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Stable Sunspot Equilibria in a Cash-in-Advance Economy

George W. Evans* Seppo M.S. Honkapohja† Ramon Marimon‡

*University of Oregon, gevans@uoregon.edu
†University of Cambridge, smsh4@cam.ac.uk
‡European University Institute and UPF-CREi, ramon.marimon@eui.eu

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Stable Sunspot Equilibria in a Cash-in-Advance Economy*

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Abstract

We analyze a monetary model with flexible labor supply, cash-in-advance constraints, and seigniorage- and tax-financed government spending. If the intertemporal elasticity of substitution of labor is greater than one, both determinate and indeterminate steady states exist. If the elasticity is less than one, there is a unique steady state, which can be indeterminate. Only in the latter case do there exist sunspot equilibria that are stable under adaptive learning. A sufficient reduction in government purchases or increase in tax rates eliminates the sunspot equilibria in many cases. However, raising taxes enough to balance the budget can fail to achieve determinacy.

KEYWORDS: indeterminacy, learnability, expectational stability, seigniorage, endogenous fluctuations

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1 Introduction

There has been considerable interest in the issue of indeterminacy in both theoretical and applied macroeconomics. The distinctive feature of indeterminacy is that there are multiple well-behaved rational expectations (RE) solutions to the model. These can take various forms, including a dependence on extraneous random variables (called "sunspots"). Such solutions correspond to self-fulfilling prophecies and have been offered as an explanation of possible real business cycle fluctuations and, in the context of monetary models, of nominal fluctuations and price instability. In as much as such fluctuations generate inefficient distortions, a central policy issue is whether well designed monetary and fiscal policies can remove these distortions.

Much of the recent research has focused on stationary sunspot equilibria (SSEs) near an indeterminate steady state, a possibility that has been examined in both extensions of Real Business Cycle (RBC) models, incorporating externalities, and in a variety of monetary models.[1] A detailed survey of this literature is provided by Benhabib and Farmer (1999), which gives extensive references.

A crucial related issue is the learnability of such solutions. Suppose agents are not assumed to have rational expectations *a priori* but instead make forecasts using a perceived law of motion with parameters that they update over time using an adaptive learning rule, such as least squares. Can such learning rules lead agents eventually to coordinate on an SSE? That it is indeed possible for sunspot solutions to be learned by agents was demonstrated by Woodford (1990) in the context of the Overlapping Generations (OG) model of money. (See also Evans (1989)). Local stability conditions for Markov SSEs in simple forward looking models were developed in Evans and Honkapohja (1994).[2] In general, it can be shown that the possibility of such coordination depends on stability conditions for the SSE, as will be discussed further below. It is also easy to develop examples in which SSEs are not stable under learning. For an extensive treatment of adaptive learning in macroeconomics, see Evans and Honkapohja (2001), who show that "expectational stability" (E-stability) conditions typically govern the local learnability of SSEs, as well as other RE

[1] In nonlinear models SSEs can also exist near multiple distinct steady states or rational deterministic cycles.

[2] Examples of learnable SSEs are the "growth cycles" studied in Evans, Honkapohja, and Romer (1998) and the "animal spirits" equilibrium provided by Howitt and McAfee (1992). These SSEs fluctuate between neighborhoods of two distinct steady states, so that nonlinearity of the model is a crucial element.

solutions.[3]

Recent work has found that in RBC type models SSEs near an indeterminate steady state fail to be stable under learning; see Chapter 10 of Evans and Honkapohja (2001), Rudanko (2002) and Evans and McGough (2005a). This raises the question whether stable SSEs under learning could exist in dynamic representative agent equilibrium models with a unique steady state. In this paper we demonstrate that this is indeed possible using a standard infinite-horizon monetary general equilibrium model.[4]

We study a model where the money demand arises from a cash-in-advance (CA) constraint, as in Svensson (1985). Svensson's CA timing requires that consumption must be purchased with existing money holdings and, therefore, any unexpected rise of the price level results in consumption losses that could have been avoided if agents had the possibility of adjusting their portfolios within the purchasing period, as in Lucas's CA timing.[5] In these models, the price elasticity of money demand corresponds to the intertemporal elasticity of substitution of labor supply. To assume an elasticity of intertemporal substitution greater than or equal to one is common in RBC models. However, microeconomic estimates of this elasticity are typically significantly smaller than one, in the range 0.5 to 0.05; see Card (1994) for a survey of the microeconometric literature.[6]

Our paper contributes to the monetary literature by providing a characterization of the set of equilibria in a CA model with seigniorage, stressing the role played by the elasticity of intertemporal substitution and, in particular, by studying the stability properties of the different equilibria under adaptive learning. Our findings are both sharp and somewhat unexpected. The model has two regimes and the results depend on the regime. In the first, characterized by an elasticity of intertemporal substitution greater than one, there are two steady states and a continuum of rational expectations equilibria (REE) with a long-run inflation corresponding to the high inflation steady state. The low inflation steady state is determinate and stable under learning, while the other is indeterminate but unstable under learning. In this regime, none of

[3] In the context of one-step forward-looking univariate models, results showing the existence of stable SSE in a neigborhood of a steady state are given in Evans and Honkapohja (2003b) and Evans and Honkapohja (2003c).

[4] Recently, it been shown that stable SSEs can exist in the sticky price New Keynesian models for specific interest rate rules; see Honkapohja and Mitra (2004) and Evans and McGough (2005b).

[5] See Woodford (1994) for an extensive discussion of indeterminacy in cash-in-advance models with Lucas's timing.

[6] Introducing indivisibilities in individual labor supply can increase the elasticity of intertemporal substitution of labor by providing an extensive as well as an intensive margin.

the SSEs near the indeterminate steady state are learnable.

In contrast, when the elasticity of intertemporal substitution is less than one, consistent with the micro evidence, there is a unique steady state that can be either determinate or indeterminate. When the elasticity is sufficiently low, the steady state is indeterminate and there are nearby Markov chain sunspot equilibria that are stable under learning. In a neighborhood of this steady state, there also exist stationary autoregressive (AR) solutions, depending on extraneous sunspot noise, but these are not stable under learning. These results show that requiring stability under learning redirects the focus of analysis to a particular type of indeterminate steady state and to particular SSEs near that steady state. The results also show that the choice of the elasticity of intertemporal substitution has important consequences in terms of the nature of the underlying equilibria.[7] With Svensson's CA timing, individual current consumption of cash goods is constrained by current real money balances and inflation. In equilibrium inflation depends on current and planned real money balances. When the elasticity of intertemporal substitution is greater than one, expected increases of future consumption are associated with increases of current consumption. In contrast, this monotonicity property is lost when the intertemporal elasticity of substitution is less than one. In the latter case, in equilibrium, expected increases of future consumption are not 'smoothed' by corresponding increases in current consumption. In fact, expected increases of future consumption can lead to reductions of current consumption and – as we show – stable cycling patterns.[8]

When stable SSEs do exist, this raises the issue of whether economic policy can be used to avoid them, either by eliminating the existence of SSEs or by rendering them unstable. In our basic model, where government finances expenditures through seigniorage, the natural policy variable is the level of such purchases. We find that for a wide range of parameter values in which the steady state has nearby stable Markov sunspot equilibria, lowering government purchases sufficiently can render the steady state determinate, making the

[7] We remark that versions of the OG model with seigniorage can also lead to the two types of regimes studied here. However, as is well known, the case of a unique indeterminate steady state in the OG model corresponds to a perverse labour supply response.

[8] The role of the price elasticity of cash goods is well understood in time-consistency problems. For example, Nicolini (1998) shows that, if the monetary authority maximizes the welfare of the representative agent, and agents have isoelastic preferences, then the Ramsey policy is time inconsistent unless the price elasticity is one (the *log* case) since the optimal policy increases current prices (i.e., reducing consumption of the 'less elastic' cash good) if the elasticity is greater than one, and reduces current prices if it is lower than one. In this paper, we abstract from time-consistency problems by assuming that there is full commitment to monetary and fiscal policies.

nearby sunspot solutions disappear.

The above results all concern the case where consumption and labor taxes are ignored. In Section 6 we show how the analysis can be extended to include taxes. Most of the results concerning the effects of changing government deficits remain unchanged for fixed tax rates: in a wide range of cases a sufficient reduction in government spending will yield determinacy. We also consider the possibility of eliminating learnable sunspot equilibria by increases in consumption and labor taxes. We show that, indeed, raising consumption and labor taxes can be sufficient for eliminating indeterminacy and learnable sunspots, making it possible to finance government expenditures that, in the absence of tax revenues, would result in indeterminacy and endogenous fluctuations. However, this does not mean that a 'balanced budget regime' will guarantee the elimination of these fluctuations. While budget balance guarantees price stability (i.e., zero mean inflation), this does not eliminate the intertemporal distortions between credit and cash goods, and expectations of inflation affect this distorted margin, as well as labor and savings decisions. In summary, a 'balanced budget regime' does not change the underlying dynamics of the system beyond the change associated with a zero level of inflation. If the intertemporal elasticity is very low, then a 'balanced budget equilibrium' can be indeterminate and learnable sunspot equilibria can exist.

2 The Model

We consider an infinite-horizon representative agent economy. There are two types of consumption goods, cash and credit goods. (Cash goods must be paid for by cash at hand.) There is also a flexible labor supply, one unit of which produces one unit of either consumption good. The unit endowment of time is split between leisure and labor. Both consumption goods are perishable and there are no capital goods.

