Optimal Monetary Policy with $r^* < 0$

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Abstract

We study the optimal monetary policy problem in a New Keynesian economy with a zero lower bound (ZLB) on the nominal interest rate, and in which the steady state natural rate (r^*) is negative. We show that the optimal policy aims to approach gradually a steady state with positive average inflation. Around that steady state, inflation and output fluctuate optimally in response to shocks to the natural rate. The central bank can implement that optimal outcome by means of an appropriate state-contingent rule, even though in equilibrium the nominal rate remains at zero most (or all) of the time. In order to establish that result, we derive sufficient conditions for local determinacy in a more general model with endogenous regime switches.

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1 Introduction

Over the past decade, a growing consensus has emerged among academic economists and policymakers pointing to a substantial decline in the *average natural rate of interest*, a variable often referred to as r^*.[1] Some of the likely sources of that decline –including lower productivity growth, demographic factors or enhanced precautionary savings induced by higher uncertainty– suggest that such a downward trend is unlikely to be reversed in the near future.[2]

A low r^* has important implications for monetary policy, due to the presence of a zero lower bound (ZLB) on the nominal interest rate. Thus, and given the inflation target, a lower r^* will generally hamper the ability of monetary policy to stabilize the economy, bringing about more frequent episodes in which the ZLB becomes binding and the economy plunges into a recession with below-target inflation. Not surprisingly, the evidence of a decline in r^* has been a key motivation behind the monetary policy strategy reviews undertaken by many central banks in recent years.

On the research front, and as discussed in the literature review below, several authors have studied the problem of optimal monetary policy in the face of shocks that drive the natural rate of interest *temporarily* into negative territory. A common finding of those analyses is that an optimizing central bank will keep the short-term nominal rate at zero during those episodes, and even for some time once the natural rate has returned to positive values. In all of those analyses, however, the natural rate tends to gravitate towards a positive mean, i.e. $r^* > 0$. By contrast, in the present paper we study the problem of optimal monetary policy under the ZLB constraint when the natural rate fluctuates around a mean that is *negative*, i.e. $r^* < 0$.

As discussed below, that environment is of particular interest since the optimal policy implies *a binding ZLB constraint in the steady state*, a feature that is absent from conventional analyses that assume a positive r^*. The reason is that the coexistence of a negative r^* with a ZLB constraint makes it impossible to support the (first-best) zero inflation outcome in the steady state. In that case, the choice of a zero steady state nominal rate minimizes the (costly)

[1] See, e.g. Brand and Mazelis (2019), Del Negro et al. (2019), Holston et al. (2017).

[2] See, e.g., Eggertsson et al. (2019). Despite the strong inflationary pressures at the time of writing this paper, we believe that the factors behind the decline in r^* not only have not disappeared, but they may have been enhanced by the impact on uncertainty of the COVID pandemic or the Ukrainian war. If that is the case, the consequences of a low r^* and its interaction with the zero lower bound constraint are likely to take again center stage in the policy debate once inflation returns to levels close to target.

deviations from price stability. By contrast, when $r^* > 0$ the optimal steady state is characterized by zero inflation and a strictly positive nominal rate, and the conventional analysis of optimal policy under a ZLB constraint applies.

While the assumption of a negative r^* is at odds with the predictions of a standard macro framework with an infinite-lived representative consumer, it can be microfounded once the latter assumption is relaxed. Thus, for instance, models with overlapping generations or heterogenous agents and idiosyncratic shocks can generate a negative r^* under suitable calibrations. Furthermore, we believe the assumption of a negative r^* is more than a theoretical curiosum: recent estimates of the evolution of the natural rate in advanced economies display a downward trend that has already attained negative territory for some of them.[3] In any event, the relevance of a negative r^* can hardly be dismissed as a real possibility in a not too distant future, if the trends in some of the fundamental forces behind the recent decline in the natural rate were to persist or even strengthen further.

As much of the related literature, we cast our analysis of the optimal monetary policy problem in the context of an otherwise standard New Keynesian model subject to a ZLB constraint.[4] A number of interesting results emerge from that exercise.

Firstly, our findings show that the optimal policy aims at steering the economy *gradually* towards a steady state characterized by positive inflation. Thus, and even though the combination of a negative r^* and the ZLB constraint rules out the first-best outcome of a zero inflation steady state, the choice of a gradual transition (rather than an immediate jump to the new steady state) makes it possible for inflation to remain closer to zero –its efficient value– for a longer period, which is welfare improving.

Secondly, once the steady state is reached, inflation and output fluctuate optimally in response to shocks to the natural rate, even though the nominal rate remains at zero most of the time (*all* the time in our baseline calibration). We show that, behind the appearance of extreme passivity, the central bank can implement the optimal outcome by means of an ap-

[3]See, e.g., Brand and Mazelis (2019)

[4]We use the textbook New Keynesian model as a framework in which we revisit the optimal policy problem in the presence of a negative r^*. This is meant to highlight in a most transparent way the key *qualitative* implications of a negative r^* for monetary policy. We believe that adding additional "realistic" features to the model (e.g. imperfect credibility, parameter uncertainty, investment, etc) would complicate the analysis without qualitatively altering or shedding additional light on those key implications.

propriate state-contingent rule which calls for one-sided adjustments in the nominal rate in response to (off-equilibrium) deviations from the desired inflation and output paths. In order to establish that result, we derive and exploit a sufficient condition for local determinacy for a relatively general class of models with endogenous regime switches, a finding which we believe has some independent interest.

The rest of the paper is organized as follows. The remaining of the present section provides a brief review of the related literature. Section 2 formulates the optimal policy problem and derives the associated optimality conditions. Section 3 analyzes the economy's (deterministic) transitional dynamics under the optimal policy. Section 4 characterizes the fluctuations of inflation and output around the steady state, in response to natural rate shocks. Section 5 discusses the implementation of the optimal plan, deriving sufficient conditions on the coefficients of a proposed interest rate rule to support the optimal plan as a unique equilibrium. Section 6 concludes.

1.1 Related Literature

Our paper is related to a branch of the literature that studies the optimal design of monetary policy in the presence of a ZLB constraint on the nominal rate. Since Krugman (1998), a number of articles have studied optimal monetary policy with an occasionally binding zero lower bound (ZLB) on the nominal interest rate. Closest to us is the work by Eggertsson and Woodford (2003), Jung, Teranishi and Watanabe (2005), Adam and Billi (2006), and Nakov (2008), who analyze the problem of optimal policy under commitment in the basic New Keynesian model with a ZLB constraint. A different line of work has focused on the implications of the ZLB for the optimal choice of an inflation target, conditional on a given interest rate rule. Relevant papers include Coibion et al. (2012), Bernanke et al. (2019), and Andrade et al. (2020, 2021). In all the papers above, however, the natural interest rate remains negative only temporarily, and the binding ZLB is a transitory phenomenon. In contrast, the analysis of the present paper assumes a negative steady state natural rate, and hence a long-lasting "secular stagnation" environment with a ZLB that is binding in the steady state (and a large fraction of the time, outside the steady state).[5]

The present paper is also related to a rather different branch of the literature, one studying the conditions for equilibrium determinacy in regime-switching models. Applications of this literature have typically focused on regime switches driven by *exogenous* stochastic variations in the coefficients of a Taylor-type interest rate rule, which are often assumed to follow a finite-state Markov process. Prominent examples include Davig and Leeper (2007), Farmer et al. (2009) and Barthélemy and Marx (2019). The main difference in our approach is that we allow for *endogeneity* in the regime switches, i.e. the regime is a function of the state.[6] That endogeneity arises as a consequence of the particular nonlinearity embedded in the interest rate rule that implements the optimal allocation, which makes the effective coefficients of the corresponding linear model depend on the levels of inflation and output.[7] We believe our finding may be of interest beyond our specific application, since it should apply to a wide range of linear stochastic models with endogenous regime switches.

