Making Sovereign Debt Safe with a Financial Stability Fund*

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Abstract

We develop an optimal design of a Financial Stability Fund that coexists with the international debt market. The sovereign can borrow long-term defaultable bonds on the private international market, while having with the Fund a long-term contingent contracts subject to limited enforcement constraints. There is a contract that minimizes the debt absorbed by the Fund, guaranteeing full debt stabilization. In equilibrium, the seniority of the Fund contract, with respect to the privately held debt, is irrelevant. We calibrate our model to the Italian economy and show it would have had a more efficient path of debt accumulation with the Fund.

Keywords: Recursive contracts, limited enforcement, debt stabilisation, debt overhang, safe assets, seniority structure

JEL Classification: E43, E44, E47, E62, F34, F36, F37

1 Introduction

In the last few years, the public debt-to-GDP ratio has reached historic levels in the world (Kose et al., 2021). For instance, in the European Union (EU), the average indebtedness

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of Member States amounted to 95% of GDP in 2020, whereas it was 66% in 2000. Some countries such as Italy, Portugal and Spain are already expected to reach a debt-to-GDP ratio above 120% by 2021, while Greece should end up with a figure around 200%[1]. This is the result of three consecutive crises — the global financial crisis of 2007–2008, the European sovereign debt crisis of 2010–2012 and the COVID-19 crisis. The question of how to efficiently stabilise the sovereign debt remains open.

We design a Financial Stability Fund (Fund) as a *constrained-efficient mechanism*, in line with Ábrahám et al. (2021). While the latter assumes that the Fund absorbs all the sovereign debt of a country and focuses on the borrower's perspective, we emphasise the lender's side of the contract and derive the optimal relationship between the private competitive lenders and the Fund. More precisely, we assume that sovereign countries can raise debt in the private international market and in the Fund.[2] While private international lenders solely offer credit (i.e. long-term non-contingent defaultable bonds), the Fund provides both credit and insurance (i.e. Arrow-type securities) in the form of long-term state-contingent bonds. The Fund's intervention is constrained to prevent default and conditional on a strict *debt sustainability analysis* (DSA), which we identify in our framework as an evaluation of the present value of the sovereign's future surpluses (net savings) from any contingency onwards. In any period and history, the sovereign has to pass this DSA if it wants to continue to receive transfers from the Fund.

The Fund provides a two-sided limited enforcement contract which is state-contingent: on the 'growth state' of the economy,[3] and on the 'binding states' (i.e. provides preventive insurance against default and non-sustainability). In designing the contract the Fund takes into account the country's indebtedness (i.e. commitments) with private lenders. This brings the issue of whether the Fund has seniority over the private lenders. We consider two regimes: *pari passu* (i.e. no seniority) and *seniority* of Fund's liabilities over private liabilities. We show that with constrained-efficient Fund contracts the equilibrium allocation is *independent of the seniority structure*. To the best of our knowledge, we are the first to present such result.

The DSA is the main focus of our analysis. It internalizes a pecuniary externality that

[1]According the IMF April 2022 *Fiscal Monitor*, General Government Gross Debt in 2021: Euro area 96%, Italy 150.9%, Portugal 127.5%, Spain 118.7% and Greece 198.9%.

[2]The adjective 'private', is used to distinguish lenders on the international market relative to the Fund.

[3]On the growth rate of the country g, also denoted γ or, if the interest rate in the international capital market r is not constant, on $(r - g)$.

competitive private lenders usually do not: the fact that marginal lending can result in debt becoming unsustainable. Instead, the DSA monitors whether additional borrowings entails expected losses. The DSA being binding results in a negative spread for the Fund-provided assets. For the private lenders this can trigger a run on the debt, unless they follow the DSA of the Fund to stop lending. By integrating the DSA together with the sovereignty constraint, our Fund prevents the possibility of default triggered by the borrower, but also accounts for the possibility that lenders withdraw before incurring any expected loss. The literature on sovereign debt has so far focused on the borrower's default decision, we contribute by characterising the lender's optimal reaction and its effect on the sovereign debt market.

Our analysis builds on the framework of Ábrahám et al. (2021). The main difference is twofold. First, we use growth shocks to better analysze the interest rate-growth differential (i.e. $r - g$). Second, we do not consider an exclusive contract between the Fund and the contracting countries. We know that if the entire position of the sovereign debt is taken over by the Fund, then the debt becomes risk free. We also know that, with default costs, for some *small* but *strictly positive* debt level, any amount of debt less than this level is free from default risk.[4] One may be tempted to infer from these two results that to stabilise the debt the Fund must absorb *most* of the country's debt. We show that, on the contrary, with its contracts the Fund only needs to absorb a *minimal* fraction of the country's debt to stabilise it all. This theoretical result is of practical relevance since, in terms of size, it brings us closer to the existing official lending practices (e.g. of the IMF or ESM).[5]

The intuition of our 'seniority irrelevance' result can be explained in a few words. In the case of a *pari passu* contract, the Fund has to ensure that the private lenders do not over-lend as the sovereign's total indebtedness might become unsustainable otherwise. To this end, it threatens the private lenders with market withdrawal in case of a deviation from the DSA. Given that, in equilibrium, the private lenders never offer debt contracts that deviate from the Fund's DSA. A negative spread arises in the security market since it is the price-signal that the Fund is restricting the provision of its insurance to the sovereign. In contrast, under seniority, in principle partial default (i.e. only default to private lenders) is possible. However, it is here that the Arrow-insurance component of the Fund contract plays

[4]Such a debt level corresponds to the smallest default threshold in the pure incomplete market economy with default risk (Zhang, 1997).

[5]Another difference with respect to Ábrahám et al. (2021) is that we do not consider moral hazard constraints as our focus is on the lending side of the contract rather than the borrowing side.

an essential role: the Fund provides insurance against 'non-sustainable debt states' at the expense of states where the DSA is not binding. As a result, when debt is not sustainable — say, in one state next period — it is not sustainable in *all* states and, therefore, private lenders deter from further lending; and the price-signal is, in this case, a positive spread. Nevertheless, the outcome of the contract remains the same as neither the sovereign nor the private lenders have an incentive to deviate from the Fund's DSA.

As we said, our analysis enables a comparison with existing lending institutions such as the ESM and the IMF. We show that the seniority structure of the Fund is irrelevant in our environment, while the ESM and the IMF usually require seniority in their lending programs.[6] Moreover, while it is true that official lending institutions conduct DSAs as a necessary condition to guarantee credits, it is not the case that their resulting debt contracts provide insurance against future DSAs, as the Fund does. In other words, international lending institutions base their lending policy on one of several scenarios — e.g. the 'most likely one,' the 'politically preferred,' or the 'worst case' scenario. In contrast, the Fund contract risk-shares among these different scenarios or paths. That is, it provides additional transfers in the worst scenario in exchange for higher payments in the best scenario.[7]

We conduct a quantitative analysis in which we calibrate the outside option of the Fund — an incomplete market economy with defaults — to Italy for the period 1992Q1–2019Q4. Unlike Greece, Portugal and Spain, Italy did not participate to an ESM programme of any sort. It therefore offers the possibility of a counterfactual analyses. Second, it faces a public indebtedness above 100% of GDP and one of the largest spreads in the Euro Area. The specificity of Italy and its debt management has already been recently studied by Bocola and Dovis (2019). Our contribution is here twofold. On the one hand, we study the impact of the introduction of a Fund on Italy's debt sustainability and welfare. On the other hand, we introduce stochastic growth and gauge the relationship between r and g.

The main results of our inquiry are also twofold. First, with the Fund, the Italian debt would have been free of default risk; i.e. its entire debt position would have been safe. This

[6]The IMF together with the World Bank have a *de facto* seniority, but it is not a formal contractual feature (see Schlegl et al., 2019). In opposition, the ESM has a *de jure* seniority with respect to the market. The only exception to this is Spain. The Spanish program was initially agreed with the EFSF with a standard *pari-passu* clause and managed to prolongate this feature into the ESM loan.

[7]Note that in fact we refer to a *Stochastic Debts Sustainability Analysis* (SDSA) since the Fund accounts for the overall debt stochastic structure. SDSA is now part of the analysis of IMF, but is not part of its contract design.

is due to the state-contigent transfers provided by the Fund. More precisely, the Fund offers a countercyclical policy subject to the enforcement constraints. This component of 'binding states' insurance is at the source of important welfare gains. We show that the sovereign benefits from a greater debt absorption capacity compared to the standard incomplete market economy with defaults. Particularly, receiving state-contingent transfers from the Fund, the sovereign can accumulate debt in states in which defaults would usually happen.

Second, we argue that by accessing the Fund, Italy would have had a more stable evolution of its indebtedness. Using the decomposition of Cochrane (2020, 2022), we show that, in the last two decades, Italy largely increased its public indebtedness despite large primary surpluses. This is due to a strongly positive interest rate-growth differential $(r - g)$ dominating the debt accumulation process. The positive differential is a combination of a relatively low, and unstable, growth of the Italian economy with an important risk premium on the Italian sovereign debt. We show that, by accessing the Fund, the Italian government would have reduced these perverse effects and therefore would have ended up with a lower indebtedness. The model predicts that the Italian indebtedness by the end of 2019 would have been around 86% of GDP rather than 135% if Italy could have joined the Fund in 2000. The Italian government's perseverance in maintaining positive primary surpluses, in spite of growth reversals, can be seen as a *commitment to debt sustainability*, in line with the European Union's fiscal policy (i.e. the Fiscal Compact). Indeed, the accumulation of large primary surpluses dampened the increase in Italian indebtedness, but this was a highly inefficient path to follow if Italy could have had access to the Fund.

Relation to the literature Our work is related to the 'sovereign debt' literature pioneered by Eaton and Gersovitz (1981) and subsequently extended by Aguiar and Gopinath (2006) and Arellano (2008) (see also Aguiar and Amador (2014) and Aguiar et al. (2016)). As in Ábrahám et al. (2021), our benchmark economy with defaultable debt builds on Chatterjee and Eyigungor (2012) who introduce long-term bonds. Within this literature, our work is closely related to Hatchondo et al. (2017), who consider the case of adding a non-defaultable bond into the otherwise standard defaultable bond economy, and show that there are welfare gains by swapping defaultable bonds into non-defaultable bonds. It is also related to the literature on optimal contracts with limited enforcement constraints such as Kehoe and Levine (2001), Kocherlakota (1996) and, in particular, Kehoe and Perri (2002) who already applied the Lagrangian-recursive approach developed by Marcet and Marimon (2019). Our Planner's problem is subsequently decentralised using the approach of Alvarez and Jermann (2000), while our focus is close to Thomas and Worrall (1994) who already studied international lending contracts, with one-sided limited commitment.

A more recent literature merges these last two strands of literature and it is the most closely related one to our work. In particular, Dovis (2019) decentralises optimal contracts through partial default and an active debt maturity management, and Müller et al. (2019) through *ex-post* state-conditionality given by default and renegotiation procedures. Our approach is not to 'rationalise' *ex-post* observed behaviour, but to account for existing constraints and, without restricting the form of the lending contracts, characterise constraint-efficient equilibria — between a sovereign debtor, private international lenders and a Fund — which are *ex-ante* and *ex-post* efficient, and assess them quantitatively in relation to a calibrated version of the benchmark defaultable debt economy. Within this approach, and in contrast to most of the literature, our specific focus is on the role of lender's Debt Sustainability Analysis (DSA), as a lender's limited-enforcement constraint.

Finally, as a theoretical foundation for the design of a — effectively running — fiscal fund, able to stabilise sovereign debt and expand the supply of safe assets, our work is related to a large literature regarding the IMF and other international institutions lending practices, and to the debate on the need to develop the Fiscal Union within the European Economic and Monetary Union (EMU) and expand its supply of eurobonds (as it has been done with the Next Generation EU (NGEU) program).[8].

Organisation of the paper We lay down the economic environment in Section 2, together with the sovereign's and the private lender's problems. We expose the Fund contract in Section 3, which also includes the decentralisation of the contract. Section 4 contains our 'seniority irrelevance' result. After this, we calibrate our model to Italy in Section 5 and present the underlying results in Section 6. Finally, we conclude in Section 7.

2 The Economy, the Sovereign Country, and the Private Lenders

We assume an infinite-horizon small open economy with a single homogenous consumption good in discrete time. The sovereign acts as a representative agent and takes the decision on behalf of the small open economy. Preference over consumption and leisure is represented by $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, n_t)$, where $\beta \in (0, 1)$ is the discount factor, n_t is the labor, $1 - n_t$ the leisure and c_t the consumption at time t. The sovereign is relatively impatient such that

[8]See Marimon and Wicht (2021) for a discussion on how our Fund proposal relates to this literature and it can be implemented within EMU.

$\beta < 1/(1+r)$, where r is the risk-free world interest rate. We adopt a specific form of utility function so as to obtain a (stochastic) balanced growth path and to simplify the expression of utility in terms detrended consumption: $U(c,n) = u(c) + h(1-n) = \log(c) + \xi\frac{(1-n)^{1-\zeta}}{1-\zeta}$. The sovereign has access to a labor technology $y = \theta f(n)$ subject to decreasing returns to scale, where $f'(n) > 0$, $f''(n) < 0$. Moreover, θ represents a growth shock to the productivity. It is the only source of uncertainty in the economy. The law of motion of the shock is given by $\theta_t = \gamma_t\theta_{t-1}$, where γ_t represents the growth rate at time t. The exact form of the growth shocks follows Aguiar and Gopinath (2006) and is detailed in Section 5.[9]

The financial market is composed of risk neutral private competitive lenders and a Financial Stability Fund. The sovereign has therefore two funding opportunities. On the one hand, it can borrow long-term defaultable bonds, b', on the private bond market.[10] On the other hand, the sovereign can trade state-contingent Arrow-type securities, $\hat{a}'(\theta')$ for all $\theta' \in \Theta$, with the Fund, and can accumulate debt, $\bar{a}'$, as it does with private lenders. The maturity structure of the aforementioned financial assets is such that a fraction $1-\delta$ of the portfolio matures every period and the remaining fraction δ is rolled-over and pays a coupon κ (Chatterjee and Eyigungor, 2012).

The sovereign cannot commit to repay its debts. In the first part of our analysis, we assume that the sovereign's liabilities with the Fund have no seniority with respect to the sovereign debt in the hands of other agents. That is, if the government decides to default, it does so on its entire debt position. In other words, every default is a *full* default as in Eaton and Gersovitz (1981). We later relax this assumption and consider the case in which the Fund possesses seniority with respect to the private bond market. There, we introduce the possibility of *partial* default in which the sovereign can solely default on its private liabilities while maintaining access to the Fund.

Under *full* default, the sovereign receives a penalty in the form of a reduced output, $\theta^p \leq \theta$, and loses access to both the private bond market and the Fund. Later, it can reintegrate the private bond market with some probability, λ, but cannot obtain the assistance of the Fund anymore. Under *partial* default, the sovereign receives the same penalty as above and is similarly excluded from the private bond market for some time. However, it maintains its access to the Fund.

[9]We present in the main text the model with the stochastic trend and keep track of θ in the state space. The detrended version is presented in Appendix A. There we only keep track of γ in the state space.

[10]Following the literature, $b' > 0$ represents an asset, while $b' < 0$ represents a debt.

The Fund contract between the Fund and the borrowing country has the following features: *i*) the only *ex-ante conditionality* is to satisfy a strict risk-assessment of the borrower (i.e. the sovereign's DSA) accounting for its debt liabilities with private investors, and *ii*) there is two-sided limited enforcement since the borrower is sovereign and can default on its debt, while the Fund has a free access to the international financial market and can withdraw whenever additional lending entails expected losses. The timing of actions is the following. After the realization of the growth shock θ, the Fund announces its lending policy and the sovereign decides whether to default or not. In the latter case, the sovereign then determines its prospective borrowing, first in the private bond market.

The sovereign's problem takes the following form. At the start of a period, the sovereign holds a portfolio a with respect to the Fund and a portfolio b with respect to the private bond market which together sum to an entire debt position of $\omega = a + b$. A fraction $1 - \delta$ of each portfolio matures today and the remaining fraction δ is rolled-over and pays a coupon κ. The sovereign can trade private bonds b' with a unit price of $q_p(\theta, \omega')$. Alternatively, it can also trade Θ state contingent securities $a'(\theta')$ with a unit price of $q_f(\theta', \omega'(\theta')|\theta)$. Both prices depend on the entire debt position and not separately on b' or $a'(\theta')$. This is due to the fact that the Fund's assets are not senior with respect to the private bonds. Thus, in the case of default, the sovereign reneges its entire debt position. This implies that the risk of default — and therefore the risk premium — is properly measured with respect to the debt held both in the Fund and in the private bond market.

Moreover, the assets provided by the Fund are state contingent, while private bonds are not. More precisely, the portfolio $a'(\theta')$ can be decomposed into a common bond $\bar{a}'$ that is independent of the next period state, traded at the implicit bond price $q_f(\theta, \omega') \equiv \sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)$, and an insurance portfolio of Θ Arrow-type securities $\hat{a}'(\theta')$. Thus we have that $a'(\theta') = \bar{a}' + \hat{a}(\theta')$ with $\bar{a}' = \frac{\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta) a'(\theta')}{q_f(\theta, \omega')}$ and $\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)\hat{a}'(\theta') = 0$, which represents the market clearing condition of the Arrow-type securities. The sovereign's problem therefore reads

$$
\begin{aligned}
W^b(\theta, a, b) = & \max_{\{c, n, b', \{a'(\theta')\}_{\theta' \in \Theta}\}} U(c, n) + \beta \mathbb{E}\left[W^b(\theta', a'(\theta'), b') \middle| \theta\right] \qquad (1) \\
& \text{s.t. } c + \sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)(a'(\theta') - \delta a) + q_p(\theta, \omega')(b' - \delta b) \\
& \leq \theta f(n) + (1 - \delta + \delta\kappa)(a + b)
\end{aligned}
$$

$$\omega'(\theta') = a'(\theta') + b' \geq \mathcal{A}_b(\theta'). \tag{2}$$

We directly notice that the sovereign is not given the choice to default. Instead, it is constrained by the endogenous borrowing limit $\mathcal{A}_b(\theta')$ which directly emanates from the Fund contract. The purpose of this limit is to prevent the occurrence of defaults on equilibrium path. It is therefore defined as

$$W^b(\theta', \ddot{a}(\theta'), \ddot{b}') = V^{af}(\theta') \text{ for all } \ddot{a}(\theta') + \ddot{b}' = \mathcal{A}_b(\theta'), \tag{3}$$

where $V^{af}(\theta)$ is the value under default and is specified below. Thus, the Fund limits the sovereign's indebtedness such that the sovereign's expected lifetime utility from repaying its debts is at least as high as that of defaulting. In other words, the sovereign is never able to accumulate a level of debt for which a default would be optimal. The borrowing limit is therefore a no-default borrowing constraint (Zhang, 1997). Particularly, it is tight enough in the sense of Alvarez and Jermann (2000) to prevent default but allows as much risk sharing as possible. The borrowing limit $\mathcal{A}_b(\theta')$ is state contingent and accounts for the risk of default. The combination of the first-order conditions of the sovereign's problem with respect to c and $a'(\theta')$ gives the sovereign's Euler equation for the Fund's securities

$$q_f(\theta', \omega'|\theta)u'(c) - \upsilon_b(\theta') = \beta\pi(\theta'|\theta)u'(c')\left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right], \tag{4}$$

where υ_b is the multipliers attached to the sovereign's endogenous borrowing limit in (2). Conversely, the first-order conditions with respect to c and b' gives the sovereign's Euler equation for the private bonds

$$q_p(\theta, \omega')u'(c) - \sum_{\theta'|\theta} \upsilon_b(\theta') = \beta \sum_{\theta'|\theta} \pi(\theta'|\theta)u'(c')\left[(1 - \delta + \delta\kappa) + \delta q_p(\theta', \omega'')\right]. \tag{5}$$

The sovereign's outside option used to determine the borrowing limit $\mathcal{A}_b(\theta')$ is given by the autarky value of the standard incomplete market model with default (Eaton and Gersovitz, 1981; Aguiar and Gopinath, 2006; Arellano, 2008). Since the Fund has no seniority with respect to the privately held sovereign debt, the sovereign reneges its entire debt position if it decides to default. In other words, every default is a *full* default. In this situation, the sovereign is excluded from the private bond market and the Fund. Once it has defaulted, it can reintegrate the former with probability λ but cannot obtain the assistance of the latter anymore. The Bellman equation for the outside option reads

$$V^{af}(\theta) = \max_n \left\{U(\theta^p f(n), n)\right\} + \beta\mathbb{E}\left[(1 - \lambda)V^{af}(\theta') + \lambda J(\theta', 0)\middle|\theta\right], \tag{6}$$

where $\theta^p \leq \theta$ contains the penalty for defaulting. Furthermore, V^{af} corresponds to the value under financial autarky and J to the value of reintegrating the private bond market without the Fund. More precisely, $J(\theta, b) = \max\{V^o(\theta, b), V^{af}(\theta)\}$, with

$$\begin{aligned} V^o(\theta, b) = & \max_{\{c,n,b'\}} U(c, n) + \beta \mathbb{E}\big[J(\theta', b')\big|\theta\big] \\ & \text{s.t. } c + q_p^o(\theta, b')(b' - \delta b) \leq \theta f(n) + (1 - \delta + \delta\kappa)b. \end{aligned} \tag{7}$$

We use the superscript f for the value function V^{af} to refer to a *full* default. Having defined the sovereign's outside option, the sovereign will renege its debt position whenever the value under default is larger than the value under repayment. In the economy with the Fund, the default policy $D(\theta, \omega) = 1$ if $W^b(\theta, a, b) < V^{af}(\theta)$ and zero otherwise. Conversely, in the economy without the Fund, $D^o(\theta, b) = 1$ if $V^o(\theta, b) < V^{af}(\theta)$ and zero otherwise. Due to equation (3), it is already clear that $D(\theta, \omega) = 0$ for all θ and for all ω. This is however not true for $D^o(\theta, b)$ as the sovereign has no access to the Fund.

The private lender's problem: private lenders are competitive and risk neutral financial intermediaries. Each period they solve a static problem. However, we express it in recursive form to later formulate the *Debt Sustainability Analysis* (DSA) of the Fund. We have

$$\begin{aligned} W^p(\theta, a_l, \bar{a}_p, b_l) = & \max_{\{c_p, b_l', \bar{a}_p'\}} c_p + \frac{1}{1+r}\mathbb{E}\big[W^p(\theta', a_l', \bar{a}_p', b_l')\big|\theta\big] \\ & \text{s.t. } c_p + q_p(\theta, \omega')(b_l' - \delta b_l) + q_f(\theta, \omega')(\bar{a}_p'(\theta') - \delta\bar{a}_p) \leq (1 - \delta + \delta\kappa)(b_p + \bar{a}_p). \end{aligned} \tag{8}$$

The private lenders also have access to the bonds issued by the Fund. This enables that the bond price in the Fund and in the private bond market coincide through arbitrage. Besides this, trade of private bonds satisfies the following transversality condition:

$$\lim_{n\to\infty} \mathbb{E}\Big\{\Big[\prod_{j=0}^n Q_p\big(\theta^{t+j}, \omega\big(\theta^{t+j+1}\big)\big)\Big] b_l\big(\theta^{t+j+1}\big)\Big|\theta^t\Big\} = 0, \quad \text{with} \tag{9}$$

$$Q_p\big(\theta^{t+j}, \omega\big(\theta^{t+j+1}\big)\big) = \frac{q_p\big(\theta^{t+j}, \omega\big(\theta^{t+j+1}\big)\big)}{1 - \delta + \delta\kappa + \delta q_p\big(\theta^{t+j+1}, \omega\big(\theta^{t+j+2}\big)\big)}. \tag{10}$$

The implicit interest rate in the private bond market is $r_p(\theta, \omega') \equiv \frac{1}{Q_p(\theta,\omega')} - 1$, where $Q_p\big(\theta, \omega'\big)$ is the intertemporal discount factor defined in (10). Taking the first-order conditions of the private lenders' problem with respect to b_l' and, $\bar{a}_p'$ gives

$$q_p(\theta, \omega') = \frac{\sum_{\theta'|\theta} \pi(\theta'|\theta)(1 - \delta + \delta\kappa + \delta q_p(\theta', \omega''))}{1+r}. \tag{11}$$

$$q_f(\theta, \omega') = \frac{\sum_{\theta'|\theta} \pi(\theta'|\theta)(1 - \delta + \delta\kappa + \delta q_f(\theta', \omega''))}{1 + r}.$$

In the economy without the Fund, there are no endogenous borrowing limits. The price of one unit of private bond boils down to

$$q_p^o(\theta, b') = \frac{\mathbb{E}\left[(1 - D^o(\theta', b'))[1 - \delta + \delta\kappa + \delta q_p^o(\theta', b'')]|\theta\right]}{1 + r}.$$

Hence, there is a default premium embedded in the private bond price depending on the default's frequency. Notice further that with one-period debt (i.e. $\delta = 0$), the price schedule simplifies to $q_p^o(\theta, b') = \frac{\mathbb{E}[1 - D_o(\theta', b')|\theta]}{1+r}$.

3 The Fund

3.1 The Fund Contract Problem

This section presents the design of the Financial Stability Fund. We first consider the Fund contract as a solution to a Ramsey-like problem with three types of agents: the two risk-neutral lenders, whose discount rate is the risk-free rate in the bond market, and the risk-averse and more impatient sovereign. Particularly, the Fund accounts for the debt in the private international market and the Fund itself. It takes as given the sovereign's borrowing and lending in the bond market, as well as the possibility that it can default on its private bonds and from the Fund. Furthermore, the Fund having access to, and commitment with, the bond market can borrow and lend at the risk-free rate (i.e. will not lend to the sovereign if the sovereign's liabilities are not sustainable). Being competitive, the Fund makes zero expected profits with the Fund contract. Now we explicitly define the Ramsey-like problem.

We formulate and solve the Fund contract using the approach of Marcet and Marimon (2019) and relies on the relative Pareto weight, x, to keep track of the binding enforcement constraints. We therefore determine this weight as part of the state space. In Appendix B, we show that there is a direct correspondence between, on the one hand, x and a, and on the other hand, ω. We thus refer to the private bond price as a function (θ, x, b) instead of (θ, a, b). As one will see, at time t, the sovereign can infer x_t from c_{t-1} given the initial value of the Pareto weight, x_0, and the initial level of indebtedness, b_0.