Let the utility function be

$$U_t = E_t \sum_{s=t}^{\infty} \delta^{s-t} \left[\frac{(c_s^1)^{1-\sigma}}{1-\sigma} + \frac{(c_s^2)^{1-\sigma}}{1-\sigma} + \alpha \frac{(1-n_s)^{1-\sigma}}{1-\sigma} \right],$$

where c_s^1, c_s^2 and n_s denote cash goods, credit goods and labor supply, respectively. δ is the discount factor and $\sigma > 0$ the coefficient of relative risk aversion. That is, $1/\sigma$ is the elasticity of intertemporal substitution. The (sub-)utilities of cash goods, credit goods and leisure are assumed identical to facilitate investigation and presentation of the results. α is the relative weight placed on leisure.[9]

The household budget constraint is

$$M_{s+1} + B_{s+1} \leq p_s(n_s - c_s^1 - c_s^2) + M_s + I_s B_s.$$

Here M_{s+1} and B_{s+1} denote the stocks of money and bonds at the beginning of period $s+1$. I_s is the nominal one-period interest rate factor on risk-free bonds earned during period s and known at the end of period $s-1$. p_s is the price of goods and labor in period s. The CA constraint takes the form[10]

$$p_s c_s^1 \leq M_s.$$

We will focus on equilibria where bonds are not held. Defining

$$m_{s+1} = M_{s+1}/p_s \text{ and } \pi_s = p_s/p_{s-1},$$

we can write the first order conditions as

$$\begin{aligned}
(c_t^2)^{-\sigma} &= \delta E_t^*[\pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}], \\
(c_t^2)^{-\sigma} &= \delta E_t^*[I_{t+1}\pi_{t+1}^{-1}(c_{t+1}^2)^{-\sigma}], \\
(c_t^2)^{-\sigma} &= \alpha(1-n_t)^{-\sigma}.
\end{aligned}$$

Here E_t^* denotes the expectations of the household, conditional on time t information, where we use the notation E_t^* to indicate that the expectations are not necessarily assumed to be fully rational, due to adaptive learning. When RE are assumed we will use the notation E_t. With the CA constraint holding with equality, it can be written as

$$c_t^1 = m_t/\pi_t. \tag{1}$$

The market clearing condition is

$$n_t = c_t^1 + c_t^2 + g_t,$$

where g_t denotes government spending on goods. We assume that g_t is an *iid* random variable with small bounded support around the mean $g > 0$. There is also a government finance constraint taking the form

$$B_{t+1} + M_{t+1} = p_t g_t + I_t B_t + M_t.$$

[9] Implicitly we are assuming that utility is additively separable in private and public consumption. Assuming instead that they are perfect substitutes, as in Prescott (2004), would alter the analysis by removing the income effect of government consumption.

[10] This is the Svensson timing, while Lucas's timing would had been $p_s c_s^1 \leq M_{s+1}$.

Note that for simplicity we are ignoring taxes at this stage. We generalize the model to include consumption and labor taxes in Section 6. If bonds are not held in positive net amount in equilibrium, then this constraint yields the familiar seigniorage equation

$$\pi_t = \frac{m_t}{m_{t+1} - g_t}. \tag{2}$$

Household optimization, market clearing and the CA constraint lead to the equations

$$n_t = 1 - \alpha^{1/\sigma}\delta^{-1/\sigma}\{E_t^*[\pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}]\}^{-1/\sigma}, \tag{3}$$

$$m_{t+1} = (1 + \alpha^{-1/\sigma})n_t - \alpha^{-1/\sigma}, \tag{4}$$

$$c_t^1 = m_{t+1} - g_t, \tag{5}$$

$$c_t^2 = n_t - c_t^1 - g_t, \tag{6}$$

$$I_{t+1} = (c_t^2)^{-\sigma}\delta^{-1}\{E_t^*[\pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}]\}^{-1}. \tag{7}$$

Note that $m_{t+1} = M_{t+1}/p_t$, the real money stock carried forward from period t to period $t+1$, is determined at time t. Similarly, I_{t+1} is determined and known in period t. Equations (3)-(7), together with (2), give the temporary equilibrium equations determining $\pi_t, n_t, m_{t+1}, c_t^1, c_t^2$ and I_{t+1} as functions of time t expectations, the exogenous government spending shock g_t, and the previous period's real money stock m_t.

We note that the labor supply response in this model is entirely standard. It can be shown that, under perfect foresight, dynamic labor supply is characterized by

$$\frac{1 - n_t}{1 - n_{t+1}} = (\delta R_{t+1})^{-1/\sigma},$$

where $R_{t+1} = I_{t+1}/\pi_{t+1}$. Thus increases in the real interest rate factor R_{t+1} lead to increases in current labor supply n_t for any value of σ. Note that $1/\sigma$ is the intertemporal elasticity of substitution of labor supply.

3 Linearized Model

Our first step is to determine the possible steady states and then to log-linearize the model around the steady states.[11] This will enable us to examine determinacy and stability under learning through an extension of the procedure developed in Evans and Honkapohja (2003c).

[11] When the shocks are sufficiently small, determinacy and stability under learning of stochastic equilibria can be examined through linearization. See Chapters 11-12 of Evans and Honkapohja (2001) for this methodology in basic nonlinear forward-looking models.

3.1 Nonstochastic Steady States

We begin by determining the non-stochastic (perfect foresight) steady states that are possible when $g_t = g$ is constant and nonstochastic. Denoting steady state values by bars over the variables, (3)-(5) and (2) imply

$$\begin{aligned} \bar{n} &= 1 - \alpha^{1/\sigma}\delta^{-1/\sigma}\bar{\pi}^{1/\sigma}\bar{c}^1, \\ \bar{m} &= (1+\alpha^{-1/\sigma})\bar{n} - \alpha^{-1/\sigma}, \\ \bar{c}^1 &= \bar{m} - g, \\ \bar{\pi} &= \bar{m}/(\bar{m} - g). \end{aligned}$$

These equations can be reduced to a single equation in the steady state inflation rate $\bar{\pi}$,

$$(1-g)\bar{\pi} = 1 + gA\bar{\pi}^{1/\sigma}, \tag{8}$$

where $A = (1+\alpha^{1/\sigma})\delta^{-1/\sigma} > 1$ since $\alpha > 0$ and $\delta < 1$.

For $g = 0$ there is a unique steady state $\pi = 1$. For $g > 0$, it can be seen that the model has two regimes, depending on σ. If $\sigma < 1$, then the right-hand side of (8) is a convex function, while the left-hand side defines a straight line and there are two cases. If $g > 0$ is below a threshold value, depending on α, σ and δ, there are two distinct steady states, $1 < \bar{\pi}_L < \bar{\pi}_H$, while if g exceeds this threshold there are no perfect foresight steady states. Below this threshold value, increases in g raise $\bar{\pi}_L$ and lower $\bar{\pi}_H$. The $\sigma < 1$ regime is standard in seigniorage models. However, this model also has a less familiar regime that arises when $\sigma > 1$. In this case the right hand side of (8) is concave, and provided $0 < g < 1$ (which we assume throughout the paper), there is a unique steady state $\bar{\pi}$. In this regime increases in g raise $\bar{\pi}$. Figures 1 and 2 illustrate the two cases $\sigma < 1$ and $\sigma > 1$.

3.2 Linearization

We begin by remarking that, in the stochastic economy, agents are assumed to make forecasts $E_t^*[\pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}]$ about a nonlinear function of future inflation and consumption of the cash good, see equation (3). When the model is log-linearized for analysis of indeterminacy and learning, we must decide on the formulation that will be used and there are two natural possibilities. First, agents might forecast separately the two variables in the linearization of the composite quantity $\pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}$. Under this assumption the model can be reduced to a bivariate system in inflation and consumption of the cash good (or equivalently employment). Second, it is possible to derive a univariate reduced form from (2)-(7) and conduct the analysis with the univariate form.

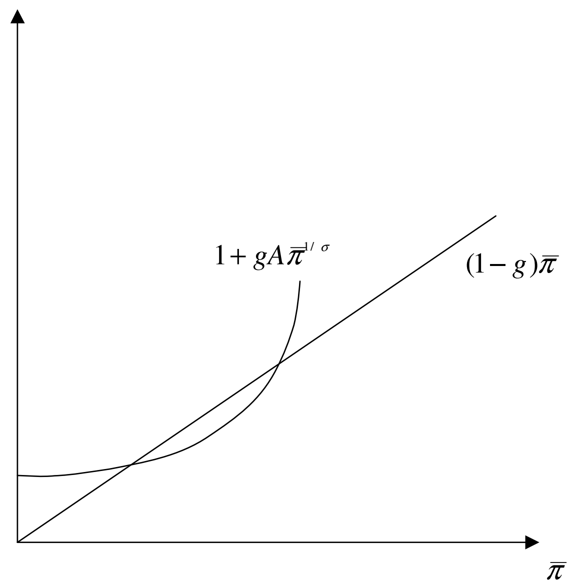

Figure 1: $\sigma < 1$

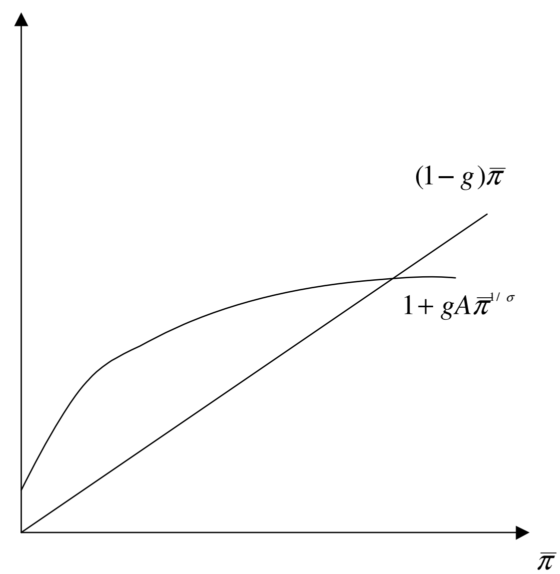

Figure 2: $\sigma > 1$

In what follows, we adopt the latter formulation for theoretical simplicity. Not surprisingly, both formulations deliver identical results.[12]

Using (1), (2), (4) and (5) we can write (3) in the form

$$c_t^1 = 1 - A \left\{ E_t^* \left[\frac{(c_{t+1}^1)^{1-\sigma}}{c_t^1 + g_t} \right] \right\}^{-1/\sigma} - g_t, \tag{9}$$

where $A = (1 + \alpha^{1/\sigma})\delta^{-1/\sigma}$ is as in equation (8). (9) is the univariate formulation that we will employ. Let $g_t = g + \hat{u}_t$, where $\hat{u}_t$ is now assumed to be white noise with mean zero.

As can be seen from (9), expected increases in future consumption are associated, in equilibrium, with increases of current consumption when $\sigma < 1$ and with reductions in current consumption when $\sigma > 1$. This lack of monotonicity of the regime with an elasticity of substitution less than one makes possible the existence of stable SSEs in the latter regime.