2 The Optimal Monetary Policy Problem

The equilibrium conditions describing the economy's non-policy block are assumed to be given by

$$\pi_t = \beta \mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t \tag{1}$$

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t^n) \tag{2}$$

for $t = 0, 1, 2, ...$where π_t denotes inflation, y_t is the output gap, i_t is the short-term nominal rate and r_t^n is the natural rate of interest. Equation (1) is the familiar New Keynesian Phillips curve, which can be derived from the aggregation of firms' price setting decisions in an environment with price rigidities à la Calvo (1983). Equation (2) is the so-called dynamic IS equation, which results from combining an Euler equation for (log) aggregate consumption, a goods market

[5] Such an environment is reminiscent of that described in Summer's celebrated speech on secular stagnation at the 2013 IMF annual Research Conference (Summers (2015)).

[6] Barthélemy and Marx (2017) also allow for endogeneity of the regime switches but only of a sort with continuous transition probabilities, which rules out the threshold switches that arise naturally in models with a ZLB constraint like ours.

[7] One drawback of our approach, of limited consequence in our particular application, is that it only allows us to derive *sufficient* conditions for determinacy, i.e. we cannot establish necessity, in contrast with the papers mentioned above.

clearing condition and an equation describing the evolution of output and the real interest rate under flexible prices.[8]

Variations in the natural rate of interest r_t^n are assumed to be described by

$$r_t^n = r^* + z_t \tag{3}$$

where $\{z_t\}$ follows an exogenous $AR(1)$ process with zero mean, autoregressive coefficient ρ_z and innovation variance σ_z^2. The unconditional mean of the natural rate is given by r^*, which coincides with the real interest rate, $r_t \equiv i_t - \mathbb{E}_t\{\pi_{t+1}\}$, in the deterministic steady state. Henceforth, we assume

$$r^* < 0 \tag{4}$$

In a companion appendix, we formally describe an environment where (1) and (2) obtain as equilibrium conditions, and where the steady state real interest rate may be negative. The proposed environment is a version of a New Keynesian model with overlapping generations (NK-OLG) à la Blanchard-Yaari, as developed in Galí (2021). In that environment the steady state real interest is not fully pinned down by the discount rate; instead it also depends on the extent to which income of any given cohort declines over time as a result of retirement or other shocks that make individuals leave employment permanently (e.g. skill obsolescence). That phenomenon tends to enhance savings, lowering the steady state real rate, which may take a negative value.[9]

The monetary authority is assumed to choose at $t = 0$ a state-contingent sequence $\{y_t, \pi_t\}_{t=0}^{\infty}$ that minimizes the welfare loss function

$$\frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \vartheta y_t^2\right)$$

subject to the sequence of constraints (1) and (2), as well the ZLB constraint

$$i_t \geq 0 \tag{5}$$

[8] See, e.g. Woodford (2003) or Galí (2015) for a derivation of (1) and (2) in a standard New Keynesian model. In a companion appendix, we show that similar equilibrium conditions obtain in an OLG version of the New Keynesian model that allows for a negative steady state real rate, as considered below.

[9] As is well known, other departures from the representative consumer assumption are also consistent with a negative steady state real rate, e.g., models with heterogenous households subject to idiosyncratic income shocks, as in Aiyagari (1994) or Huggett (1993). In contrast with the NK-OLG model, those models do not generally yield an aggregate Euler equation like (2), though the latter has been shown to constitute a good approximation under plausible calibrations (see, e.g, Debortoli and Galí (2022)).

all for $t = 0, 1, 2, ..$[10]

Note that the ZLB constraint can be rewritten in terms of inflation and the output gap as:

$$r_t^n + \mathbb{E}_t\{\pi_{t+1}\} + \sigma(\mathbb{E}_t\{y_{t+1}\} - y_t) \geq 0 \tag{6}$$

for $t = 0, 1, 2, ..$

The (discounted) Lagrangian is given by:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2} \left(\pi_t^2 + \vartheta y_t^2 \right) - \xi_{1,t}(\pi_t - \kappa y_t - \beta \pi_{t+1}) - \xi_{2,t}[\pi_{t+1} + \sigma(y_{t+1} - y_t)] \right]$$

The associated optimality conditions are:

$$\pi_t = \xi_{1,t} - \xi_{1,t-1} + \beta^{-1} \xi_{2,t-1} \tag{7}$$

$$\vartheta y_t = -\kappa \xi_{1,t} - \sigma \xi_{2,t} + \sigma \beta^{-1} \xi_{2,t-1} \tag{8}$$

$$\xi_{2,t} \geq 0 \tag{9}$$

$$\xi_{2,t} \ \ [r_t^n + \mathbb{E}_t\{\pi_{t+1}\} + \sigma(\mathbb{E}_t\{y_{t+1}\} - y_t)] = 0 \tag{10}$$

which should be interpreted as holding for each time period and each possible state. The previous conditions, combined with (1), (2), (3), (6) and the initial conditions $\xi_{1,-1} = \xi_{2,-1} = 0$, describe the economy's equilibrium under the optimal policy.[11]

In the next two sections, we characterize that equilibrium and provide simulations for a calibrated version of the model. First we study the transitional dynamics. Then we look at the economy's response to shocks in a neighborhood of the steady state.

3 Transitional Dynamics under the Optimal Monetary Policy

In the present section we focus on the equilibrium implied by the optimal policy in the absence of shocks, with $r_t^n = r^* < 0$ for $t = 0, 1, 2, ...$

[10] As discussed in the companion appendix, the previous loss function can be microfounded as the second order approximation to the expected welfare losses of individuals currently alive in a New Keynesian model with overlapping generations.

[11] The initial conditions reflect formally the fact that past constraints are irrelevant when solving for the optimal policy at time 0.

We start by characterizing the perfect foresight steady state under the optimal policy. In that steady state we must have $i = \pi + r^* \geq 0$ or, equivalently, $\pi \geq -r^* > 0$. In addition, it follows from (7)-(10) that under the optimal policy:

$$\pi = \beta^{-1}\xi_2 \geq 0$$

$$\vartheta y = -\kappa \xi_1 + \sigma(\beta^{-1} - 1)\xi_2$$

$$\xi_2 \geq 0 \; ; \; r^* + \pi = 0 \; ; \; \xi_2(r^* + \pi) = 0$$

It is easy to check that the optimal policy requires that $i = 0$. To see this, note that if $i > 0$ then $\xi_2 = 0$ implying $\pi = 0$, which is inconsistent with a steady state. Thus the steady state under the optimal policy must satisfy:

$$\pi = -r^* > 0$$

$$y = \frac{1-\beta}{\kappa}\pi = -\frac{1-\beta}{\kappa}r^* > 0$$

$$\xi_2 = \beta\pi = -\beta r^* > 0$$

$$\begin{aligned} \xi_1 &= -\frac{\vartheta}{\kappa}y + \frac{\sigma(\beta^{-1} - 1)}{\kappa}\xi_2 \\ &= -\frac{(1-\beta)}{\kappa}\left(\sigma - \frac{\vartheta}{\kappa}\right)r^* \end{aligned}$$

Note that the previous steady state is (globally) unique. This contrasts with the multiplicity of steady states that generally arise in the presence of the ZLB constraint when the central bank follows a Taylor-type interest rate rule (see, e.g., Benhabib et al. (2001)).

