker should be such that (from the Hamilton–Jacobi–Bellman equation)

$$(r+\delta)\; W_y(p) = w_y(p) + \lambda_y p W_y'(p).$$

Suppose that there is a cut-off $\underline{p}$ such that workers with $p > \underline{p}$ are matched with H firms, and vice versa. Then the absence of deviation implies that a $p > \underline{p}$ worker has no incentive to deviate, rematch with an L firm, and switch back after time $\mathrm{d}t$:

$$W_H(p) > \widetilde{W}_L(p) = \mathbb{E}\left\{\int_t^{t+\mathrm{d}t} \mathrm{e}^{-(r+\delta)(s-t)}\; w_L(p_s)\,\mathrm{d}s + \mathrm{e}^{-(r+\delta)\,\mathrm{d}t}\; W(p_{t+\mathrm{d}t})\right\}.$$

For $\mathrm{d}t$ sufficiently small, $p_{t+\mathrm{d}t}$ is still close to p such that it is optimal for a $p_{t+\mathrm{d}t}$ worker to choose firm H as well. It is immediate to see that

$$\lim_{\mathrm{d}t\to 0} \frac{W_H(p) - \widetilde{W}_L(p)}{\mathrm{d}t} = w_H(p) - w_L(p) + (\lambda_H - \lambda_L) p W_H'(p),$$

hence no deviation implies that

$$w_H(p) - w_L(p) + (\lambda_H - \lambda_L) p W_H'(p) > 0$$

for all $p > \underline{p}$. Let $p \to \underline{p}+$, and we have by applying the value-matching condition that

$$w_H(\underline{p}+) - w_L(\underline{p}-) + (\lambda_H - \lambda_L)\underline{p}\; W_H'(\underline{p}+) = \lambda_L \underline{p}\left(W_L'(\underline{p}-) - W_H'(\underline{p}+)\right) \geq 0,$$

or equivalently, $W_L'(\underline{p}-) \geq W_H'(\underline{p}+)$. On the other hand, a $p < \underline{p}$ worker also has no incentive to deviate, rematch with an H firm, and switch back after time dt. Similarly, no deviation implies that

$$w_L(p) - w_H(p) + (\lambda_L - \lambda_H)pW_L'(p) > 0$$

for all $p < \underline{p}$. Let $p \to \underline{p}-$, and it can be shown that

$$w_L(\underline{p}-) - w_H(\underline{p}+) + (\lambda_L - \lambda_H)\underline{p}\ W_L'(\underline{p}-) = \lambda_H \underline{p}\left(W_H'(\underline{p}+) - W_L'(\underline{p}-)\right) \geq 0,$$

or equivalently, $W_H'(\underline{p}+) \geq W_L'(\underline{p}-)$. Therefore at $\underline{p}$, it must be the case that $W_H'(\underline{p}) = W_L'(\underline{p})$, and the no-deviation condition coincides with the smooth-pasting condition.

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NOTES

1. Of course, the search model also inherently exhibits turnover, but with observable types, turnover is constant over the lifecycle. Moscarini (2005) brings together search and learning in the Jovanovic framework.
2. For example, Anderson and Smith (2010) find that positive assortative matching may fail under supermodularity due to the concern of dynamic learning in a different model setup.
3. See, for example, Chernoff (1968). Only with multiple *independent* arms are reservation strategies guaranteed, and the Gittins index policy (in discrete time) is optimal.
4. Papageorgiou (2014) analyses a learning model with heterogeneity. He estimates the version of Moscarini's search model with two-sided heterogeneity. With search frictions, wage setting is non-competitive, and as a result, the no-deviation condition is not imposed in addition to value matching and smooth pasting. Nonetheless, his findings provide us with realistic estimates of the labour market characteristics of our model. See also Groes *et al.* (2015) for estimates of a different learning model.
5. This result is generalized further in Anderson (2015).
6. Our model is more closely related to the standard firm–worker model to which they compare their two-sided model in the discussion. There is only a one-sided inference problem in that model, and they find that positive assortative matching arises for extreme beliefs $p = 0$ and $p = 1$, but conjecture that it does not in the interior.
7. The difficulty is to account for agents switching partners. Anderson and Smith (2010) resolve this by assuming symmetric learning in discrete time. Both sides of the market update in an identical fashion, and under positive assortative matching (PAM) their new matched partner coincides exactly with the updated type of their old partner. As a result, in a candidate PAM equilibrium there is never any switching.
8. This substantially simplifies the problem at hand. With private signals, Cripps *et al.* (2008) show that with a finite signal space there will be common learning, but not necessarily with an infinite signal space as is the case in our model here.
9. Without death, we know that the posterior belief will converge with probability 1 to $p = 1$ or $p = 0$. Death here actually acts as a shuffling device to guarantee a non-trivial stationary distribution of posterior beliefs.
10. However, we can allow σ to be firm-specific. In the first subsection of Section IV, we analyse the general case of firm-dependent σ_y.
11. Bergemann and Välimäki (1996) and Felli and Harris (1996) consider a two-firm, one-worker/buyer model with strategic price setting in a world with independent arms. With *ex ante* heterogeneous firms and workers, and correlated arms, we instead focus on competitive price setting that is closest in spirit to the Beckerian benchmark.
12. Notice that since there is no free entry, V_y need not be zero. We could model free entry as long as in equilibrium there is a non-degenerate distribution of firm types in the economy. We consider that this does not add to the insights of our model.
13. And there is limited liability, that is, workers and firms cannot receive negative payoffs.
14. Note that critically, we need the assumption that the worker does not have any private information about his type. If this assumption is violated, then the worker's value functions could not be written like this.