The log-linearization of (9) is

$$z_t = \beta_0 E_t^* z_t + \beta_1 E_t^* z_{t+1} + u_t, \tag{10}$$

where we have introduced the notation $z_t = \log c_t^1 - \log \bar{c}^1$ and

$$\begin{aligned} \beta_0 &= -\frac{A}{\sigma}(\bar{\pi})^{(1-\sigma)/\sigma}, \\ \beta_1 &= \frac{A(1-\sigma)}{\sigma}(\bar{\pi})^{1/\sigma} \text{ and} \\ u_t &= -\frac{g}{\bar{c}^1}\hat{u}_t. \end{aligned}$$

Note that $\beta_0 < 0$ while $\beta_1 \gtrless 0 \Leftrightarrow \sigma \lessgtr 1$. We make the regularity assumptions that $\beta_0 \neq 1$, $\beta_1 \neq 0$ and $\beta_0 + \beta_1 \neq 1$. In deriving (10) we have made the assumption that current exogenous, but not current endogenous variables, are known when expectations are formed.[13] This strikes us as the most natural assumption within the univariate framework, but we will comment on an alternative assumption below. (Under RE, knowledge of the exogenous and predetermined variables is equivalent to knowledge of all endogenous variables, but this equivalence does not hold under learning.)

We next assess the determinacy of the steady state and consider the different types of REE using the linearized model (10).

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[12] The results for the bivariate formulation are available on request.

[13] Knowledge of aggregate c_t^1 is equivalent to knowledge of the current price level and inflation rate.

3.3 Noisy Steady State and AR Solutions

The method of undetermined coefficients can be used to show that the log-linearized reduced form (10) has RE solutions taking the form of "noisy steady states,"

$$z_t = (1 - \beta_0)^{-1} u_t. \tag{11}$$

These solutions are often called minimal state variable (MSV) solutions in the terminology of McCallum (1983) and are the solutions most typically adopted in applied work.

If the steady state, around which we have linearized the model, is determinate then, as is well known, this is the unique stationary solution near the steady state. See, for example, Blanchard and Kahn (1980), Farmer (1999) and Evans and Honkapohja (2001). On the other hand, if the steady state is indeterminate there exist SSEs in a neighborhood of that steady state.[14] The possibility of modeling business cycle fluctuations as SSEs has been emphasized by Cass and Shell (1983), Azariadis (1981), Farmer (1999) and Guesnerie and Woodford (1992). The following result is easily verified:

Proposition 1 *The general linear model (10) exhibits indeterminacy if and only if*

$$\left| \frac{\beta_1}{1 - \beta_0} \right| > 1.$$

This follows since, under RE, (10) is equivalent to the standard one-step forward-looking model $z_t = \beta_1 (1 - \beta_0)^{-1} E_t z_{t+1} + (1 - \beta_0)^{-1} u_t$ studied e.g. in Farmer (1999). Some intuition for the indeterminacy condition in the Proposition is provided by noting that it requires strong feedback from expectations $E_t z_{t+1}$ to the current value z_t of the endogenous variable. As is well known, strong expectational feedbacks are needed for the existence of self-fulfilling prophecies.

A familiar feature of the seigniorage model is that, in the case of two steady states, the high inflation equilibrium is indeterminate. As we will see in Section 4.2, this result also holds in our model. In addition, it can be shown that for some parameter values we have indeterminacy in the regime $\sigma > 1$ when there is a single steady state. In fact we have:

Corollary 2 *In the case $\sigma > 1$ the steady state in the monetary model is indeterminate when σ is sufficiently large.*

[14] The terminology "regular" and "irregular" is often used synonymously with "determinate" and "indeterminate."

This result can be seen as follows. First note from (8) that $\lim_{\sigma\to\infty}\bar{\pi} = (1+2g)/(1-g)$, implying that $\lim_{\sigma\to\infty}\bar{c}^1 = (1-g)/3$. Second, we have $\lim_{\sigma\to\infty}\beta_0 = 0$ and $\lim_{\sigma\to\infty}\beta_1 = -2$. With these values the condition in Proposition 1 is satisfied.

In the case of Figure 1, the two steady states $\bar{\pi}_H$ and $\bar{\pi}_L$ have the following determinacy properties:

Corollary 3 *In the case $\sigma < 1$ the high inflation steady state $\bar{\pi}_H$ is indeterminate and the low inflation steady state $\bar{\pi}_L$ is determinate.*

This result is proved in the Appendix.

We next consider the form of SSEs in cases of indeterminate steady states. Using the method of undetermined coefficients one can show that there exist stochastically stationary solutions of the form

$$z_t = a + bz_{t-1} + d_0u_t + d_1u_{t-1} + k\eta_t, \tag{12}$$

where η_t, the sunspot variable, is an arbitrary (observable) exogenous variable satisfying $E_t\eta_{t+1} = 0$. Computing conditional expectations under (12) and substituting into (10), it is seen that the REE of the form (12) consist of the steady state (with $a = b = d_1 = k = 0$ and $d_0 = (1-\beta_0)^{-1}$) or a continuum with $a = 0$, $b = (1-\beta_0)/\beta_1$, $d_1 = -\beta_1^{-1}$ and d_0, k free. For convenience we will refer to the latter as autoregressive (AR) SSEs, though they also include moving average dependencies on the intrinsic disturbance. This is the type of solution that is emphasized by much of the applied indeterminacy literature, see e.g. Benhabib and Farmer (1999). For the case at hand, when the steady state is indeterminate, there are stationary AR SSEs. Stationarity stipulates that $|b| = |(1-\beta_0)/\beta_1| < 1$, which holds precisely when the steady state is indeterminate, see Proposition 1.

Although the AR solutions are the form of SSEs that have recently received the most attention, the literature has also drawn attention to the existence of solutions generated by finite state Markov chains. We next show that this type of solution can exist in our monetary model.

3.4 Markov Sunspot Solutions

When a steady state is indeterminate it follows from the theoretical literature that there will also exist SSEs around the steady state for which the sunspot process is a Markov chain with a finite number of states; see Chiappori, Geoffard, and Guesnerie (1992) for the case without intrinsic shocks. We will call such solutions Markov SSEs to distinguish them from the AR SSEs discussed above. Evans and Honkapohja (2003c) examine the relation between these two types of SSEs in the basic one-step forward looking model without intrinsic shocks.

For simplicity, we focus on SSEs driven by a 2-state Markov chain. Thus assume that s_t is a two state exogenous process, taking values $s_t = 1$ or $s_t = 2$. The transition probabilities are p_{ij}, $j = 1, 2$, so that $p_{12} = 1 - p_{11}$ and $p_{21} = 1 - p_{22}$. Let P be the 2×2 matrix $P = (p_{ij})$. We look for solutions of the form:

$$z_t = z(j) + Ku_t \text{ if } s_t = j.$$

To satisfy (10) under RE the values of $z(j)$ and K must satisfy

$$z(j) = \frac{\beta_1}{1-\beta_0}[p_{j1}z(1) + p_{j2}z(2)], \text{ and } K = (1-\beta_0)^{-1},$$

the first equation of which can be rewritten in the vector form

$$\theta = \mathcal{T}\theta, \text{ where } \theta' = (z(1), z(2)),\ \mathcal{T}(\theta) = \frac{\beta_1}{1-\beta_0}P. \qquad (13)$$

A Markov SSE θ exists if there exist $0 < p_{11} < 1$ and $0 < p_{22} < 1$ and $\theta \neq 0$ for which θ satisfies the equation (13). Note that if an SSE exists then, in our linearized model, $k\theta$ is also an SSE for any real k for the same transition probabilities, so that the "size" of the sunspot fluctuations is indeterminate. Formally, SSEs exist if and only if $\mathcal{T} - I$ is singular for some $0 < p_{11}, p_{22} < 1$. Noting that $\mathcal{T}$ depends on p_{11} and p_{22}, we can solve the equation

$$\det(\mathcal{T}(p_{11}, p_{22})) = 0$$

and make use of the fact that P is a $2-$dimensional transition probability matrix. In linearized models such as the current one, SSEs exist only for very particular transition probabilities, namely those for which $\frac{\beta_1}{1-\beta_0}$ is the inverse of an eigenvalue of P. Since P has one eigenvalue equal to 1 and the other in the open interval $(-1, 1)$, when $\left|\frac{\beta_1}{1-\beta_0}\right| > 1$, i.e. the steady state is indeterminate, we always have existence of Markov SSEs. The required condition on P can be thought of as a *resonant frequency* condition that makes possible the excitation of the SSE.[15]

[15] For other transition probabilities the matrix $\mathcal{T}$ is nonsingular and the equation $\theta = \mathcal{T}\theta$ has only the trivial solution $\theta = 0$, which corresponds to the steady state.

4 Stability Under Learning

We now take up the question of stability of the RE solutions under adaptive learning rules. In the case of indeterminate steady states we separately assess each of the three types of solution for their stability under learning. Least squares and related learning dynamics have been widely studied and shown to converge to the usually employed REE in many standard models. This is true of the stationary solutions of, for example, the Cagan model of inflation, the Sargent-Wallace IS-LM-PC model, the Samuelson overlapping generations model and the RBC model. A recent overview of the literature is provided in Evans and Honkapohja (2001).

The starting point for analysis of learning is the temporary equilibrium in the model, given by equation (9) or its linearization (10). To complete the description of the agents' behavior we must supplement this equation with a rule for forecasting the required state variables next period. The agents are assumed to have perceptions about the (in general stochastic) equilibrium process of the economy. This is usually called the perceived law of motion (PLM) and depends on parameters that are updated as new data become available over time. At each period t, agents form expectations by making forecasts using the estimated PLM. This leads to a temporary equilibrium, called the actual law of motion (ALM), which provides the agents a new data point of the key variables. Estimated parameters are updated in each period according to least squares and the new data. The issue of interest is the stability under learning of some rational expectations solution, i.e. whether the estimated parameters of the PLM converge to REE values over time.

It is well known that, for a wide range of models, stability under least squares learning is governed by E-stability conditions, which under a regularity assumption are necessary and sufficient for convergence of least squares learning. The E-stability conditions are developed by computing, for given values of the parameters of the PLM, the resulting ALM and a differential equation in notional time in which the parameters adjust in the direction of the ALM parameter values. The regularity assumption stipulates that at the REE of interest, the linearized differential equation defining E-stability of the REE does not have any eigenvalue with zero real part. It should be noted that the precise form of the regularity assumption depends on the REE of interest.[16] For brevity, in what follows we do not explicitly state the regularity assumption in connection with each REE and the E-stability conditions. See the Evans and Honkapohja (2001) book for an extensive discussion of these

[16]For example, in Proposition 4 the regularity assumption is ($\beta_0 + \beta_1 \neq 1$ and $\beta_0 \neq 1$) for part (i) whereas for part (ii) it is ($\beta_1 \neq 0$ and $\beta_0 \neq 1$).

concepts and analytical techniques.