Next we study the transitional dynamics, i.e. we characterize the equilibrium paths that satisfy

$$\widehat{\pi}_t = \beta\widehat{\pi}_{t+1} + \kappa\widehat{y}_t$$

$$\widehat{\pi}_t = \widehat{\xi}_{1,t} - \widehat{\xi}_{1,t-1} + \beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\vartheta\widehat{y}_t = -\kappa\widehat{\xi}_{1,t} - \sigma\widehat{\xi}_{2,t} + \sigma\beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\widehat{\xi}_{2,t} + \xi_2 \geq 0$$

$$\widehat{\pi}_{t+1} + \sigma(\widehat{y}_{t+1} - \widehat{y}_t) \geq 0$$

$$(\widehat{\xi}_{2,t} + \xi_2)\ \ [\widehat{\pi}_{t+1} + \sigma(\widehat{y}_{t+1} - \widehat{y}_t)] = 0$$

for $t = 0, 1, 2, ...$with initial conditions $\widehat{\xi}_{1,-1} = -\xi_1$ and $\widehat{\xi}_{2,-1} = -\xi_2$, and such that $\lim_{t\to\infty} \widehat{x}_t = 0$ for $\widehat{x}_t \in \{\widehat{\pi}_t, \widehat{y}_t, \widehat{\xi}_{1,t}, \widehat{\xi}_{2,t}\}$, where a "$\widehat{\ }$" symbol on a variable denotes deviations from the corresponding steady state value.

Figure 1 illustrates the transitional dynamics for a calibrated version of our economy.[12] In particular, we assume $\sigma = 1$, $\beta = 0.99$, $\kappa = 0.1717$, $\vartheta = 0.0191$, which are values consistent with the baseline calibration in Galí (2015). In addition, we set $r = -0.0025$, implying an annualized steady state natural rate of minus 1 percent. Interest rates and the inflation rate are shown in annualized terms in all figures.

As shown in Figure 1, the transition to the steady state under the optimal policy is not immediate. Instead, the initial values of inflation and the output gap are significantly below their long run values of 1 and 0.058 percent, respectively, and adjust only gradually towards that steady state. In fact, inflation is negative for a few periods under our baseline calibration.[13] By choosing a path like the one depicted in Figure 1, the central bank succeeds in keeping inflation close to the first best temporarily, even though it is at the cost of a persistently negative output gap. Given the relative small weight of the latter under our baseline calibration ($\vartheta \simeq 0.02$), that choice turns out to be more desirable than jumping immediatly to the steady state (which would be perfectly feasible). The persistent low inflation and output gaps are consistent with the observed path for the real rate, which remains above its long run value r during the transition. Most interestingly, the path for the real rate is entirely driven by expected inflation, since the nominal rate remains at the ZLB throughout the transition. Thus, the central bank manages to implement its nontrivial optimal plan while keeping the setting for its policy instrument unchanged. In section 5 below, we discuss how the central bank may implement the optimal outcome, given the multiplicity of equilibrium paths consistent with a constant nominal rate.

[12]We use Dynare's perfect foresight solver, based on Kanzow and Petra (2004), to compute the transition paths.

[13]The result of an optimal negative inflation in the short run is not general. In particular, it doesn't obtain when the weight on the output gap is raised sufficiently (e.g. when $\vartheta = 1$).

4 Aggregate Fluctuations under the Optimal Monetary Policy

In this section, we characterize the behavior of inflation and the output gap under the optimal policy in a neighborhood of the steady state, in response to shocks to the natural rate (i.e. fluctuations in z_t). The (local) equilibrium dynamics are described by the system of stochastic difference equations given by:

$$\widehat{\pi}_t = \beta \mathbb{E}_t\{\widehat{\pi}_{t+1}\} + \kappa \widehat{y}_t$$

$$\widehat{\pi}_t = \widehat{\xi}_{1,t} - \widehat{\xi}_{1,t-1} + \beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\vartheta \widehat{y}_t = -\kappa \widehat{\xi}_{1,t} - \widehat{\xi}_{2,t} + \beta^{-1}\widehat{\xi}_{2,t-1}$$

$$\widehat{\xi}_{2,t} + \xi_2 \geq 0$$

$$\sigma(\mathbb{E}_t\{\widehat{y}_{t+1}\} - \widehat{y}_t) + \mathbb{E}_t\{\widehat{\pi}_{t+1}\} + z_t \geq 0$$

$$[\widehat{\xi}_{2,t} + \xi_2][\sigma(\mathbb{E}_t\{\widehat{y}_{t+1}\} - \widehat{y}_t) + \mathbb{E}_t\{\widehat{\pi}_{t+1}\} + z_t] = 0$$

for $t = 0, 1, 2, ...$with initial conditions now given by $\widehat{\xi}_{1,-1} = 0$ and $\widehat{\xi}_{2,-1} = 0$. Appendix A describes our approach to determining the solution to the system above.

Figure 2 displays the equilibrium path for inflation and the output gap under the optimal policy, given a sequence of realized values of the shock $\{z_t\}$, drawn from an $AR(1)$ process with $\rho_z = 0.5$ and $\sigma_z = 0.0025$. The remaining parameters are kept at their baseline settings. The top-left box of the Figure displays the simulated path of the natural rate (in black) and the actual real rate (in blue). Note that the latter is much smoother than the former, which reflects the central bank's inability to match one-for-one fluctuations in the natural rate, due to the ZLB constraint. As a result, the central bank can't prevent some fluctuations in inflation and the output gap around the steady state, as illustrated in the two bottom plots. Furthermore the nominal rate remains at the ZLB throughout the simulation, as shown on the top-right plot. Thus, the central bank manages to steer the economy along the optimal path without changing the settings for its policy instrument, and keeping it instead constant at its steady state level. The reason why it does not lower the nominal rate in the face of negative natural rate shocks is clear: the ZLB prevents it from doing so. Perhaps less obvious is why it keeps the nominal rate at zero even when the natural rate lies above its steady state value. Intuitively, the anticipation that the central bank will keep the interest rate lower than the natural rate when the latter is high helps stabilize inflation and the output gap when the natural rate is low (and can thus not be matched due to the ZLB). That policy, which relies on the forward looking nature of aggregate demand and inflation, can thus be viewed as a form of forward guidance. In the simulation shown in Figure 2, the contemporaneous stabilizing gains from raising the nominal rate above zero, in order to bring it closer to the natural rate when the latter is high do not compensate the gains in earlier periods with a low natural rate from the anticipation of a constant zero nominal rate in the future. As a result, the nominal rate remains at the ZLB throughout the simulation.

The property of a constant nominal rate at zero is not general, however. In particular, the central bank may find it desirable to deviate from the constant zero nominal rate policy in response to an increase in the natural rate of interest that is sufficiently large and which may thus induce very high inflation if not counteracted at least partly by an increase in the nominal rate. This is illustrated in Figure 3, which shows a simulation of equilibrium fluctuations in a calibrated economy identical to that underlying the simulations of Figure 2 except for a higher shock volatility, with $\sigma_z = 0.0075$. Thus, in the simulation shown in Figure 3 there are three episodes in which the central bank optimally chooses to raise the nominal rate above zero, even if only briefly. Roughly speaking, those episodes can be seen to take place when two conditions are met simultaneously: (i) the natural interest rate is unusually high, and (ii) this has not been preceded by a recent episode with an unusually low natural rate, for in the latter case it would be desirable to keep the nominal rate "low for longer" for the reasons discussed above. Note, however, that the nominal rate remains unchanged at the ZLB for much of the simulation.