The Fund contract in sequential form For simplicity, we assume that, in state $\theta^t = (\theta_0, \ldots, \theta_t) = (\theta^{t-1}, \theta_t)$, $b(\theta^t)$ is the amount of outstanding private bonds. The contracting problem between the sovereign, the private lenders and the Fund takes into account the existence of a sequence of private bond positions $\{b(\theta^t)\}_{t=0}^{\infty}$, together with the price sequence

$\{q_p(\theta^t, x(\theta^{t+1}), b(\theta^{t+1}))\}_{t=0}^{\infty}$ in the market. The private bond sequence is determined by the borrower's choice in the private bond market as highlighted in the previous section. Moreover, the private bond sequence satisfies the transversality condition given by equation (9). Hence, given $\{b(\theta^t)\}_{t=0}^{\infty}$ and $\{q_p(\theta^t, x(\theta^{t+1}), b(\theta^{t+1}))\}_{t=0}^{\infty}$, the Fund's contacting problem in *sequential form* reads

$$\max_{\{c(\theta^t),n(\theta^t)\}_{t=0}^{\infty}} \mathbb{E}\left[\mu_{b,0} \sum_{t=0}^{\infty} \beta^t U(c(\theta^t), n(\theta^t)) + \mu_{l,0} \sum_{t=0}^{\infty} \left(\tfrac{1}{1+r}\right)^t \tau(\theta^t) \middle| \theta_0\right] \tag{12}$$

$$\text{s.t. } \mathbb{E}\left[\sum_{j=t}^{\infty} \beta^{j-t} U(c(\theta^j), n(\theta^j)) \middle| \theta^t\right] \geq V^{af}(\theta_t), \tag{13}$$

$$\mathbb{E}\left[\sum_{j=t}^{\infty} \left(\tfrac{1}{1+r}\right)^{j-t} \tau(\theta^j) \middle| \theta^t\right] \geq \theta_{t-1} Z - b(\theta^t), \tag{14}$$

$$\tau(\theta^t) = \theta(\theta^t) f(n(\theta^t)) - c(\theta^t), \quad \forall \theta^t, t \geq 0,$$

$$\text{with } \mu_{b,0},\ \mu_{l,0},\ \{b(\theta^t)\}_{t=0}^{\infty}, \{q_p(\theta^t, x(\theta^{t+1}), b(\theta^{t+1}))\}_{t=0}^{\infty},\ \text{given.}$$

The sovereign consumes $c(\theta^t)$ and provides labor $n(\theta^t)$, and $\tau(\theta^t) \equiv \tau_f(\theta^t) + \tau_p(\theta^t)$ is the sum of the sovereign's net savings in the private bond market and in the Fund, where

$$\tau_p(\theta^t) = q_p(\theta^t, x(\theta^{t+1}), b(\theta^{t+1}))[b(\theta^{t+1}) - \delta b(\theta^t)] - (1 - \delta + \delta\kappa) b(\theta^t), \quad \text{and}$$

$$\tau_f(\theta^t) = \theta(\theta^t) f(n(\theta^t)) - c(\theta^t) - \tau_p(\theta^t).$$

Whenever $\tau(\theta^t) < 0$ the sovereign is a net borrower with respect to the rest of the world. The Fund's transfer, $\tau_f(\theta^t)$, is defined in a tautological way. Only when we come to the decentralisation will we be able to properly define it in terms of asset positions and prices.

Equations (13) and (14) represent the *limited enforcement constraints* of the borrower and the lenders, respectively. The borrower's outside option is to default and is given by $V^{af}(\theta_t)$, which only depends on the current state θ_t. As shown previously, the underlying assumption is that if the sovereign defaults from the Fund, it also defaults on its sovereign debt liabilities and then is never allowed to return to the Fund in the future. Alternatively, whenever the sovereign defaults on its sovereign debt in the private international capital market it also defaults from the Fund, since Fund's liabilities are not senior to privately held debt. In order to prevent that the Fund provides permanent transfers to a sovereign — e.g. in order to prevent debt mutualisation — we will assume that $Z = 0$, i.e. that in no state the Fund contract has expected losses. Then (14) shows the second aspect that makes the Fund contract different from an uncontingent defaultable debt contract: in states where the sovereign's indebtedness becomes financially unsustainable — say, when (14) is binding at θ^t — both lenders provide less resources to avoid losses that would go beyond the contract's terms. In other words, the Fund contract anticipates these states and limits the amount of lending, while with defaultable debt these states are anticipated by positive spreads.

Our present formulation is close to the current rules of international lending institutions — such as the IMF or the ESM. The Fund takes into account all the sovereign's debt liabilities — within and outside the Fund — that satisfy the *Debt Sustainability Analysis* (DSA) in every possible state. The difference with current practices is that the DSA is usually only conducted at the beginning of the contract, or at certain time intervals, while in our characterisation of the Fund contract, DSA, i.e. our (14), is contingent to all states that the contract specifies, including those where limited enforcement constraints are binding.

Another difference is that in this framework, the Fund has no seniority over privately owned debt. This is in general not the case when multilateral lending institutions intervene (cf. footnote 6). In Section 4, we consider an alternative formulation where Fund liabilities has seniority over privately held sovereign debt. However, it is a case that relies on a strong *non bailout* commitment from the Fund: the Fund does not act as a *crisis resolution* mechanism, when privately held sovereign debt is at risk of default. Nevertheless, we show that the seniority structure of the Fund does not impact the outcome of the contract in equilibrium.

With $\tau(\theta^t) \equiv \tau_f(\theta^t) + \tau_p(\theta^t)$, the contract accounts for both the private international lenders and the Fund. In other words, it takes into consideration the sovereign's entire debt position. While the Fund *directly* specifies $\tau_f(\theta^t)$ taking as given $\tau_p(\theta^t)$, effectively the contract is taking into account the total surplus $\tau_f(\theta^t) + \tau_p(\theta^t)$ when evaluating the participation constraint, since only in this way it is capable of consistently stabilising the borrower's entire debt position. An equivalent interpretation is that the Fund stands ready to absorb the debt position of the borrower in the form of private bonds, and effectively there is complete credit (risk) transfer from the private bond investors to the Fund, up to certain limits implied by the participation constraints both from the Fund and the borrower.

To have a better idea of the link between the private lender's and the Fund's value, observe that, conditional on θ^t,

$$\begin{aligned} V^l(\theta^t) &= \mathbb{E}\Big[\textstyle\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j \tau(\theta^{t+j}) \Big| \theta^t \Big] = \mathbb{E}\Big[\textstyle\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j \big(\tau_f(\theta^{t+j}) + \tau_p(\theta^{t+j})\big) \Big| \theta^t \Big] \\ &= \mathbb{E}\Big[\textstyle\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j \big(\tau_f(\theta^{t+j}) + \big[q(\theta^t, x(\theta^{t+1}), b(\theta^{t+1}))\big[b(\theta^{t+1}) - \delta b(\theta^t)\big] \\ &\quad - (1-\delta+\delta\kappa) b(\theta^t)\big]\big) \Big| \theta^t \Big]. \end{aligned} \tag{15}$$

Using the transversality condition in (9), equation (15) simplifies into

$$V^l(\theta^t) = \mathbb{E}\Big[\sum_{j=0}^{\infty}\left(\tfrac{1}{1+r}\right)^j \tau_f(\theta^{t+j})\Big|\theta^t\Big] - b(\theta^t).$$

The present value constraint on Fund's lending is $\mathbb{E}\big[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \tau_f(\theta^{t+j})\big|\theta^t\big] \geq \theta_{t-1}Z$, thus the overall participation constraint of the Fund is given by

$$V^l(\theta^t) = \mathbb{E}\Big[\sum_{j=0}^{\infty}\left(\tfrac{1}{1+r}\right)^j \tau(\theta^{t+j})\Big|\theta^t\Big] \geq \theta_{t-1}Z - b(\theta^t).$$

Following Marcet and Marimon (2019), solutions to the Fund's contacting problem are homogenous of degree one in $\mu = (\mu_b, \mu_l)$ and the initial relative Pareto weight $x_0 \equiv \frac{\mu_{b,0}}{\mu_{l,0}}$ is pinned down by the initial break-even condition for the Fund $\mathbb{E}\big[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \tau(\theta^t)\big|\theta_0\big] = \theta_{-1}Z - b_0$, given the initial debt position in the private international market b_0. As said, we will assume that $Z = 0$, which is a (expected) zero profit condition for the Fund; given this, x_0 depends on (θ_0, b_0) and the present value of all the expected net savings. If without private debt there is an interior solution to the Fund's contracting problem, then an optimal solution exists and there are feasible paths of private debt, starting at b_0, subject to an upper bound on how large the initial debt $-b_0$ can be. We elaborate more on this in Appendix B.

The Fund contract in recursive form Using the approach of Marcet and Marimon (2019), we can state the *recursive Fund contract* with $s \equiv \{\theta^-, \gamma\}$ as follows:

$$\begin{aligned} FV(s,x,b) = \mathcal{SP} \min_{\{\nu_b,\nu_l\}} \max_{\{c,n\}} \; & x\big[(1+\nu_b)U(c,n) - \nu_b V^{af}(\theta)\big] \\ & + \big[(1+\nu_l)\tau - \nu_l(\theta^- Z - b)\big] + \tfrac{1+\nu_l}{1+r}\mathbb{E}\big[FV(s',x',b')\big|\theta\big] \\ \text{s.t. } \; & \tau = \theta f(n) - c \text{ and } x' = \tfrac{1+\nu_b}{1+\nu_l}\eta x, \end{aligned} \tag{16}$$

where $\eta \equiv \beta(1+r) < 1$, and ν_b and ν_l are the normalized multipliers attached to the sovereign's and the lender's limited enforcement constraints, respectively.[11] As the problem depends on θ and θ^-, we keep track of $s \equiv \{\theta^-, \gamma\}$ in the state space. The private bond policy of the sovereign, $b' = B(\theta, x, b)$ is taken as given.[12] The value function satisfies:

$$FV(s,x,b) = xV^b(\theta,x,b) + V^l(s,x,b), \text{ with}$$
$$V^b(\theta,x,b) = U(c,n) + \beta\mathbb{E}[V^b(\theta',x',b')|\theta] \quad \text{and} \quad V^l(s,x,b) = \tau + \tfrac{1}{1+r}\mathbb{E}[V^l(s',x',b')|\theta].$$

[11]The normalization of the Pareto weights is the same as the one in Ábrahám et al. (2021).

[12]In this (Nash) specification of the Fund contract the effect of τ_f on $B(\theta, x, b)$ is not taken into account.

We obtain the optimal consumption and leisure policies, $c(\theta, x, b)$ and $n(\theta, x, b)$ by taking the first-order conditions of problem (16), $u'(c) = \frac{1+\nu_l}{1+\nu_b}\frac{1}{x}$, and $\theta f'(n) = \frac{h'(1-n)}{u'(c)}$, which results in a transfer policy $\tau(\theta, x, b) = \tau_f(\theta, x, b) + [q(\theta, x(\theta'), b(\theta'))(b(\theta') - \delta b) - (1 - \delta + \delta\kappa)b]$. The first-order condition with respect to consumption tells us that the sovereign can infer x_t from $u'(c_{t-1})$ given (x_0, b_0). As highlighted before, the initial value of the Pareto weight is given by the Fund's break-even condition.

The relative Pareto weight evolves according to the binding limited enforcement constraints (Marcet and Marimon, 2019). Particularly, it increases when the sovereign's constraint binds (i.e. $\nu_b > 0$) and decreases when the lender's constraint binds (i.e. $\nu_l > 0$). In the former case, the sovereign's consumption increases not to generate default incentives, while in the latter case, the sovereign's consumption decreases to avoid expected losses from the lenders' perspective.

3.2 The Properties of the Fund Contract

This subsection demonstrates the main properties of the Fund contract. Other properties of the contract such as existence, partial risk sharing and the inverse Euler Equation, as well as the welfare theorems associated with its competitive implementation, are presented in Appendix B.

The long-run property of the Fund contract is related to the definition of an ergodic set of relative Pareto weights. The term ergodic refers to the fact that the relative Pareto weights in this set are aperiodic and recurrent with non-zero probability. In other words, the economy will move around the same set of relative Pareto weights over time and over histories. The following definition relies on the model in detrended form presented in Appendix A.

Definition 1 (Steady State). *Given a Markov chain of γ with a unique ergodic set in Γ, a* Steady State Equilibrium *is defined by an ergodic set in X with a lower bound $\underline{x} = \min_{\gamma\in\Gamma}\{x : \widetilde{V}^b(\gamma, x, b) = \widetilde{V}^{af}(\gamma)\}$ and an upper bound $\overline{x} = \max_{\gamma\in\Gamma}\{x : \widetilde{V}^b(\gamma, x, b) = \widetilde{V}^{af}(\gamma)\}$, satisfying $\underline{x} < \overline{x}$, for the relative Pareto weights.*[13]

The lower bound of the ergodic set is determined by the lowest achievable relative Pareto weight in the contract. It represents the lowest value that the sovereign accepts in the contract, which keeps it away from immiseration. The upper bound represents the highest relative Pareto weight that makes the sovereign's constraint bind; therefore it is the highest

[13] The value functions marked with $\widetilde{V}$ are the detrended value functions presented in Appendix A.

weight that the lender may need to accept.[14] We can further characterise the bounds of the ergodic set with the following lemma, validating their independence on b.[15]

Lemma 1 (Bounds of the Ergodic set)**.** *The bounds of the ergodic set solely depend on the current growth state, θ, thus for the detrended form, solely depend on γ.*[16]

This lemma states that the bounds of the ergodic set are independent of b. In other words, the sovereign's participation constraint is solely determined by the realised growth state. This is because, in the Fund, the sovereign's participation constraint is always satisfied meaning that the sovereign is guaranteed to receive a minimal level of utility irrespective of its indebtedness.

Proposition 1 (No Default)**.** *In a Fund contract, the sovereign does not default.*

This proposition directly follows from (13). The Fund always provides state-contingent transfers to the sovereign. This sustains the chosen sequence of private bond, $\{b(\theta^t)\}_{t=0}^{\infty}$ and ensures that the sovereign obtains at least the value of its outside option in any sate. Hence, given the transfer, the sovereign is at most indifferent between reneging the contract or not and finds it optimal not to default at all.

This shows the importance of the state contingency of the Fund's transfer. Without this feature, the Fund would not be capable of accounting for the possibility of default and its intervention would be ineffective. The next section elaborates more on that.

3.3 The Decentralised Fund Contract

Following the work of Alvarez and Jermann (2000) and Krueger et al. (2008), we can further decentralise the Fund contract. It therefore completes the exposition of the decentralised economy which started in Section 2 with the presentation of the sovereign's and the private lenders' problems. The main aim of the decentralisation is to obtain the asset positions and the price schedule relating to the Fund's transfers τ_f. We show that, given the realization of the state, the Fund formulates a DSA recommendation stating the level of indebtedness that remains sustainable in all future states. Deviations from this recommendation have

[14]Note that if $\underline{x} = \overline{x}$ there would be a steady state with perfect risk-sharing, but this is a knife-edge case which can only exists with strong restrictions on the structure of the model.

[15]It should be noted that if the sovereign and the Fund are equally patient (i.e. $\eta = 1$), then the upper bound would be determined by $\min_{\gamma \in \Gamma}\{x : \widetilde{V}^l(\gamma, x, b) = Z - \tilde{b}\}$, which depends on the endogenous b.

[16]See Appendix C for the proofs of lemmas and propositions.

direct repercussions provided that the Fund does not possess seniority over private debt. Nevertheless, moving after the private lenders, the Fund can credibly enforce its recommendation by the threat of exiting the market. As a result, the private lenders always follow the Fund's prescriptions. This further implies that the price of debt in the private bond market and the Fund coincides in equilibrium. The share of debt held by the Fund might thus be indeterminate. Nonetheless, there is one contract that minimises the debt absorbed by the Fund. This is what we call the *effective intervention* of the Fund.

The Fund's transfer is decentralised as a competitive equilibrium with endogenous borrowing constraints following Alvarez and Jermann (2000).[17] The aim is to obtain the current and future asset positions (a and a', respectively) and the underlying asset price, q_f, that corresponds to the Fund's transfer. Formally, we seek the following relationship

$$\tau_f(\theta, x, b) = \textstyle\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)(a'(\theta') - \delta a) - (1 - \delta + \delta\kappa)a,$$

where $\omega = a + b$ records the entire debt position. We need to account for both a and b jointly because the Fund aims at stabilizing the sovereign's entire debt position.

We have already exposed the sovereign's and the private lenders' problems in Section 2. The last problem in the decentralised economy is now the one of Fund which is given by

$$W^f(s, a_l, b_l) = \max_{\{c_f, \{a_l'(\theta')\}_{\theta' \in \Theta}\}} c_f + \tfrac{1}{1+r}\mathbb{E}\left[W^f(s', a_l'(\theta'), b_l')\middle|\theta\right] \tag{17}$$

$$\begin{aligned} \text{s.t. } & c_f + \textstyle\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)(a_l'(\theta') - \delta a_l) \leq (1 - \delta + \delta\kappa)a_l, \\ & b_l' = B_l(\theta, a_l, b_l) \\ & a_l'(\theta') + b_l' \geq \mathcal{A}_f(\theta', b_l'), \end{aligned} \tag{18}$$

where a_l the amount of assets provided by the Fund, b_l is the amount of assets provided by the private lenders, $B_l(\theta, a, b)$ is the lending policy of the private lenders and $s \equiv \{\theta^-, \gamma\}$. The variable $\mathcal{A}_f(\theta', b_l)$ represents an endogenous limit defined as

$$W^f(s', \mathcal{A}_f(\theta', b_l') - b_l', b_l') = \theta Z. \tag{19}$$

This condition restricts the extent of losses. Particularly, it ensures that the present discounted value of the Fund's assets are at least equal to $\theta Z \leq 0$. Specifically, when $Z = 0$,

[17] Other decentralisations are possible. For example, Dovis (2019) obtains state-contingent contracts by means of an active debt structure management and partial defaults. Conversely, Müller et al. (2019) propose a decentralisation using preemptive sovereign debt restructurings, GDP-linked and defaultable bonds.

$\mathcal{A}_f(\theta', b'_l)$ ensures that the total level of the sovereign's liabilities can be absorbed by the Fund without incurring permanent losses. Adding equations (19) to the value of the lender (8) and applying the transversality condition (9), we obtain

$$W^f(s', \mathcal{A}_f(\theta', b'_l) - b'_l, b'_l) + W^p(\theta', a'_l(\theta'), \bar{a}'_p, b'_l) = \theta Z + b'_l.$$

This gives the decentralised counterpart of the lenders' participation constraint in (14),

$$W^l(s', a'_l(\theta'), b'_l) \equiv W^f(s', a'_l(\theta'), b'_l) + W^p(\theta', a'_l(\theta'), \bar{a}'_p, b'_l) \geq \theta Z + b'_l, \tag{20}$$

We interpret condition (20) as a proper DSA since it links the value of the current lending with its prospective stream of cashflow. This DSA takes into account the sovereign's entire debt position — within and outside the Fund — in every possible state. Moreover, owing to the trade of Arrow-type securities, it is contingent in all the states that the contract specifies, including those states where limited enforcement constraints are binding.

In addition, $\mathcal{A}_f(\theta', b'_l)$ represents the Fund's DSA recommendation. We later show that the Fund is capable of credibly punishing the private lenders in the case of deviations from its recommendations. As a result, the private lenders will never deviate from the Fund's DSA. Note that the market clearing condition in the Fund is given by $a(\theta) + \bar{a}_p(\theta) + a_l(\theta) = 0$ for all θ. In addition, the initial asset holdings of the sovereign in the Fund, $a(\theta_0) + \bar{a}_p(\theta_0) = -a_l(\theta_0) = 0$, are given.

Taking the first-order conditions of the decentralised Fund's problem with respect to c and $a'_l(\theta')$ gives the Fund's Euler equation

$$q_f(\theta', \omega'(\theta')|\theta) - \varphi_f(\theta') = \tfrac{1}{1+r}\pi(\theta'|\theta)\big[(1-\delta+\delta\kappa) + \delta \textstyle\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\big], \tag{21}$$

where φ_f is the multipliers attached to the Fund's endogenous limit.

Given value functions for the outside value options of the sovereign, $V^{af}(\theta')$, and of the lenders, $\theta Z - b$, where $-b$ is the debt of the sovereign with the private lenders, a *recursive competitive equilibrium* (RCE) consists of: prices $q_f(\theta', \omega'(\theta')|\theta)$ and $q_p(\theta, \omega')$; value functions $W^b(\theta, a, b)$, $W^f(s, a_l, b_l)$, and $W^p(\theta, a_l, \bar{a}_p, b_l)$; endogenous limits, $\mathcal{A}_b(\theta')$ and $\mathcal{A}_f(\theta', b'_l)$, and policy functions $c(\theta, a, b)$, $c_f(\theta, a_l, b_l)$, $c_p(\theta, a_l, b_l)$, $n(\theta, a, b)$, $a'(\theta') = A(\theta', \theta, a, b)$, $a'_l = A_l(\theta', \theta, a_l, b_l)$, $b' = B(\theta, a, b)$, $\bar{a}'_p = A_p(\theta, a_l, \bar{a}_p, b_l)$ and $b'_l = B_l(\theta, a_l, b_l)$, with $\omega'(\theta') = a'(\theta') + b'$, which are solutions to the problems of the sovereign, private lenders and the Fund, and all markets clear.[18]

[18] Appendix B, provides a more detailed definition and determines the correspondence between the Fund contract and the decentralised Fund contract (i.e. Second Welfare Theorem). We also show that the First Welfare Theorem holds.

We can further characterise the equilibrium in the decentralised economy. The sovereign faces two alternatives to purchase debt: the Fund and the private bond market. Besides bonds and unlike private lenders, the Fund also trades Arrow-type securities. We therefore establish the price dynamic and the optimal holdings of assets in the decentralised environment. Using the fact that the borrowing constraints of the borrower and the lenders do not bind at the same time,[19] the price is determined by the agent whose constraint is not binding (Krueger et al., 2008). It then follows that

$$q_f(\theta',\omega'(\theta')|\theta) = \tfrac{\pi(\theta'|\theta)}{1+r}\Big[(1-\delta+\delta\kappa)+\delta\textstyle\sum_{\theta''|\theta'} q_f(\theta'',\omega''(\theta'')|\theta')\Big]\max\Big\{\tfrac{u'(c(\theta',a'(\theta'),b'))}{u'(c(\theta,a,b))}\eta, 1\Big\}. \quad (22)$$

Given the above price schedule, the intertemporal discount factor is defined by

$$Q_f\big(\theta',\omega'(\theta')|\theta\big) \equiv \frac{q_f\big(\theta',\omega'(\theta')|\theta\big)}{1-\delta+\delta\kappa+\delta\sum_{s''|\theta'} q_f\big(\theta'',\omega''(\theta'')|\theta'\big)}.$$

The implicit interest rate in the Fund is then defined by $r_f(\theta,\omega') \equiv \frac{1}{Q_f(\theta,\omega')} - 1$ with $Q_f(\theta,\omega') \equiv \sum_{\theta'|\theta} Q_f\big(\theta',\omega'(\theta')|\theta\big)$. Given the definition of the price in (22), if the lenders' constraint is binding in one future state θ', the price of one unit of Arrow-type security in that state is $q_f(\theta',\omega'(\theta')|\theta) > \frac{1}{1+r}\pi(\theta'|\theta)\big[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''(\theta'')|\theta')\big]$, while the price of a bond reads $q_f(\theta,\omega') > \frac{1}{1+r}\sum_{\theta'|\theta}\pi(\theta'|\theta)\big[(1-\delta+\delta\kappa)+\delta q_f(\theta',\omega'')\big]$, or equivalently $Q_f(\theta,\omega') > \frac{1}{1+r}$ implying that $r_f(\theta,\omega') < r$. In words, when the lenders' constraint bind, a negative spread appears. The lenders' binding constraint has therefore two opposite effects. On the one hand, accumulating debt, $\bar{a}' < 0$, is cheaper for the sovereign owing to the fact that $q_f(\theta,\omega')$ is above the risk-free price. On the other hand, buying insurance, $\hat{a}'(\theta') > 0$, for the state in question becomes more expensive. This effect is even stronger provided that the trade of Arrow-type securities has to be such that $\sum_{\theta'|\theta} q_f(\theta',\omega'(\theta')|\theta)\hat{a}'(\theta') = 0$.

In our environment, Arrow-type securities are the sole financial instruments capable of preventing default. They complement the issuance of debt especially in the states in which the sovereign could not sustain its liabilities otherwise. In view of this, the Fund never allows the sovereign to accumulate some level of debt for which it cannot insure.

Being unable to obtain more debt from the Fund when the lenders' constraint bind, can the sovereign accumulate more debt in the private bond market instead? The answer is negative. As we already explained, the sovereign has to determine its prospective borrowing in the private bond market before going to the Fund. Hence, the Fund can always react to

[19] If both constraints would bind at the same time, no agreement could be reached between the sovereign and the lenders. In other words, no contract would exist.

deviations from its DSA recommendation, $\mathcal{A}_f(\theta', b'_l)$. Most notably, if the private lenders lend more than what the Fund prescribes, the latter withdraws from the market. The rationale behind this is that, having no seniority, the Fund would be as penalised as the private lenders if the risk of default realises. The Fund has therefore the ability to make credible threats to private lenders.

Given that the private lenders have access to the Fund's securities, no arbitrage is possible between the Fund and the private bond market for the borrower. Hence, the bond prices in the Fund and the private bond market are alike unless the Fund's constraint binds.

Proposition 2 (Bond Price)**.** *In a RCE, for all (θ, ω').*

$$\sum_{\theta'|\theta} q_f\left(\theta', \omega(\theta')|\theta\right) = q_p(\theta, \omega').$$

Moreover, whenever, (20) binds, $\sum_{\theta'|\theta} q_f\left(\theta', \omega(\theta')|\theta\right) > \frac{1-\delta+\delta\kappa}{1+r-\delta}$.

When the lenders' constraint given in (20) is binding, the sovereign would like to borrow more today with the promise that it would pay back tomorrow, but this is a non credible promise, from the perspective of the Fund. If the sovereign borrows more today, there is a chance that its debt becomes unsustainable tomorrow as its level of insurance is limited by the negative spread.[20] The Fund therefore provides just enough resources such that the sovereign's indebtedness remains sustainable in all future states. The negative spread also impacts private lending as debt accumulation becomes cheaper in the Fund. As a result, it not only restricts the provision of the Fund's insurance to the sovereign, it also sustains a no-trade equilibrium in the private bond market.

Proposition 3 (Effective Private Lending)**.** *In a RCE, in the states in which (20) binds $b' \geq \delta b$.*

However, when the lenders' constraint in (20) does not bind, the sovereign can equally access the private bond market and the Fund. As a matter of fact, in this case, debt is as expensive in the Fund as in the private bond market and the sovereign can accumulate debt in both locations. Therefore, the sovereign is indifferent between holding debt in the private bond market or in the Fund. This results into an indetermination.