We provide E-stability conditions for the different REE in the general linear model (10). These results extend Evans and Honkapohja (2003c) to models with intrinsic shocks and by including the $E_t^* z_t$ term, which arises under our information assumptions. We remark that the stability results in our linearized monetary model are unaffected if instead full $t-$dating of expectations is assumed.

4.1 Steady State and AR Solutions

Consider the noisy steady state solutions (11). These solutions exist whether or not a steady state is indeterminate. We outline the technique for determining stability under learning for this simple case. For other cases details are relegated to the Appendix.

Agents have a PLM of the form

$$z_t = a + Ku_t.$$

Note that, like the steady state REE, the PLM is a linear function of the shock and it naturally incorporates an intercept since log-linearization around the steady state was done only for analytical purposes. Under adaptive learning agents are assumed to estimate a and K by recursive least squares.

Given estimates a_t and K_t in period t, the temporary equilibrium is then given by (10) with $E_t^* z_t = a_t + K_t u_t$ and $E_t^* z_{t+1} = a_t$. The question of interest is whether $a_t \to \bar{a}, K_t \to \bar{K}$ as $t \to \infty$, where $\bar{a} = 0$, $\bar{K} = (1 - \beta_0)^{-1}$ is the REE. The answer is that convergence is governed by E-stability.

To determine E-stability one assumes expectations $E_t^* z_t = a + Ku_t$, $E_t^* z_{t+1} = a$, based on the above PLM, for an arbitrary a and K (intuitively, a_t and K_t evolve asymptotically slowly under adaptive learning). Substituting these into (10) the implied ALM takes the form

$$z_t = (\beta_0 + \beta_1)a + (\beta_0 K + 1)u_t.$$

This gives rise to a mapping $T(a, K) = ((\beta_0 + \beta_1)a, 1 + \beta_0 K)$, and E-stability is defined as the local asymptotic stability of the fixed point $(\bar{a}, \bar{K})$ of the differential equations

$$\frac{da}{d\tau} = (\beta_0 + \beta_1)a - a,\ \frac{dK}{d\tau} = 1 + \beta_0 K - K, \tag{14}$$

where τ is virtual or notional time. (Note that this is simply a partial adjustment formula in the virtual time.) Clearly, the general stability conditions for (14) are $\beta_0 + \beta_1 < 1$ and $\beta_0 < 1$.

Consider next the AR SSEs. Agents are assumed to have the PLM (12) and to make forecasts based on the estimated PLM. E-stability conditions are derived in the Appendix. Collecting the results together, we have:

Proposition 4 *For the general linear model (10) we have:*
(i) the E-stability condition for the noisy steady state is

$$\beta_0 + \beta_1 < 1 \textit{ and } \beta_0 < 1;$$

(ii) conditions for E-stability of the AR SSEs are

$$\beta_1 < 0 \textit{ and } \beta_0 > 1.$$

Here and throughout the paper stated E-stability conditions should be interpreted as necessary and sufficient conditions. This requires that the regularity assumptions mentioned in footnote 15 are satisfied. From now on we assume that the appropriate regularity assumptions hold and refrain from stating this explicitly.

Intuition for the steady-state E-stability condition (i) is fairly straightforward. Under this PLM, agents are estimating the mean of the process. If $\beta_0 + \beta_1 > 1$ then estimates of the mean that are higher than the steady state value will be revised *upward*, i.e. away from the equilibrium value. In essence too much positive feedback from expectations to current endogenous data destabilizes the system. The additional requirement $\beta_0 < 1$ is needed because the PLM also depends on the exogenous shock u_t. We omit the more elaborate intuition that could be developed for (ii).

We remark that the E-stability condition in (i) is not the same as the determinacy condition. The condition in (i) of Proposition 4 can be examined for different values of the parameters in the monetary model. Recalling that $\beta_0 < 0$ and that $\sigma > 1$ implies $\beta_1 < 0$, we have:

Corollary 5 *For the linearized monetary model if $\sigma > 1$ the unique steady state is E-stable irrespective of its determinacy/indeterminacy.*

When $\sigma < 1$ and there are two steady states we have:

Corollary 6 *For the linearized monetary model if $\sigma < 1$ the steady state $\bar{\pi}_L$ is E-stable and the steady state $\bar{\pi}_H$ is not E-stable.*

This result follows directly from the observation that $\beta_0 + \beta_1 > 1$ at $\bar{\pi}_H$ and $\beta_0 + \beta_1 < 1$ at $\bar{\pi}_L$, which is shown in the proof of Corollary 3 in the Appendix.

Using part (ii) of Proposition 4, we get:

Corollary 7 *The AR SSEs are not E-stable in the linearized monetary model.*

We remark that in related models, the AR paths can be E-stable in some cases. See, for example, the monetary OG model of Duffy (1994).

4.2 Markov Solutions

Finally, we consider the Markov SSEs that exist when the steady state is indeterminate (see the beginning of Section 3.4 for references). Suppose that in addition to the shock u_t, agents observe a sunspot s_t satisfying the resonant frequency condition and that they consider also conditioning their actions on the values of the sunspot. A simple learning rule is that agents run least squares regressions on the sunspot state and u_t. Agents then make forecasts using these estimates and the transition probabilities for the observed sunspot (these probabilities can also be estimated if they are unknown).[17] The stability of SSEs under adaptive learning depends upon the corresponding E-stability condition and we now examine E-stability of the Markov SSEs. In the Appendix we discuss the definition of E-stability for this set-up. The results are:

Proposition 8 *Assume* $\left|\frac{\beta_1}{1-\beta_0}\right| > 1$*, which implies that Markov SSEs exist. The E-stability conditions for Markov SSEs in the general linear model (10) are* $\beta_0 < 1$ *and* $\beta_0 + \beta_1 < 1$*.*

We remark that the E-stability conditions for the Markov SSEs are identical to those for steady states and the intuition is analogous. A key feature of E-stable Markov SSEs is that they arise in a neighborhood of an indeterminate steady state that is stable under steady state learning. The phenomenon of SSEs inheriting E-stability properties of nearby steady-state solutions arises more generally, see Evans and Honkapohja (1994) and Chapter 12 of Evans and Honkapohja (2001).

Corollary 9 *In the linearized monetary model we have: (i) if* $\sigma \geq 1$ *and the single steady state* $\bar{\pi}$ *is indeterminate, the Markov SSEs are E-stable and (ii) if* $\sigma < 1$ *the Markov SSEs near an indeterminate steady state are not E-stable.*

Part (i) of Corollary 9 follows at once from Proposition 8 by noting that $\sigma \geq 1$ implies $\beta_1 \leq 0$ while always $\beta_0 < 0$. Part (ii) follows from noting that $\sigma < 1$ implies $\beta_1 > 0$ and then indeterminacy requires $\beta_0 + \beta_1 > 1$, which contradicts the E-stability condition in Proposition 8.

[17] In terms of the original nonlinear model, an alternative learning procedure would be for agents to directly estimate the mean of $\pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}$ conditional on the current sunspot state s_t. In the simulations below we adopt this alternative scheme when examining the nonlinear model numerically. The stability properties of the two approaches appear to be identical.

To gain intuition to Corollary 9 we note that with $\sigma \geq 1$ the E-stability conditions of Proposition 8 necessarily hold and thus the existence of stable SSEs depends only on the indeterminacy condition in the proposition. Since in this case $\beta_1 \leq 0$ and $\beta_0 < 0$ we require $-\beta_1 > 1 - \beta_0$. Stable SSEs therefore arise when there is strong negative feedback from future expectations to the current period. In contrast, if $\sigma < 1$ the feedback from future expectations is positive, which in cases of indeterminacy leads to instability under learning. The importance of negative expectational feedback for stability of SSEs under learning is emphasized in Evans and Honkapohja (2003b).

We now give a numerical example of a Markov SSE satisfying the resonant frequency condition in the linearized model.

Example of a Markov SSE: Suppose that $\delta = 0.95$, $\alpha = 1, \sigma = 3.5$ and $g = .19$.[18] The steady state is indeterminate: $\beta_1/(1 - \beta_0) = -1.2408$. There are Markov SSEs with, for example, $p_{11} = 0.1041$ and $p_{22} = 0.09$, and these SSEs are stable under learning.

Two important remarks must be made at this point. First, it is possible for a steady state to be stable under learning and at the same time for nearby SSEs to be stable under learning. When this occurs, the solution to which the economy converges depends on the form of the PLM, i.e. on whether or not agents in their learning allow for a possible dependence on sunspots.

Second, the results of this section are subject to the qualification that our linearized monetary model has been derived from an underlying nonlinear model. Therefore, we investigate the existence and stability of SSEs directly for the original nonlinear model using numerical computations and simulations when $\sigma \geq 1$.

There are several important differences between the nonlinear system (2)-(5) and the linearized system (10) with respect to the Markov SSEs. In the linearized system the resonant frequency condition (equation (24) in the Appendix) must be satisfied exactly and the "size" of the SSE is indeterminate, as earlier discussed. In the exact nonlinear system there are Markov SSEs for transition probabilities close to the resonant frequency condition and the value of equilibrium is in part determined by these probabilities. This issue is fully analyzed for the univariate forward-looking model in Evans and Honkapohja (2003b). Thus, although the linearized model is convenient for obtaining existence and stability conditions for Markov SSEs, it is important to establish

[18] We have not tried to obtain the parameter values by calibrating the model to data, since our main goal is to provide numerical illustrations.

further details using the nonlinear model.[19]

Consider, therefore, the learning dynamics in the original nonlinear model (2)-(5). The key variable that agents must forecast is $X_{t+1} = \pi_{t+1}^{-1}(c_{t+1}^{1})^{-\sigma}$. Because the sunspot variable is assumed to be first-order Markov, the conditional expectation of this variable depends only on the current state. A simple learning rule is thus to estimate the mean value of X_{t+1} conditional on the current sunspot state at t, e.g. by state contingent averaging:

$$\hat{X}_{i,t} = (\#N_i(t))^{-1} \sum_{\substack{1 \le \ell \le t, \\ s_{t-\ell-1}=i}} X_{t-\ell},$$

$i = 1, 2$, where $\#N_i(t)$ denotes the number of data points in which $s_{t-\ell-1} = i$ for $1 \le \ell \le t$. Thus agents at time t estimate the mean value that X_{t+1} will take, next period, as $\hat{X}_{i,t}$, when the sunspot in period t is in state $s_t = i$, $i = 1, 2$.[20] Accordingly they form expectations at t as $E_t^* X_{t+1} = \hat{X}_{i,t}$. Over time the estimates $\hat{X}_{i,t}$ are revised in accordance with observed values of X_t following each of the two different states.