A similar qualitative finding emerges when we replace shocks to the natural rate with cost-push shocks, i.e. exogenous disturbances to the New Keynesian Phillips curve (1). As is well known, in that case a trade-off between inflation stabilization and output gap stabilization emerges independently of the presence of a ZLB (see, e.g. Clarida et al. (1999)), with the optimal policy calling for output gap variations in order to dampen fluctuations in inflation. As in the environment analyzed above, with a negative r^*, and in the absence of very large shocks,the (second-best) management of output and inflation fluctuations is consistent with a nominal rate that remains at zero all (or most of) the time (simulations not shown).

How the central bank manages to steer the economy as required by the solution to its optimal policy problem while keeping the nominal rate unchanged at zero is the subject of the next section.

5 Implementing the Optimal Monetary Policy when the ZLB Constraint is Nearly-Always Binding

Let (i_t^*, y_t^*, π_t^*) denote the central bank's optimal plan, i.e. the solution to the policy problem analyzed in the previous sections. Consider next deviations from the optimal plan satisfying the equilibrium conditions (1), (2) and (5). Formally, and letting $\widetilde{\pi}_t \equiv \pi_t - \pi_t^*$, $\widetilde{y}_t \equiv y_t - y_t^*$ and $\widetilde{i}_t \equiv i_t - i_t^*$, we have

$$\widetilde{\pi}_t = \beta \mathbb{E}_t\{\widetilde{\pi}_{t+1}\} + \kappa \widetilde{y}_t \tag{11}$$

$$\widetilde{y}_t = \mathbb{E}_t\{\widetilde{y}_{t+1}\} - \frac{1}{\sigma}(\widetilde{i}_t - \mathbb{E}_t\{\widetilde{\pi}_{t+1}\}) \tag{12}$$

as well as the ZLB constraint

$$\widetilde{i}_t \geq -i_t^* \tag{13}$$

for all t.[14]

We complement the previous equations with the following interest rate rule

$$\widetilde{i}_t = \phi_\pi \left|\widetilde{\pi}_t\right| + \phi_y \left|\widetilde{y}_t\right| \tag{14}$$

where $(\phi_\pi, \phi_y) \geq 0$. According to the rule, the central bank commits to deviating from the nominal rate path prescribed by the optimal plan whenever inflation and/or the output gap deviate from their corresponding optimal paths. Note that (14) guarantees that $i_t \geq i_t^* \geq 0$ for all t, thus meeting the ZLB constraint (13) at all times, even on any off-equilibrium path.

Note that $\widetilde{\pi}_t = \widetilde{y}_t = \widetilde{i}_t = 0$ for all t is always a solution to the system (11)-(14). Our objective is to study the conditions (if any) on (ϕ_π, ϕ_y) that guarantee that the previous solution is (locally) unique or, equivalently, that the optimal plan is effectively implemented.

[14] Note that the previous representation in terms of equilibrium deviations from the optimal plan holds independently of the underlying source of fluctuations (natural rate shocks or cost-push shocks). More generally, (i_t^*, y_t^*, π_t^*) can be interpreted as the central bank's desired equilibrium path, which may or may not coincide with the solution to the optimal policy problem analyzed above.

We tackle this problem in two stages. First we rewrite the interest rate rule (14) in a way that allows us to reformulate our model of deviations from the optimal plan as a regime switching model. In particular we rewrite (14) as a piecewise linear rule given by

$$\widetilde{i}_t = \begin{cases} \phi_\pi \widetilde{\pi}_t + \phi_y \widetilde{y}_t & \text{if } \widetilde{\pi}_t \geq 0 \text{ and } \widetilde{y}_t \geq 0 \\ -\phi_\pi \widetilde{\pi}_t - \phi_y \widetilde{y}_t & \text{if } \widetilde{\pi}_t < 0 \text{ and } \widetilde{y}_t < 0 \\ \phi_\pi \widetilde{\pi}_t - \phi_y \widetilde{y}_t & \text{if } \widetilde{\pi}_t \geq 0 \text{ and } \widetilde{y}_t < 0 \\ -\phi_\pi \widetilde{\pi}_t + \phi_y \widetilde{y}_t & \text{if } \widetilde{\pi}_t < 0 \text{ and } \widetilde{y}_t \geq 0 \end{cases} \tag{15}$$

where $(\phi_\pi, \phi_y) \geq 0$. The resulting system, consisting of (11), (12) and (15), can be viewed as a regime switching model, with *endogenous* regime switches.

In a second stage, to which we turn next, we apply a novel result that allows us to establish sufficient conditions for the (local) uniqueness of the solution of an endogenous regime switching model. Given its potential interest beyond the problem at hand, we first state our result for a more general setting before we apply it to the model above.

5.1 A Sufficient Condition for Equilibrium Determinacy of an Endogenous Regime Switching Model

Consider a regime switching model whose equilibrium is described by a system of difference equations of the form:

$$\mathbf{x}_t = \mathbf{A}_t \mathbb{E}_t\{\mathbf{x}_{t+1}\} \tag{16}$$

where $\mathbf{x}_t$ is an $(n \times 1)$ vector of non-predetermined variables and A_t is an $(n \times n)$ matrix. We assume $\mathbf{A}_t \in \mathcal{A}$ where $\mathcal{A} \equiv \{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, ..., \mathbf{A}^{(Q)}\}$ is a finite set of $(n \times n)$ nonsingular matrices. The evolution of $\mathbf{A}_t$ over time is left unspecified. It may evolve exogenously, e.g. according to a Markov process. Alternatively, $\mathbf{A}_t$ may vary endogenously, i.e. it may be a function of current and lagged values of $\mathbf{x}_t$.

It is clear that $\mathbf{x}_t = 0$ for all t is a solution to (16). Our goal is to establish *sufficient* conditions on $\mathcal{A}$ that guarantee that $\mathbf{x}_t = 0$ all t is the only bounded solution to (16). We take this to be the case if $\lim_{T\to+\infty} \mathbb{E}_t\{\|\mathbf{x}_{t+T}\|\} > M\|\mathbf{x}_t\|$ for any scalar $M > 0$ and $\mathbf{x}_t \neq 0$, and where $\|\cdot\|$ is the usual L^2 norm.

Let us define the induced matrix norm $\|\mathbf{A}\| \equiv \max_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|$ subject to $\|\mathbf{x}\| = 1$. In addition, define $\alpha \equiv \max\{\left\|\mathbf{A}^{(1)}\right\|, \left\|\mathbf{A}^{(2)}\right\|, ... \left\|\mathbf{A}^{(Q)}\right\|\}$. Note that nonsingularity of $\mathbf{A}^{(q)}$ for $q = 1, 2, ...Q$ implies $\alpha > 0$.