[20] We say that the sovereign's debt might become unsustainable as the lenders' constraint does not necessarily bind for all possible θ'. Hence, if the sovereign is lucky enough, it could end up in a state tomorrow where lenders could sustain its indebtedness even if it borrowed more today.

Proposition 4 (Debt Indetermination)**.** *In all states θ' in which* (20) *does not bind, the division of ω' between b' and $\bar{a}'$ is indeterminate.*

One way to address this indetermination is to minimize the Fund's intervention. Hence, in opposition to Ábrahám et al. (2021), the amount of debt held in the Fund shall be here as small as possible. This is what we call the *effective intervention* of the Fund.

Corollary 1 (Effective Intervention of the Fund)**.** *If there is a RCE then, in any state (θ, b) with $b \leq 0$, there exists $\underline{\underline{a}}(\theta, b) \in [\delta b, 0]$ such that $\bar{a}' = \underline{\underline{a}}(\theta, b)$ is the Fund's minimal intervention.*

Under Corollary 1, the Fund's credit line is set to its minimal level when (20) does not bind. This does not necessarily imply that the Fund solely provides Arrow-type securities in this situation.[21] As the level of private debt appears on the right-hand side of (20), one cannot always set $\bar{a}' = 0$. If the private lenders would absorb today the entire debt position of the sovereign, they face the danger of violating the constraint $W_p(\theta', a'_l, b'_l) \geq b'_l$ tomorrow. To avoid that the debt burden becomes too large for private lenders, the Fund needs to take over part of the sovereign's debt.[22]

4 The Seniority Structure of the Fund

So far, we assume that the Fund had no seniority with respect to the privately held sovereign debt. We therefore consider that a default always implicates the sovereign's entire debt position. We now relax this assumption and introduce the state of *partial* default in which the sovereign can default solely on its private liabilities while remaining in the Fund.

We show that the seniority structure of the Fund does not impact the outcome of the model. The mechanism underlying the result might be different, though. Notably, under seniority, positive spreads substitute negative spreads. That is, the private lenders charge a risk premium for all borrowing that exceeds Fund's lending policy. This prevents the sovereign to deviate from the Fund's prescription in terms of debt accumulation. In light of

[21]This is a major difference compared to the case of one-period bonds. With short-term debt, when the lenders' participation constraint does not bind, the entire debt position of the sovereign is located in the private market. The Fund solely trades Arrow-type securities.

[22]In light of this, the effective intervention is possible only if the Fund moves after the private lenders. Hence, the timing of actions is crucial not only for the Fund to credibly punish the private lenders from deviating but also to implement a level of intervention that is minimal in terms of absorbed debt.

this, the public announcement of the Fund's policy at the beginning of each period becomes crucial.

The Sovereign and the Private Lenders under Seniority. Compared to the case without seniority, the sovereign possesses two outside options. On the one hand, it can default on its entire debt position. This represents the case of *full* default considered previously. Having seniority, the Fund will only seek to prevent this type of default. On the other hand, the sovereign can repudiate its private debt while remaining in the Fund. We refer to this situation as a *partial* default because the sovereign solely defaults on the private lenders. The value in this case is given by

$$\begin{aligned} V^{ap}(\theta, a) = \max_{\{c,n,\{a'(\theta')\}_{\theta' \in \Theta}\}} \quad & U(c, n) + \beta \mathbb{E}\big[(1-\lambda) V^{ap}(\theta', a'(\theta')) + \lambda W^b(\theta', a'(\theta'), 0) \big| \theta\big] \\ \text{s.t.} \quad & c + \sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)(a'(\theta') - \delta a) \leq \theta^p f(n) + (1 - \delta + \delta\kappa) a, \\ & \omega'(\theta') = a'(\theta') \geq \mathcal{A}_b^{ap}(\theta'), \end{aligned}$$

where the endogenous borrowing limit is defined as

$$V^{ap}(\theta', \mathcal{A}_b^{ap}(\theta')) = V^{af}(\theta').$$

The default penalty and the expected market exclusion are the same in *partial* and *full* defaults. Despite this, the value under *partial* default can be greater than the value under *full* default. In the former situation, the sovereign continues to receive the support from the Fund, whereas, in the latter situation, it cannot obtain assistance from the Fund anymore. This implies, as we will see, that the sovereign's participation constraint alone is not sufficient to prevent for the occurrence of *partial* defaults on equilibrium path.

From the perspective of the lenders, the Fund's seniority implies that the private lenders are not anymore accounted for by the Fund. Thus, the private lender's problem becomes a static problem without endogenous limit as in Arellano (2008). As a result, the private bond price reads

$$q_p(\theta, \omega') = \frac{\mathbb{E}\left[(1 - D'(\theta, a', b'))[1 - \delta + \delta\kappa + \delta q_p(\theta', \omega'')]|\theta\right]}{1 + r}, \tag{23}$$

where $D(\theta, a, b) = D_p(\theta, a, b) + D_f(\theta, a, b)$, and

$$D_p(\theta, a, b) = \begin{cases} 1, & \text{if } V^{ap}(\theta, a) > W^b(\theta, a, b) \text{ and } V^{ap}(\theta, a) \geq V^{af}(\theta), \\ 0, & \text{else;} \end{cases}$$

$$D_f(\theta, a, b) = \begin{cases} 1, & \text{if } V^{af}(\theta) > W^b(\theta, a, b) \text{ and } V^{af}(\theta) \geq V^{ap}(\theta, a), \\ 0, & \text{else.} \end{cases}$$

The value under *full* default might coincide with the value under *partial* default due to the continuous access to the Fund in the latter case. Hence, if the sovereign is indifferent between the two types of default, on considers it selects the *partial* default.

The price still depends on the total level of debt despite the relaxation of the seniority assumption. This is because the Fund continues to announces its optimal lending policy in terms of total indebtedness. The sovereign's entire debt position therefore remains the relevant statistic for the private lenders to measure the appropriate risk premium.

The Fund Contract under Seniority. The timing of actions remain unchanged. Moreover, the Fund still aims at making the sovereign's debt safe and sustainable for the future. Thus, even though it possesses now seniority, its lending policy continues to relate to the sovereign's entire indebtedness as in the case without seniority.[23] In terms of the decentralised Fund contract, the lending policy relates to a certain level of total debt $\bar{\omega}' = b' + \bar{a}'$.[24] In every period, given the state, the Fund computes and announces $\bar{\omega}'$. As the Fund moves after the private lenders, it defines its credit line as the residual of $\bar{\omega}'$ after the private borrowing b'. That is, if the sovereign chooses $b' > \bar{\omega}'$, the Fund is willing to provide some $\bar{a}' = \bar{\omega}' - b' < 0$. However, if the sovereign selects $b' < \bar{\omega}'$, the Fund does not offer any additional credit as the sovereign exceeded the limit set by the Fund. Moreover, it does not further react to the over-accumulation of debt and only insures $\bar{\omega}'$. As we will see, it is the private lenders that have to properly react here.

Given this environment, the sovereign's participation constraint remains the same as in the case without seniority. In the contract with seniority, the sovereign defaults on the Fund only in the case of a *full* default. Thus, the Fund does not need to insure against *partial* defaults as it is sheltered against repudiations of private debts. This means that if the value under *partial* default is greater than the value of *full* default in some states, *partial* defaults

[23]Note that in the case of a *partial* default, the entire debt position is held in the Fund.

[24]Note that the Fund's DSA recommendation, $\mathcal{A}_f(\theta', b'_l, D'_p)$, shall not be confused with the Fund's lending policy, $\bar{\omega}'$. The former represents the limit for which the lenders do not record permanent losses (for $Z = 0$), while the latter gives the maximal level of debt the sovereign can safely accumulate in each period and in each state. The DSA recommendation solely matters in the states in which the lenders' constraint bind. In opposition, the Fund's lending policy pertains to all the states that the contract specifies and not only those states where limited enforcement constraints are binding.

can arise on equilibrium path. On the contrary, in the contract without seniority, the Fund also aimed at insuring against *full* default but the paradigm was different. The sovereign could not select on which creditor it desired to default. This is because the contract was *pari passu* meaning that neither the Fund nor the private lenders enjoyed a (implicit or explicit) preferred creditor status.

The Funds's participation constraint changes in the case with seniority. Most notably, the Fund does not need to extensively account for the private lenders as it did without seniority. The Fund's participation constraint therefore becomes

$$\mathbb{E}\Big[\sum_{j=t}^{\infty}\left(\tfrac{1}{1+r}\right)^{j-t}\tau^f(\theta^j)\Big|\theta^t\Big] \geq \theta_{t-1}Z,$$

while in decentralised Fund contract, the Fund's endogenous limit now reads

$$W^f(s', \mathcal{A}_f(\theta', b_l', D_p') - b_l', b_l', D_p') = \theta Z.$$

The state space contains the sovereign's *partial* default policy defined above. Indeed, the Fund needs to keep track of the occurrence of *partial* default to properly define its transfer policy, due to the default penalty on output. Hence, despite its seniority, the Fund indirectly accounts for the sovereign's decisions in the private bond market as this impacts the level of resource it provides. Moreover, as

$$\tau_{p,t} = q_p(\theta_t, x_{t+1}, b_{t+1})b_{t+1} - (1 - \delta + \delta\kappa + \delta q_p(\theta_t, x_{t+1}, b_{t+1}))b_t(1 - D_t).$$

using the definition of $\tau^f = \tau - \tau^p$ and the transversality condition (9), we obtain

$$\mathbb{E}\Big[\sum_{j=t}^{\infty}\left(\tfrac{1}{1+r}\right)^{j-t}\tau(\theta^j)\Big|\theta^t\Big] + b_t \geq \theta_{t-1}Z,$$

Depending on whether there are partial default, b_t might not be the same as in the Fund's participation constraint with seniority (14). The difference comes from the output penalty and the haircut following a default. Hence, only without default does the participation constraint with and without seniority coincide.

The Irrelevance of the Seniority Structure. To demonstrate the irrelevance of the seniority assumption, we need to check whether the sovereign is willing to follow the Fund's prescription in terms of borrowing when we impose seniority. We show this in three steps. First, the sovereign never enters in *partial* default for some level of private debt it can accumulate in the Fund. In other words, the sovereign defaults on its private liabilities only if it borrows more than what the Fund prescribes. Second, if the sovereign desires to enter in *partial* default in a particular state θ', it does so for all θ' for which $\pi(\theta'|\theta) > 0$. Finally, following the two previous points, the price of private debt goes to zero for all level of debt above the Fund's lending policy. As a result, the sovereign cannot profitably deviate from the Fund's prescriptions in terms of borrowing. We establish the first point in the following proposition.

Proposition 5 (Fund's Lending Policy and Partial Default)**.** *In equilibrium, the sovereign never enters in partial default for some level of private debt it can accumulate in the Fund.*

The sovereign gets insured for all level of debt it accumulates within the Fund's lending policy. In this regard, the split of the debt between the Fund and the private bond market is irrelevant. Hence, the value under *partial* default can never be greater than the value of staying in the contract and repaying both the Fund and the private lenders.

Having established that the sovereign does not want to partially default when it holds debt that it can accumulate in the Fund, a *partial* default might be optimal only if the sovereign overborrows (from the perspective of the Fund). In what follows we show that if the sovereign desires to default in a state θ', then it does so in all $\theta' \in \Theta$. That is the default decision is not state contingent. Thus we establish the second point in the following proposition.

Proposition 6 (Overborrowing and Partial Default)**.** *In equilibrium, if the sovereign desires to enter in partial default in a particular state θ', it does so for all θ' for which $\pi(\theta'|\theta) > 0$.*

The rationale behind this result is the following. The sovereign does not get insured by the Fund for the part of debt it accumulated in excess of $\bar{\omega}'$. Hence, irrespective of the realization of the state in the next period, the sovereign will be overindebted. That is it will not be able to reach the level of consumption the Fund aimed at providing initially. The sovereign will therefore find it optimal to renege its private liabilities as this enables a greater consumption, with the advantage of remaining in the Fund. Thus, the insurance provided by the Fund makes it beneficial for the sovereign to repudiate any amount of debt it holds in excess of $\bar{\omega}'$.

We have established two points so far. On the one hand, the sovereign does not enter in *partial* default as long as it accumulates a level of private debt within the Fund's lending policy $\bar{\omega}'$. This means that partially defaulting is only desirable when the sovereign borrows more than what the Fund prescribed. On the other hand, as soon as the sovereign overborrows, it defaults with probability one in the next period. Those two elements taken together have a direct impact on the price charged by private lenders.

Corollary 2 (Private Lenders and Partial Default). *In equilibrium, for a given Fund's lending policy* $\bar{\omega}'$*, the private lenders set* $q_p(\theta, \omega') = 0$ *for all* $\omega' < \bar{\omega}'$*.*

The private lenders anticipate that as soon as the sovereign overborrows, it defaults on its private liabilities with probability one in the next period. Accordingly, given the private bond pricing equation, they set $q_p(\theta, \tilde{\omega}') = 0$ for all $\tilde{\omega}' < \bar{\omega}'$. As a result, the sovereign cannot deviate from the Fund's policy. That is why it is crucial that the Fund properly announces its lending policy to stabilise sovereign debt. A consequence of this constrained-efficient policy is that Fund's seniority is irrelevant.[25]

5 Calibration

We calibrate the parameters of the model economy by fitting the sovereign debt model (6)–(7), i.e. the one without the Fund, to quarterly data of Italy over the period 1992Q1 to 2019Q4.[26] Table 1 summarizes the value of each parameters.

We calibrate the productivity growth rate shock γ_t with a Markov regime switching AR(1) process to the sample productivity series of Italy. We choose a specification of 2 regimes, with the worst regime capturing the crisis period (i.e the Great Financial crisis) observed in the data. Specifically, we estimate the following model for the (net) growth rate $\gamma_t - 1$ with the expectation maximization (EM) algorithm of Hamilton (1990):

$$\gamma_t - 1 = (1 - \rho(\varsigma_t))\mu(\varsigma_t) + \rho(\varsigma_t)(\gamma_{t-1} - 1) + \sigma(\varsigma_t)\epsilon_t, \tag{24}$$

where ς_t denotes the regime at t, and $\rho(\varsigma_t)$, $\mu(\varsigma_t)$, and $\sigma(\varsigma_t)$ denote regime specific parameters. As shown in Appendix D, such a regime switching process can capture the sudden drop in productivity dynamics around crisis periods. In the computation, we further discretize the shock process using the method of Liu (2017) with 15 grid points for each regime. Aguiar and Gopinath (2006) show that given a CRRA utility in consumption $\frac{c^{1-\sigma}}{1-\sigma}$, one requires that $\lim_{t\to\infty} \mathbb{E}_0 \beta^t (\theta_{t-1}^{1-\sigma} - 1)/(1-\sigma) = 0$, so that the discount utility can be well defined with

[25] In that logic, Wicht (2021) shows that in an environment without state-contingent claims, seniority matters for multilateral lending institutions such as the IMF.

[26] The calibration starts in 1992 due to data availability and ends in 2019 owing to the pandemic. Appendix D contains detailed explanations on data sources, measurement, and additonal information on shock process estimation.

Table 1: Parameter Values

Parameter	Value	Definition	Targeted Moment
A. Direct measures from data			
α	0.5295	labor share	labor share
λ	0.032	return probability	average exclusion period
r	0.0132	risk-free rate	annual real short-term rate
δ	0.9297	bond maturity	bond maturity
κ	0.0543	bond coupon rate	bond coupon rate
B. Based on model solution			
β	0.956	discount factor	$\overline{b'/y_{\text{annual}}}$
ψ	0.73	productivity penalty	$\rho(\text{spread}, y)$ and $\max(\text{spread})$
ζ	0.34	labor elasticity	$\overline{n}$, $\rho(n, y)$ and $\sigma(n)/\sigma(y)$
ξ	1.734	labor utility weight	
C. By assumption			
Z	0	Fund's outside option	

stochastic trend. For the case of log utility, this amounts to $\lim_{t\to\infty} \mathbb{E}_0\beta^t \log\theta_{t-1} = 0$, which holds automatically in our setup. We subsequently detrend an 'allocation' variable x_t by θ_{t-1}: $\tilde{x}_t = x_t/\theta_{t-1}$.

The preference parameters for labor supply are set to $\zeta = 0.34$ and $\xi = 1.734$. These are used to match the average fraction of working hours, together with the volatility of labor relative to GDP. The risk free interest rate is set to $r = 1.32\%$, the average real short-term interest rates of the Euro area. We further set $\delta = 0.9297$ and $\kappa = 0.0543$ to match the average Italian bond maturity and coupon rate (coupon payment to debt ratio), respectively. Finally, we fix $\beta = 0.956$ to match the average indebtedness relative to annual output. The production function is Cobb-Douglas $f(n) = n^\alpha$, and we set $\alpha = 0.5295$ to match the average labor share in Italy.

The default penalty is assymetric as in Arellano (2008). To ensure that we can properly detrend the penalty, we consider

$$\theta_t^p = \theta_{t-1}\psi\mathbb{E}\gamma_t \text{ if } \theta_t \geq \theta_{t-1}\psi\mathbb{E}\gamma_t \quad \text{and} \quad \theta_t^p = \theta_t \text{ if } \theta_t < \theta_{t-1}\psi\mathbb{E}\gamma_t.$$

One sets $\psi = 0.73$ to match the maximum spread and the correlation of spread with respect to output. Furthermore, we fix $\lambda = 0.032$ which corresponds to an exclusion between 7 and 8 years. This is consistent with the average exclusion period Italy recorded during its defaults on external debt in the 1930s and the 1940s (Reinhart and Rogoff, 2011).

6 Quantitative Analysis

6.1 Model Fit and Comparison

The fit of the model with respect to the data is depicted in Table 2. As we calibrate the model to Italy, the relevant benchmark is the economy without the Fund. To compute the moments we run 10,000 simulations of the model with 400 periods each, and we discard the first 100. For the volatilities and correlation statistics, we filter the simulated data through the HP filter with a smoothness parameter of 1600.

As one can see, the model replicates well the average indebtedness of Italy owing to the long-term debt structure (Chatterjee and Eyigungor, 2012). We are also matching the share of hours worked given the specification of the shocks. The same holds true for the volatility of hours worked and its correlation with output. In addition, the model replicates well the correlation of the spread with output and the maximal spread.[27] However, it cannot match the average spread observed in the data.[28] In terms of other non-targeted moments, the model is capable of capturing the business cycle dynamic of consumption and the primary surplus observed in the data.

Table 2 also compares the economy with and without the Fund. The difference between the two is important. First, the Fund enables a greater accumulation of debt in total. Particularly, the Fund almost doubles the debt capacity of the economy. Second, we observe low volatility and negative spreads with the Fund. The highest level of spread is zero with the Fund while it attains 8% without. Hence, the Fund achieves the goal of making sovereign debt safe — i.e. without default risk. Third, consumption is much less volatile in the presence of the Fund. This means that there is a greater risk sharing across states. This comes from the highly pro-cyclical surplus. In other words, in periods of distress, the Fund provides resources to sustain consumption. Such mechanism is less marked in the economy without the Fund owing to the risk-premium attached on the debt and the lack of state contingency.

[27] As the spread during default is not clearly determined in the model, we compute the spread at the default's start as the spread related to a borrowing at the peak of the borrowing Laffer curve.

[28] Models of sovereign defaults in the spirit of Aguiar and Gopinath (2006) and Arellano (2008) have difficulty to match the average spreads. Chatterjee and Eyigungor (2012) manage to match an average spread of 8% by means of long-term debt and quadratic output penalty but do not use growth shocks. Bocola and Dovis (2019) also match the average spread using multiple maturities but target an average spread of 0.61%.

Table 2: Data and Models

Targeted Moments				Non-Targeted Moments			
Variable	Data	w/o Fund	w/ Fund	Variable	Data	w/o Fund	w/ Fund
			A. First Moments				
$b'/y_{\text{annual}}\%$	117.64	116.50	204.80	$ps/y\%$	2.09	6.68	8.83
$n\%$	38.64	38.62	40.21	spread%	2.50	0.40	-0.04
max(spread)%	6.76	8.13	0.00				
			B. Second Moments				
$\sigma(n)/\sigma(y)$	0.75	0.75	0.64	$\sigma(\text{spread})$	0.96	0.05	0.00
$\rho(n, y)$	0.68	0.57	0.99	$\sigma(c)/\sigma(y)$	1.27	0.78	0.25
$\rho(\text{spread}, y)$	-0.16	-0.07	-0.81	$\rho(c, y)$	0.53	0.41	0.96
				$\sigma(ps/y)/\sigma(y)$	1.09	1.13	0.77
				$\rho(ps/y, y)$	0.29	0.67	0.99

6.2 Policy Functions and Financial Variables

To gain better understanding of the working of the Fund, we first present the numerical solutions of the policy functions of the Fund under our calibration. Figure 1 depicts the the different policy functions for zero private debt as a function of (γ, z), while Figures 2 depicts the main financial variables.[29] All figures relate to the detrended version of the model. We focus on three main values of the growth rate: the smallest one, γ_{min}, the median one, γ_{med}, and the highest one, γ_{max}.

Figure 1 presents the optimal policies with respect to the future relative Pareto weights, consumption and labor as function of (γ, z). With a logarithmic utility, one has that $\tilde{c} = \tilde{x}' \frac{\gamma}{\eta}$. Both $\tilde{c}$ and $\tilde{x}'$ are increasing, while n is decreasing in the current relative Pareto weight $\tilde{x}$. In each panel, the horizontal line on the left hand side is determined by the sovereign's binding participation constraint, while the horizontal line on the right hand side is determined by the lenders' binding participation constraint. The line rejoining both horizontal lines is determined by the first best allocation and has a slope of $\eta < 1$. Consistent with Lemmas 1 the borrower's binding constraint does not depend on the level of private debt. However, the lenders' binding constraint does.

We now turn to the financial variables depicted in Figure 2.[30] The first row of the figure

[29] In appendix G, Figure G.4 and G.7 present the main policy functions and financial variables as a function of (γ, ω) and (γ, b), respectively.

[30] In appendix G, Figure G.6 presents the same financial variables but for different levels of private debt.

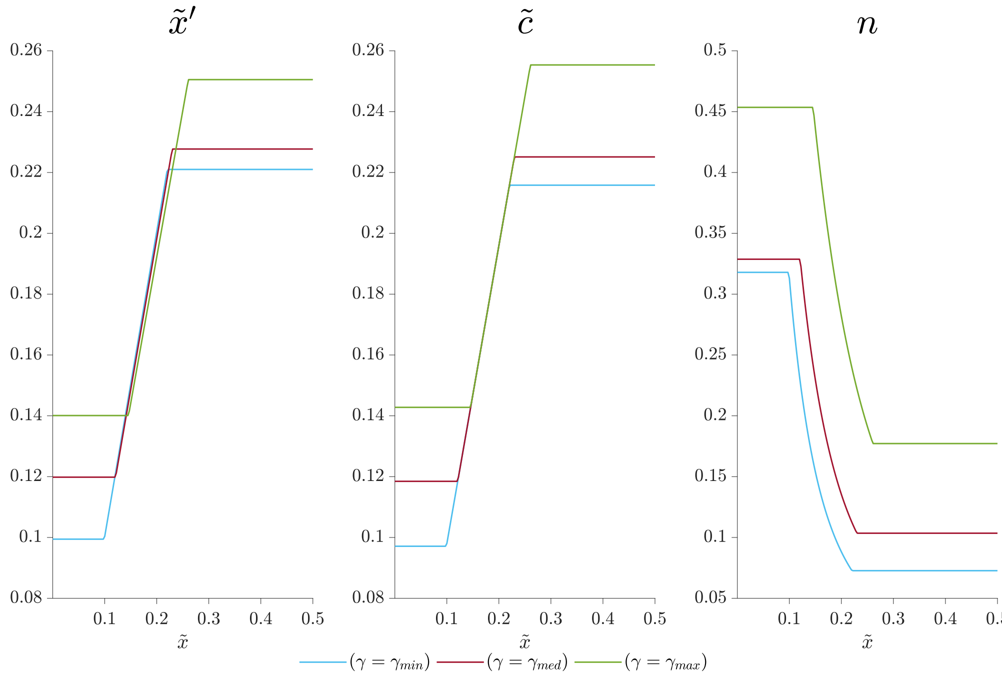

Figure 1: Optimal Policies with Zero Private Debt as Function of $(\gamma, \tilde{x})$

represents the prospective debt holdings of the sovereign. Consistent with the definition of effective intervention in Corollary 1, when the lenders' constraint does not bind, the credit line of the Fund is minimal. This does not necessarily mean that the majority of the debt is held in the private bond market, though. Conversely, when the lenders' participation constraint binds, private lenders do not roll-over the sovereign's debt. With zero initial private debt this translates into a complete stop of private lending activities. In this case, the debt accumulation is largely reduced. As we will see this is because the sovereign has a limited access to Arrow-type securities when the lenders' participation constraint binds.

The second row of Figure 2 depicts the current asset holdings and the interest spreads. One sees that when the lenders' participation constraint is binding, ω is very close to zero because of Proposition 3 and the fact that $Z = 0$ and $\tilde{b} = 0$. It might not exactly be equal to zero depending on the value of the total surplus and how large the negative spread is. This nonetheless tells us that if the lenders' participation constraint is binding today then the value of the sovereign's debt is in great part offset by the value of the realized Arrow-type security. Hence, when the lenders' participation constraint binds, the sovereign is limited in the trade of Arrow-type securities and bonds. This limitation ensures that the sovereign

Also, Figures G.9 and G.10 present the holdings of Arrow-type securities and the transfers from the Fund, respectively.