Under adaptive learning the model consists of these learning dynamics together with the equations (2)-(7). We remark that under this formulation the realized value of X_t depends on both $E_t^* X_{t+1}$ and $E_{t-1}^* X_t$ because π_t depends on both m_{t+1} and m_t. From the results for the linearized model, we anticipate convergence to a stationary sunspot solution, for transition probabilities close to satisfying the resonant frequency condition, when $\sigma > 1$ and the steady state is indeterminate.

We have simulated the nonlinear system and the corresponding E-stability differential equation using the parameter values from the above example. Figure 3 illustrates convergence to the sunspot equilibrium for the choice of transition probabilities $p_{11} = 0.07$ and $p_{22} = 0.05$ and initial conditions near the steady state. (The vertical axis shows deviations of $\hat{X}_1$ and $\hat{X}_2$ from the steady state value of $\bar{X}$ and the initial deviations were $\hat{X}_1 = \bar{X} + 0.01$ and $\hat{X}_2 = \bar{X} - 0.01$). The simulation clearly shows convergence to a Markov SSE. In this SSE the ratio of output in the two states is $n_1/n_2 = 1.11$ and expected inflation $E_t^* \pi_{t+1}$ in the two states are 1.20 and 2.86.

[19] We remark that, with $\sigma > 1$, indeterminacy of the steady state does not imply the existence of deterministic cycles, which is in contrast to the monetary OG models analysed e.g. by Grandmont (1985) and Azariadis and Guesnerie (1986). The numerical example above does not have 2−cycles. Thus these SSEs are not local to pure cycles. However, pure cycles exist for some other parameter configurations (details are available on request).

[20] Because in the nonlinear model agents at time t make only one-step ahead forecasts of X_{t+1}, it is unnecessary to directly estimate the dependence of X_t on g_t.

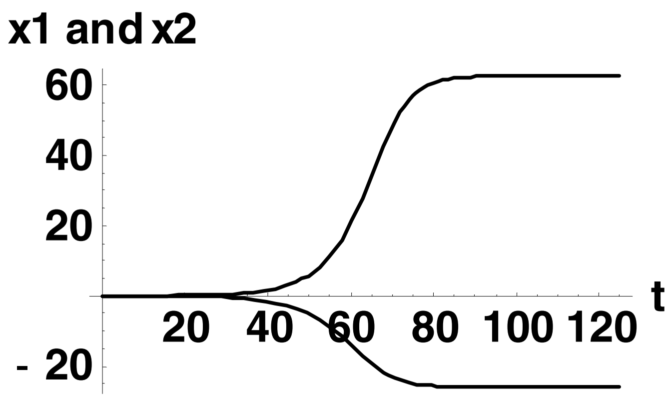

Figure 3: Convergence of adaptive learning to sunspot solution

The results of this section show that, in the case in which $\sigma > 1$ and the steady state is indeterminate, endogenous fluctuations due to expectational indeterminacy are a real concern. For exogenous sunspots near the resonant frequency, rational expectation SSEs exist and are stable under simple learning rules. These considerations raise the question of whether policy is able to avoid such expectational volatility.

5 Changes in Policy

We have seen that in this model there are two steady states, $\bar{\pi}_L$ and $\bar{\pi}_H$, when $\sigma < 1$ and $g > 0$. There has been much discussion of the issue of whether the economy might converge to the indeterminate steady state $\bar{\pi}_H$ in related seigniorage models. While we do take seriously the potential economic instability of the economy due to the multiplicity of equilibria in this case, we believe that the primary concern in this case is divergent paths (with π_t increasing beyond $\bar{\pi}_H$ to unsustainable levels) if for some reason π_t escapes from the basin of attraction of $\bar{\pi}_L$. Reductions in g tend to stabilize the economy in this case, making convergence to $\bar{\pi}_L$ more likely. Fiscal constraints on deficits and debt can also play an important role. The stability results of this and other papers suggest that $\bar{\pi}_H$ and SSEs near $\bar{\pi}_H$ are not locally stable under learning, though divergent paths are a concern.[21]

[21] The standard seigniorage model under learning was first studied by Marcet and Sargent (1989). The role of fiscal constraints is discussed in Evans, Honkapohja, and Marimon (2001).

However, a new case appears in our model of seigniorage finance. In the case $\sigma > 1$ there is a unique steady state that can be determinate or indeterminate and for σ sufficiently large it will necessarily be indeterminate. For values of $\sigma > 1$ the (noisy) steady state is stable under steady state learning. Furthermore, if the steady state is indeterminate and agents condition their actions on an exogenous sunspot near the resonant frequency, then they will converge to a noisy Markov SSE. Can policy help to avoid the endogenous fluctuations that can arise in this case?[22]

That this is indeed possible is shown by the following result (see the Appendix for a proof).[23] Recalling that $A = (1 + \alpha^{1/\sigma})\delta^{-1/\sigma}$, we have:

Proposition 10 *Assume that $\sigma > 1$ and consider the determinacy of the unique steady state $\bar{\pi}$ for different values of g.*
(i) If $\sigma < 2A/(A-1)$, there exists a critical value $\hat{g} > 0$ such that the steady state $\bar{\pi}$ is determinate if $g < \hat{g}$ and indeterminate if $g \geq \hat{g}$.
(ii) If $\sigma > 2A/(A-1)$, the steady state $\bar{\pi}$ is indeterminate for all $g \geq 0$.

Part (i) of Proposition 10 implies that, for $1 < \sigma < 2A/(A-1)$, E-stable Markov SSEs are eliminated by sufficient reductions in government spending g. Note that if $\alpha \geq 1$ then for $\sigma \geq 4$ all positive values of g are consistent with stable Markov SSEs.

The results of Proposition 10 are illustrated in Table 1. We set $\alpha = 1$ and fix the discount rate at $\delta = 0.95$. For each value of σ we examine the implications of different choices of g.

TABLE 1: Critical values of g for $\alpha = 1$.

σ	1.1	1.5	2.0	2.5	3.0	3.5	3.7	3.9	≥ 4
$\hat{g}$	0.36	0.34	0.27	0.19	0.12	0.05	0.03	0.01	0
$\bar{\pi}$	16.89	4.39	2.56	1.84	1.44	1.18	1.09	1.02	1

For each value of σ reported in Table 1, a critical value $\hat{g}$ is reported, together with the associated value of the steady state inflation rate. At the stated or larger values of g the steady state is indeterminate, there exist Markov SSEs, and these SSEs are stable under adaptive learning. For lower values of g the

[22] Avoidance of endogenous fluctuations is a major concern in the design of interest-rate rules for monetary policy; see Evans and Honkapohja (2003a) for a survey.

[23] These results are similar in spirit, but different in detail, from those obtained for the monetary OG model by Grandmont (1986). The latter emphasizes policy to eliminate cycles, and thus the SSEs that are associated with cycles in the OG model.

steady state becomes determinate and SSEs near the steady state no longer exist. For any σ in the range $1 < \sigma < 4$ stable Markov SSEs exist for high values of g but not for sufficiently low values. In this region a reduction in g can bring about a double benefit by both reducing steady state inflation and eliminating SSEs.[24]

In the model of this paper we have focussed on the seigniorage case in which government purchases are financed entirely by printing money. Seigniorage models have been used most commonly as a potential explanation of hyperinflation, and our numerical example indeed emphasized the possibility of endogenous fluctuations arising at high levels of inflation when the level of seigniorage is large. It is immediate from Table 1, however, that stable Markov SSEs near indeterminate steady states can also arise at low levels of inflation. We have made no attempt to calibrate the model to actual economies, and this would only be more appropriate with more elaborate versions of the model. However, the theoretical results of this paper show that monetary models of this type indeed have the power to explain business cycle fluctuations.

6 The Model with Taxes

In the previous section we have seen how, if $\sigma > 1$ but not too large, when government expenditures are low enough, the equilibrium is determinate and therefore no endogenous fluctuations arise. However, in other cases with $\sigma > 1$ the equilibrium is indeterminate and the Markov SSEs are stable under learning. In considering policy changes, another natural question is whether indeterminacy and endogenous fluctuations can be avoided by financing government expenditures through direct and indirect taxes. More generally, we are interested in how the introduction of taxes changes the characterization of equilibria. This is a natural question since, historically, most economies have ceased using seigniorage as a major source of government's revenue.

In this section we show that the introduction of consumption and labor taxes, while it raises revenues to finance government expenditures, does not change the qualitative features of the previous analysis, although it leads to nontrivial quantitative effects of fiscal policy on the stability of equilibria. Loosely speaking, we show that for 'reasonable' levels of expenditures it is often possible to avoid endogenous fluctuations by financing an important fraction of them with taxes. More precisely, the critical level of government

[24] These results are quantitatively, but not qualitatively, sensitive to the choice of α. For example, for $\alpha < 1$ determinacy can be obtained, with sufficiently low g, for values of σ less than an upper limit that now exceeds 4.

expenditures $\hat{g}$ defined in Proposition 10 is much larger with taxes than without them. However, this does not mean that a balanced budget is a guarantee for avoiding endogenous fluctuations, since even when the budget is – on average – balanced, the equilibrium can be indeterminate for some parameter values.