Theorem *[sufficient condition for determinacy]*: If $\alpha < 1$, then $\mathbf{x}_t = 0$ for all t is the only bounded solution to (16)

Proof: See Appendix B

Remark: the previous condition is sufficient but not necessary. As a counterexample consider a switching regime model given by (16) with $\mathbf{A}_t = \mathbf{A}^{(1)}$ for odd t and $\mathbf{A}_t = \mathbf{A}^{(2)}$ for even t, where

$$\mathbf{A}^{(1)} = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.5 \end{bmatrix} \quad ; \quad \mathbf{A}^{(2)} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.1 \end{bmatrix}$$

Note that the previous model does not satisfy the sufficiency condition since $\alpha = 1.1 > 1$. Yet, $\mathbf{x}_t = 0$ can be shown to be the only bounded solution. See Appendix C for a proof.[15]

Remark: note that $\|\mathbf{A}\| < 1$ implies that all the eigenvalues of $\mathbf{A}$ lie within the unit circle, though the converse is not true. See Appendix D for a proof. Hence our sufficient condition $\alpha < 1$ also implies that $\mathbf{x}_t = 0$ is the unique bounded solution to each of the "single regime" models $\mathbf{x}_t = \mathbf{A}^{(q)}\mathbb{E}_t\{\mathbf{x}_{t+1}\}$, for $q = 1, 2, ..., Q$. The previous result is consistent with the finding in Barthélemy and Marx (2019), in the context of a New Keynesian model with exogenous switches in the interest rate rule coefficients, showing that indeterminacy may emerge even if each of the regimes adheres to the Taylor principle (i.e. it satisfies the eigenvalue condition for uniqueness in the corresponding single regime economy).

Remark: an alternative sufficient condition for determinacy is given by $\rho(\mathcal{A}) < 1$, where $\rho(\mathcal{A}) \equiv \lim_{T\to+\infty} \max\{||A_{i_1}A_{i_2}\cdots A_{i_T}||^{\frac{1}{T}} : A_i \in \mathcal{A}\}$ is the joint spectral radius of $\mathcal{A}$. The proof is almost identical to that in Appendix B. Note that this alternative condition is weaker than $\alpha < 1$ but is not necessary either. In particular, the counterexample above also applies, since $\rho(\mathcal{A}) > 1.1$. We prefer to work with the norm condition since it is easier to check computationally.

5.2 Application to the Problem of Optimal Policy Implementation

Next, we apply the result of the previous subsection to the problem of implementation of the optimal monetary policy analyzed above. Recall that feasible deviations from the optimal outcome are described by (11), (12) and (15), with the latter effectively defining four regimes.

[15] We thank Danila Smirnov for suggesting this counterexample.

Plugging (15) into (12) to eliminate $\widetilde{i}_t$, and after some straightforward substitutions, we can represent the dynamics for $\mathbf{x}_t \equiv [\widetilde{y}_t, \widetilde{\pi}_t]'$ as in (16), with

$$\mathbf{A}^{(1)} \equiv \frac{1}{\sigma + \phi_y + \kappa\phi_\pi} \begin{bmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{bmatrix}$$

$$\mathbf{A}^{(2)} \equiv \frac{1}{\sigma - \phi_y - \kappa\phi_\pi} \begin{bmatrix} \sigma & 1 + \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma - \phi_y) \end{bmatrix}$$

$$\mathbf{A}^{(3)} \equiv \frac{1}{\sigma - \phi_y + \kappa\phi_\pi} \begin{bmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma - \phi_y) \end{bmatrix}$$

$$\mathbf{A}^{(4)} \equiv \frac{1}{\sigma + \phi_y - \kappa\phi_\pi} \begin{bmatrix} \sigma & 1 + \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{bmatrix}$$

corresponding to the four regimes defined above (i.e., $Q = 4$).

The green (dark) area in Figure 4 displays the configurations of (ϕ_π, ϕ_y) values for which $\alpha < 1$, i.e. for which $\left\|\mathbf{A}^{(q)}\right\| < 1$, for $q \in \{1, 2, 3, 4\}$. Thus, to the extent that the central bank adopts rule (15) with coefficients that fall within the depicted determinacy region, no deviations from the desired allocation will be consistent with a (bounded) equilibrium, and hence the rule will indeed implement the desired allocation (y_t^*, π_t^*), while satisfying the ZLB constraint.

Finally, a word about some of the rule's implications. The rule instructs the central bank to deviate from the interest rate i_t^* implied by the optimal policy if and only if inflation and/or output deviate from their optimal values, π_t^* and y_t^*. If the rule coefficients satisfy the sufficient condition for a unique equilibrium (as assumed in our simulations), those deviations remain off-equilibrium, i.e. they never materialize ex-post. While the previous feature is often found in interest rate rules that implement a desired feasible allocation,[16] a specific characteristic of our nonlinear rule is that all its implied off-equilibrium deviations are *positive*, i.e. they involve raising the nominal interest rate above i_t^*. That property guarantees that that the ZLB constrained is never violated, not even on off-equilibrium paths, given that $i_t^* \geq 0$ for all t (with $i_t^* = 0$ most of the time in our simulations). Needless to say, some of the off-equilibrium interest rate movements called for by the rule may be perceived ex-post as being suboptimal (e.g. raising the interest rate if inflation falls below its desired level), but this sort of time inconsistency is inherent to optimal policies under commitment even in the absence of the ZLB

[16] See, e.g. the discussion in Galí (2015, chapters 4 and 5) regarding the implementation of optimal policies through interest rate rules, in the context of a baseline New Keynesian model without a ZLB constraint.

constraint, their benefits arising from the (desirable) effects of their anticipation (as it is the case here).[17]

6 Concluding Remarks

The analysis in the present paper has shown that a central bank may feel compelled to keep the policy rate at zero for an indefinite period in response to a permanent decline in the natural rate of interest, so that the latter's mean, r^*, becomes negative. We have also shown that in such an environment, and despite the possible constancy of the policy rate, a fully credible central bank operating under commitment can keep influencing macro outcomes and implement the constrained-efficient allocation in the face of continuous shocks that may impinge on the economy.

More specifically, we have studied the optimal monetary policy problem in a New Keynesian economy with a zero lower bound (ZLB) on the nominal interest rate, and in which the natural rate of interest has a negative mean, i.e. $r^* < 0$. We have shown that the optimal policy in that environment aims to approach *gradually* a steady state with positive average inflation. A gradualist approach minimizes welfare losses by keeping inflation close to zero for longer.

Around that steady state, inflation and the output gap have been shown to fluctuate in response to shocks to the natural rate, since the central bank is unable to fully stabilize those variables at their (first-best) zero value due to the ZLB constraint. Under the optimal policy, persistent fluctuations in the output gap and inflation coexist with a nominal rate that remains at its ZLB most (or all) of the time.

Finally we have shown that the central bank can implement the optimal policy as a (locally) unique equilibrium by means of an appropriate nonlinear state-contingent rule consistent with the ZLB. In order to establish that result, we derive a sufficient condition for local determinacy in a more general model of endogenous regime switches. That result may be of interest beyond the problem studied in the present paper.

[17] Departures from the assumption of full credibility adopted here will generally have implications on the optimal policy outcomes. Given the absence of a widely accepted model of imperfect credibility we do not pursue this avenue here.