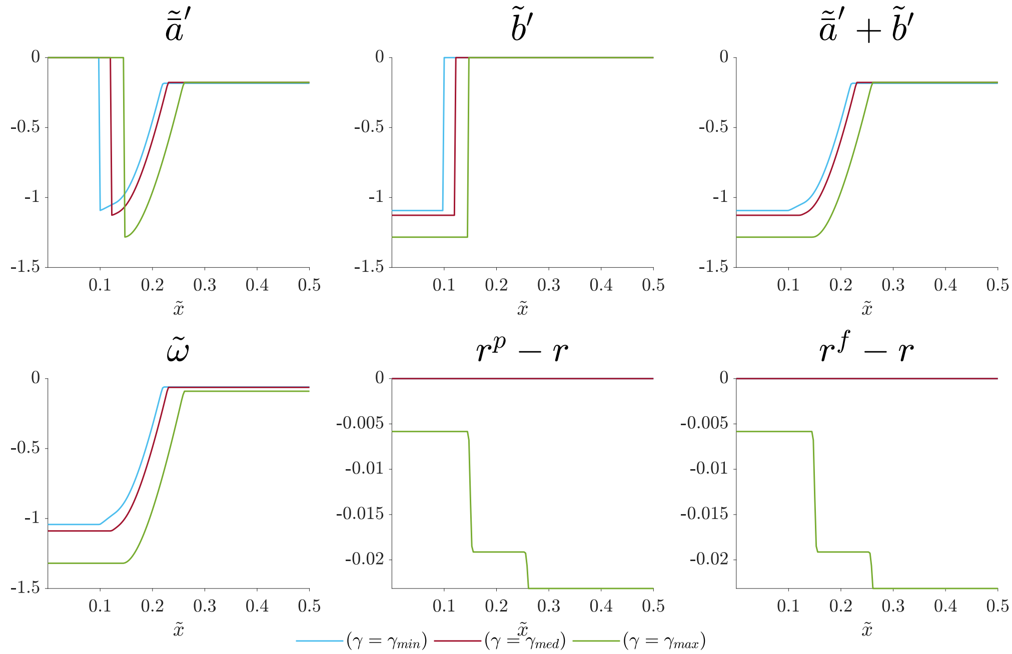

Figure 2: Financial Variables with Zero Private Debt as Function of $(\gamma, \tilde{x})$

does not violate the constraint of the lenders.

Regarding interest rates, the Fund's and private market's spreads are nil when the lenders' constraint is not binding consistent with Proposition 1.[31] In contrast, spreads are negative when the lenders' constraint is binding. As one can see, the negative spread remains relatively modest. Furthermore, it is larger the more constraints are binding in γ' for which $\tilde{\pi}(\gamma'|\gamma) > 0$. Hence, it relates to the extent of insurance required in each future state. A negative spread reduces the trade of Arrow-type securities in the binding states γ'. To see why, recall that the holdings of Arrow-type securities are defined such that $\sum_{\gamma'|\gamma} q_f(\gamma', \omega'|\gamma)\tilde{\tilde{a}}'(\gamma') = 0$. Hence, when $Q_f(\gamma', \omega'|\gamma) > \frac{\pi(\gamma'|\gamma)}{1+r}$ for some γ' with $\pi(\gamma'|\gamma) > 0$, the sovereign has to reduce its holdings of Arrow-type securities in the binding states to satisfy this condition. Thus, if the lenders' constraint binds in many future states, little hedge is offered by the Fund limiting the accumulation of debt.

6.3 Steady State Analysis

We simulate the economy within the ergodic set of relative Pareto weights.[32] For this purpose, we generate one history of shocks for 400 periods in steady state starting with the lowest

[31] The default set of the economy with and without the Fund is presented in Figure G.8 in Appendix G.

[32] More discussion on the policy function for the law of motion of the relative Pareto weight is in Appendix G, and Figure G.3 depicts the policy function under our calibration.

Pareto weight in the ergodic set. To avoid that the initial conditions blur the results, the first 250 periods are discarded. To gauge the impact of the Fund's intervention in this exercise, we simulate both the economy with and without the Fund in parallel.

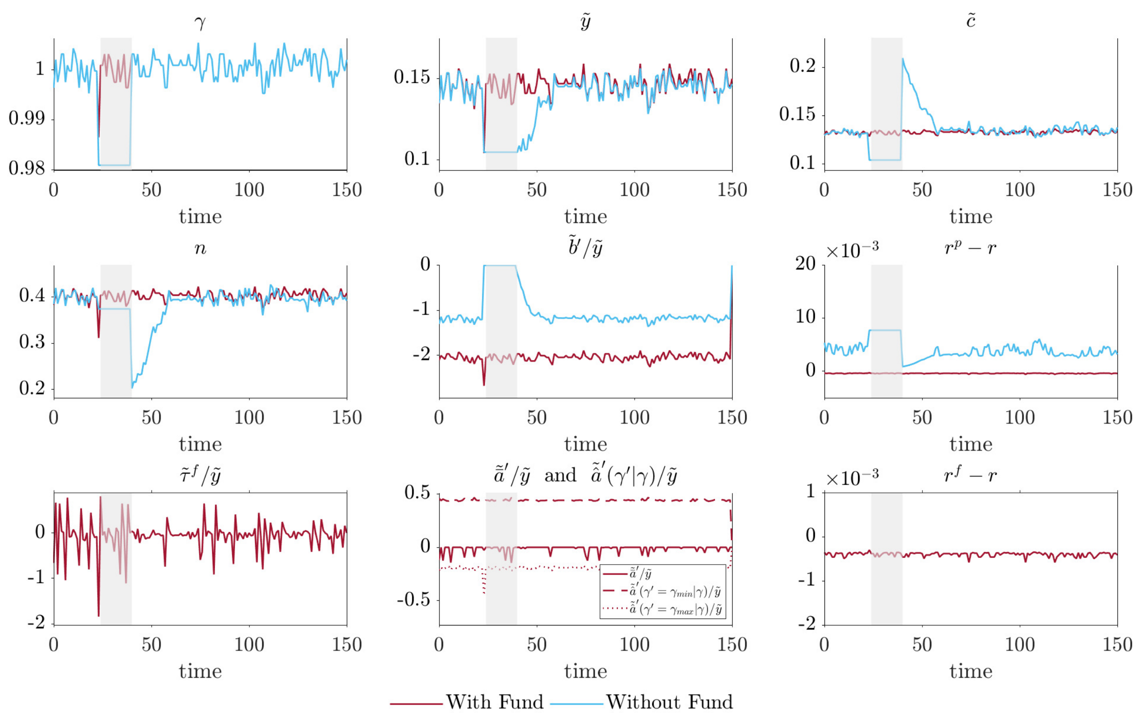

Figure 3: Simulation of a Typical Path

Figure 3 depicts the simulation result, with the grey region represents the periods in which the economy without the Fund is in default. With the Fund's intervention, the economy has a more stable consumption path over time. Hence, the sovereign avoids the major fluctuations of consumption that characterises the standard incomplete market economy with defaults. Moreover, the sovereign is able to accumulate private debt at the risk-free rate in regions where it would normally default without the Fund. This is entirely due to the fact that debt positions are hedged by Arrow-type securities. To get a sense of the insurance component, we display the Arrow-type securities purchased today for the highest and the lowest states tomorrow. Two points deserve to be noted. First, the portfolio of Arrow-type securities is procyclical as it closely follows the shock process. Second, the positions taken in Arrow-type securities are substantial. If one focuses on $\tilde{\tilde{a}}'(\gamma'|\gamma)$ for $\gamma' = \gamma_{min}$, we see that it amounts on average 50% of annual GDP. Instead of looking at the Arrow-type securities one can observe the Fund's primary surplus, $\tilde{\tau}_f$, which also moves procyclically and largely oscillates around zero since $Z = 0$.

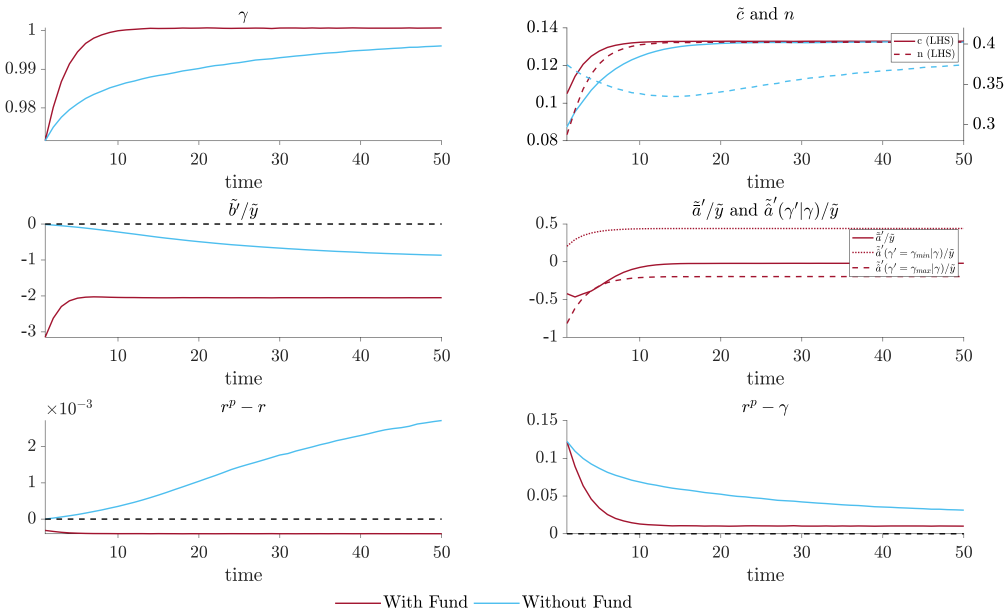

Figure 4: Impulse Response Functions to a Negative γ Shock

Figure 4 depicts the impulse response functions resulting from a stark negative growth shock on selected key variables.[33] The responses are computed as the mean of 10,000 independent shock histories starting with the lowest growth shock as well as initial debt holdings and relative Pareto weights drawn from the ergodic set. In the very first periods following the negative shock's realization, the Fund transfers resources to the sovereign. Especially, the Fund complements the provision of debt from the private bond market in the immediate outbreak of the shock. This prevents a large decrease in consumption and a large increase in labor supply. Hence, without the Fund's intervention, the sovereign repudiates its debt and is obliged to provide more labor to avoid a massive reduction in consumption. Thus, the immediate impact of a sudden low growth shock is more severe in the absence of the Fund. In the long run, the sovereign without the Fund is likely to repudiate debt again and therefore reaches a lower level of steady state consumption and indebtedness. Besides this, the economy with the Fund avoids the positive spread in the private bond market. It can therefore reach more quickly a low level of $r^p - \gamma$ easing debt management.

[33]Figures G.11 and G.12 in Appendix G present the impulse response functions to a negative and a to a positive shock of all relevant variables in the model.

6.4 Welfare Analysis

Sharp difference in the dynamics of the economy with and without Fund translates into superior welfare implications of the Fund. The first column of Table 3 represents the welfare gains of the Fund's intervention in consumption equivalent terms at zero initial debt holdings. Recall that the sovereign which has access to the Fund can hold debt in the Fund or in the private bond market. Thus, to adequately compare the two economies, we compare them for the same *total* debt holdings. That is, the welfare comparisons are computed at the points where $\omega = 0$ for the economy in the Fund and at $b = 0$ for the economy outside the Fund. The welfare computation is presented in Appendix E.

Table 3: Welfare Comparison at Zero Initial Debt

State	Welfare Gains (%)	Maximal Debt Absoption (% of GDP)	
		With Fund	Without Fund
$\gamma = \gamma_{min}$	10.17	433	190
$\gamma = \gamma_{med}$	9.86	187	106
$\gamma = \gamma_{max}$	9.79	185	104
Average	9.94		

Welfare gains are significant with the Fund's intervention. With zero initial debt, the consumption-equivalent welfare gains are on average 9.94%. Moreover, the largest welfare gains are recorded in low growth states. Thus, the Fund's intervention is mostly valued when the sovereign is in a difficult economic situation. As mentioned above, welfare gains are the consequence of two main features of the Fund's intervention. First, the Fund provides state-contingent transfers and therefore enhances consumption smoothing. Second, it enables a greater accumulation of debt in general. In Appendix E we provide a decomposition of the welfare gains showing that they are mostly due (circa 90%) to the greater debt capacity and the insurance component; among these two factors 'debt capacity' represents the largest share of total gains (circa 80%). Nevertheless, to a large extent, this is due to the insurance capacity of the Fund.

6.5 Debt Dynamic Decomposition

We further decompose the evolution of the debt according to Cochrane (2020, 2022): sovereign debt (with respect to GDP) at the end of the year, v_{t+1}, is equal to its value at the beginning of the year, v_t, plus the net cost of keeping debt, $r_t^p - \gamma_t$, and the year's primary deficit (excluding interest payment), $-s_t$, so that $v_{t+1} = v_t + r^p_t - \gamma_t - s_t$, assuming no discounting for simplification. In our environment $s_t = \frac{b_{t+1}-b_t}{\theta f(n)}$ for the economy without Fund and $s_t = \frac{\omega_{t+1}-\omega_t}{\theta f(n)}$ for the economy with Fund.

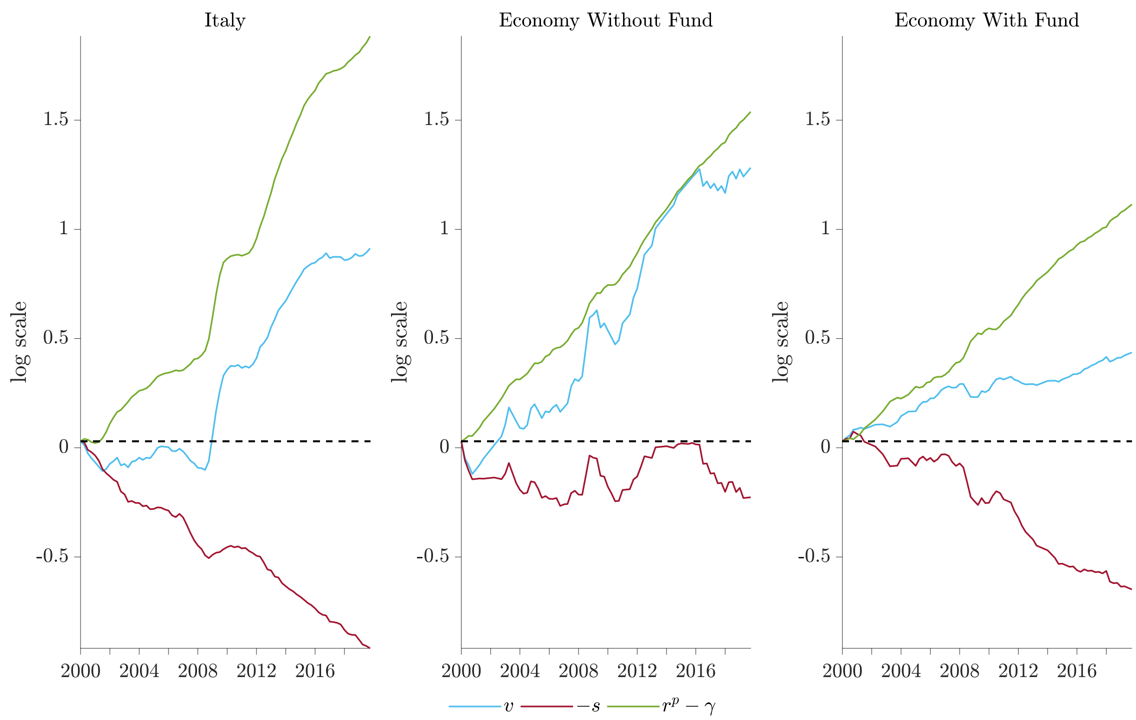

Figure 5: Cochrane Decomposition

Figure 5 depicts the decomposition for Italy as well as the model economy with and without the Fund in logarithmic scale. We generate the two panels for the model economy by feeding the growth path of Italy over 2000Q1–2019Q4 into the model and start with the same level of debt of Italy in 2000Q1. We then obtain the path of debt and interest rate through the optimal policy functions. The blue line represents the evolution of the value of debt which is the combination of the green line (i.e. $r^p - \gamma$) and the red line (i.e. $-s$). In view of this, had the accumulation of debt been costless (i.e. $r^p - \gamma = 0$), then the blue line would coincide with the red line.

We observe that the evolution of Italy's debt is the result of two conflicting forces: a remarkable history of increasing accumulated primary surpluses and two decades of growth decline resulting in accumulated costs $r^p - \gamma$. The model without the Fund replicates well the dynamic of the Italian public indebtedness. It nonetheless minimises the positive impact of primary surpluses and the negative impact of the interest rate-growth differential.

Turning to the economy with the Fund, we see that the evolution of debt is flatter than in the economy without. This comes from two components. On the one hand, the rate at which the sovereign issues debt is at most risk free. This therefore largely reduces the $r^p - \gamma$ cost compared to the economy without the Fund. On the other hand, the Fund provides insurance through Arrow securities. This eases debt management by making fiscal policy countercyclical as shown previously. As a result, the debt path is more smooth. Particularly, the model predicts that the Italian indebtedness by the end of 2019 would have been around 86% of GDP rather than 135% if Italy could have joined the Fund in 2000.[34] This shows that the path followed by the Italian economy in the last two decades was highly inefficient. Even though the accumulation of large primary surpluses dampened the increase in the Italian indebtedness, it prevented a proper countercyclical policy which could have corrected the interest rate-growth differential.[35]

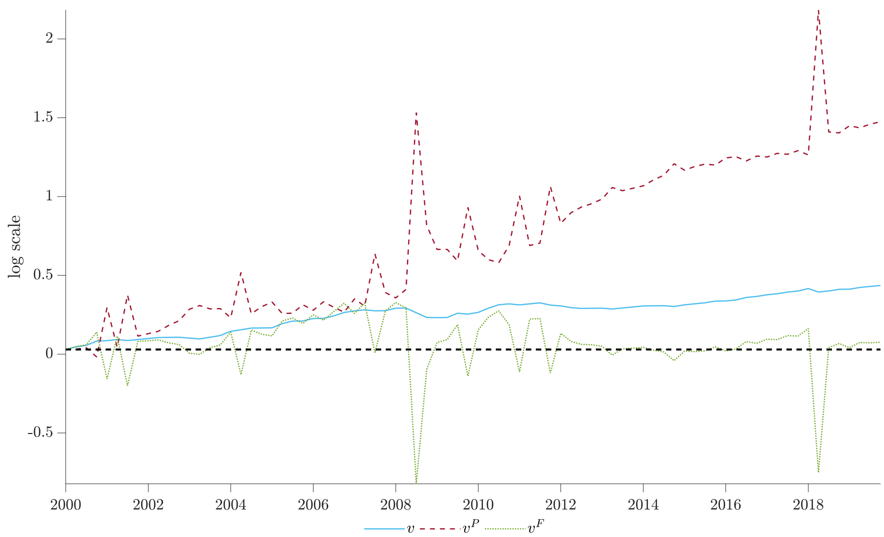

Figure 6: Fund vs. Private Debt

With the intervention of the Fund, debt can be located both in the private bond market and the Fund itself. We can further decompose the value of the debt as $v_t = v_t^P + v_t^F$, where v_t^P and v_t^F are the value of the debt held in the private bond market and the Fund, respectively. Figure 6 presents the above decomposition. We note two elements. On the one hand, given that the Fund's intervention is minimal, most of the value of the debt emanates

[34]We obtain this figure by computing the model implied debt-to-GDP ratio at the end of the sample period using the decomposition of Cochrane (2020, 2022). We then rescale this figure according to the true debt-to-GDP ratio in the data and the one obtained in the decomposition for Italy.

[35]Given the importance of the $r^p - \gamma$ component, we extend the baseline contract with a stochstic risk-free rate. Results are presented in Appendix F.

from the private bond market consistent with Corollary 1. On the other hand, the Fund dampens the dynamic of debt over time. In other words, it counterbalances the large level of indebtedness in the private bond market. Particularly, the spikes observed in the figure correspond to episodes in which growth suddenly drops. In such circumstances the level of private debt per GDP increases but is counter-acted by the realization of Arrow-type securities which insure the sovereign against such adverse shocks.[36]

7 Conclusion

We design the optimal interaction between a Financial Stability Fund, private competitive international lenders and a sovereign. The Fund's long-term contract is shaped by two-sided limited enforcement constraints. We interpret the sovereign's constraint as a sovereignty constraint and the lenders' constraint as a DSA. The sovereign can borrow long-term defaultable bonds on the private international market, while receiving state-contingent transfers from the Fund. The Fund prevents the sovereign from defaulting on its entire debt position, regardless of its seniority with respect to private lenders, increasing the supply of safe assets.

As we show in our calibration to the Italian economy and subsequent simulations and computations, important welfare gains can be achieved by improving existing official lending practices offering long-term state-contingent Fund contract. Imbedded in these welfare gains is the effect of the Fund in transforming (defaultable) sovereign debts into safe assets, even they are mostly held by private investors, and inducing a (constrained) efficient countercyclical fiscal policy, even when there is debt accumulation or $r - g$ uncertainty, as most countries nowadays face.

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[36]We observe three regions with major spikes in Figure 6. The first one corresponds to the global financial crisis of 2008, the second to the European sovereign debt crisis between 2010 and 2012 and the third one to the political and economic crisis of 2018. Note in the first panel of Figure 5 how these spikes are the three main debt accumulation episodes that the Fund contract prevents.

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Online Appendix
(Not For Publication)

A Detrended Model

As in Aguiar and Gopinath (2006), we consider a growth shock to the productivity of the following form $\theta_t = \gamma_t \theta_{t-1}$, where γ_t represents the growth rate and θ_t the trend at time t. We detrend the variables for allocations (except for labor n where we normalise the time endowment to 1) of the model by dividing them by θ_{t-1}. We therefore denote by $\tilde{c}_t$ the detrended form of c_t such that $\tilde{c}_t = \frac{c_t}{\theta_{t-1}}$ represents the deviation from the trend. It follows that $U(c_t, n_t) = \ln(\theta_{t-1}) + U(\tilde{c}_t, n_t)$, and clearly, $\ln(\theta_{t-1})$ does not affect optimal choice. By the homogeneity of the sovereign's recursive problem, we have the detrended formulation as

$$\widetilde{W}^b(\gamma, \tilde{a}, \tilde{b}) = \max_{\left\{\tilde{c}, n, \tilde{b}', \{\tilde{a}'(\gamma')\}_{\gamma' \in \Gamma}\right\}} U(\tilde{c}, n) + \beta \mathbb{E}\left[\widetilde{W}^b(\gamma', \tilde{a}'(\gamma'), \tilde{b}') \middle| \gamma\right] \tag{A.1}$$
$$\text{s.t. } \tilde{c} + \sum_{\gamma'|\gamma} q_f(\gamma', \tilde{\omega}'(\gamma')|\gamma)(\gamma \tilde{a}'(\gamma') - \delta \tilde{a}) + q_p(\gamma, \tilde{\omega}')(\gamma \tilde{b}' - \delta \tilde{b})$$
$$\leq \gamma f(n) + (1 - \delta + \delta\kappa)(\tilde{a} + \tilde{b})$$
$$\tilde{\omega}'(\gamma') = \tilde{a}'(\gamma') + \tilde{b}' \geq \tilde{\mathcal{A}}_b(\gamma').$$

Similarly, the private lender's problem in detrended form reads

$$\widetilde{W}^p(\gamma, \tilde{a}_l, \tilde{b}_l) = \max_{\tilde{b}_l'} (1 - \delta + \delta\kappa + \delta q_p(\theta, \tilde{\omega}'))\tilde{b}_l + \frac{1}{1+r} \mathbb{E}\left[\widetilde{W}^p(\gamma', \tilde{a}_l'(\gamma'), \tilde{b}_l') \middle| \theta\right]. \tag{A.2}$$

The sovereign's outside option in detrended form takes the following form

$$\widetilde{V}^{af}(\gamma) = \max_n \left\{U(\gamma^p f(n), n)\right\} + \beta \mathbb{E}\left[(1-\lambda)\widetilde{V}^{af}(\gamma') + \lambda \tilde{J}(\gamma', 0) \middle| \gamma\right],$$

The detrended Fund's problem in sequential form is given by

$$\max_{\{\tilde{c}(\gamma^t), n(\gamma^t)\}_{t=0}^{\infty}} \mathbb{E}\left[\mu_{b,0} \sum_{t=0}^{\infty} \beta^t U(\tilde{c}(\gamma^t), n(\gamma^t)) + \mu_{l,0} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left(\prod_{i=0}^{t-1} \gamma_i\right) \tilde{\tau}(\gamma^t) \middle| \theta_{-1}\right] \tag{A.3}$$
$$\text{s.t. } \mathbb{E}\left[\sum_{j=t}^{\infty} \beta^{j-t} U(\tilde{c}(\gamma^j), n(\gamma^j)) \middle| \gamma^t\right] \geq \widetilde{V}^{af}(\gamma_t), \tag{A.4}$$
$$\mathbb{E}\left[\sum_{j=t}^{\infty} \left(\frac{1}{1+r}\right)^{j-t} \left(\prod_{i=t}^{j-1} \gamma_i\right) \tilde{\tau}(\gamma^j) \middle| \gamma^t\right] \geq Z - \tilde{b}(\gamma^t), \tag{A.5}$$
$$\tilde{\tau}(\gamma^t) = \gamma_t f(n(\gamma^t)) - \tilde{c}(\gamma^t), \quad \forall \gamma^t, t \geq 0,$$

$$\text{with } \mu_{b,0},\ \mu_{l,0},\ \{\tilde{b}(\gamma^t)\}_{t=0}^{\infty}, \{q_p(\gamma^t, x(\gamma^{t+1}), \tilde{b}(\gamma^{t+1}))\}_{t=0}^{\infty} \text{ given.}$$

And in recursive form

$$\begin{aligned}
\widetilde{FV}(\gamma, \tilde{x}, \tilde{b}) = \mathcal{SP} \min_{\{\nu_b,\nu_l\}} \max_{\{\tilde{c},n\}}\ & \tilde{x}\Big[(1+\nu_b)U(\tilde{c}, n) - \nu_b \widetilde{V}^{af}(\gamma)\Big] \qquad (A.6)\\
& + \big[(1+\nu_l)\tilde{\tau} - \nu_l(Z - \tilde{b})\big] + \frac{1+\nu_l}{1+r}\gamma\mathbb{E}\big[\widetilde{FV}(\gamma', \tilde{x}', \tilde{b}')\big|\gamma\big]\\
\text{s.t.} \quad & \tilde{\tau} = \gamma f(n) - \tilde{c},\\
& \tilde{x}' = \frac{1+\nu_b}{1+\nu_l}\frac{\eta}{\gamma}\tilde{x}, \qquad (A.7)
\end{aligned}$$

The value function takes the form of

$$\begin{aligned}
&\widetilde{FV}(\gamma, \tilde{x}, \tilde{b}) = \tilde{x}\widetilde{V}^b(\gamma, \tilde{x}, \tilde{b}) + \widetilde{V}^l(\gamma, \tilde{x}, \tilde{b}), \text{ with}\\
&\widetilde{V}^b(\gamma, \tilde{x}, \tilde{b}) = U(\tilde{c}, n) + \beta\mathbb{E}[\widetilde{V}^b(\gamma', \tilde{x}', \tilde{b}')|\gamma], \text{ and}\\
&\widetilde{V}^l(\gamma, \tilde{x}, \tilde{b}) = \tilde{\tau} + \frac{1}{1+r}\gamma\mathbb{E}[\widetilde{V}^l(\gamma', \tilde{x}', \tilde{b}')|\gamma].
\end{aligned}$$

Taking the first-order conditions with respect to c and n leads to

$$u'(\tilde{c}) = \frac{1+\nu_l}{1+\nu_b}\frac{1}{\tilde{x}} \quad \text{and} \quad \gamma f'(n) = \frac{h'(1-n)}{u'(\tilde{c})},$$

The consumption is therefore equal to $\tilde{c} = \tilde{x}'\frac{\gamma}{\eta} \equiv \tilde{z}'\gamma$. From this, we see that whenever the growth rate of the economy settles below one, the relative Pareto weight increases. However, the consumption does not react to changes in γ. In fact, the consumption is affected only when one of the limited enforcement constraints binds.