We now briefly describe the model with taxes. Let τ_s^c and τ_s^n be the consumption and labor tax rates in period s. The household budget constraint becomes

$$M_{s+1} + B_{s+1} \leq M_s + I_s B_s + p_s(1-\tau_s^n)n_s - p_s(1+\tau_s^c)(c_s^1 + c_s^2)$$

and the CA constraint takes the form

$$p_s(1+\tau_s^c)c_s^1 \leq M_s.$$

The *effective marginal tax rate on labor income,* τ, is the relevant tax for our analysis. This is the fraction of additional labor income taken away through taxes – in terms of effective consumption – holding savings fixed. This is given by

$$\tau_t = \frac{\tau_t^n + \tau_t^c}{1+\tau_t^c}, \text{ i.e., } (1-\tau_t) = \frac{1-\tau_t^n}{1+\tau_t^c}.$$

Keeping the same notation as before, let

$$\begin{aligned} m_{s+1} &= M_{s+1}/p_s(1+\tau_s^c),\ b_{s+1} = B_{s+1}/p_s(1+\tau_s^c), \\ \pi_s &= p_s/p_{s-1}, \text{ and } \upsilon_s = (1+\tau_s^c)/(1+\tau_{s-1}^c). \end{aligned}$$

We can now rewrite the household budget constraint as

$$m_{s+1} + b_{s+1} \leq m_s \pi_s^{-1} \upsilon_s^{-1} + I_s b_s \pi_s^{-1} \upsilon_s^{-1} + (1-\tau_s)n_s - (c_s^1 + c_s^2)$$

and the CA constraint as

$$c_s^1 = m_t \pi_s^{-1} \upsilon_s^{-1}. \tag{15}$$

The first order conditions of the household's maximization problem become

$$\begin{aligned} (c_t^2)^{-\sigma} &= \delta E_t^*[\pi_{t+1}^{-1} \upsilon_{t+1}^{-1} (c_{t+1}^1)^{-\sigma}], \\ (c_t^2)^{-\sigma} &= \delta E_t^*[I_{t+1} \pi_{t+1}^{-1} \upsilon_{t+1}^{-1} (c_{t+1}^2)^{-\sigma}], \\ (c_t^2)^{-\sigma} &= \alpha(1-\tau_t)^{-1}(1-n_t)^{-\sigma}. \end{aligned}$$

Similarly, the government finance constraint is

$$M_{s+1} + B_{s+1} \geq M_s + I_s B_s + p_s g_s - p_s \tau_s^n n_s - p_s \tau_s^c (c_s^1 + c_s^2).$$

Using $c_s^1 + c_s^2 + g_s = n_s$, this can be expressed as

$$m_{s+1} + b_{s+1} \geq m_s \pi_s^{-1} \upsilon_s^{-1} + I_s b_s \pi_s^{-1} \upsilon_s^{-1} + d_s,$$

where

$$d_s = g_s - \tau_s n_s \tag{16}$$

is the period s fiscal deficit. As before, if bonds are not held in positive net amount in equilibrium, then this constraint yields the following modified seigniorage equation

$$\pi_t = \frac{m_t}{m_{t+1} - d_t} \quad {}_t^{-1}. \tag{17}$$

Household optimization, market clearing and the CA constraint lead now to the equations

$$\begin{aligned}
n_t &= 1 - \alpha^{1/\sigma}(1-\tau_t)^{-1/\sigma}\delta^{-1/\sigma}\{E_t^*[\pi_{t+1}^{-1}\upsilon_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}]\}^{-1/\sigma}, \\
m_{t+1} &= [(1-\tau_t) + \alpha^{-1/\sigma}(1-\tau_t)^{1/\sigma}]n_t - \alpha^{-1/\sigma}(1-\tau_t)^{1/\sigma}, \\
c_t^1 &= m_{t+1} - d_t, \\
c_t^2 &= n_t - c_t^1 - g_t, \\
I_{t+1} &= (c_t^2)^{-\sigma}\delta^{-1}\{E_t^*[\pi_{t+1}^{-1}\upsilon_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}]\}^{-1}.
\end{aligned} \tag{18}$$

To keep the current version close to the earlier framework, we make the assumption that the government has a constant mean deficit d that is subject to shocks, i.e. $d_t = d + \tilde{d}_t$ where $\tilde{d}_t$ is exogenous white noise. Tax rates are taken to be constant, so that $\tau_t = \tau$, and government spending is given endogenously by $g_t = d_t + \tau n_t$. Steady states are determined by the following equations:

$$\begin{aligned}
\bar{n} &= 1 - \alpha^{1/\sigma}(1-\tau)^{-1/\sigma}\delta^{-1/\sigma}\bar{\pi}^{1/\sigma}\bar{c}^1, \\
\bar{m} &= [(1-\tau) + \alpha^{-1/\sigma}(1-\tau)^{1/\sigma}]\bar{n} - \alpha^{-1/\sigma}(1-\tau)^{1/\sigma}, \\
d &= g - \tau\bar{n}, \\
\bar{c}^1 &= \bar{m} - d, \\
\bar{\pi} &= \bar{m}/(\bar{m} - d).
\end{aligned}$$

These equations can be reduced to a single equation in the steady state inflation rate $\bar{\pi}$,[25]

$$(1-\tau-d)\bar{\pi} = 1 - \tau + d\tilde{A}(\tau)\bar{\pi}^{1/\sigma}, \tag{19}$$

$$\text{where } \tilde{A}(\tau) = (1 + \alpha^{1/\sigma}(1-\tau)^{(\sigma-1)/\sigma})\delta^{-1/\sigma}. \tag{20}$$

[25] This is done by (i) substituting for $\bar{c}^1$ in the first equation giving $\bar{n}$, (ii) substituting the result into the second equation giving $\bar{m}$, (iii) solving the resulting equation to get $\bar{m}$ in terms of parameters and $\bar{\pi}$. Then use this last equation also to compute $\bar{m} - d$. Finally, substitute the expressions for $\bar{m}$ and $\bar{m} - d$ into the seigniorage equation and simplify.

Note that if $\tau = 0$ then $g = d$ and thus (19) generalizes the earlier single equation for computing the steady state inflation rate. We need to assume that $\tau + d < 1$ to have interior and finite steady states. As in the model without taxes, it can easily be seen that there are two steady states when $\sigma < 1$. In contrast, for $\sigma > 1$ there is always a unique steady state for all admissible values of τ and d. This follows from (19) since then the right-hand side is a increasing concave function of $\bar{\pi}$. The comparative statics with respect to d, for constant τ, are the same as with respect to g in the model without taxes.

It will also be useful below to have a version of (19) written in terms of τ and g instead of τ and d:[26]

$$(1-g)\bar{\pi} = 1 - \tau + [(g-\tau) + g\alpha^{1/\sigma}(1-\tau)^{(\sigma-1)/\sigma}]\delta^{-1/\sigma}\bar{\pi}^{1/\sigma}. \tag{21}$$

For changes in the tax rate τ, one has to decide whether to keep d or g constant because of the government budget constraint $d = g - \tau\bar{n}$.[27] Holding g fixed, the comparative statics are straightforward when $\sigma > 1$: increases in τ with g fixed reduce the unique steady state inflation rate. For $\sigma < 1$ the comparative statics affect the two steady states differently.

To analyze determinacy and stability under learning, we require the full dynamic system. Assuming that tax rates are constant over time at rate $\tau_s = \tau$, the equations be combined to yield a univariate reduced form

$$c_t^1 = 1 - \tau - \tilde{A}(\tau)\left\{E_t^*\left[\frac{(c_{t+1}^1)^{1-\sigma}}{c_t^1 + d_t}\right]\right\}^{-1/\sigma} - d_t,$$

which has a linearization taking the same form (10) as before, but with coefficients β_0 and β_1 that now also depend on τ:

$$z_t = \beta_0 E_t^* z_t + \beta_1 E_t^* z_{t+1} + u_t, \tag{22}$$

where

$$\begin{aligned} \beta_0 &= -\frac{\tilde{A}(\tau)}{\sigma}(\bar{\pi})^{(1-\sigma)/\sigma}, \\ \beta_1 &= \frac{\tilde{A}(\tau)(1-\sigma)}{\sigma}(\bar{\pi})^{1/\sigma}. \end{aligned}$$

[26] To get this equation, replace $\bar{c}^1$ in the $\bar{n}$ equation by $\bar{m} - g + \tau\bar{n}$ and solve this and the $\bar{m}$ equation simultaneously to get $\bar{m}$ and $\bar{n}$ in terms of $\bar{\pi}$. These expressions are then inserted into $\bar{\pi} = \bar{m}/(\bar{m} - g + \tau\bar{n})$ and rearranged.

[27] Holding g fixed is more natural if one wants to examine the impact of a change in how g is financed: increases in τ with d fixed will increase g when tax revenues increase.

We focus on the case of a unique steady state ($\sigma > 1$), for which we earlier demonstrated that stable Markov sunspot equilibria can arise. When also $\sigma < 2A/(A-1)$ it was shown above that sufficient reductions in g will eliminate these sunspot solutions, see Proposition 10 above. We now show that the analysis generalizes in a very natural way:

Proposition 11 *Let $G(\tau) = 2\tilde{A}(\tau)/(\tilde{A}(\tau) - 1)$. Assume that the tax rate τ and mean deficit d are constant and non-negative. Then*
(i) If $1 < \sigma < G(\tau)$, there exists a critical value $\hat{d} > 0$ such that $d \geq \hat{d}$ implies indeterminacy and existence of E-stable Markov SSEs and $d < \hat{d}$ implies determinacy.
(ii) If $\sigma > G(\tau)$ the steady state is indeterminate and E-stable Markov SSEs exist for all values of d.

Since $G'(\tau) > 0$, this proposition implies that, with partial financing through taxes, there is a wider range of σ for which determinacy can be guaranteed by a sufficient reduction in d. The proposition does not, however, say that higher values of τ raise the critical value $\hat{d}$ above which indeterminacy occurs. Numerical calculations show that in fact $\hat{d}$ can be either increasing or decreasing in τ.

It is more revealing to consider the critical values of government spending for different tax rates. Given a tax rate τ, for every critical value of deficit $\hat{d}$ there is an associated critical value of government expenditures $\hat{g} = \hat{d} + \tau\bar{n}$ such that $g > \hat{g}$ implies indeterminacy. Here $\bar{n}$ is the endogenously determined steady state employment. The following table provides these critical values $\hat{g}$. It can be seen that, provided σ is not too high, sufficiently low levels of government expenditures can be sustained without generating E-stable sunspot equilibria. Furthermore, higher levels of τ raise the critical value $\hat{g}$. Notice that the first row corresponds to Table 1 and, as we have seen, for $\sigma \geq 4$ any positive level of government expenditures results in indeterminate equilibria and the existence of endogenous fluctuations. Yet, as Table 2 shows, moderate expenditures may be sustainable – without generating E-stable equilibria – provided that there is enough tax financing.

TABLE 2: Critical values $\hat{g}$ with taxes for $\alpha = 1$.