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APPENDIX A: Solving for the local equilibrium dynamics under the optimal policy

We use the numerical algorithm for solving rational expectations models as implemented in the CompEcon toolkit of Miranda and Fackler (2002). In particular, we solve for the optimal policy x as a function of the state s, when equilibrium is governed by a system of the form

$$f[s_t, x_t, E_t h(s_{t+1}, x_{t+1})] = \xi_t$$

where s follows the state transition function

$$s_{t+1} = g(s_t, x_t, \varepsilon_{t+1})$$

and x_t and ξ_t in our case satisfy the following Kuhn-Tucker condition

$$i_t \geqslant 0, \quad \xi_{2t} \geqslant 0, \quad i_t > 0 \Rightarrow \xi_{2t} = 0.$$

The solution is obtained with the collocation method, which consists of approximating the expectation functions by linear combinations of known basis functions, θ_j. The corresponding coefficients, c_j, are determined by requiring the approximating function to satisfy the equilibrium equations *exactly* at n collocation nodes:

$$h[s, x(s)] \approx \sum_{j=1}^{n} c_j \theta_j(s)$$

For a given value of the coefficient vector c, the equilibrium policies x_i are computed at the n collocation nodes s_i by solving a standard root-finding problem. The coefficient vector c is updated solving the n-dimensional linear system

$$\sum_{j=1}^{n} c_j \theta_j\left(s_i\right) = h\left(s_i, x_i\right)$$

The previous iterative procedure is repeated until the distance between successive values of c becomes sufficiently small. To approximate the expectation functions, we discretize the innovation to r_t^n using a K-node Gaussian quadrature scheme:

$$Eh[s, x(s)] \approx \sum_{k=1}^{K} \sum_{j=1}^{n} \omega_k c_j \theta_j\left[g\left(s_i, x, \varepsilon_k\right)\right]$$

where ε_k and ω_k are Gaussian quadrature nodes and weights chosen so that the discrete distribution approximates the continuous univariate normal distribution $N(0, \sigma^2)$. We use linear splines on a uniform grid of 200 points for values of the natural rate of interest between -10 percent and $+10$ percent, so that each point on the grid corresponds to 10 basis points.

APPENDIX B: Proof of Theorem [sufficiency conditions for determinacy]

By the law of iterated expectations

$$
\begin{aligned}
\mathbf{x}_t &= \mathbf{A}_t \mathbb{E}_{t+T-1}\{\mathbf{x}_{t+1}\} \\
&= \mathbb{E}_t\{\mathbf{A}_t \mathbf{A}_{t+1} \cdots \mathbf{A}_{t+T-1} \mathbf{x}_{t+T}\}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\mathbf{x}_t\| &= \|\mathbb{E}_t\{\mathbf{A}_t \mathbf{A}_{t+1} \cdots \mathbf{A}_{t+T-1} \mathbf{x}_{t+T}\}\| \\
&\leq \mathbb{E}_t\{\|\mathbf{A}_t \mathbf{A}_{t+1} \cdots \mathbf{A}_{t+T-1} \mathbf{x}_{t+T}\}\|\} \\
&\leq \mathbb{E}_t\{\|\mathbf{A}_t \mathbf{A}_{t+1} \cdots \mathbf{A}_{t+T-1}\| \, \|\mathbf{x}_{t+T}\|\} \\
&\leq \alpha^T \, \mathbb{E}_t\{\|\mathbf{x}_{t+T}\|\}
\end{aligned}
$$

where the last inequality uses the fact that

$$
\|A_{i_1} A_{i_2} \cdots A_{i_T}\| \leq \|A_{i_1}\| \, \|A_{i_2}\| \cdots \|A_{i_T}\| \leq \alpha^T
$$

where $A_i \in \mathcal{A}$.

Accordingly, $\alpha < 1$ implies that $\lim_{T \to +\infty} \mathbb{E}_t\{\|\mathbf{x}_{t+T}\|\} > M \|\mathbf{x}_t\|$ for any arbitrarily large $M > 0$ and $\mathbf{x}_t \neq 0$. *QED.*

APPENDIX C [A Counterexample]

Letting $\mathbf{A} \equiv \mathbf{A}^{(1)} \mathbf{A}^{(2)} = \mathbf{A}^{(2)} \mathbf{A}^{(1)}$ we can write

$$
\mathbf{x}_t = \mathbf{A}^T \mathbb{E}_t\{\mathbf{x}_{t+2T}\}
$$

Thus,

$$
\begin{aligned}
\|\mathbf{x}_t\| &\leq \left\|\mathbf{A}^T\right\| \; \mathbb{E}_t\{\|\mathbf{x}_{t+2T}\|\} \\
&= \|\mathbf{A}\|^T \; \mathbb{E}_t\{\|\mathbf{x}_{t+2T}\|\}
\end{aligned}
$$

In our numerical example $\|\mathbf{A}\| = 0.55 < 1$. Accordingly,

$$\mathbb{E}_t\{\|\mathbf{x}_{t+2T}\|\} = 0.55^{-T}\,\|\mathbf{x}_t\|$$

whichi implies $\lim_{T\to+\infty}\mathbb{E}_t\{\|\mathbf{x}_{t+T}\|\} > M\,\|\mathbf{x}_t\|$ for any arbitrarily large $M > 0$ and $\mathbf{x}_t \neq 0$. *QED.*

APPENDIX D [Eigenvalue vs. Norm Criteria]

Let $\mathbf{A}$ be a nonsingular matrix with $\|\mathbf{A}\| < 1$. Thus, $0 < \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} < 1$ for all $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$. Let $\mathbf{Q}$ be the matrix of (orthonormal) eigenvectors of $\mathbf{A}'\mathbf{A}$ and let Υ be the corresponding (diagonal) matrix with (real) eigenvalues on its diagonal. Thus, $\mathbf{A}'\mathbf{A}\mathbf{Q} = \mathbf{Q}\Upsilon$ with $\mathbf{Q}'\mathbf{Q} = \mathbf{I}$. Hence $\mathbf{Q}'\mathbf{A}'\mathbf{A}\mathbf{Q} = \Upsilon$, with all diagonal elements of Υ between zero and one. Thus we can write $\mathbf{A}'\mathbf{A} = \mathbf{Q}\Upsilon\mathbf{Q}'$ or, equivalently, $\mathbf{A}'\mathbf{Q}\mathbf{Q}'\mathbf{A} = (\mathbf{Q}\Upsilon^{\frac{1}{2}})(\boldsymbol{\Upsilon}^{\frac{1}{2}}\mathbf{Q}')$ implying $\mathbf{A}'\mathbf{Q} = \mathbf{Q}\Upsilon^{\frac{1}{2}}$. Thus the eigenvalues of $\mathbf{A}'$ (and, hence, of $\mathbf{A}$, since both share the same characteristic polynomial) are given by the diagonal elements of $\Upsilon^{\frac{1}{2}}$ and are thus real and between zero and one. This is precisely the condition for determinacy in a single regime model.