For completeness, the decentralised Fund problem in detrended form is given by

$$\begin{aligned}
\widetilde{W}^f(\gamma, \tilde{a}_l, \tilde{b}_l) = \max_{\{\tilde{c}_f, \{\tilde{a}_l'(\gamma')\}_{\gamma'\in\Gamma}\}}\ & \tilde{c}_f + \frac{1}{1+r}\gamma\mathbb{E}\big[\widetilde{W}^f(\gamma', \tilde{a}_l'(\gamma'), \tilde{b}_l')\big|\gamma\big] \qquad (A.8)\\
\text{s.t. } & \tilde{c}_f + \sum_{\gamma'|\gamma} q_f(\gamma', \omega'(\gamma')|\gamma)(\gamma\tilde{a}_l'(\gamma') - \delta\tilde{a}_l) \le (1-\delta+\delta\kappa)\tilde{a}_l,\\
& \tilde{b}_l' = \tilde{B}_l(\gamma, \tilde{a}_l, \tilde{b}_l)\\
& \tilde{a}_l'(\gamma') + \tilde{b}_l' \ge \tilde{\mathcal{A}}_f(\gamma', \tilde{b}_l'). \qquad (A.9)
\end{aligned}$$

B Further Theory Development

In this section we present other properties of the Fund contract. We start with the inverse Euler Equation which is a key concept determining the dynamic of consumption in the contract.

Proposition B.1 (Insurance)**.** *In the Fund contract, the inverse Euler equation is given by*

$$\mathbb{E}\left[\frac{1}{u'(c(\theta',x',b'))}\frac{1+\nu_l(\theta',x',b')}{1+\nu_b(\theta',x',b')}\middle|\theta\right]=\eta\frac{1}{u'(c(\theta,x,b))},$$

and risk sharing is imperfect.

Proof. See Appendix C □

We obtain the inverse Euler equation by means of the first-order condition on consumption and the law of motion of the relative Pareto weight. This equation gives the intertemporal dynamic of consumption. If none of the constraints are ever binding (i.e. $\nu_b = \nu_l = 0$), it becomes

$$\mathbb{E}\left[\frac{1}{u'(c(\theta',x',b'))}\middle|\theta\right]\leq\frac{1}{u'(c(\theta,x,b))},$$

with strict inequality if $\eta < 1$, in our case. We therefore obtain a positive martingale, which by the supermartingale theorem, converges almost surely to $-\infty$. This is what the literature has called immiseration.

Thus, with $\eta < 1$, when none of the constraints are binding, consumption decreases. However, this reduction cannot go on indefinitely given the sovereign's limited enforcement constraint. This constraint puts a lower bound to the supermartingale and therefore acts as a stopper for immiseration. Conversely, the lender's constraint puts an upper bound to the supermartingale which prevents consumption to increase indefinitely. As a result, in a contract with tow-sided limited enforcement constraints and impatient borrower, risk sharing is only partial. The contract cannot converge to the first-best allocation characterised by constant consumption over time.

Having determined the inverse Euler Equation, we can now show existence. To ensure the existence of the Lagrange multipliers — and therefore of the above contract, we need to the following technical assumption (Marcet and Marimon, 2019).

Assumption B.1 (Interiority)**.** *There is an $\epsilon > 0$, such that, for all $\theta \in \Theta$, there is a sequence $\{\ddot{c}(\theta^t), \ddot{n}(\theta^t)\}$ satisfying equations (13) and (14) in which each outside option is replaced by $V^{af}(\theta_t) + \epsilon$ and $Z + \epsilon$, respectively.*

This assumption ensures the uniform boundedness of the Lagrange multipliers. For equations (13) and (14), it requires that, in spite of the enforcement constraints, there are strictly positive rents to be shared among the contracting parties. Otherwise there may not exist a constrained-efficient risk-sharing agreement. Given this, we can show that, under general conditions, a Fund contract exists.

Proposition B.2 (Existence of Fund Contract)**.** *For every $\theta \in \Theta$ there is a $\underline{b}(\theta) < 0$ such that if $b_0(\theta) \geq \underline{b}(\theta)$, then there exist a Fund contract with initial condition $(\theta, b_0(\theta))$. Furthermore, there is a $\underline{t}(\theta, \underline{b}(\theta))$ such that for $t > \underline{t}(\theta, \underline{b}(\theta))$ the Fund contract is at steady state.*

Proof. See Appendix C □

Proposition B.2 is made of two parts. First, a Fund contract exists if — among other requirements — the initial level of private indebtedness is not too high.[37] Thus, if an economy is in an initial state $(\theta, b_0(\theta))$ but $b_0(\theta) < \underline{b}(\theta)$ then the private debt will need to be restructured — i.e. to a $\ddot{b}_0(\theta) \geq \underline{b}(\theta)$ — for a Fund contract to exist. Second, the Fund contract is characterised by an ergodic distribution. Hence, in the long-run, the relative Pareto weight moves within the same set of values over and over again. The exact shape of the ergodic distribution is the purpose of the next definition and lemma.

Having shown existence of the Fund contract, we can now determine the correspondence between the Fund contract established in Section 3.1 and the decentralised Fund contract presented in section 3.3.

Proposition B.3 (Second Welfare Theorem)**.** *Given initial conditions $\{\theta_0, b_0, x_0\}$, a Fund's allocation can be decentralised as a competitive equilibrium with endogenous borrowing limits.*

Proof. See Appendix C □

This proposition states that there is a direct correspondence between, on the one hand, a and, on the other hand, x given by

$$u'(c(\theta, a, b)) = \frac{1 + \nu_l(\theta, x, b)}{1 + \nu_b(\theta, x, b)} \frac{1}{x}.$$

In words, for a given θ, if a and x satisfy the above correspondence, then $B(\theta, x, b) = B(\theta, a, b)$, $c(\theta, a, b) = c(\theta, x, b)$, $c_p(\theta, a, b) = \tau_p(\theta, x, b)$, $c_f(\theta, a, b) = \tau_f(\theta, x, b)$, $c_p(\theta, a, b) + c_f(\theta, a, b) = \tau(\theta, x, b)$ and $n(\theta, a, b) = n(\theta, x, b)$. In that same logic, we have that $W^b(\theta, a, b) = V^b(\theta, x, b)$ and $W_p(\theta, a, b) + W_f(s, a, b) = V^l(s, x, b)$. Thus, the endogenous limits (3) and (20) are exactly and uniquely binding when they are binding in the Fund contract.

Properly speaking the correspondence relates to, on the one hand, x and b and, on the other hand, ω as only the entire sovereign's debt matter in the Fund. The split of ω between a and b is irrelevant for the Fund as the sovereign defaults ω and not selectively on a or b.

[37] Note that $\underline{b}(\theta) = \min_b\{b : \theta^- Z - b \geq V^b(\theta, \underline{x}, b)\}$.

For completeness of the argument, we also show that the First Welfare Theorem holds. That is, a recursive competitive equilibrium allocation with borrowing limits implements the constrained efficient allocation of the Fund. For that purpose, we define

Definition 2 (High Implied Interest Rates)**.** *An allocation has high implied interest rates if for all t and s^t*

$$\mathbb{E}_0 \sum_{t=0}^{\infty} Q_f\big(s^t, \omega(s^t)|s_0\big) \left[c\left(s^t, a(s^t), b(s^t)\right) + c_l\left(s^t, a(s^t), b(s^t)\right)\right] < \infty,$$

The intertemporal discount factor, $Q_f\big(s^t, \omega(s^t)|s_0\big)$, is defined below. This condition ensures that the present value of the total transfer is finite. It rules out autarky as an equilibrium.

Proposition B.4 (First Welfare Theorem)**.** *Given initial conditions $\{\theta_0, b_0, a_0\}$, under high-implied interest rates, a competitive equilibrium with endogenous borrowing limits implements the constrained efficient allocation of the Fund.*

Proof. See Appendix C □

We end this subsection with a result relating to the endogenous borrowing limits. Using the intertemporal budget constraints, we can construct the asset holdings that make the consumption allocations in the Fund contract satisfy the present value of the budget. This leads to the following proposition.

Lemma B.1 (Borrowing and Net Present Value Constraints)**.** *At some period t and n with $t \neq n$, if the participation constraint of one of the contracting parties is binding, the borrowing limit for of the constrained agent in the decentralised economy is determined by*

$$\mathcal{A}_b(\theta^t) = \mathbb{E}_t \sum_{j=0}^{\infty} Q_f(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))], \tag{B.1}$$

$$\mathcal{A}_f(\theta^n) = \mathbb{E}_t \sum_{j=0}^{\infty} Q_f(\theta^{n+j}, \omega(\theta^{n+j})|\theta^n) c_l(\theta^{n+j}, a(\theta^{n+j}), b(\theta^{n+j}))), \tag{B.2}$$

with $Y(\theta^t, x(\theta^t), b(\theta^t)) \equiv \theta(\theta_t) f(n(\theta^t, x(\theta^t), b(\theta^t)))$ for all t and θ^t.

Proof. See Appendix C □

Given this, (18) truly represent a net present value (NPV) constraint in equilibrium. In any state, the decentralised asset portfolio between the sovereign and the Fund is a whole plan of contingent asset position to the indefinite future. The whole contingent plan of asset holdings corresponds to the whole plan of transfers $\{\tau(\theta^t)\}_{t=0}^{\infty}$, which is clearly not a one period decision. The fact that the whole plan can be determined recursively does not mean that the asset positions in θ^{t+1} — that is $\omega(\theta^{t+1})$ — refer only to a set of contingent payoffs at $t+1$. Rather, $\omega(\theta^{t+1})$ represents the NPV of all future Fund's transfers starting from θ^{t+1}. Therefore when (18) binds with strictly positive probability, the Fund refuses to grant an alternative plan embedded in some other $\ddot{\omega}(\theta^{t+1})$, which would render the NPV negative. Equivalently, this means that the Fund should not lend too much at too low a price or it would end up loosing money. Hence, the lender's constraint is a present value — or more lively, a no bailout — constraint, which is conceptually distinct from the borrower's borrowing constraint, (i.e. a sovereignty constraint).

C Proofs

Proof of Proposition B.1

The first order condition on consumption reads $u'(c) = \frac{1+\nu_l}{1+\nu_b}\frac{1}{x}$. The law of motion of the relative Pareto weight is given by $x' = \frac{1+\nu_b}{1+\nu_l}\eta x$. Combining those two equations one obtains

$$x' = \frac{1+\nu_b(\theta,x,b)}{1+\nu_l(\theta,x,b)}\eta x = \frac{1}{u'(c(\theta',x',b'))}\frac{1+\nu_l(\theta',x',b')}{1+\nu_b(\theta',x',b')}. \tag{C.1}$$

Moreover, observe that using the above first-order condition

$$\frac{1+\nu_b(\theta,x,b)}{1+\nu_l(\theta,x,b)}\eta x = \eta\left[\frac{1}{u'(c(\theta,x,b))}\frac{1+\nu_b(\theta,x,b)}{1+\nu_l(\theta,x,b)}\frac{1+\nu_l(\theta,x,b)}{1+\nu_b(\theta,x,b)}\right] = \eta\frac{1}{u'(c(\theta,x,b))}.$$

Hence, one can rewrite (C.1) as

$$\eta\frac{1}{u'(c(\theta,x,b))} = \frac{1}{u'(c(\theta',x',b'))}\frac{1+\nu_l(\theta',x',b')}{1+\nu_b(\theta',x',b')}.$$

Taking expectations on both sides with respect to θ' leads to

$$\eta\frac{1}{u'(c(\theta,x,b))} = \mathbb{E}\left[\frac{1}{u'(c(\theta',x',b'))}\frac{1+\nu_l(\theta',x',b')}{1+\nu_b(\theta',x',b')}\middle|\theta\right].$$

This equation is the inverse Euler equation. It gives the dynamic of consumption over time and therefore the extent of insurance. If none of the constraint ever binds and $\eta = 1$, then the contract achieves full insurance. However, whenever one of those two point is no true, consumption is not constant across states. Insurance is thus only partial in our environment. □

Proof of Proposition B.2

Consider the model in detrended form presented in Appendix A. If one has that $\{\tilde{b}(\theta^t)\}_{t=0}^{\infty} = 0$, we are back to the standard model of Ábrahám et al. (2021) without moral hazard.

To show existence, one needs to determine whether the assumptions to apply Theorem 3(i) in Marcet and Marimon (2019) are met: A1 well defined Markovian process for γ, A2 continuity in $\{c, n\}$ and measurability in γ, A3 non-empty feasible sets, A4 uniform boundedness, A5 convex technologies, A6 concavity for the lender and strict concavity for the sovereign, A7 interiority. Assumption A1, A2, A5 and A6 are trivially met as elicited in Section 2. Since c and n are bounded, payoffs functions are bounded as well. This combined with the fact that the outside options are finite ensure that A4 is met. Assumption B.1 ensures A7.

One is left to show that A3 is met. If one assumes that the sequence of debt is different than zero for some $t > 0$ and especially for $t = 0$, it is the initial $\tilde{b}_0$ that is crucial for existence. If $\tilde{b}_0$ is such that the following break even condition holds:

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \tilde{\tau}(\theta^t)\middle|\theta_0\right] = Z - \tilde{b}_0,$$

then a contract exists. The break even condition is a consequence of the homogeneity of degree 1 of the problem's solution (Marcet and Marimon, 2019, Lemma 1A). Whenever the break even condition holds, one obtains for all t and θ^t, $V^l(\gamma^t, \tilde{x}(\gamma^t), \tilde{b}(\gamma^t)) \geq Z - \tilde{b}(\theta^t)$. However, should it not be the case, the initial debt is too large to be absorbed by the Fund. The debt has to be restructured until the above break even condition holds.

The homegeneity of degree one in $\mu = (\mu_b, \mu_l)$ allows us to redefine the contracting problem using x as a co-state variable. This combined with Assumption B.1 ensures that there exists a $C > 0$ such that for the Lagrange multiplier ϑ, $||\vartheta|| \leq ||x||C$. Accounting for the lender's participation constraint there is a $\bar{C}$ such that $||\vartheta|| \leq \bar{C}$. We can therefore define the set of of feasible Lagrange multipliers by $L = \{\vartheta \in \mathbb{R}_+^2 : ||\vartheta|| \leq \bar{C}\}$ and the set of feasible consumption and labor by $A = \{(c, n) \in \mathbb{R}_+^2 : n \leq 1\}$.

With this, one can use Theorem 3(i) in Marcet and Marimon (2019). That is the correspondence $SP : A \times L \to A \times L$ mapping non-empty, convex, and compact sets to themselves, is non-empty, convex-valued, and upper hemicontinuous. We can therefore apply Kakutani's fixed point theorem and existence immediately follows.

Regarding the steady state, the lower bound of the ergodic set is determined by the lowest achievable relative Pareto weight in the contract. It represents the lowest value that the sovereign accepts in the contract. The upper bound represents the highest relative Pareto weight that makes the sovereign's constraint bind; therefore it is the highest weight that the lender may need to accept. This means that every time the highest productivity shock hits (i.e. γ_{max}), the sovereign climbs to the top of the ergodic set. In opposition, for a sufficiently long string of lowest productivity shock (i.e. γ_{min}), the sovereign eventually hits the bottom of the set — owing to immiseration.

To show the existence of a unique stationary equilibrium, one shows that the dynamic of the contract satisfies the conditions given by Stokey et al. (1989, Theorem 12.12). Set $\ddot{x}$ as the midpoint of $[\tilde{\underline{x}}, \tilde{\overline{x}}]$ and define the transition function $Q : [\tilde{\underline{x}}, \tilde{\overline{x}}] \times \mathcal{X}([\tilde{\underline{x}}, \tilde{\overline{x}}]) \to \mathbb{R}$ as

$$Q(x, G) = \sum_{\theta'|\theta} \pi(\theta'|\theta)\mathbb{I}\{x' \in G\}$$

We want to show is that $\ddot{x}$ is a mixing point such that for $N \geq 1$ and $\epsilon > 0$ one has that $Q(\tilde{\underline{x}}, [x, \tilde{\overline{x}}])^N \geq \epsilon$ and $Q(\tilde{\overline{x}}, [\tilde{\underline{x}}, x])^N \geq \epsilon$. Starting at $\tilde{\overline{x}}$, for a sufficiently long but finite series of γ_{min}, the relative Pareto weight transit to $\tilde{\underline{x}}$. Hence for some $N < \infty$, $Q(\tilde{\overline{x}}, [\tilde{\underline{x}}, \ddot{x}])^N \geq \pi(\gamma_{min})^N > 0$ where $\pi(\gamma_{min})$ is the stationary probability of drawing γ_{min}. Moreover, starting at $\tilde{\underline{x}}$, after drawing $N < \infty$ γ_{max}, the relative Pareto weight transit to $\tilde{\overline{x}}$ meaning that $Q(\tilde{\underline{x}}, [\ddot{x}, \tilde{\overline{x}}])^N \geq \pi(\gamma_{max})^N > 0$. Setting $\epsilon = \min\{\pi(\gamma_{min})^N, \pi(\gamma_{max})^N\}$ makes $\ddot{x}$ a mixing point and the above theorem applies. □

Proof of Lemma 1

Recall that, in the detrended version of the model, the lower bound is defined by $\underline{x} = \min_{\gamma\in\Gamma}\{x : \widetilde{V}^b(\gamma, x, b) = \widetilde{V}^{af}(\gamma)\}$, while the upper bound corresponds to $\overline{x} = \max_{\gamma\in\Gamma}\{x : \widetilde{V}^b(\gamma, x, b) = \widetilde{V}^{af}(\gamma)\}$.

The key insight is to see that the sovereign's outside option is independent of the level of indebtedness, while the sovereign's value increases with the relative Pareto weight by definition. Assume now by contradiction that the lower bound $\underline{x}(\gamma, b)$ is a function of γ and the level of debt b. That is for some $\ddot{b} \neq b$, $\underline{x}(\gamma, b) \neq \underline{x}(\gamma, \ddot{b})$. This implies that either $\widetilde{V}^b(\gamma, \underline{x}(\gamma, b), b) > \widetilde{V}^b(\gamma, \underline{x}(\gamma, \ddot{b}), \ddot{b})$ or $\widetilde{V}^b(\gamma, \underline{x}(\gamma, b), b) < \widetilde{V}^b(\gamma, \underline{x}(\gamma, \ddot{b}), \ddot{b})$ depending on which of the two relative Pareto weight is the largest. The former case leads to the fact that $\widetilde{V}^b(\gamma, \underline{x}(\gamma, b), b) > \widetilde{V}^{af}(\gamma)$, while the latter case leads to $\widetilde{V}^b(\gamma, \underline{x}(\gamma, b), b) < \widetilde{V}^{af}(\gamma)$. Both cases contradict the fact that $\underline{x}(\gamma, b)$ is the relative Pareto weight for which the sovereign's constraint binds. It must therefore be that for all $\ddot{b} \neq b$, $\underline{x}(\gamma) = \underline{x}(\gamma, b) = \underline{x}(\gamma, \ddot{b})$. The same reasoning applies to the upper bound. □

Proof of Proposition 1

We conduct a proof by contradiction. The present proof only considers the economy in equilibrium. It might be that default occurs off equilibrium path. This situation is, however, outside the scope of the proposition. The proof follows the argument of Thomas and Worrall (1994) and Zhang (1997). The participation constraint of the borrower ensures that the value of the borrower is at most equal to its outside option. Hence, the borrower is at most indifferent between defaulting or not. □

Proof of Proposition B.3

Following Alvarez and Jermann (2000) we prove the proposition by construction. First, define the Fund's asset price as

$$q_f(\theta', x', b'|\theta) = \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', x'', b''|\theta')\right] \max\left\{\frac{u'(c(\theta', x', b'))}{u'(c(\theta, x, b))}\eta, 1\right\}.$$

Second, as shown in Lemma B.1, iterating over the budget constraint of the sovereign gives

$$\begin{aligned} &a(\theta^t) + b(\theta^t) = \\ &\mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j}))], \end{aligned} \tag{C.2}$$

where, $Y(\theta^t, x(\theta^t), b(\theta^t)) = \theta(\theta_t) f(n(\theta^t, x(\theta^t), b(\theta^t)))$ for all t and θ^t. Similarly, iterating over the consolidated budget constraint of the two lenders leads to

$$\begin{aligned} a_l(\theta^t) + \bar{a}_p(\theta^t) + b_l(\theta^t) &= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})|\theta^t) c_l(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})) \qquad \text{(C.3)} \\ &= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})|\theta^t)[Y(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})) \\ &\quad - c(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j}))] \\ &= -a(\theta^t) - b(\theta^t). \end{aligned}$$

The market clearing conditions in the Fund and the private bond market implies that $a_l(\theta^t) + \bar{a}_p(\theta^t) + a(\theta^t) = 0$ and $b(\theta^t) + b_l(\theta^t) = 0$, respectively, for all t and θ^t.

We now need to establish the correspondence between the initial conditions, (x_0, b_0), in the Fund contract and the initial conditions in the recursive competitive equilibrium, $(a_0, a_{l,0}, b_0, b_{l,0})$. Given (C.2) and (C.3) evaluated at $t = 0$, one can determine $\bar{a}(\theta_0, a_0, b_0)$ using the budget constraint

$$c(\theta_0, a_0, b_0) + q_f(\theta_0, \omega_1)(\bar{a}' - \delta a_0) + \sum_{\theta_1|\theta_0} q_f(\theta_1, \omega_1(\theta_1)|\theta_0)\hat{a}'(\theta_1) + q_p(\theta_0, \omega_1)(b' - \delta b_0)$$
$$\leq \theta_0 f(n) + (1 - \delta + \delta\kappa)(a_0 + b_0).$$

and the fact that $\sum_{\theta_1|\theta_0} q_f(\theta_1, \omega_1(\theta_1)|\theta_0)\hat{a}'(\theta_1) = 0$. Once, $\bar{a}(\theta_0, a_0, b_0)$ is determined, one can find the holdings of Arrow-type securities $\hat{a}'(\theta', \theta_0, a_0, b_0)$ for all $\theta' \in \Theta$. We can then retrieve the entire portfolio recursively for $t > 0$.

Third, define the endogenous borrowing limits such that

$$\mathcal{A}_b(\theta) = a(\theta, \underline{x}(\theta, b), b) + b(\theta, \underline{x}(\theta, b), b),$$
$$\mathcal{A}_f(\theta, b) = a_l(\theta, \overline{x}(\theta, b), b) + b_l(\theta, \overline{x}(\theta, b), b).$$

This definition implies that $a'(\theta', \theta, a, b) + b' \geq \mathcal{A}_b(\theta')$ and $a_l'(\theta', \theta, a, b) + b_l' \geq \mathcal{A}_f(\theta', b')$. Hence, the constructed asset holdings satisfy the competitive equilibrium constraints for both the lenders and the sovereign.

Fourth, defining $I(\theta, a, b)$ as the Lagrange multiplier attached to the sovereign's budget constraint, one ensures optimality of the policy functions by setting

$$I(\theta, a, b) = \frac{1 + \nu_l(\theta, x, b)}{1 + \nu_b(\theta, x, b)} \frac{1}{x}.$$

Hence, since $c(\theta, x, b)$ and $n(\theta, x, b)$ satisfy the optimality conditions in the Fund, $c(\theta, a, b)$ and $n(\theta, a, b)$ are also optimally determined in the competitive equilibrium. For the lenders, consumption is optimal if the asset portfolio is optimally determined. For this observe that

$$q_f(\theta', \omega'(\theta')|\theta) = \frac{1}{1+r}\pi(\theta'|\theta)\frac{u'(c(\theta', a'(\theta'), b'))}{u'(c(\theta, a, b))}\eta\left[(1 - \delta + \delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right]$$
$$\geq \frac{1}{1+r}\pi(\theta'|\theta)\left[(1 - \delta + \delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right]$$
$$\text{if } a'(\theta', \theta, a, b) + b' > \mathcal{A}_b(\theta'),$$
$$q_f(\theta', \omega'(\theta')|\theta) = \frac{1}{1+r}\pi(\theta'|\theta)\left[(1 - \delta + \delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right]$$
$$\geq \frac{1}{1+r}\pi(\theta'|\theta)\frac{u'(c(\theta', a'(\theta'), b'))}{u'(c(\theta, a, b))}\eta\left[(1 - \delta + \delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right]$$
$$\text{if } a_l'(\theta', \theta, a, b) + b_l' > \mathcal{A}_f(\theta', b')$$

Hence the portfolio is optimally determined. It then directly follows that $W^b(\theta, a, b) = V^b(\theta, x, b)$ and $W_p(\theta, a, b) + W_f(s, a, b) = V^l(s, x, b)$.

We therefore obtain a one-to-one map between (x, b) and $\omega = a + b$ for a given θ. More precisely, $B(\theta, x, b) = B(\theta, a, b)$, $c(\theta, a, b) = c(\theta, x, b)$, $c_p(\theta, a, b) = \tau_p(\theta, x, b)$, $c_f(\theta, a, b) = \tau_f(\theta, x, b)$, $c_p(\theta, a, b) + c_f(\theta, a, b) = \tau(\theta, x, b)$ and $n(\theta, a, b) = n(\theta, x, b)$. Moreover the endogenous limits of the sovereign and the lenders bind uniquely and exclusively when the participation constraints of the sovereign and the lenders bind, respectively. □

Proof of Proposition B.4

Following Alvarez and Jermann (2000) we prove the proposition by construction. As for the proof of Proposition B.3, one establishes a one-to-one mapping from (x, b) to $\omega = a+b$. The key equation linking those two objects is

$$I(\theta, a, b) = u'(c(\theta, a, b)) = u'(c(\theta, x, b)) = \frac{1 + \nu_l(\theta, x, b)}{1 + \nu_b(\theta, x, b)} \frac{1}{x},$$

where $I(\theta, a, b)$ is the Lagrange multiplier attached to the sovereign's budget constraint in the competitive problem. Given the initial bond holdings $(a_0, a_{l,0}, b_0, b_{l,0})$, the above condition enables to identify $x(\theta_0)$ if none of the enforcement constraint binds. We can subsequently determine consumption and labor.