$\tau \diagdown \sigma$	1.5	2.0	2.5	3.0	3.5	4	5
0	0.34	0.27	0.19	0.12	0.05	0	0
0.1	0.38	0.32	0.26	0.19	0.13	0.08	0.07
0.25	0.45	0.41	0.36	0.31	0.26	0.21	0.18
0.5	0.57	0.56	0.54	0.51	0.49	0.46	0.41
0.75	0.71	0.73	0.74	0.74	0.73	0.73	0.71

In light of these results, one might be tempted to infer that learnable endogenous fluctuations can always be avoided through sufficient fiscal discipline. However, this need not be the case if σ is sufficiently high. We first calculate the unique tax rate τ^b, which balances the budget in the steady state (i.e. $d = 0$) for a given $g > 0$. Setting $d = 0$ in (19) implies $\bar{\pi} = 1$ and substituting these values in the steady state system we obtain that the required tax rate τ^b satisfies

$$\tau(\delta^{1/\sigma} + 1) = g\left[\delta^{1/\sigma} + 1 + \alpha^{1/\sigma}(1-\tau)^{(\sigma-1)/\sigma}\right],$$

which clearly has a unique solution $0 < \tau^b < 1$ when $0 < g < 1$. We have:

Corollary 12 *Let $G(\tau)$ be as in Proposition 11 and consider a balanced-budget steady state with $g > 0$ and $\tau^b > 0$. If $\sigma > G(\tau^b)$ the steady state is indeterminate and there exist E-stable Markov SSEs.*

Corollary 12 can be illustrated using Table 2. When $\sigma \geq 4$ and $\tau = 0$ there is indeterminacy for all values of g (or d). By continuity it follows that there are cases of indeterminacy for positive g and τ and it can be verified that this is so even if taxes are set at level τ^b for $g > 0$ but sufficiently small. It should be noted that this result arises because of intertemporal distortions that are present even when $\bar{\pi} = 1$.

The finding that imposing budget balance using distortionary taxes does not eliminate indeterminacy is in itself not new: Schmitt-Grohe and Uribe (1997) showed a similar result in the context of the standard RBC model. However, sunspot solutions in the RBC model with taxes are almost never stable under learning, as shown by Evans and McGough (2005a) and Rudanko (2002).

We now provide some intuition for the various factors that affect the existence of stable SSEs. Recall from the discussion in Section 4.2 that existence of stable Markov SSEs requires $\sigma > 1$ with strong negative feedback from future expectations. More specifically, for stable indeterminacy we need $\beta_0 - \beta_1 > 1$ (here $\beta_0 < 0$ and $\beta_1 < 0$). Using the expressions after (22) we have

$$\beta_0 - \beta_1 = \frac{-\tilde{A}(\tau)}{\sigma}[(\bar{\pi})^{(1-\sigma)/\sigma} + (1-\sigma)(\bar{\pi})^{1/\sigma}].$$

The effects of changes in government variables d, g or τ on determinacy vs. indeterminacy work through their effects on inflation and, in the case of τ, also through $\tilde{A}(\tau)$. (We recall that d, g and τ are related through the government budget constraint.) Holding the tax rate τ constant, $\bar{\pi}$ is increasing in d and $\lim_{d\to 1-\tau} \bar{\pi} = \infty$. In other words, $\beta_0 - \beta_1$ is an increasing function of d and also tends to infinity in the limit. Thus, increases in d holding τ constant raise the susceptibility to indeterminacy, and stable SSEs arise for sufficiently large values of d. The source of this is the higher negative expectational feedback when the (steady-state) inflation rate is higher. Similarly, increases in τ for given g reduce the inflation rate, and since $\tilde{A}'(\tau) < 0$ the direct effect through $\tilde{A}(\tau)$ reinforces the indirect effect that works through inflation. Consequently increases in τ for given g reduce the susceptibility to indeterminacy. More formally, we have:

Corollary 13 *Assume $\sigma > 1$.*
(i) Consider changes in d with τ fixed. Increases in d leave indeterminate steady states indeterminate, while determinate steady states may become indeterminate. Similarly, decreases in d leave determinate steady states determinate, while indeterminate steady states may become determinate.
(ii) Consider changes in τ with g fixed. Decreases in τ leave indeterminate steady states indeterminate, while determinate steady states may become indeterminate. Similarly, increases in τ leave determinate steady states determinate, while indeterminate steady states may become determinate.
(iii) Consider an increase in d. If financed by increasing $\bar{\pi}$, maintaining τ fixed, the increase in d leaves indeterminate states indeterminate, while determinate steady states may become indeterminate. If financed by increasing τ, maintaining $\bar{\pi}$ fixed, the increase in d leaves determinate steady states determinate, while indeterminate states may become determinate.

Our results for this section can be summarized as follows. The availability of partial tax financing contributes to determinacy of equilibria. Comparing Tables 1 and 2, when tax rates are positive the critical value of g is higher than when pure seigniorage financing is used. This is a central point of our model with taxes: lowering government expenditures with tax rates kept constant can help price stability by promoting determinacy. Correspondingly, given some level of government expenditures g, increasing the tax rate can render determinate the corresponding steady state equilibrium, provided that σ is not too large. However, for large enough $\sigma > 1$ there may be indeterminacy, and hence learnable Markov SSEs, even if there is a 'balanced budget,' for a range of $g \geq 0$. The reason for this result is that, even at an equilibrium with $\bar{\pi} = 1$, intertemporal distortions persist and result in strong negative feedbacks – between expected future consumption and current consumption – when σ is very large.

7 Conclusions

Indeterminacy of equilibria has been a major issue in both business cycle analysis and monetary economics. Most of the applied research has examined the question of the existence of self-fulfilling fluctuations in a neighborhood of an indeterminate steady state. This paper has imposed the additional discipline of asking whether there exist RE solutions, of this type, that are stable under adaptive learning dynamics. If solutions are not learnable, they may well be just theoretical artifacts. On the other hand, if they are stable under learning dynamics, then agents could plausibly coordinate on such solutions. We remark that, in experiments, agents appear able to condition their expectations on sunspots if, as a group, they have previously experienced fluctuations driven by fundamentals. In such cases, it is possible that even once the 'fundamental uncertainty' vanishes, belief driven fluctuations persist (see Marimon, Spear, and Sunder (1993)).

We have examined these issues in the context of a standard infinite-horizon representative agent framework, in which money demand is generated by CA constraints and exogenous government spending is financed by a mixture of taxes and seigniorage. The model allows for both cash and credit goods as well as for a flexible labor supply for which the response to increased interest rates is always positive irrespective of the degree of curvature of the utility functions.

We have shown that, for some regions of the parameter space, there do exist learnable sunspot equilibria in a neighborhood of an indeterminate steady state. The learnable SSEs take a particular form, driven by a sunspot process following a finite state Markov chain with transition probabilities close to the resonant frequency property. The other types of SSEs examined are not learnable. In our model, the case of learnable SSEs arises for the regime in which there is a single steady state.

In many cases, an appropriate change in fiscal policy can eliminate stable sunspot equilibria by rendering the steady state determinate. Achieving this goal requires some care in the choice and magnitude of the fiscal instruments. An important contribution of our paper is to show that a sufficient reduction of government spending will in many cases lead to determinacy and hence to the elimination of SSEs. Likewise, an increase in tax financing will reduce the region of indeterminacy. In other words, existing proposals, such as the EMU Growth and Stability Pact that constrains the deficit as a fiscal measure to enhance price stability, can in many cases help to attain this goal.

However, our results also show that these measures *may* not be sufficient to eliminate stable endogenous price fluctuations. In particular, fiscal policies that balance the budget on average are consistent with stable sunspot fluctuations for some parameter values. This result may not be very worrisome, on empirical grounds, if one takes as reference the estimates for high aggregate substitution effects, based on changes in the extensive employment margin. These estimates would seem to rule out the range of parameters in which endogenous fluctuations can exist with a 'balanced budget.' Furthermore, on theoretical grounds, one must take into account that in our set-up a 'balanced budget' is not optimal: instead the Friedman rule of maintaining persistent deflation, at a rate consistent with zero nominal interest rates, would be the optimal policy. The question would then be whether such a policy eliminates endogenous fluctuations. More generally, one can ask whether a fiscal-monetary policy mix that aims to ensure that money growth – or the inflation rate – is steady over time in the presence of shocks – i.e., period by period as well as on average – eliminates endogenous fluctuations. In our monetary economy, such policies would result in having endogenous tax rates, deficits or debt, whereas we have treated tax rates and deficits as exogenous and assumed that no bonds are traded in equilibrium. Incorporating debt financing into the framework of this paper, and a complete treatment of alternative fiscal-monetary rules, are important extensions that are left for future research.

A Appendix: Technical Details

Proof of Corollary 3: By Proposition 1 we need to show that $\beta_0 + \beta_1 > 1$ at $\bar{\pi}_H$ and $\beta_0 + \beta_1 < 1$ at $\bar{\pi}_L$ when $\sigma < 1$.

Using (8), we have at a steady state $\bar{\pi}$

$$\beta_0 + \beta_1 - 1 = \frac{[(1-g)\bar{\pi} - 1]\,[(1-\sigma)\bar{\pi} - 1]}{\sigma g \bar{\pi}} - 1.$$

Thus, we consider the quadratic function

$$f(\pi) = [(1-g)\pi - 1]\,[(1-\sigma)\pi - 1] - \sigma g \pi.$$

The equation $f(\pi) = 0$ has two roots $\pi = 1$ and $\tilde{\pi} = (1-\sigma)^{-1}(1-g)^{-1}$. Clearly, $\beta_0 + \beta_1 \gtrless 1$ when $\bar{\pi} \gtrless \tilde{\pi}$.

On the other hand, consider the value $\breve{\pi}$ at which the graph of the function $1 + gA\pi^{1/\sigma}$ has the tangent parallel to the straight line $(1-g)\pi$. (Note that the two functions are the right- and left-hand sides of (8), respectively.) $\breve{\pi}$ solves the equation

$$\frac{gA}{\sigma}\pi^{1/\sigma - 1} = 1 - g.$$

If we require $\breve{\pi}$ to be also a steady state, we get $\breve{\pi} = \tilde{\pi}$ and g is at a maximal value for which steady states exist under the restriction $\sigma < 1$. When g is reduced from the maximal level, the comparative static properties of $\bar{\pi}_H$ and $\bar{\pi}_L$ imply that $\bar{\pi}_L < \tilde{\pi}$ and $\bar{\pi}_H > \tilde{\pi}$ and thus $\beta_0 + \beta_1 > 1$ at $\bar{\pi}_H$ and $\beta_0 + \beta_1 < 1$ at $\bar{\pi}_L$.

