# Online Appendix
# (Not For Publication)

## A Detrended Model

As in Aguiar and Gopinath (2006), we consider a growth shock to the productivity of the following form $\theta_t = \gamma_t \theta_{t-1}$, where $\gamma_t$ represents the growth rate and $\theta_t$ the trend at time $t$. We detrend the variables for allocations (except for labor $n$ where we normalise the time endowment to 1) of the model by dividing them by $\theta_{t-1}$. We therefore denote by $\tilde{c}_t$ the detrended form of $c_t$ such that $\tilde{c}_t = \frac{c_t}{\theta_{t-1}}$ represents the deviation from the trend. It follows that $U(c_t, n_t) = \ln(\theta_{t-1}) + U(\tilde{c}_t, n_t)$, and clearly, $\ln(\theta_{t-1})$ does not affect optimal choice. By the homogeneity of the sovereign's recursive problem, we have the detrended formulation as

$$
\begin{aligned}
\widetilde{W}^b(\gamma, \tilde{a}, \tilde{b}) = & \max_{\left\{\tilde{c}, n, \tilde{b}', \{\tilde{a}'(\gamma')\}_{\gamma' \in \Gamma}\right\}} U(\tilde{c}, n) + \beta \mathbb{E}\big[\widetilde{W}^b(\gamma', \tilde{a}'(\gamma'), \tilde{b}') \big| \gamma\big] \qquad \text{(A.1)} \\
& \text{s.t. } \tilde{c} + \sum_{\gamma'|\gamma} q_f(\gamma', \tilde{\omega}'(\gamma')|\gamma)(\gamma \tilde{a}'(\gamma') - \delta \tilde{a}) + q_p(\gamma, \tilde{\omega}')(\gamma \tilde{b}' - \delta \tilde{b}) \\
& \leq \gamma f(n) + (1 - \delta + \delta\kappa)(\tilde{a} + \tilde{b}) \\
& \tilde{\omega}'(\gamma') = \tilde{a}'(\gamma') + \tilde{b}' \geq \tilde{\mathcal{A}}_b(\gamma').
\end{aligned}
$$

Similarly, the private lender's problem in detrended form reads

$$
\begin{aligned}
\widetilde{W}^p(\gamma, \tilde{a}_l, \tilde{b}_l) = & \max_{\{c_p, \tilde{b}_l'\}} c_p + \frac{1}{1+r} \gamma \mathbb{E}\big[\widetilde{W}^p(\gamma', \tilde{a}_l'(\gamma'), \tilde{b}_l') \big| \gamma\big] \qquad \text{(A.2)} \\
& \text{s.t. } \tilde{c}_p + q_p(\gamma, \omega')(\gamma \tilde{b}_l' - \delta \tilde{b}_l) \leq (1 - \delta + \delta\kappa)\tilde{b}_l, \\
& \tilde{a}_l'(\gamma') = \tilde{A}_l(\gamma', \gamma, \tilde{a}, \tilde{b}), \\
& \tilde{a}_l'(\gamma') + \tilde{b}_l' \geq \tilde{\mathcal{B}}_l(\gamma', \tilde{a}_l'(\gamma')).
\end{aligned}
$$

The sovereign's outside option in detrended form takes the following form

$$
\widetilde{V}^{af}(\gamma) = \max_n \left\{U(\gamma^p f(n), n)\right\} + \beta \mathbb{E}\big[(1-\lambda)\widetilde{V}^{af}(\gamma') + \lambda \tilde{J}(\gamma', 0) \big| \gamma\big],
$$

The detrended Fund's problem in sequential form is given by

$$
\max_{\{\tilde{c}(\gamma^t), n(\gamma^t)\}_{t=0}^{\infty}} \mathbb{E}\left[\mu_{b,0} \sum_{t=0}^{\infty} \beta^t U(\tilde{c}(\gamma^t), n(\gamma^t)) + \mu_{l,0} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left(\prod_{i=0}^{t-1} \gamma_i\right) \tilde{\tau}(\gamma^t) \middle| \theta_{-1}\right] \qquad \text{(A.3)}
$$

$$
\text{s.t.} \quad \mathbb{E}\left[\sum_{j=t}^{\infty} \beta^{j-t} U(\tilde{c}(\gamma^j), n(\gamma^j)) \middle| \gamma^t\right] \geq \widetilde{V}^{af}(\gamma_t), \qquad \text{(A.4)}
$$

$$\mathbb{E}\left[\sum_{j=t}^{\infty}\left(\frac{1}{1+r}\right)^{j-t}\left(\prod_{i=t}^{j-1}\gamma_i\right)\tilde{\tau}(\gamma^j)\middle|\gamma^t\right] \geq Z - \tilde{b}(\gamma^t), \tag{A.5}$$

$$\tilde{\tau}(\gamma^t) = \gamma_t f(n(\gamma^t)) - \tilde{c}(\gamma^t), \quad \forall\gamma^t, t \geq 0,$$

$$\text{with } \mu_{b,0},\ \mu_{l,0},\ \{\tilde{b}(\gamma^t)\}_{t=0}^{\infty}, \{q_p(\gamma^t, x(\gamma^{t+1}), \tilde{b}(\gamma^{t+1}))\}_{t=0}^{\infty} \text{ given.}$$