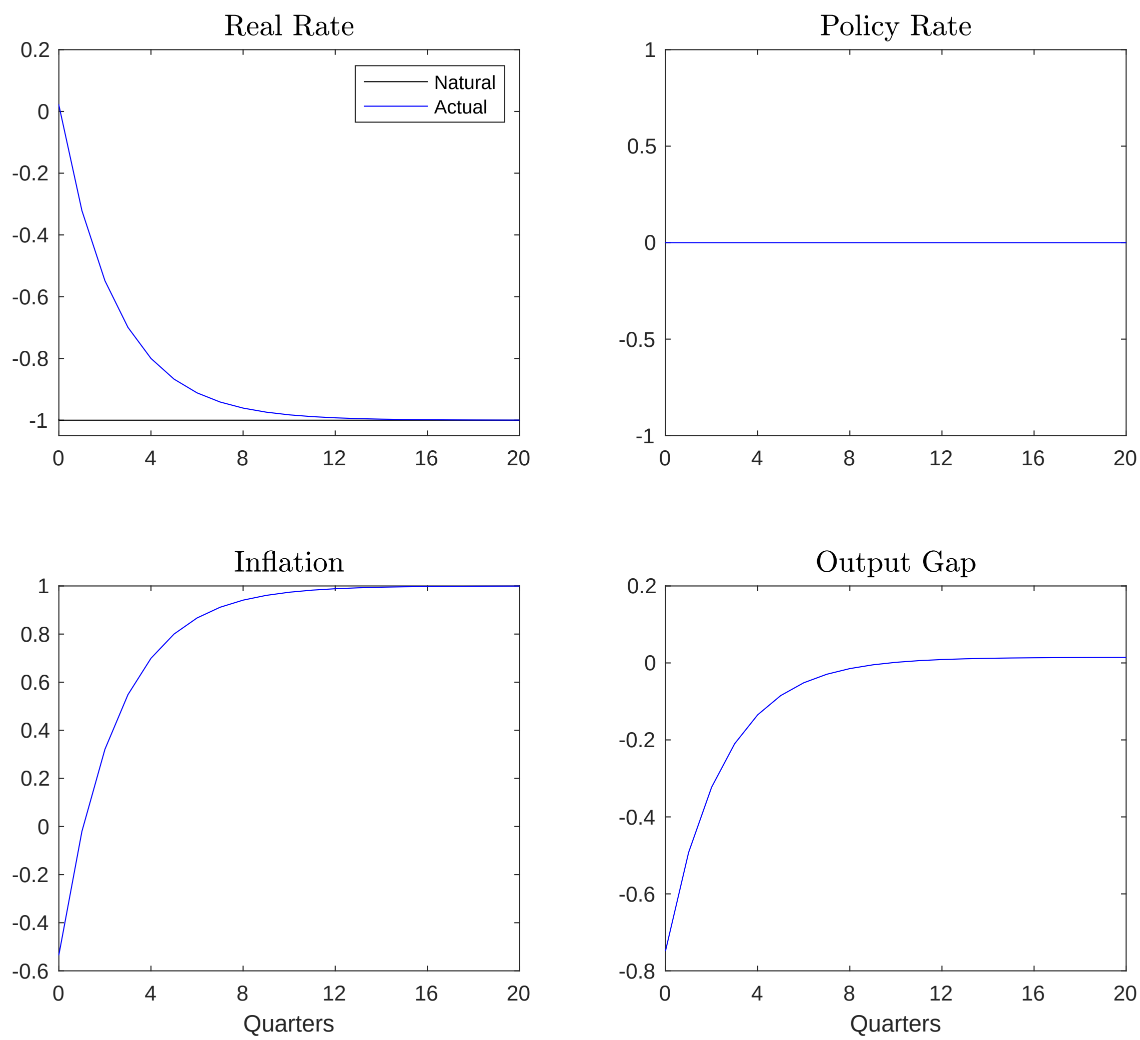

Figure 1: Transitional dynamics under the optimal monetary policy. Percent deviations from steady state in annualized terms.

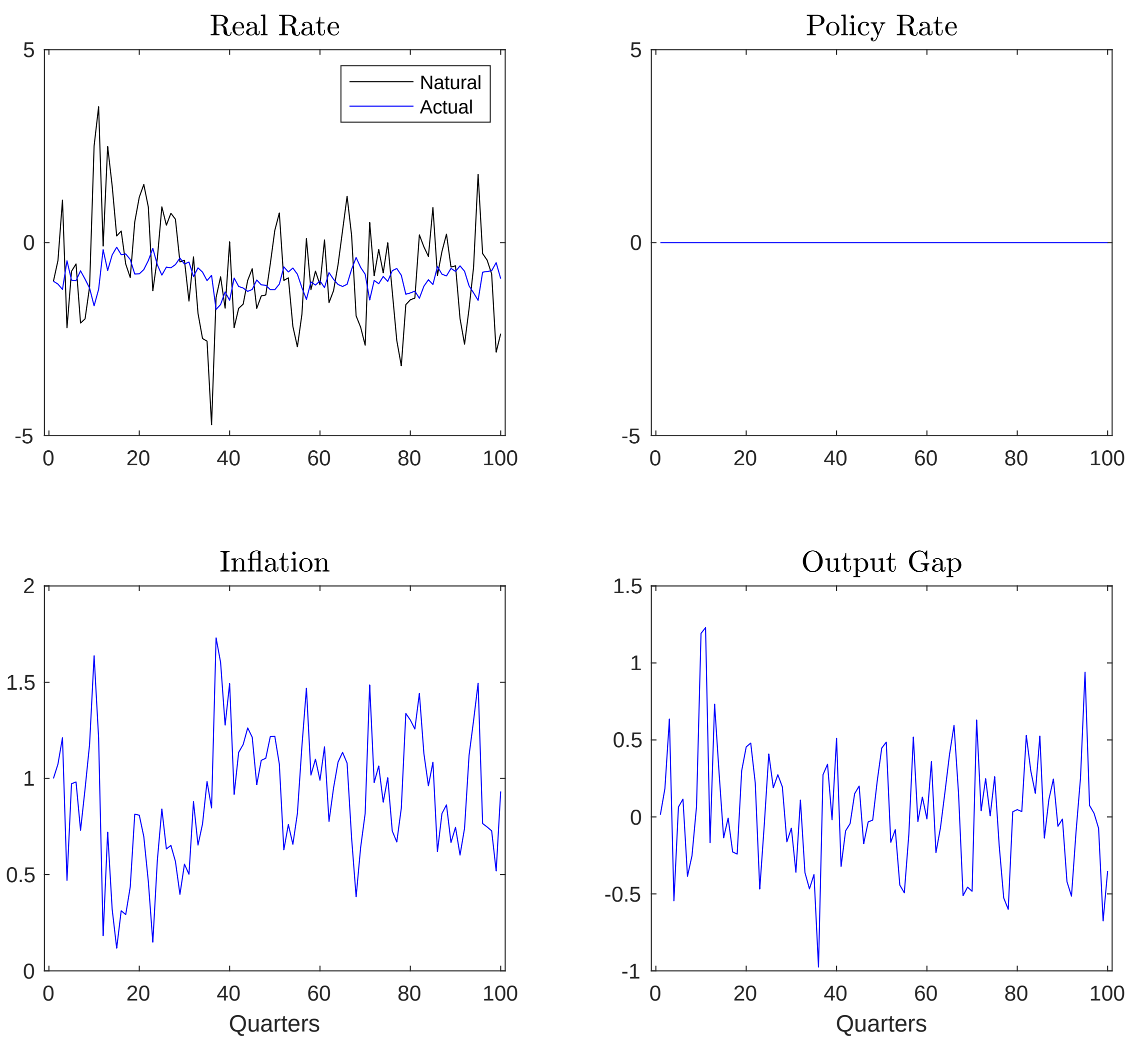

Figure 2: Aggregate fluctuations under the optimal monetary policy with baseline calibration. Percent deviations from steady state in annualized terms.