For the participations constraints (13) and (14), one lets $\nu_b(\theta, x, b) = 0$ if $a'(\theta', \theta, a, b) + b' > \mathcal{A}_b(\theta', b')$ and $\nu_l(\theta, x, b) = 0$ if $a'_l(\theta', \theta, a, b) + b'_l > \mathcal{A}_f(\theta', b')$. Furthermore, given that the sovereign's and lenders' intertemporal budget constraints are satisfied, the resource feasibility constraints are also satisfied.

Note that autarky is ruled out as an equilibrium under high-implied interest rates. In other words. the high implied interest rate assumption ensures interiority of the solution. □

Proof of Proposition 2

We conduct a proof by construction. From Proposition 1, we have that

$$q_p(\theta, \omega') = \frac{\mathbb{E}\left[(1 - \delta + \delta\kappa + \delta q_p(\theta', \omega''))\middle|\theta\right]}{1 + r}.$$

This gives, for all (θ, ω'),

$$q_p(\theta, \omega') = \frac{1 - \delta + \delta\kappa}{1 + r - \delta}.$$

Regarding the Fund-provided assets, we distinguish three cases:

1. The sovereign's and the Fund' participation constraints are not binding. The lenders' Euler equation reads

$$q_f(\theta', \omega'|\theta) = \frac{\pi(\theta'|\theta)}{1 + r}\left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta')\right],$$

$$q_p(\theta,\omega') = \sum_{\theta'|\theta} \frac{\pi(\theta'|\theta)}{1+r}[(1-\delta+\delta\kappa) + \delta q_p(\theta',\omega'')],$$

and the sovereign's Euler equations are

$$q_f(\theta',\omega'|\theta) = \beta\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right]$$

$$q_p(\theta,\omega') = \beta\sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}[(1-\delta+\delta\kappa) + \delta q_p(\theta',\omega'')]$$

If none of the two constraints is ever binding,

$$\begin{aligned}
\sum_{\theta'|\theta} q_f(\theta',\omega'|\theta) &= \beta\sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right] \\
&= \sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{1}{1+r}\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right], \\
q_p(\theta,\omega') &= \beta\sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}[(1-\delta+\delta\kappa) + \delta q_f(\theta',\omega'')] \\
&= \sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{1}{1+r}[(1-\delta+\delta\kappa) + \delta q_p(\theta',\omega'')],
\end{aligned}$$

It then follows that $Q_p(\theta,\omega') = \sum_{\theta'|\theta} Q_f(\theta',\omega'|\theta)$.

2. The sovereign's participation constraint binds and the lenders' participation constraint is not binding.

   The lenders' Euler equation is

$$q_f(\theta',\omega'|\theta) = \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right],$$

$$q_p(\theta,\omega') = \sum_{\theta'|\theta}\frac{\pi(\theta'|\theta)}{1+r}[(1-\delta+\delta\kappa) + \delta q_p(\theta',\omega'')],$$

   and the sovereign's Euler equations are

$$q_f(\theta',\omega'|\theta)u'(c(\theta,\omega)) - \varphi_b(\theta') = \beta\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right],$$

$$q_p(\theta,\omega')u'(c(\theta,\omega)) - \sum_{\theta'|\theta}\varphi_b(\theta') = \beta\sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}[(1-\delta+\delta\kappa) + \delta q_p(\theta',\omega'')].$$

   If the lenders' participation constraint never binds,

$$\sum_{\theta'|\theta} q_f(\theta',\omega'|\theta) > \beta\sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right] \quad \text{and}$$

$$\sum_{\theta'|\theta} q_f(\theta', \omega'|\theta) = \sum_{\theta''|\theta'} \frac{\pi(\theta'|\theta)}{1+r} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta') \right],$$

Moreover, $Q_p(\theta, \omega') = \sum_{\theta'|\theta} Q_f(\theta', \omega'|\theta)$.

3. The sovereign's participation constraint is not binding and the Fund's participation constraint binds.

   The lenders' Euler equation reads

$$q_f(\theta', \omega'|\theta) - \varphi_f(\theta') = \frac{\pi(\theta'|\theta)}{1+r} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta') \right],$$

$$q_p(\theta, \omega') = \sum_{\theta'|\theta} \frac{\pi(\theta'|\theta)}{1+r} [(1 - \delta + \delta\kappa) + \delta q_p(\theta', \omega'')],$$

   The sovereign's Euler equations are

$$q_f(\theta', \omega'|\theta) = \beta\pi(\theta'|\theta) \frac{u'(c(\theta', \omega'))}{u'(c(\theta, \omega))} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta') \right],$$

$$q_p(\theta, \omega') = \beta \sum_{\theta'|\theta} \pi(\theta'|\theta) \frac{u'(c(\theta', \omega'))}{u'(c(\theta, \omega))} [(1 - \delta + \delta\kappa) + \delta q_p(\theta', \omega'')].$$

   If the sovereign's participation constraint never binds,

$$\sum_{\theta'|\theta} q_f(\theta', \omega'|\theta) = \beta \sum_{\theta'|\theta} \pi(\theta'|\theta) \frac{u'(c(\theta', \omega'))}{u'(c(\theta, \omega))} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta') \right] \quad \text{and}$$

$$\sum_{\theta'|\theta} q_f(\theta', \omega'|\theta) > \sum_{\theta''|\theta'} \frac{\pi(\theta'|\theta)}{1+r} [(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta')],$$

   However, $Q_p(\theta, \omega') < \sum_{\theta'|\theta} Q_f(\theta', \omega'|\theta)$ cannot be an equilibrium. At this price the private lender is willing to hold an infinite amount of debt in the Fund and provide an infinite amount of assets to the sovereign. To avoid this, it must be that $Q_p(\theta, \omega') = \sum_{\theta'|\theta} Q_f(\theta', \omega'|\theta)$.

Hence, in all possible states, $Q_p(\theta, \omega') = \sum_{\theta'|\theta} Q_f(\theta', \omega'|\theta)$. □

Proof of Proposition 3

When the Fund's constraint binds, $Q_f > \frac{1}{1+r}$. This means that it is cheaper to accumulate debt. However, at this price only the Fund is willing to provide debt. There is therefore no trade in the private bond market. However, the private lenders cannot force the sovereign to repay in advance. As a result, the private lenders can only refuse to roll-over the maturing portion of the debt — that is $b' > \delta b$.

Given this, it is clear that in the case of short term debt (i.e. $\delta = 0$), the binding constraint of the lenders directly translate to a complete shutdown of private lending. $\square$

Proof of Proposition 4

We conduct a proof by construction. When (20) does not bind, the budget constraint reads

$$c + q_p(\theta, \omega')(b' - \delta b) + \sum_{\theta'|\theta} q_f(\theta', \omega'|\theta)(a'(\theta') - \delta a) = \theta f(n) + (1 - \delta + \delta\kappa)(b + a).$$

Given that $\sum_{\theta'|\theta} q_f(\theta', \omega'|\theta)\hat{a}(\theta') = 0$ and Proposition 2, it can be rewritten as

$$c + q_f(\theta, \omega')(b' - \delta b) + q_f(\theta, \omega')(\bar{a}' - \delta\bar{a}) = \theta f(n) + (1 - \delta + \delta\kappa)(b + a),$$
$$c + q(\theta, \omega')(\bar{\omega}' - \delta(b + a)) = \theta f(n) + (1 - \delta + \delta\kappa)(b + a).$$

Having the same price and being equally accessible, private and Fund-provided bonds are prefect substitute, so that the decomposition of $\bar{\omega}'$ between b' and $\bar{a}'$ is indeterminate. $\square$

Proof of Corollary 1

We conduct a proof by construction. Setting $\bar{a}' = 0$ implies that the sovereign exclusively accumulates debt in the private bond market, resolving the indetermination. None of the debt is located in the Fund which solely provides insurance. However, it is not always possible to set $\bar{a}' = 0$ if one does not want the constraint $W_p(\theta', a'_l, b'_l) \geq b'_l$ to be violated. More precisely, the maximal level of debt the private lender can absorb is given by

$$b' = \min_{\theta' \in \ddot{\Theta}} \{\theta Z - W^l(\theta', \mathcal{A}_f(\theta'))\},$$

where $W^l = W_p + W_f$. Moreover, $\ddot{\Theta}$ designate the set of all θ' such that $\pi(\theta'|\theta) > 0$. Then define

$$\underline{\underline{a}}(\theta, b) = \bar{\omega}'(\theta, a, b) - \min_{\theta' \in \ddot{\Theta}} \{\theta Z - W^l(\theta', \mathcal{A}_f(\theta'))\},$$

as the minimal level of debt the Fund can absorb in a given state (θ, b). Such a threshold value exists given Propositions B.2 and B.3.

Obviously, $\underline{\underline{a}}(\theta, b) \leq 0$ as $\min_{\theta' \in \ddot{\Theta}} \{\theta Z - W^l(\theta', \mathcal{A}_f(\theta'))\} \geq \bar{\omega}'(\theta, a, b)$ by definition of the lenders' participation constraint. Furthermore, $\underline{\underline{a}}(\theta, b) \geq \delta b$ given Proposition 3. $\square$

Proof of Lemma B.1

Under Proposition 2, define

$$q(\theta^t, \omega(\theta^{t+1})) \equiv \sum_{s^{t+1}|\theta^t} q_f(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t) = q_p(\theta^t, \omega(\theta^t)),$$
$$Q(\theta^t, \omega(\theta^{t+1})) \equiv \sum_{s^{t+1}|\theta t} Q_f(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t) = Q_p(\theta^t, \omega(\theta^t)),$$

for all t and θ^t. Furthermore, the transversality condition of the borrower is:[38]

$$\lim_{j\to\infty} \mathbb{E}_t Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[a(\theta^{t+j}) + b(\theta^{t+j})] = 0,$$

where

$$Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t) = Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^{t+j-1}) \cdots Q(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t).$$

Using the borrower's budget constraint and the price relationship, one gets

$$(a(\theta^t) + b(\theta^t))(1 - \delta + \delta\kappa + \delta q(\theta^t, \omega(\theta^{t+1}))) =$$
$$c(\theta^t, a(\theta^t), b(\theta^t)) + q(\theta^t, \omega(\theta^{t+1}))a(\theta^{t+1}) + q(\theta^t, \omega(\theta^{t+1}))b(\theta^{t+1}) - Y(\theta^t, a(\theta^t), b(\theta^t)),$$

where, $Y(\theta^t, a(\theta^t), b(\theta^t)) = \theta(\theta_t) f(n(\theta^t, a(\theta^t), b(\theta^t)))$ for all t and θ^t. Iterating forward the budget constraint and using the transversality condition as well as the equilibrium price relationship, one obtains

$$a(\theta^t) + b(\theta^t) =$$
$$\mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))].$$

Similarly, the transversality condition of the lender is:

$$\lim_{t\to\infty} \mathbb{E}_t Q(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t)[a_l(\theta^{t+1}) + b_l(\theta^{t+1})] = 0.$$

Using the consolidated budget constraint of both lenders, one gets

$$(a_l(\theta^t) + b_l(\theta^t))(1 - \delta + \delta\kappa + \delta q(\theta^t, \omega(\theta^{t+1}))) =$$
$$c_l(\theta^t, a(\theta^t), b(\theta^t)) + q(\theta^t, \omega(\theta^{t+1}))a_l(\theta^{t+1}) + q(\theta^t, \omega(\theta^{t+1}))b_l(\theta^{t+1}).$$

[38] The differentiability and strict concavity and convexity assumptions of the functional forms guarantee the local uniqueness of the policy and value functions. This in turn implies that the transversality conditions are satisfied.

Iterating forward the budget constraint and using the transversality condition as well as the equilibrium price relationship, one obtains

$$a_l(\theta^t) + \bar{a}_p(\theta^t) + b_l(\theta^t) = \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t) c_l(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))$$
$$= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))] = a(\theta^t$$

The market clearing conditions in the Fund and the private bond market implies that $a_l(\theta^t) + \bar{a}_p(\theta^t) + a(\theta^t) = 0$ and $b(\theta^t) + b_l(\theta^t) = 0$, respectively, for all t and θ^t.

If the participation constraint of one of the contracting parties is binding, the borrowing limit for of the constrained agent in the decentralised economy is determined by

$$\mathcal{A}_b(\theta^t) = \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))], \quad \text{(C.4)}$$

$$\mathcal{A}_l(\theta^n) = \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{n+j}, \omega(\theta^{n+j})|\theta^n) c_l(\theta^{n+j}, a(\theta^{n+j}), b(\theta^{n+j}))), \quad \text{(C.5)}$$

Further note that one distinguishes between t and n with $t \neq n$ as the sovereign's and the lenders' constraints cannot bind at the same time if the contract is feasible. □

Proof of Proposition 5

Given the definitions of the sovereign's endogenous borrowing limits, it holds that for all θ and for all level of private debt b within the Fund's prescription $\bar{\omega}$, $V^b(\theta, \mathcal{A}_b(\theta), b) = V^{ap}(\theta, \mathcal{A}_b^{ap}(\theta)) = V^{af}(\theta)$. There is therefore no *partial* default incentive when the borrower's constraint binds and $b \geq \bar{\omega}$.

Turning now to the case in which the Fund's participation constraint binds, assume there exists a level of private debt $\bar{\bar{b}} \geq \bar{\omega}'$ such that for all θ

$$V^f(\theta, \mathcal{A}_f(\theta, \bar{\bar{b}}_l) - \bar{\bar{b}}_l, \bar{\bar{b}}_l, 0) = V^f(\theta, \mathcal{A}_f(\theta, 0, 1), 0, 1),$$

which implies that for all $b < \bar{\bar{b}}$

$$V^f(\theta, \mathcal{A}_f(\theta, b) - b_l, b_l, 0) > V^f(\theta, \mathcal{A}_f(\theta, 0, 1), 0, 1),$$
$$W^b(\theta, -\mathcal{A}_f(\theta, \bar{\bar{b}}) - \bar{\bar{b}}, \bar{\bar{b}}) < V^{ap}(\theta, -\mathcal{A}_f(\theta, 0, 1)).$$

In that situation, the sovereign will gain from repudiating its private debt when the lender's participation constraint binds with $b < \bar{\bar{b}} < 0$. The private lenders anticipate this behavior.

They impose a risk premium for all $b' < \bar{\bar{b}}$ whenever the lender's participation constraint binds with strictly positive probability in the next period. Even if the risk premium might be relatively small, this directly reduces the amount of debt the sovereign can raise from the private lenders. Under the assumption that the sovereign desires to accumulate no more debt than the Fund can provide, the sovereign does not accumulate more than $-\bar{\bar{b}}$ in the private bond market to avoid this risk premium and simply accumulates more debt in the Fund. This in turn implies that *partial* defaults never occur on equilibrium path as the sovereign never accumulates a sufficient level of private debt in the states in which *partial* defaults would be attractive. Conversely if for all θ and $b \geq \bar{\omega}$

$$V^f(\theta, \mathcal{A}_f(\theta, b) - b, b, 0) < V^f(\theta, \mathcal{A}_f(\theta, 0, 1), 0, 1),$$

then, there is no advantage in entering in *partial* default. □

Proof of Proposition 6

In light of Proposition 5, the sovereign will enter in *partial* default only if it overborrowed beforehand. In what follows, one refers to the decentralised Fund contract as it enables a better exposition of the argument. Let's focus first on the sovereign's participation constraint and consider that there are three productivity states in the economy. Assume further that for a given Fund's lending policy $\bar{\omega}' = \bar{a}' + b'$,

$$\begin{aligned} \bar{a}' + b' + \hat{a}'(1) > \mathcal{A}_b(1) \quad &\text{and} \quad \bar{a}_l' + b_l' + \hat{a}_l'(1) = \mathcal{A}_f(1, b_l'), \\ \bar{a}' + b' + \hat{a}'(2) > \mathcal{A}_b(2) \quad &\text{and} \quad \bar{a}_l' + b_l' + \hat{a}_l'(2) > \mathcal{A}_f(2, b_l'), \\ \bar{a}' + b' + \hat{a}'(3) = \mathcal{A}_b(3) \quad &\text{and} \quad \bar{a}_l' + b_l' + \hat{a}_l'(3) > \mathcal{A}_f(3, b_l'). \end{aligned}$$

The borrower decides to overborrow the amount $\delta a + \grave{b}' < \bar{a}' + b'$ with $\delta a \geq \bar{a}'$ and $\grave{b}' < b'$. If it keeps the same level of insurance, it gets

$$\begin{aligned} \delta a + \grave{b}' + \hat{a}'(1) \geq \mathcal{A}_b(1) \quad &\text{and} \quad \delta a_l + \grave{b}_l' + \hat{a}_l'(1) > \mathcal{A}_f(1, b_l'), \\ \delta a + \grave{b}' + \hat{a}'(2) \geq \mathcal{A}_b(2) \quad &\text{and} \quad \delta a_l + \grave{b}_l' + \hat{a}_l'(2) > \mathcal{A}_f(2, b_l'), \\ \delta a + \grave{b}' + \hat{a}'(3) < \mathcal{A}_b(3) \quad &\text{and} \quad \delta a_l + \grave{b}_l' + \hat{a}_l'(3) > \mathcal{A}_f(3, b_l'). \end{aligned}$$

If the borrower decides to default on its private debt, it gets

$$\begin{aligned} \delta a + \hat{a}'(1) > \bar{a}' + b' + \hat{a}'(1) > \mathcal{A}_b(1) = \mathcal{A}_b^{ap}(1) \quad &\text{and} \quad \delta a_l + \hat{a}_l'(1) \geq \mathcal{A}_f^{ap}(1), \\ \delta a + \hat{a}'(2) > \bar{a}' + b' + \hat{a}'(2) > \mathcal{A}_b(2) = \mathcal{A}_b^{ap}(2) \quad &\text{and} \quad \delta a_l + \hat{a}_l'(2) > \mathcal{A}_f^{ap}(2), \end{aligned}$$

$$\delta a + \hat{a}'(3) > \bar{a}' + b' + \hat{a}'(3) = \mathcal{A}_b(3) = \mathcal{A}_b^{ap}(3) \quad \text{and} \quad \delta a_l + \hat{a}_l'(3) > \mathcal{A}_f^{ap}(3),$$

which is clearly a better option than repaying the private debt. Thus, with this level of insurance, the borrower will default in all states. In other words, the default decision is not state contingent. Instead, the borrower can decide to reshuffle the insurance such that $\delta a + \grave{b}' + \grave{\hat{a}}'(3) = \mathcal{A}_b(3)$, meaning that the borrower would not default in the third state. For that purpose, the Arrow-type securities become

$$\grave{\hat{a}}'(3) = \hat{a}'(3) - [(\delta a + \grave{b}') - (\bar{a}' + b')] \equiv \hat{a}'(3) - \Delta,$$

and for all $i \in \{1, 2\}$ and a given $\theta \in \{1, 2, 3\}$,

$$\grave{\hat{a}}'(i) = \hat{a}'(i) + \Delta \frac{\pi(3|\theta)}{\sum_{j=1}^{2} \pi(j|\theta)} < \hat{a}'(i).$$

Basically, the borrower takes more insurance in the third state and less in the other two states. Notice that in the states in which the borrower takes less insurance, one has a double burden: more debt and less insurance. Now the question is: can the Fund sustain this reshuffle of Arrow-type securities? To answer that question, define $\ddot{\hat{a}}_l'(3)$ such that $\delta a_l + \ddot{\hat{a}}_l'(3) = \mathcal{A}_f^{ap}(3)$. In words, $\ddot{\hat{a}}'(3)$ represents the highest level of insurance the Fund can provide in state 3. Given this definition, one gets that

$$\delta a_l + \hat{a}_l'(3) \geq \delta a_l + \ddot{\hat{a}}_l'(3) = \mathcal{A}_f^{ap}(3),$$

leading to $\hat{a}_l'(3) \geq \ddot{\hat{a}}_l'(3)$. Using the definition of $\grave{\hat{a}}_l'(3)$,

$$\begin{aligned} \delta a_l + \grave{\hat{a}}_l'(3) &= \delta a_l + \hat{a}_l'(3) - [(\delta a_l + \grave{b}_l') - (\bar{a}_l' + b_l')] \\ &= \bar{a}_l' + \hat{a}_l'(3) - (\grave{b}_l' - b_l') \\ &\geq \delta a_l + \ddot{\hat{a}}_l'(3) - (\grave{b}_l' - b_l'), \end{aligned}$$

where the inequality comes from the fact that $\hat{a}_l'(3) \geq \ddot{\hat{a}}_l'(3)$ and $\bar{a}_l' \geq \delta a_l$. Rearranging the expression leads to $(\grave{b}_l' - b_l') \geq \ddot{\hat{a}}_l'(3) - \grave{\hat{a}}_l'(3)$. As one assumed that $\grave{b}_l' > b_l'$, for the above inequality to hold it must be that $\grave{\hat{a}}_l'(3) < \ddot{\hat{a}}_l'(3)$. This in turn implies that $\delta a_l + \grave{\hat{a}}_l'(3) < \mathcal{A}_f^{ap}(3)$. The Fund will therefore not accept this reshuffle as its participation constraint is violated in the third state if the borrower defaults on its private debt. Moreover, notice that if the reshuffling of Arrow-type securities is such that for at least one of the two states $i \in \{1, 2\}$, $\delta a + \grave{\hat{a}}'(i) < \mathcal{A}_b^{ap}(i)$, then it is not optimal for the borrower to perform the reshuffling of Arrow-type securities. Hence, the mechanism is the following. The sovereign cannot reshuffle because it is either not optimal for itself (as it would loose too much if the third state does not realize) or because the Fund refuses this reshuffle (as it would violate its constraint). As a result, being unable to insure its overaccumulation of debt, the borrower will partially default in all future states as soon as it accumulates more debt than what the Fund prescribes.

The previous case was focusing on the sovereign's participation constraint. We now pass to the states in which the Fund's participation constraint binds. As before consider there exists a level of private debt $\bar{\bar{b}}' \geq \bar{\omega}'$ such that $\mathcal{A}_f(\theta', \bar{\bar{b}}') = \mathcal{A}_f^{ap}(\theta')$. It then holds for all $\grave{b}' < \bar{\omega}' \leq \bar{\bar{b}}'$ and for all θ', $\mathcal{A}_f(\theta', \grave{b}') > \mathcal{A}_f^{ap}(\theta')$. If this is not the case, this means that there is an arbitrage opportunity. Consider that among all θ', there exists a single $\ddot{\theta}'$ for which

$$\begin{aligned} \hat{a}_l'(\theta') + \bar{a}' &= \mathcal{A}_f^{ap}(\theta') < \mathcal{A}_f(\theta', \grave{b}') \quad \forall \theta' \in \Theta \setminus \ddot{\theta}', \\ \hat{a}_l'(\ddot{\theta}') + \bar{a}' &= \mathcal{A}_f^{ap}(\ddot{\theta}') = \mathcal{A}_f(\ddot{\theta}', \grave{b}'). \end{aligned}$$

The Fund can then reshuffle the Arrow-type securities. More precisely, it can sufficiently increase $\hat{a}_l'(s')$ by $\epsilon > 0$ such that $\hat{a}_l'(s') + \bar{a}' + \epsilon < \mathcal{A}_f(\grave{s}', \grave{b}')$ for all $\theta' \in S \setminus \ddot{\theta}'$. Given this increase, it can now slightly decrease $\hat{a}_l'(\ddot{\theta}')$. As a result,

$$\begin{aligned} \hat{a}_l'(s') + \bar{a}' + \epsilon &< \mathcal{A}_f(s', \grave{b}') \quad \forall \theta' \in \Theta \setminus \ddot{\theta}', \\ \hat{a}_l'(\ddot{\theta}') + \bar{a}' - \frac{\sum_{\theta' \in S \setminus \ddot{\theta}'} \pi(\theta'|\theta)\epsilon}{\pi(\ddot{\theta}'|\theta)} &< \mathcal{A}_f(\ddot{\theta}', \grave{b}'), \end{aligned}$$

contradicting our initial assumption. To complete the argument, note that the reshuffling is such that

$$\sum_{\theta' \in \Theta \setminus \ddot{\theta}'} \pi(\theta'|\theta)(\hat{a}_l'(\theta') + \epsilon) + \pi(\ddot{\theta}'|\theta)\left(\hat{a}_l'(\ddot{\theta}') - \frac{\sum_{\theta' \in S \setminus \ddot{\theta}'} \pi(\theta'|\theta)\epsilon}{\pi(\ddot{\theta}'|\theta)}\right) = \sum_{\theta'|\theta} \pi(\theta'|\theta)\hat{a}_l'(\theta').$$

Given this, one has that for all $\grave{b}' < \bar{\omega}' \leq \bar{\bar{b}}$ and for all $\theta' \in \Theta$,

$$W^b(\theta', -\mathcal{A}_f(\theta', \grave{b}') - \grave{b}', \grave{b}') < V^{ap}(\theta', -\mathcal{A}_f^{ap}(\theta')).$$

In words, as soon as the sovereign overborrows and the Fund's participation constraint binds, it will enter in *partial* default. Now, if for all $b' \geq \bar{\omega}'$ and for all $\theta' \in S$, $\mathcal{A}_f(\theta', b') < \mathcal{A}_f^{ap}(\theta')$, then the sovereign simply never overborrows. □

Proof of Corollary 2

Given Proposition 5, it holds that for all θ, and a and b such that $a + b \geq \bar{\omega}$, $D_p(\theta, a, b) = 0$, and under Proposition 1, $D_f(\theta, a, b) = 0$. Moreover, given Proposition 6, for all θ and for all

a and b such that $a + b < \bar{\omega}$, $D_p(\theta, a, b) = 1$ and $D_f(\theta, a, b) = 0$, which implies that for all θ and for all $\omega' < \bar{\omega}'$, $q_p(\theta, \omega') = 0$. □

D Additional Details of the Calibration

D.1 Data Sources and Measurement

We calibrate the model for Italy. The main data sources and definitions of data variables are listed in Table D.1. The data frequency is quarterly, and the time periods are from 1992Q1 to 2019Q4, avoiding the interruption caused by COVID-19. Whenever the data souces contain the seasonally adjusted series for the relevant data variables, we use the them directly; otherwise, we seasonally adjust the data series using X11 algorithm with R package `seasonal`. For debt service and average maturity, we use annual series since quarterly ones are unavailable meanwhile we only need the sample avearge for our calibration.

To map the data to the model, we construct model consistent data measures as below.

Labor input For the aggregate labor input n_t, we use two series, the aggregate working hours H_t and the total employment E_t. We calculate the normalized labor input as $n_t = H_t/(E_t \times 5200)$, assuming 100 hours of allocatable time per worker per week. However, for second order data moment computations, we use H_t directly, since the per worker annual working hours do not show a significant cyclical pattern and both the level and the trend do not affect the computation of the moments.