15. In that case, p can take both values 0 and 1. So the boundedness of the value function requires that both k_{y1} and k_{y2} are zero, and hence $W_y''(p) = 0$ for every p.
16. We slightly abuse notation here since W_L is not defined on p_2. A more precise way of writing the equations is $W_L(p_2+) = W_H(p_2)$ and $W_L'(p_2+) = W_H'(p_2)$. In what follows, we will continue to use the expression in the text in order to economize on notation.
17. This notion is also used implicitly in Sannikov (2007, Prop. 2), and also in Cohen and Solan (2013), who consider deviations from Markovian strategies in bandit problems.
18. For example, in our model, assume $\mu_{HL} = \mu_{LL}$ and the return in the low-type firm is deterministic.
19. In a model of option pricing by Dumas (1991), there does exist a condition on the second derivative called the 'super contact' condition, which is of a very different nature. It arises as the optimal solution to the option pricing problem with proportional cost.
20. Monotonicity is just to help us to find one particular way to divide the surplus. The whole construction of equilibrium also goes through if we do not make this assumption.
21. Here the assumption that there is no heterogeneity in the prior p_0 substantially simplifies the solution of this differential equation. While there is no solution for a general distribution of priors, we have been able to solve the stationary distribution if the priors are drawn from a beta distribution. See also Papageorgiou (2014).
22. Observe that with more unknowns than variables, the solution to our system is indeterminate. In fact, there is potentially a continuum of wages that can be supported in equilibrium, though the allocation will be unique. This indeterminacy is as in Becker (1973): the allocation is unique, but there may be multiple ways to split the surplus. In all that follows, when we use the term uniqueness of equilibrium, we refer to the allocation, not to the wages.
23. This multiplicity is standard in the Beckerian model of assortative matching. For example, in a model with a continuum of firm and worker types, the wages are determined by a differential equation. By assigning different initial conditions, we can derive multiple equilibrium wage schedules, each solving for the same differential equation with a different initial condition.
24. Generally, value-matching and no-deviation conditions imply that

$$(r+\delta)\ W_H(\underline{p}) = w_H(\underline{p}) + \Sigma_H(\underline{p})\ W_H''(\underline{p}) = (r+\delta)\ W_L(\underline{p}) = w_L(\underline{p}) + \Sigma_L(\underline{p})\ W_L''(\underline{p})$$

and

$$W_H''(\underline{p}) = W_L''(\underline{p}).$$

These immediately give that $w_H(\underline{p}) < w_L(\underline{p})$ and $rV_H - rV_L > \mu_H(\underline{p}) - \mu_L(\underline{p})$.
25. The sufficiency of the no-deviation condition is also extended to include all of the combinations of (σ_H, σ_L) as we prove generalized versions of Claim 2 and Lemma 6 in the Appendix.
26. The fully general planner's problem should be dynamic, where the distribution of beliefs is the state variable. But that problem is very complicated, if not impossible to solve. For our purpose therefore, we restrict attention to the problem under stationarity.
27. This property is also established in the one-sided model of Anderson and Smith (2010). Our results show that not only at the extremes but also at the interior, the planner's (and the equilibrium) allocation exhibits PAM.
28. Having a continuous relation between tenure and human capital renders the system of differential equations into a system of partial differential equations. Typically, there is no solution. In the current setup, there is an additional state (experienced versus inexperienced) and the model remains tractable.
29. Observe that experience is worker-dependent, but not firm-dependent. While it is likely a realistic feature to have experience dependent on job type, we would have a different level of experience for different histories, which makes the problem non-tractable.
30. In this section, we maintain the earlier assumption that $\sigma_H = \sigma_L = \sigma$.
31. We can write the value of a worker of type p in firm y as $W_y(p) = w_y(p)\,\mathrm{d}t + (1-(r+\delta)\,\mathrm{d}t)\ W_y(p+\mathrm{d}p)$. Using a Taylor expansion $W_y(p+\mathrm{d}p) = W_y(p) + W_y'(p)\,\mathrm{d}p + o(\mathrm{d}t)$ and the fact that $\mathrm{d}p = \lambda_y p\,\mathrm{d}t$, we obtain the expression for $W_y(p)$.
32. This comes from the fact that $W_L(\cdot)$ is a strictly convex function.
33. Actually, we are using the result that if $G_2(p)/G_1(p)$ is decreasing in p, then $(\int_0^{\underline{p}} G_2(p)\,\mathrm{d}p)/(\int_0^{\underline{p}} G_1(p)\,\mathrm{d}p)$ will also be decreasing in $\underline{p}$. This is true because by the definition of the Riemann integral, $\int_0^{\underline{p}} G_1(p)\,\mathrm{d}p$ and $\int_0^{\underline{p}} G_2(p)\,\mathrm{d}p$ could be written as limits of Riemann sums. The ratio of two Riemann sums is always decreasing in $\underline{p}$ since $G_2(p)/G_1(p)$ is decreasing in p.
34. Things are slightly different if we have $p_0 \in (p_2', p_1')$. Then we have four new distribution coefficients but we also have two more equations: $\widehat{f}_L(p_0-) = \widehat{f}_L(p_0+)$ and $\Sigma_L(p_0)\ (\widehat{f}_L'(p_0-) - \widehat{f}_L'(p_0+)) = \delta$. We can use this system of four linear equations to pin down the four parameters.

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