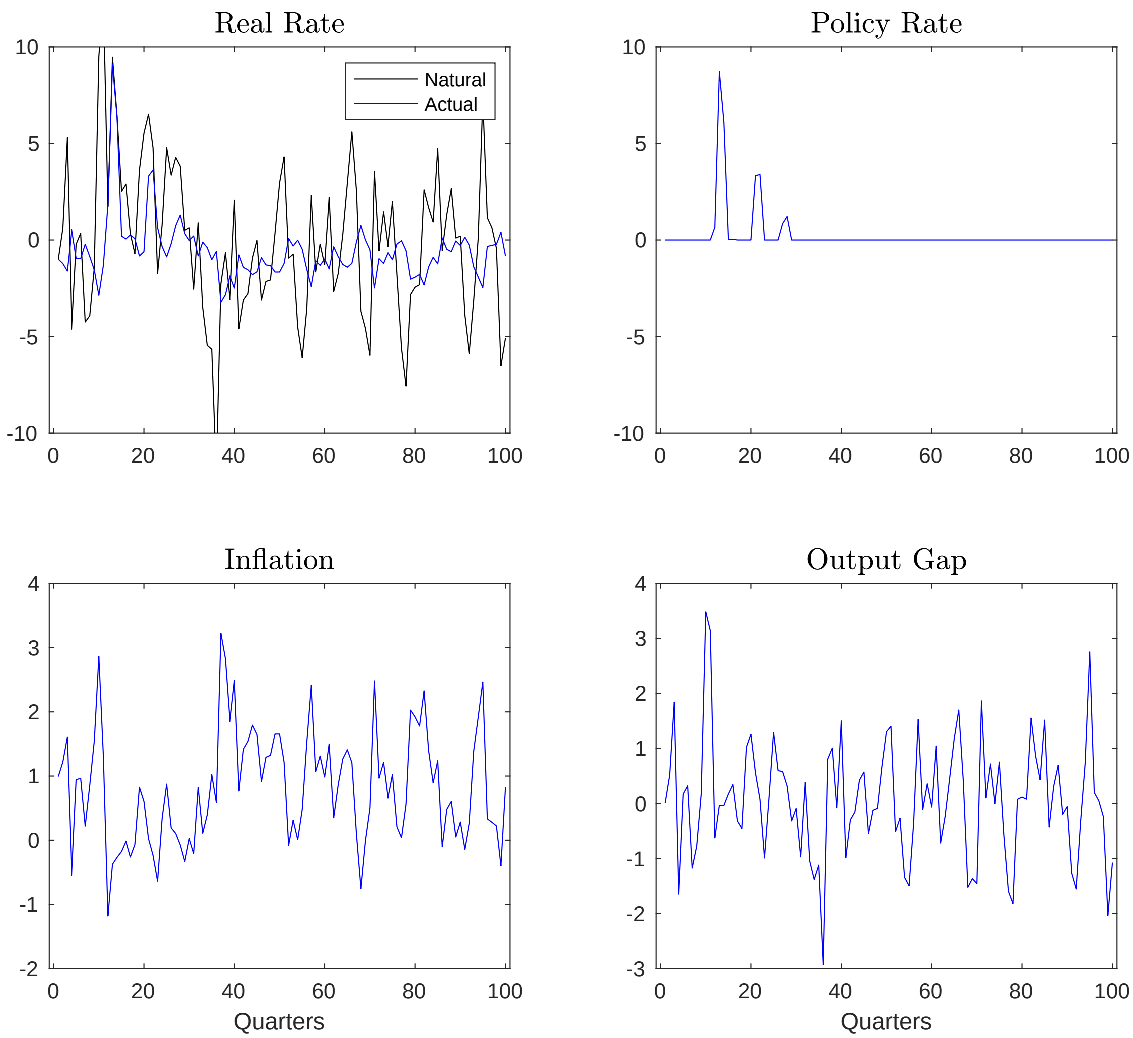

Figure 3: Aggregate fluctuations under the optimal monetary policy with higher shock volatility. Percent deviations from steady state in annualized terms.

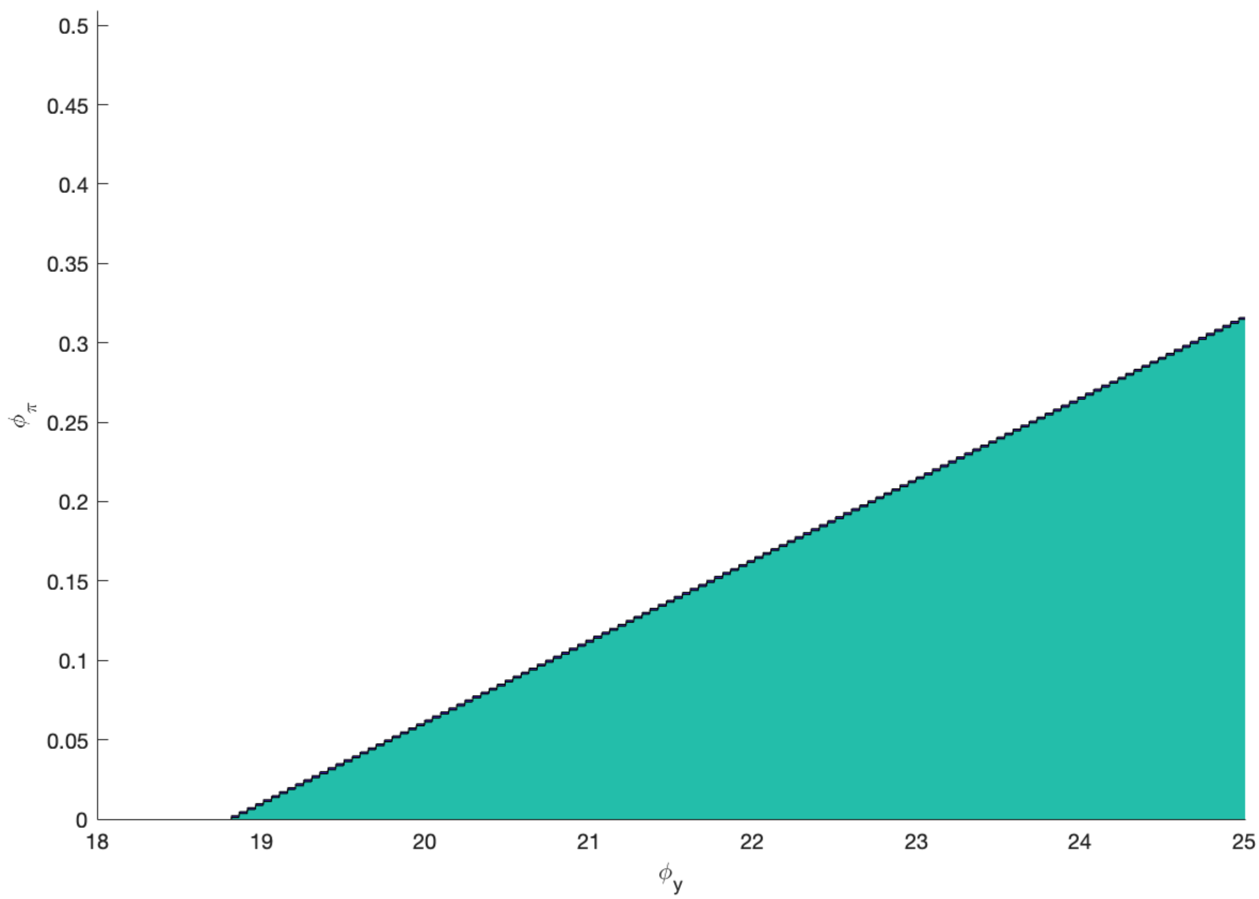

Figure 4: Implementation of the optimal monetary policy with state-contingent interest rate rule. Green (dark) area show values of the rule coefficients consistent with the sufficient condition for determinacy.