Proof of Proposition 4: Part (i) was derived in the text. For part (ii) concerning the AR solutions one starts with PLM of the form (12) and computes that, under the PLM

$$\begin{aligned} E_t^* z_t &= a + bz_{t-1} + d_0 u_t + d_1 u_{t-1} + k\eta_t, \\ E_t^* z_{t+1} &= a + bE_t^* z_t + d_1 u_t, \end{aligned}$$

where it is assumed that the current values u_t, η_t of the exogenous shocks, but not the endogenous variable z_t are in the information set for period t. Substituting these into the linear model (10) we have

$$\begin{aligned} z_t = {} & (\beta_0 + \beta_1 b)(a + bz_{t-1} + d_0 u_t + d_1 u_{t-1} + k\eta_t) \\ & + \beta_1 a + \beta_1 d_1 u_t + u_t. \end{aligned}$$

This yields the mapping from PLM to ALM, which, component by component, is given by

$$\begin{aligned} a &\rightarrow \beta_1 a + (\beta_0 + \beta_1 b)a \equiv T_a(a, b), \\ b &\rightarrow (\beta_0 + \beta_1 b)b \equiv T_b(b), \\ d_0 &\rightarrow (\beta_0 + \beta_1 b)d_0 + \beta_1 d_1 + 1 \equiv T_{d0}(b, d_0, d_1), \\ d_1 &\rightarrow (\beta_0 + \beta_1 b)d_1 \equiv T_{d1}(b, d_1), \\ k &\rightarrow (\beta_0 + \beta_1 b)k \equiv T_k(b, k). \end{aligned}$$

Looking first at possible fixed points we see that $b = 0$ or $b = (1 - \beta_0)/\beta_1$. If $b = 0$, we have $k = d_1 = 0$, $a = 0$ and $d_0 = (1 - \beta_0)^{-1}$. This corresponds to the noisy steady state. If $b = (1 - \beta_0)/\beta_1$ we get $d_1 = \beta_1^{-1}$, $a = 0$ while k and d_0 can take any values, which is the indeterminate case. This latter class of solutions will be stationary if $|(1 - \beta_0)/\beta_1| < 1$.

Introducing the notation $T(a, b, d_0, d_1, k) = (T_a, T_b, T_{d0}, T_{d1}, T_k)$, for E-stability one examines the local stability of

$$\frac{d(a, b, d_0, d_1, k)}{d\tau} = T(a, b, d_0, d_1, k) - (a, b, d_0, d_1, k)$$

at the REE of interest. In the indeterminate case we say that the solution class is E-stable if the dynamics of the E-stability differential equations converge to some member of the class from each nearby initial condition. For the indeterminate case the E-stability conditions are easily seen to be $\beta_1 < 0$ and $\beta_0 > 1$.

Proof of Proposition 8: The PLM of the agents is equivalent to

$$z_t = z(j) + Ku_t \text{ if } s_t = j.$$

Given the informational assumptions the ALM is

$$\begin{aligned} z(j)^* &= \beta_0 z(j) + \beta_1[p_{j1}z(1) + p_{j2}z(2)],\ j = 1, 2, \\ K^* &= \beta_0 K + 1. \end{aligned}$$

where $z(j)^*$ is the actual mean of z_t and K^* is the actual u_t coefficient, under this PLM, if $s_t = j$. The T map for K is the same as in the case of steady state learning and the corresponding condition for E-stability is satisfied when $\beta_0 < 1$. The equations for $z(j)$ can be written in the matrix form

$$\theta^* = T\theta, \text{ where } \theta^* = \begin{pmatrix} z(1)^* \\ z(2)^* \end{pmatrix}, T = \beta_0 I + \beta_1 P \text{ and } \theta = \begin{pmatrix} z(1) \\ z(2) \end{pmatrix}. \quad (23)$$

We remark that the T map in (23) is different from $\mathcal{T}$ in (13), due to our information assumption for the system under learning. However, they have the same fixed points and thus a Markov SSE exists if and only if there are transition probabilities $0 < p_{jj} < 1$, $j = 1, 2$, that satisfy

$$\frac{1 - \beta_0}{\beta_1} = p_{11} + p_{22} - 1. \quad (24)$$

The mapping from the PLM to the corresponding ALM is here linear and given by the matrix in (23). E-stability is determined by the stability of the differential equation

$$d\theta/d\tau = T\theta - \theta.$$

We need to compute the eigenvalues of $T - I = (\beta_0 - 1)I + \beta_1 P$. If λ is an eigenvalue of $T - I$ then we must have

$$0 = \det((\beta_0 - 1)I + \beta_1 P - \lambda I) = \beta_1^2 \det(P - \frac{\lambda + 1 - \beta_0}{\beta_1} I)$$

so that

$$\frac{\lambda + 1 - \beta_0}{\beta_1} = 1 \text{ or } p_{11} + p_{22} - 1.$$

In the latter case we get $\lambda = 0$ while the former yields $\lambda = \beta_0 + \beta_1 - 1$ implying the requirement $\beta_0 + \beta_1 < 1$. We remark that $T - I$ always has a zero eigenvalue as a result of the resonant frequency condition (24) and there being a continuum of Markov due to the linearity of the model. The E-stability differential equation

$$\frac{d\theta}{d\tau} = (T - I)\theta$$

has a zero eigenvalue and a negative eigenvalue if $\beta_0 + \beta_1 < 1$ holds and (24) is satisfied. Stability of the set of Markov SSEs then follows from the mathematical Lemma in the Appendix of Honkapohja and Mitra (2004).

Proof of Proposition 10: This is a special case of the proof of Proposition 11 when $\tau = 0$.

Proof of Proposition 11: If $\sigma > 1$, we have $\beta_1 < 0$ while always $\beta_0 < 0$. Indeterminacy of the steady state requires $\beta_0 - \beta_1 > 1$. In general, we have

$$H(\bar{\pi}, \tau) \equiv \beta_0 - \beta_1 = -\frac{\tilde{A}(\tau)}{\sigma}[(\bar{\pi})^{(1-\sigma)/\sigma} + (1-\sigma)(\bar{\pi})^{1/\sigma}] \tag{25}$$

and we note that for constant τ, $H(\bar{\pi}, \tau)$ is strictly increasing in $\bar{\pi}$ with

$$\begin{aligned} \lim_{\bar{\pi}\to\infty} H(\bar{\pi}, \tau) &= \infty \\ \lim_{\bar{\pi}\to 1} H(\bar{\pi}, \tau) &= \frac{\tilde{A}(\tau)(\sigma - 2)}{\sigma}. \end{aligned}$$

We couple these limits with the observations (a) $\bar{\pi} \to \infty$ as $d \to 1 - \tau$ and (b) $\bar{\pi} \to 1$ as $d \to 0$, which follow from (19).

To prove part (i), we note that $\lim_{\bar{\pi}\to 1} H(\bar{\pi}, \tau) < 1$ if $\sigma < G(\tau)$, which by continuity and $\lim_{\bar{\pi}\to\infty} H(\bar{\pi}, \tau) = \infty$ imply the existence of $\hat{d} > 0$ as required. Part (ii) follows by noting that $\lim_{\bar{\pi}\to 1} H(\bar{\pi}, \tau) > 1$ when $\tilde{A}(\tau)(\sigma - 2) > \sigma$ i.e. when $\sigma > G(\tau)$.

Proof of Corollary 12: When $\bar{\pi} = 1$ we have

$$H(1, \tau) = \frac{\tilde{A}(\tau)(\sigma - 2)}{\sigma},$$

where $H(\bar{\pi}, \tau)$ is defined in (25). The indeterminacy condition $H(1, \tau) > 1$ is easily seen to be equivalent to the requirement $\sigma > G(\tau)$. In this case we must have $\sigma > 1$ and so indeterminacy implies the existence of stable Markov SSEs.

Proof of Corollary 13: Part (i) follows directly from Proposition 11. Consider then part (ii) in which g is fixed and τ is changed. The steady state is indeterminate if $\beta_0 - \beta_1 > 1$ and determinate if $\beta_0 - \beta_1 < 1$, and we have

$$\begin{aligned}\beta_0 - \beta_1 &= \frac{-\tilde{A}(\tau)}{\sigma}\phi(\bar{\pi}(\tau)) \text{ where} \\ \phi(\pi) &= (\pi)^{(1-\sigma)/\sigma} + (1-\sigma)(\pi)^{1/\sigma}.\end{aligned}$$

It is easily verified that $\tilde{A}(\tau) > 0$ and $\tilde{A}'(\tau) < 0$. Furthermore

$$\phi'(\pi) = \frac{1-\sigma}{\sigma}\pi^{1/\sigma-1}(\pi^{-1}+1) < 0,$$

and it was earlier shown that holding g fixed $\bar{\pi}'(\tau) < 0$. We now break the proof into pieces.

(a) Consider first a steady state that is indeterminate. Then $\beta_0 - \beta_1 > 1$, which implies that $\phi(\bar{\pi}(\tau)) < 0$. Thus

$$\frac{d(\beta_0 - \beta_1)}{d\tau} = -\sigma^{-1}\left\{\tilde{A}' \cdot \phi + \tilde{A} \cdot \phi' \cdot \bar{\pi}'\right\} < 0$$

so that reductions in τ must further increase $\beta_0 - \beta_1$. This establishes that reductions in τ must leave indeterminate steady states indeterminate.

(b) Next, consider a steady state that is determinate at tax rate τ_0. Suppose, contrary to the claim, that when τ is increased to $\tau_1 > \tau_0$ the steady state becomes indeterminate. But then, by part (a) of the proof, if the tax rate is reduced from τ_1 to τ_0 the steady state at τ_0 must be indeterminate, contradicting our initial assumption. This proves that increases in τ leave determinate steady states determinate.

(c) Furthermore, it can be easily verified, e.g. see Table 2, that, for a range of $\sigma > 1$, increases in τ, with g fixed, can make indeterminate steady states determinate and that decreases in τ, with g fixed, can make determinate steady states indeterminate. The proof of part (ii) is now complete.

Finally, notice that the first part of (iii) follows directly from the first part of (i), while the second part of (iii) follows directly from the previous proof of part (ii) with $\bar{\pi}'(\tau) = 0$.

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