Fiscal position and private consumption We hold the premise of fitting the *observed* fiscal behavior of Italy, so that we use directly the *data measures* of primary surplus to calibrate the model, and correspondingly, define the model consistent measure of consumption as the difference between output and primary surplus, since in the model, primary surplus ps is equal to output y minus consumption c. We have raw data on quarterly fiscal *surplus* instead of *primary surplus*. To arrive the latter from the former, we add back interest payment of the government to fiscal surplus. To be more precise, we first calculate fiscal suplus to GDP ratio (nominal quarterly GDP obtained from CEIC for Italy). Second, we obtain quarterly interest payment to GDP ratio from Eurostat (label `gov_10q_ggnfa`) for 1999Q1 onwards, and use the end-of-year annual value (obtained from AMECO and European Commission *General Government Data*) for each quarter in the year as a proxy for 1992Q1–1998Q4. Third, we add fiscal surplus to GDP and interest payment to GDP to arrive at primary surplus to GDP, and conduct seasonal adjustment to the series. And finally, we obtain the level of quarterly (*real*) primary surplus by multiplying the seasonally

Table D.1: Data Sources and Definitions

Series	Sources	Unit
Output	ECB[a]	1 million 2010 constant euro
Total working hours	ECB[b]	1 thousand hours
Employment	Eurostat[c]	1000 persons
Government debt	Eurostat[d]	end-of-quarter percentage
Debt service	AMECO[e]	end-of-year percentage of GDP, annual
Fiscal surplus	Eurostat, Bank of Italy[f]	million euro
Long-term bond yields	Eurostat[g]	percentage, nominal
Debt maturity	OECD, EuroStat, ESM[h]	years, annual
Labor share	AMECO[i]	percentage, annual

[a] Real GDP, chain linked volume; data in 1991Q1–2014Q2 under ESA95, and data in 2014Q3–2019Q4 under ESA10, with the latter series adjusted to match the former in the overlapping periods 1995Q1–2014Q2.

[b] Hours for total employment; same adjustment to data under ESA95 and ESA10 as for output.

[c] Total employment (Eurostat label `lfsi_emp_q_h`).

[d] General government consolidated gross debt (Eurostat label `gov_10q_ggdebt`); quarterly series available for 2000Q1 onwards, and for 1992Q1-1999Q4, interpolate annual series instead; measured as end-of-quarter debt stock to total GDP of previous 4 quarters.

[e] AMECO (label `UYIGE`) for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1992–1995.

[f] Eurostat (net lending, label `gov_10q_ggnfa`) 1999Q1–2019Q4; Bank of Italy (financing of the gross borrowing requirement, including privatization receipts) 1992Q1–1998Q4.

[g] EMU convergence criterion bond yields (label `irt_lt_mcby_q`).

[h] See text below; ESM data are obtained from private correspondance.

[i] Compensation of employees (UWCD) plus gross operating surplus (UOGD) minus gross operating surplus adjusted for imputed compensation of self-employed (UQGD), then divided by nominal GDP (UVGD).

adjusted primary surplus to GDP ratio to (*real*) output in the same quarter.

Government debt, spread, and maturity Following Bocola et al. (2019) and Ábrahám et al. (2021), we calibrate the model to match the total public debt of Italy.

For the nominal risk free rate, we use the annualized short-term (3M) interest rates in the Euro money market (obtaied from EuroStat with label `irt_st_q`) for 1999Q1–2019Q4, and the annulized short-term (3M) bond return of Germany (obtained from EuroStat with label `irt_h_mr3_q`) for 1992Q1–1998Q4, before the start of Euro. To convert the nominal risk-free rate into real rate, we subtract GDP deflator of Germany from the former series. To arrive at a meaningful measure of the *real* spread, i.e., a spread unaffected by expected inflation hence rightly reflecting credit risk, we split the sample into to two parts. After the introduction of Euro, we can directly use the spread between the long-term nominal bond yields and the nominal risk-free rate, since all rates are denominated in euro and thus subject to the same inflation expectation. For the period before Euro, we follow Ábrahám et al. (2021) and use spot and forward exchange rates (retrieved from Datastream) to convert the German nominal risk free rate into Italy's local currency, hence deriving a synthetic local currency risk free rate, and finally take the difference between the local nominal long-term bond yield with the synthetic risk free rate.

The information on the maturity structure of the government debt for Italy is not comprehensive. We manage to obtain government debt maturity data over 1990–2015 for Italy from all sources listed in Table D.1.

D.2 Estimation Results

Panel (a) of Figure D.1 plots the sample productivity series for Italy used for our calibration of the productivity shock process. It is clear that the during the 2008 Global Financial Crisis, there was prominent negative growth in productivities. This distinctive feature in the productivity dyanmics is also the main motivation for the use of Markov regime switching model (24) to calibrate the productivity shock. Correspondingly, Panel (b) shows that a 2-regime specification capture the crisis dynamics very well, with the smoothed regime probabilities reach almost 1 during the sudden drop periods observed in Panel (a).

The final estimation results are summarized in Table D.2. Note that we identify regime 1 as the crisis regime, and regime 2 as the normal regime. To overcome the local maximum problem of the highly nonlinear likelihood function, we randomize initializations of the EM algorithm of 1,000 times.

Table D.2: Parameters of the regime switching productivity process

	$\mu(\varsigma)$	$\rho(\varsigma)$	$\sigma(\varsigma)$	P	$\varsigma' = 1$	$\varsigma' = 2$	invariant dist.
$\varsigma = 1$	-0.0336	0.9018	0.0009	$\varsigma = 1$	0.6633	0.3367	0.0372
$\varsigma = 2$	0.0009	0.2167	0.0020	$\varsigma = 2$	0.0130	0.9870	0.9628

Notes: ς denotes the current regime of productivity shock, and ς' denotes that of the next period.

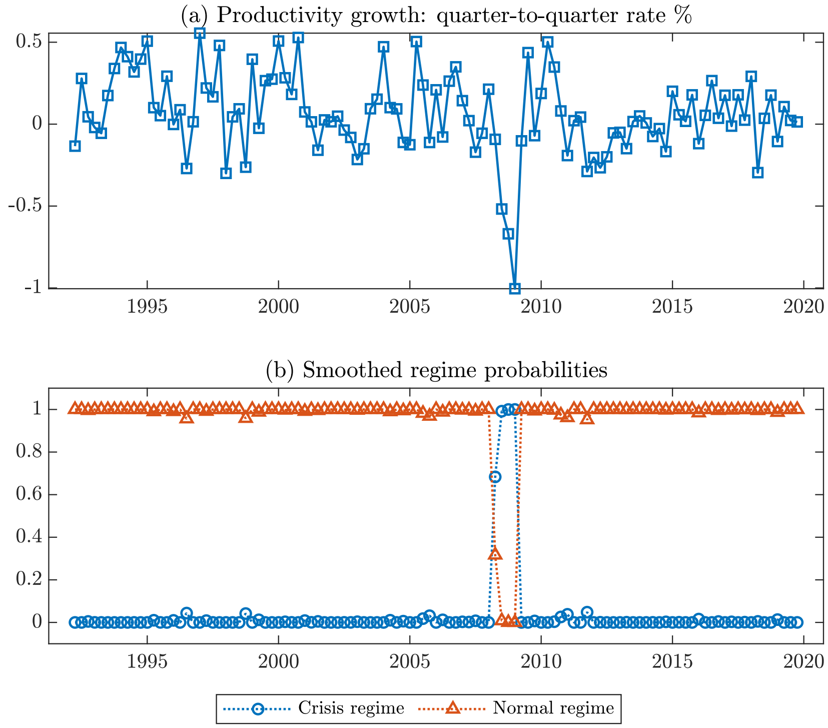

Figure D.1: Data sample and the estimated smoothed regime probabilities

E Welfare Calculations

This section describes how the welfare gains depicted in Table 3 are computed. Similar to Ábrahám et al. (2021), define value of the sovereign for a sequence $\{c(\theta^t), n(\theta^t)\}$ starting from an initial state at $t = 0$ as

$$V^b(\{c(\theta^t), n(\theta^t)\}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c(\theta^t), n(\theta^t)) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log(c(\theta^t)) + \gamma \frac{(1 - n(\theta^t))^{\sigma_n} - 1}{1 - \sigma_n} \right],$$

where the last equality is obtained from the functional form considered in Section 5. We denote the sovereign's allocations with the Fund by $\{c^f(\theta^t), n^f(\theta^t)\}$ and the allocations without the Fund by $\{c^i(\theta^t), n^i(\theta^t)\}$. The value for the borrower with and without the Fund is given by $W^{bf}(\theta, \omega) = W^{bf}(\{c^f(\theta^t), n^f(\theta^t)\})$ and $V^{bi}(\theta, b) = V^{bi}(\{c(\theta^t), n(\theta^t)\})$, respectively. To properly compare the two economies, we consider the point where $\omega = b =: o$. Thus (θ, o) represents the initial state for both economies. Now define $V^{bi}(\theta, o; \chi) = V^{bi}(\{(1+\chi)c(\theta^t), n(\theta^t)\})$, where $\chi(\theta, o)$ represents the consumption-equivalent welfare gain of the Fund's intervention. It then directly follows that the welfare gain is computed in the following way $V^{bi}(\theta, o; \chi) = W^{bf}(\theta, o)$. Given the above functional form, we have that $\frac{\log(1+\chi)}{1-\beta} + V^{bi}(\theta, o) = V^{bf}(\theta, o)$. The welfare gain therefore boils down to $\chi(\theta, o) = \exp\left[(V^{bf}(\theta, o) - V^{bi}(\theta, o))(1 - \beta)\right] - 1$.

We concentrate our analysis to the case in which $o = 0$.

Welfare decomposition

Following Ábrahám et al. (2021), we can decompose the welfare gains into four main components. As the Fund avoids default, it avoids the output penalty and the market exclusions. Those are the first two sources of welfare gains. In addition, as one can see from the two last columns of Table 3, the Fund enlarges the debt capacity of the sovereign. Finally, the Fund provides state-contingent transfer, whereas the economy without the Fund only has access to non-contingent bonds. Table E.3, presents the decomposition of the welfare gains for each of the depicted growth states and zero initial debt. As one can see, the main source of welfare gains is the larger debt capacity followed by the state contingency and the circumvention of output penalty. Note that debt capacity and state contingency are closely linked one another. Without state-contingent transfers, the sovereign could not sustain a larger indebtedness.

Table E.3: Welfare Decomposition at Zero Initial Debt

State	No penalty (%)	Immediate return to market (%)	Greater debt capacity (%)	State-contingent insurance (%)
$\gamma = \gamma_{min}$	8.49	2.62	80.02	8.87
$\gamma = \gamma_{med}$	8.79	2.33	81.66	7.22
$\gamma = \gamma_{max}$	8.41	1.88	78.49	11.22

F Interest Rate-Growth Differential

Given the importance of the interest rate-growth differential highlighted in our study, we add to the benchmark model a shock to the risk-free rate r. This enables an analysis of the insurance component related to the direct change in r^p and γ. We consider a two-state Markov process for the risk-free rate. More precisely, $r \in \{r_H, r_L\}$ with probability $\pi_r(r|r_-)$. We set $r_H = 0.0132$ as in the benchmark calibration and $r_L = 10^{-4}$ with $\pi_r(r_H|r_H) = 0.995$ and $\pi_r(r_L|r_L) = 0.985$.

The stochastic risk-free rate directly affect the bond price — and therefore r^p — as the lender discount the future differently. When r reduces, q^p increases as the lender gives less importance to future outcomes. In what follows we analyze the main difference between the economy with and without the Fund in steady state.

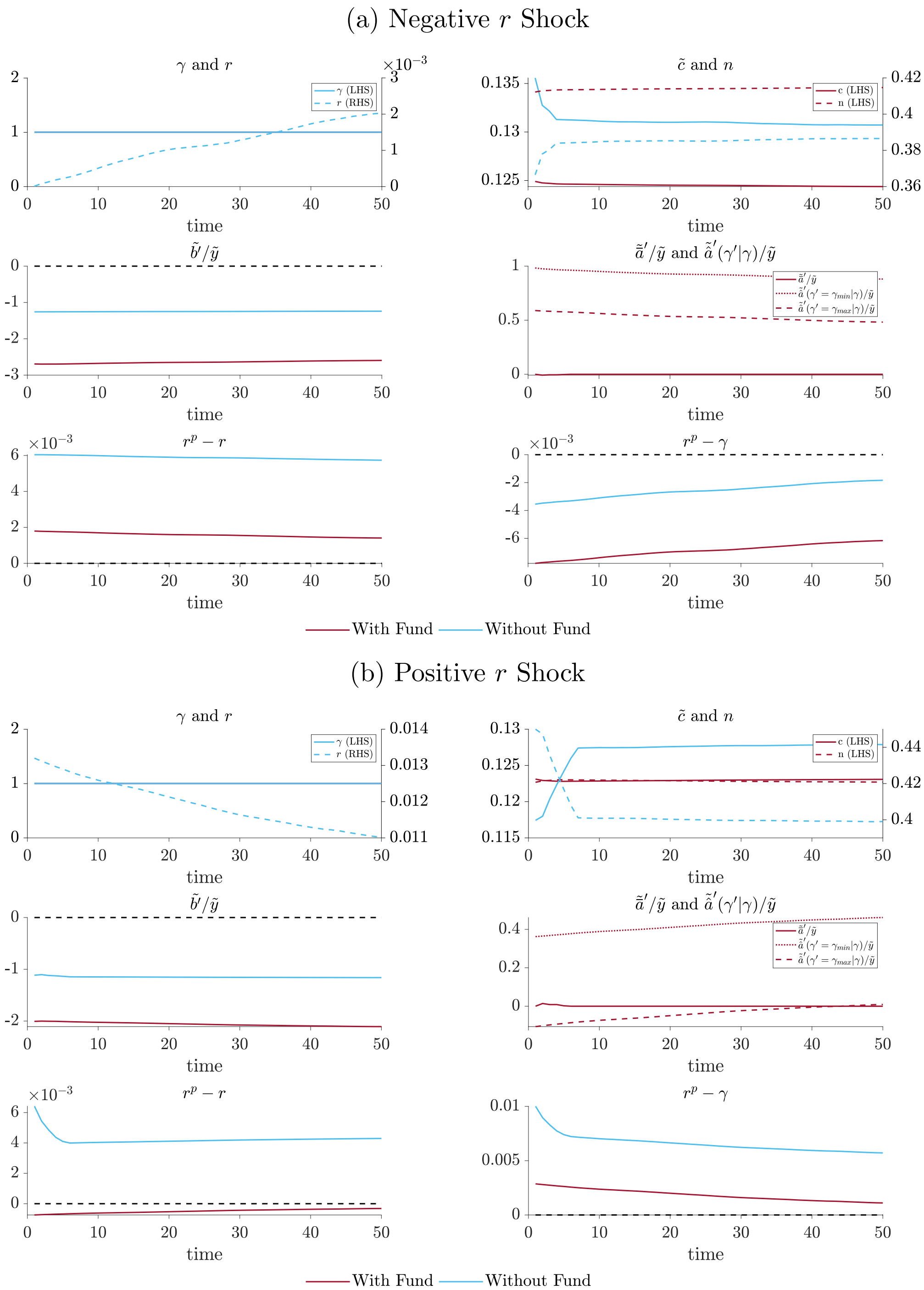

Figure F.2: Impulse Response Functions

Figure F.2 depicts the impulse response function following a negative and positive interest rate shock.[39] The construction of the impulse responses follows the exact same step as

[39]Figures G.13 and G.14 in Appendix G present the impulse response function for all relevant variables.

highlighted previously. As one can see, the negative r shock reduces consumption in the economy without the Fund. At a lower r, the price of debt is larger enabling a greater consumption per unit of issued debt. The effect is however very short-lived. Moreover, consumption in the economy with the Fund moves very little. One observes a slight increase in the debt held in the Fund as the lender's participation constraint might bind in some states. The opposite happens in the case of a positive interest rate shock. In the economy without the Fund, consumption is reduced as the price of debt is low. However, it quickly recovers to its steady state level. Again, the level of consumption remains very stable in the economy with the Fund. This avoids the large shift in the economy without the Fund.

Table F.4: Welfare Comparison at Zero Initial Debt

State	Welfare Gains (%)	Maximal Debt Absoption (% of GDP)	
		With Fund	Without Fund
$(\gamma, r) = (\gamma_{min}, r_H)$	7.17	398	177
$(\gamma, r) = (\gamma_{med}, r_H)$	6.53	195	111
$(\gamma, r) = (\gamma_{max}, r_H)$	6.76	198	113
$(\gamma, r) = (\gamma_{min}, r_L)$	22.54	588	224
$(\gamma, r) = (\gamma_{med}, r_L)$	21.31	275	124
$(\gamma, r) = (\gamma_{max}, r_L)$	21.44	277	127
Average	10.28		

Table F.4 presents the welfare gains in consumption equivalent between the economy with and without the Fund. The welfare computation is the same as in section 6.3 and is exposed in Appendix E. Again, welfare gains are important. This is due to the large jumps in consumption and labor that the stochastic r generates in the economy without the Fund. Thus, even though consumption can be larger and labor can be lower in the economy without the Fund, jumps in those variables are very costly in terms of consumption smoothing. One sees that welfare gains are the highest when the risk-free rate is low. This is because debt is much cheaper to accumulate in the Fund in this situation.

Table F.5 depicts the decomposition of welfare gains. As before, most of the welfare gains are concentrated towards the greater debt capacity. The state contingency is also at the source of a large part of the welfare gains especially when the risk-free rate is low. This should not come as a surprise. As we noted in Figure F.2, consumption largely oscillates in the economy without the Fund in such case.

Table F.5: Welfare Decomposition at Zero Initial Debt

State	No penalty (%)	Immediate return to market (%)	Greater debt capacity (%)	State-contingent insurance (%)
$(\gamma, r) = (\gamma_{min}, r_H)$	6.35	3.33	75.08	15.23
$(\gamma, r) = (\gamma_{med}, r_H)$	6.38	3.28	76.34	14.00
$(\gamma, r) = (\gamma_{max}, r_H)$	6.29	3.36	76.51	13.84
$(\gamma, r) = (\gamma_{min}, r_L)$	4.42	2.33	59.64	33.60
$(\gamma, r) = (\gamma_{med}, r_L)$	4.32	2.30	60.80	32.57
$(\gamma, r) = (\gamma_{max}, r_L)$	4.32	2.31	60.97	32.39

G Additional Tables and Figures

The relative Pareto weight is the key to the dynamics of the model economy. Figure G.3 displays its law of motion. The dark grey region represents the ergodic set given in Definition 1. It is delimited by a lower bound of $\underline{\tilde{x}} = 0.09$ and an upper bound of $\tilde{\bar{x}} = 0.145$. The light grey region represents the basin of attraction of the ergodic set. As one can clearly see the upper and lower bounds of the set do not coincide. Thus, we are in the case of an imperfect risk sharing steady state. As noted earlier, the line characterizing the first best in our economy is below the 45° line as the sovereign is relatively more impatient than the lenders. This means that whenever none of the constraints is binding, the relative Pareto weight decreases. It continues to do so until it hits the value at which the sovereign's participation constraint is binding avoiding immiseration. This is different than the case of equally patient agents where the relative Pareto weight remains constant when none of the constraints is binding.

We can illustrate the movement of the relative Pareto weights in the ergodic set with the following example. Suppose we start in the ergodic set on the first best line of the median shock (red non-horizontal dots) with a relative Pareto weight of say $\underline{\tilde{x}} = 0.13$ and $\tilde{b} = 0$. There, neither of the two participation constraints binds. If the economy remains in this state with that level of private debt, the relative Pareto weight decreases until it reaches the sovereign's binding constraint at around $\underline{\tilde{x}} = 0.12$. At this point, consider that the economy moves to the highest growth state. There, the value $\underline{\tilde{x}} = 0.12$ is now too low for the sovereign — its participation constraint therefore binds irrespective of its indebtedness level. The Planner will then increase the relative weight and set it to the minimum level to make the sovereign indifferent between reneging the contract or not — that is $\underline{\tilde{x}} = 0.145$. As

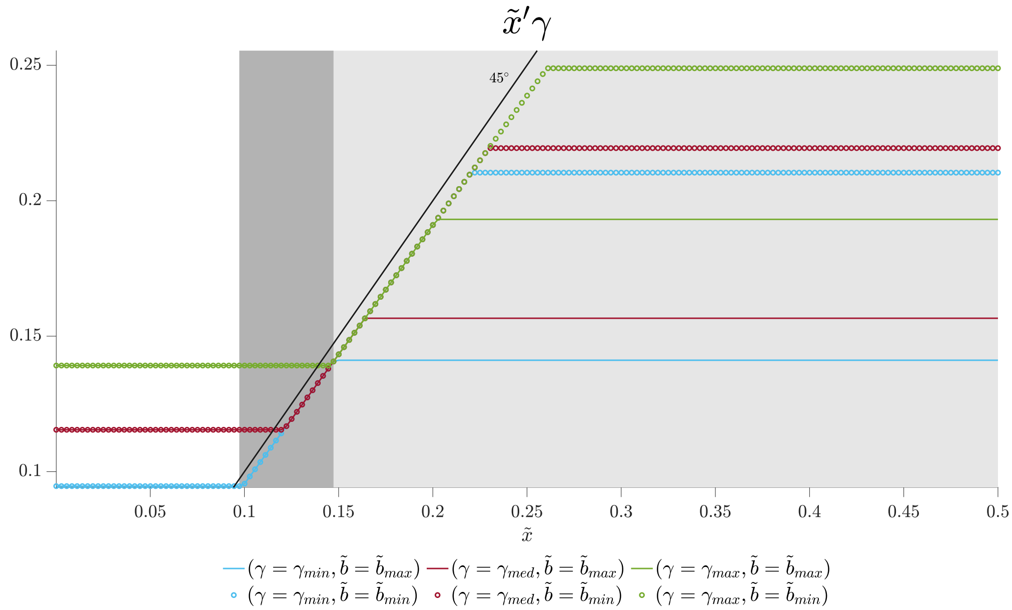

Figure G.3: Evolution of the Relative Pareto Weight in Steady State as a Function of $(\gamma, \tilde{b}, \tilde{x})$

long as the growth state does not change, the economy remains there.

Figures G.4 and G.5 depict the main policy functions and financial variables as a function of $(\gamma, \tilde{\omega})$ for zero debt and different levels of debt, respectively. More precisely, they both present the aforementioned statistics for the largest, the median and the lowest growth shocks. The dynamic is fairly similar to what we have highlighted in Section 6. This is because there is a direct correspondence between $\tilde{\omega}$ and $(\tilde{x}, \tilde{b})$ as discussed in Appendix B.

Figure G.7 depicts the main policy functions and financial variables as a function of $(\gamma, \tilde{b})$. Most notably, it present the aforementioned statistics for the largest and lowest growth shocks γ_{max} and γ_{min}, as well as, the largest and lowest relative Pareto weights $\tilde{z}_{max}$ and $\tilde{z}_{min}$, respectively.

Figure G.8 presents the default set of the economy with and without the Fund's intervention. The former is depicted on the right hand side and the latter on the left hand side of the figure. Without the Fund's intervention, the sovereign defaults at different levels of labor productivity and different levels of debt depending on the labor productivity regime. In regimes of greater average growth, the sovereign defaults on relatively higher debt levels or even decides not to default. With the Fund's intervention, the sovereign never defaults consistent with Proposition 1.

Figure G.9 presents the holdings of Arrow-type securities. This figure is key in explaining the insurance mechanism provided by the Fund. First, we clearly see that the sovereign goes long in the transition between a relatively high growth state to a relatively low growth state. The opposite is true for short positions. Hence, Arrow-type securities prevent large drops in consumption when growth suddenly decreases. That is, the holding of Arrow-type securities is procyclical. In other words, the prospective insurance is large when the current growth state is high. Second, one observes that the insurance taken when $\gamma' = \gamma_{min}$ decreases when the lender's participation constraint binds, while the repayment (i.e. negative holdings) when $\gamma' = \gamma_{max}$ largely increases. This is due to the negative spread.

Figure G.10 presents the transfers from the Fund and the private lenders. The Fund's primary surplus, $\tilde{\tau}_f$, represents the net savings of the sovereign in the Fund. As the relative Pareto weight increases towards the value at which the lenders' participation constraint binds, the surplus becomes negative. The opposite is true when the relative Pareto weight is decreasing. Thus, the surplus is procyclical or if one prefers the deficit is countercyclical. As already mentioned, this procyclicality is the key mechanism preventing default. Next to the net savings in the Fund, one has the net savings in the private bond economy, $\tilde{\tau}_p$. The pattern here is the opposite of the one observed before, reflecting the hedging property of the Fund. The last panel of Figure G.10 depicts the total net savings, $\tilde{\tau}_f + \tilde{\tau}_p = \tilde{\tau}$. It follows the same pattern as $\tilde{\tau}_f$. The total surplus is therefore procyclical (or countercyclical if one refers to primary deficits) as well. Furthermore, it remains modest compared to $\tilde{\tau}_f$ or $\tilde{\tau}_p$, reflecting the fact that positions in the private bond market are counterbalanced by positions in the Fund.

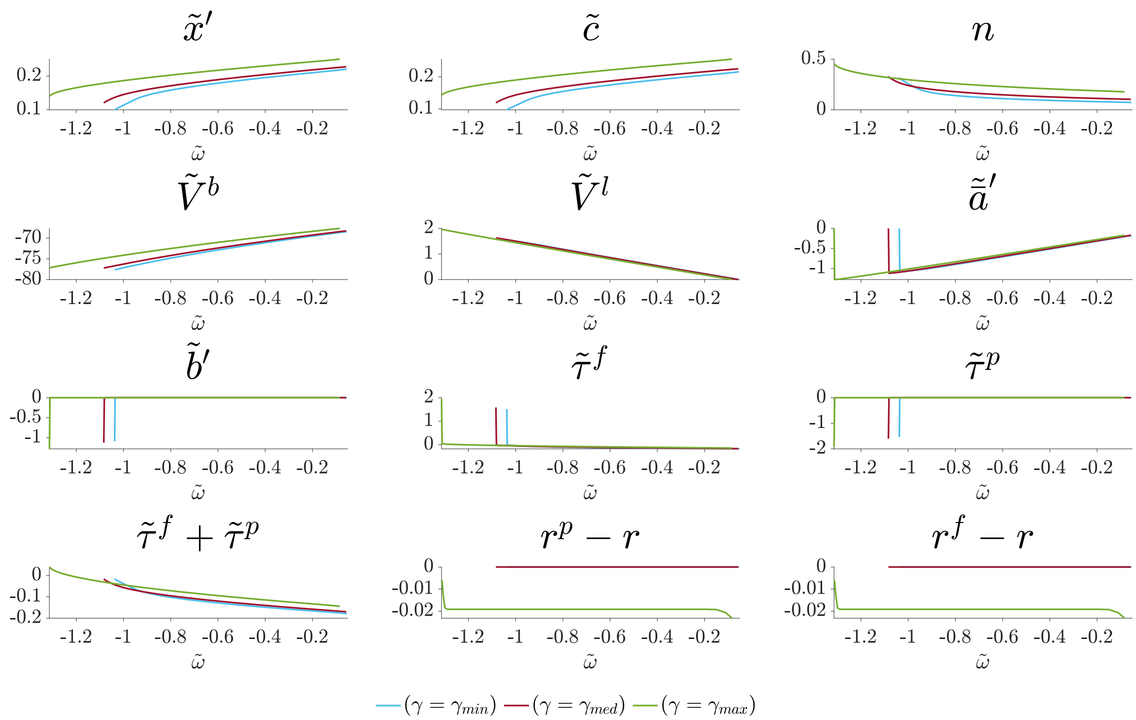

Figure G.4: Optimal Policies with Zero Private Debt as Function of $(\gamma, \tilde{\omega})$

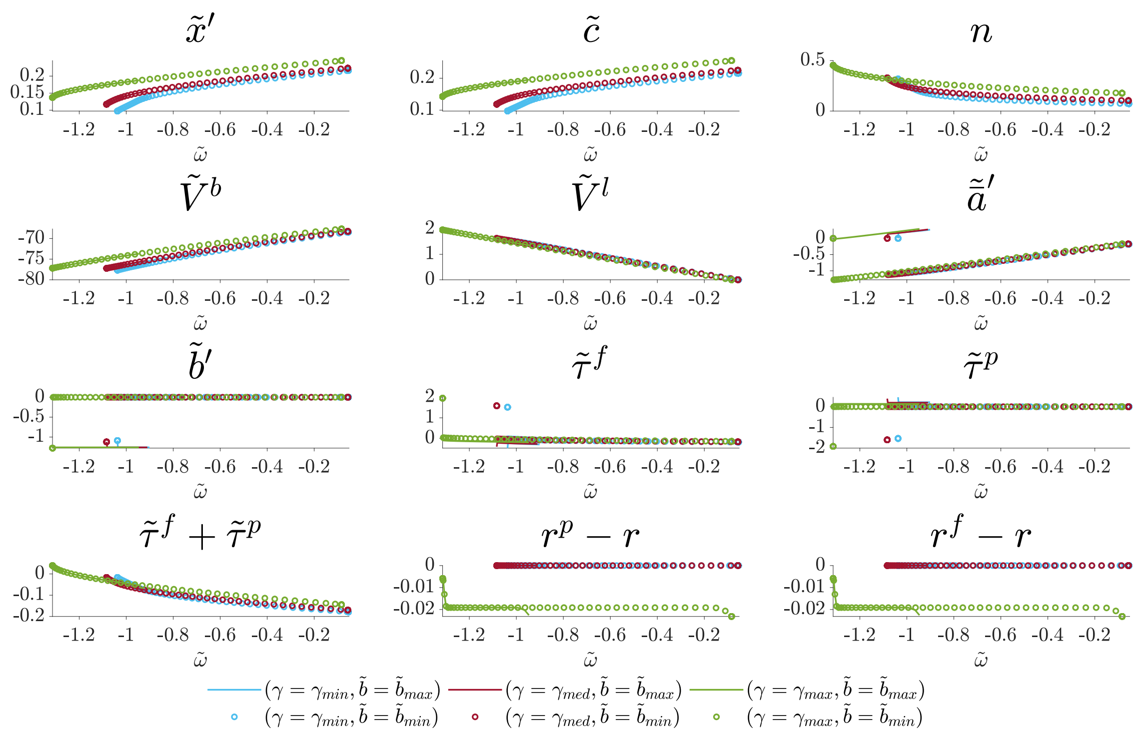

Figure G.5: Optimal Policies for Different Levels of Private Debt as Function of $(\gamma, \tilde{\omega})$

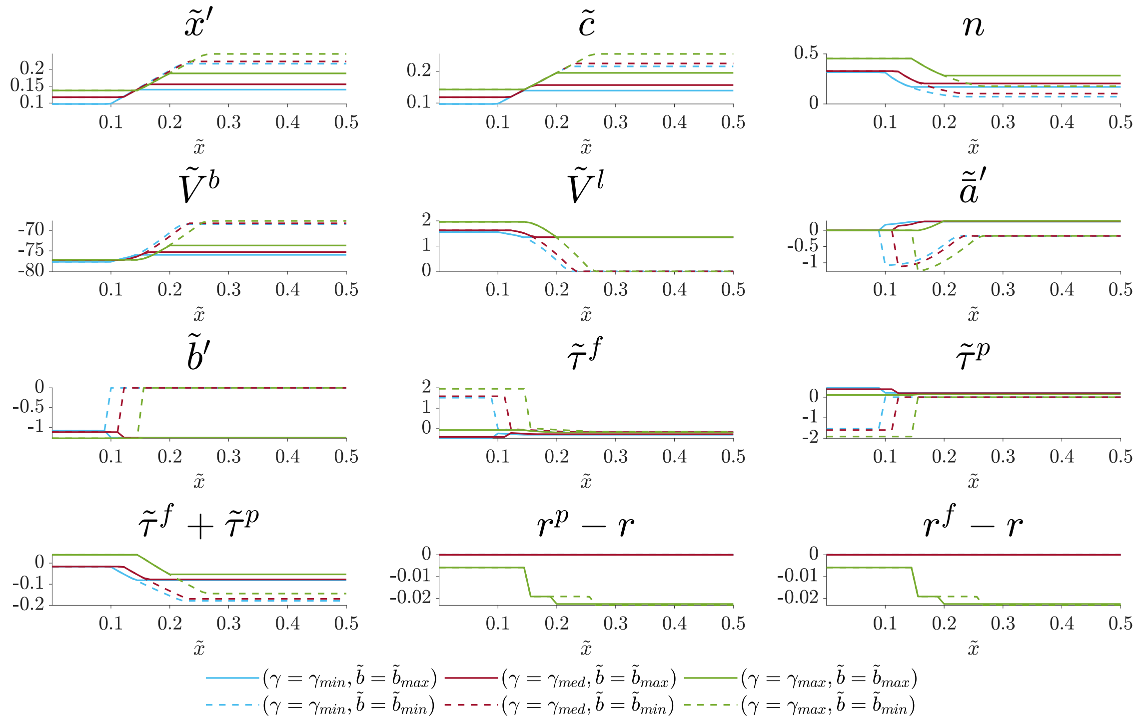

Figure G.6: Optimal Policies for Different Levels of Private Debt as Function of $(\gamma, \tilde{z})$

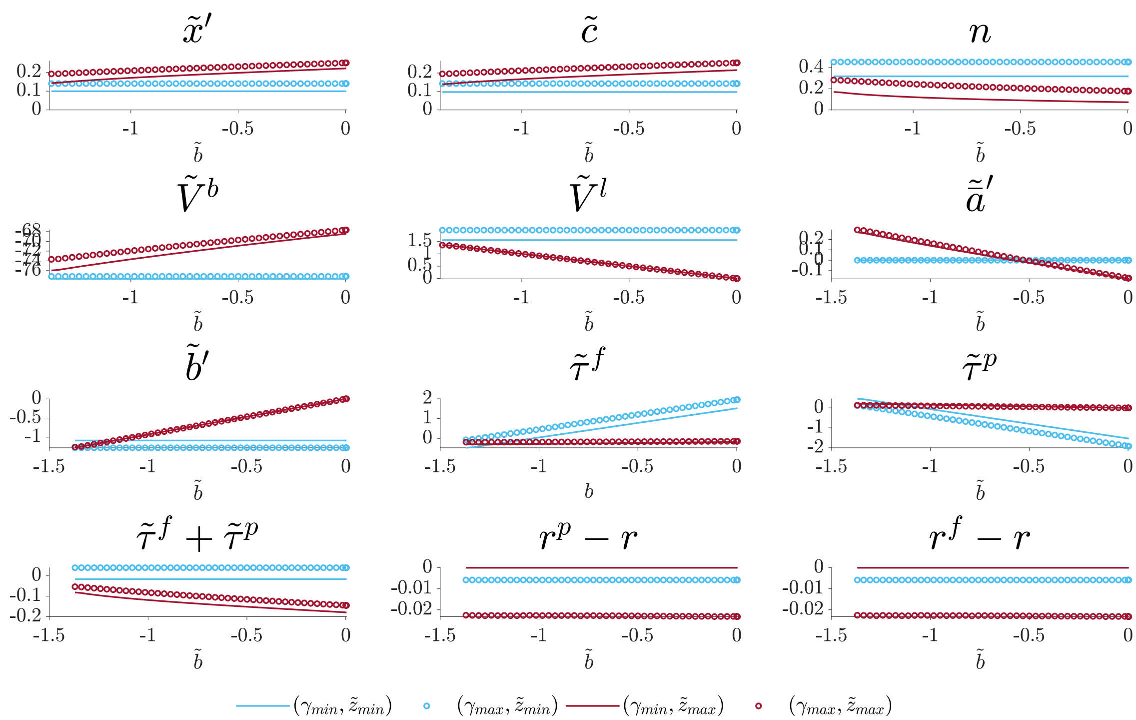

Figure G.7: Optimal Policies as Function of $(\gamma, \tilde{b})$

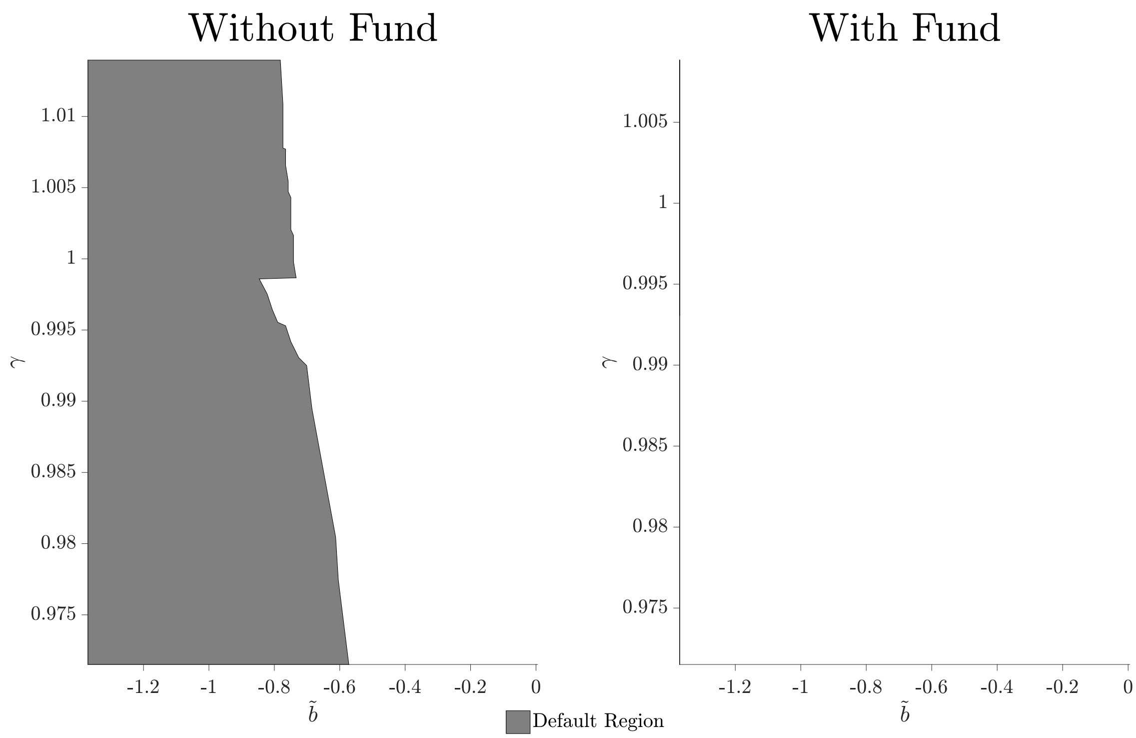

Figure G.8: Default Set as a Function of $(\gamma, \tilde{b})$

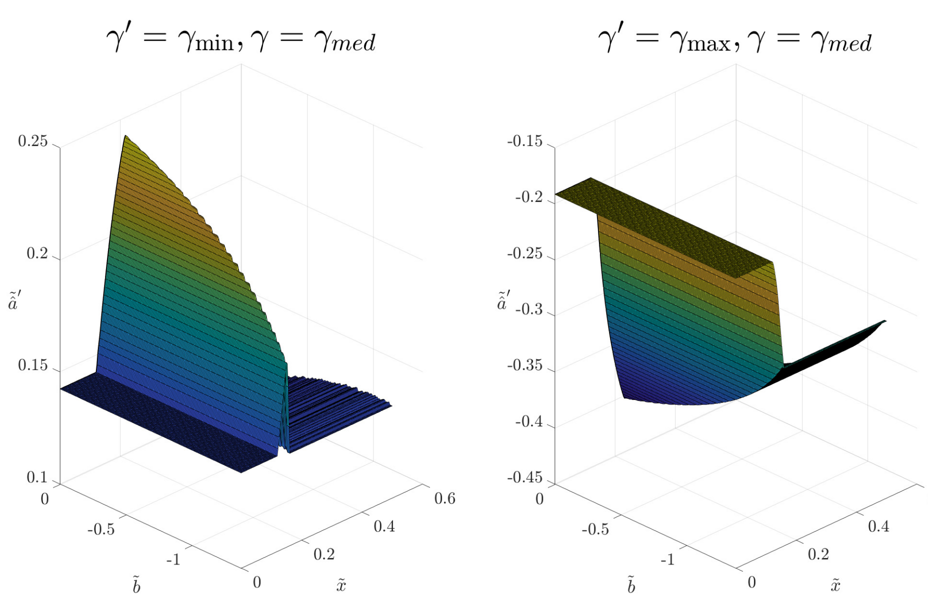

Figure G.9: Arrow-type Securities with Zero Private Debt as Function of $(\gamma, \tilde{x})$

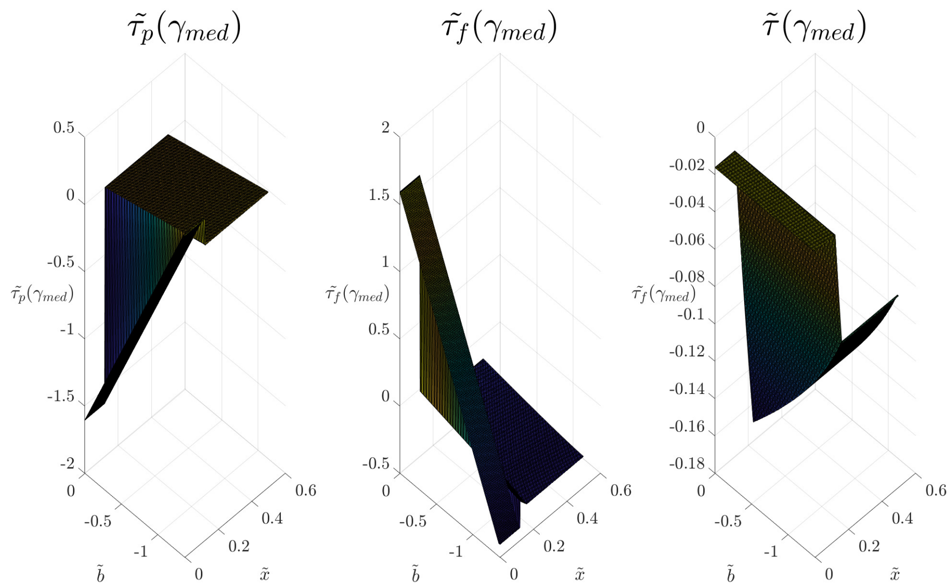

Figure G.10: Transfers as Function of $(\gamma, \tilde{b}, \tilde{x})$

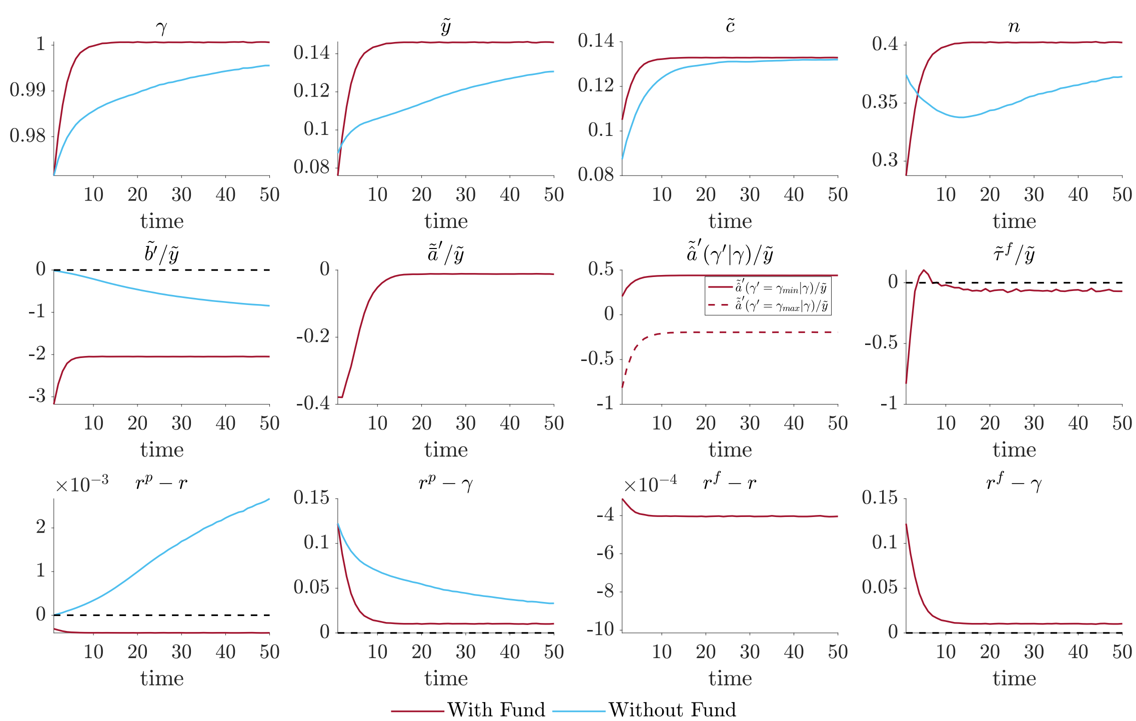

Figure G.11: Impulse Response Functions — Negative γ Shock

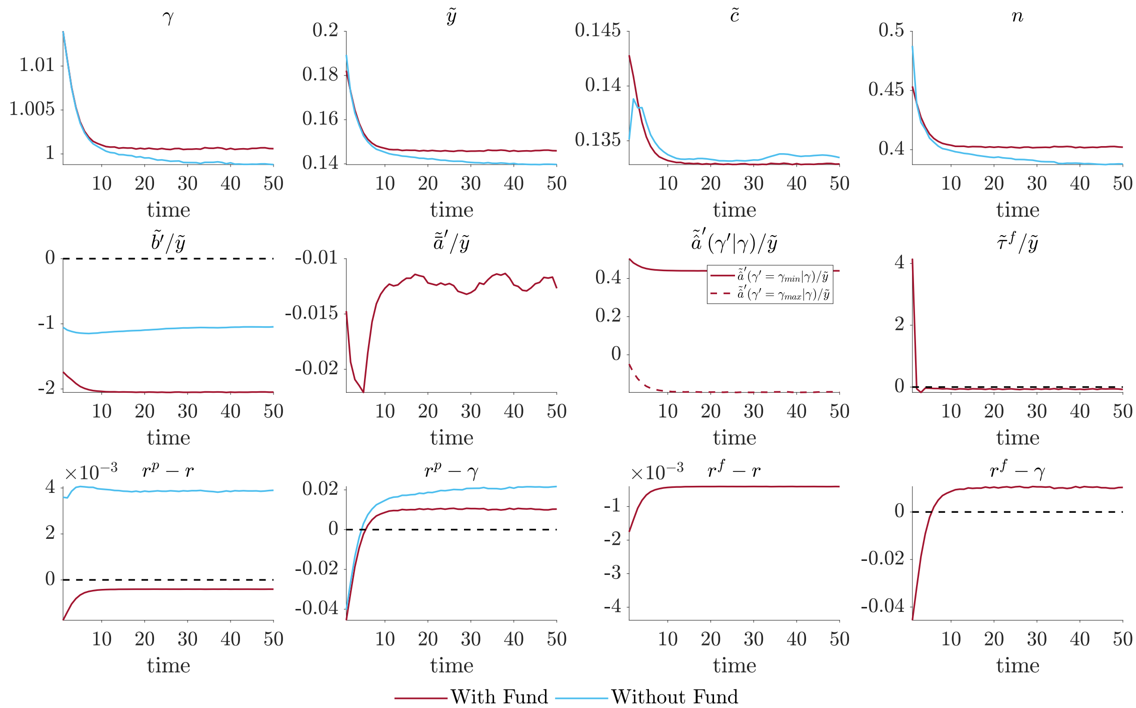

Figure G.12: Impulse Response Functions — Positive γ Shock

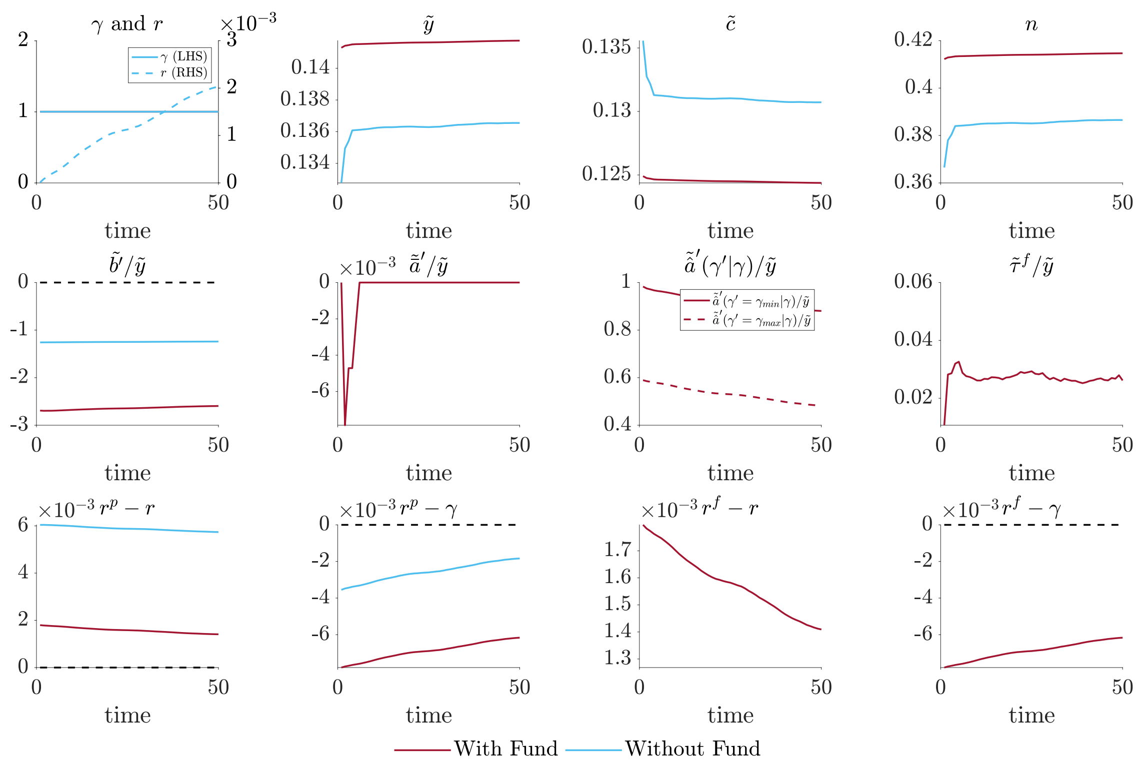

Figure G.13: Impulse Response Functions — Negative r Shock

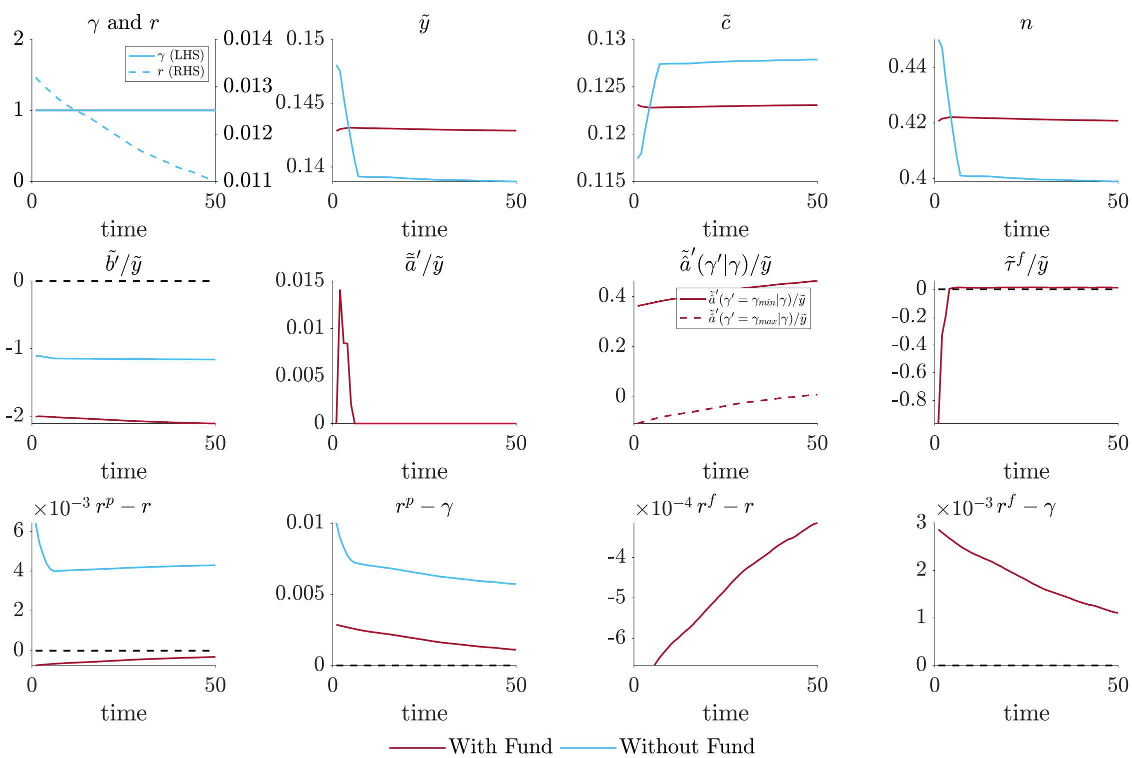

Figure G.14: Impulse Response Functions — Positive r Shock