And in recursive form

$$\begin{aligned}
\widetilde{FV}(\gamma,\tilde{x},\tilde{b}) = \mathcal{SP} \min_{\{\nu_b,\nu_l\}} \max_{\{\tilde{c},n\}}\ & \tilde{x}\Big[(1+\nu_b)U(\tilde{c},n) - \nu_b \widetilde{V}^{af}(\gamma)\Big] \\
& + \big[(1+\nu_l)\tilde{\tau} - \nu_l(Z-\tilde{b})\big] + \frac{1+\nu_l}{1+r}\gamma\mathbb{E}\big[\widetilde{FV}(\gamma',\tilde{x}',\tilde{b}')\big|\gamma\big]
\end{aligned} \tag{A.6}$$

$$\text{s.t.} \quad \tilde{\tau} = \gamma f(n) - \tilde{c},$$

$$\tilde{x}' = \frac{1+\nu_b}{1+\nu_l}\frac{\eta}{\gamma}\tilde{x}, \tag{A.7}$$

The value function takes the form of

$$\begin{aligned}
&\widetilde{FV}(\gamma,\tilde{x},\tilde{b}) = \tilde{x}\widetilde{V}^b(\gamma,\tilde{x},\tilde{b}) + \widetilde{V}^l(\gamma,\tilde{x},\tilde{b}), \text{ with} \\
&\widetilde{V}^b(\gamma,\tilde{x},\tilde{b}) = U(\tilde{c},n) + \beta\mathbb{E}[\widetilde{V}^b(\gamma',\tilde{x}',\tilde{b}')|\gamma], \text{ and} \\
&\widetilde{V}^l(\gamma,\tilde{x},\tilde{b}) = \tilde{\tau} + \frac{1}{1+r}\gamma\mathbb{E}[\widetilde{V}^l(\gamma',\tilde{x}',\tilde{b}')|\gamma].
\end{aligned}$$

Taking the first-order conditions with respect to $c$ and $n$ leads to

$$u'(\tilde{c}) = \frac{1+\nu_l}{1+\nu_b}\frac{1}{\tilde{x}} \quad \text{and} \quad \gamma f'(n) = \frac{h'(1-n)}{u'(\tilde{c})},$$

The consumption is therefore equal to $\tilde{c} = \tilde{x}'\frac{\gamma}{\eta} \equiv \tilde{z}'\gamma$. From this, we see that whenever the growth rate of the economy settles below one, the relative Pareto weight increases. However, the consumption does not react to changes in $\gamma$. In fact, the consumption is affected only when one of the limited enforcement constraints binds.

For completeness, the decentralised Fund problem in detrended form is given by

$$\widetilde{W}^f(\gamma,\tilde{a}_l,\tilde{b}_l) = \max_{\{\tilde{c}_f,\{\tilde{a}_l'(\gamma')\}_{\gamma'\in\Gamma}\}} \tilde{c}_f + \frac{1}{1+r}\gamma\mathbb{E}\big[\widetilde{W}^f(\gamma',\tilde{a}_l'(\gamma'),\tilde{b}_l')\big|\gamma\big] \tag{A.8}$$

$$\text{s.t. } \tilde{c}_f + \sum_{\gamma'|\gamma} q_f(\gamma',\omega'(\gamma')|\gamma)(\gamma\tilde{a}_l'(\gamma') - \delta\tilde{a}_l) \leq (1-\delta+\delta\kappa)\tilde{a}_l,$$

$$\tilde{b}_l' = \tilde{B}_l(\gamma,\tilde{a}_l,\tilde{b}_l)$$

$$\tilde{a}_l'(\gamma') + \tilde{b}_l' \geq \tilde{\mathcal{A}}_f(\gamma',\tilde{b}_l'). \tag{A.9}$$

## B Further Theory Development

In this section we present other properties of the Fund contract. We start with the inverse Euler Equation which is a key concept determining the dynamic of consumption in the contract.

**Proposition B.1** (Insurance). *In the Fund contract, the inverse Euler equation is given by*

$$\mathbb{E}\left[\frac{1}{u'(c(\theta',x',b'))}\frac{1+\nu_l(\theta',x',b')}{1+\nu_b(\theta',x',b')}\bigg|\theta\right]=\eta\frac{1}{u'(c(\theta,x,b))},$$

*and risk sharing is imperfect.*

*Proof.* See Appendix C □

We obtain the inverse Euler equation by means of the first-order condition on consumption and the law of motion of the relative Pareto weight. This equation gives the intertemporal dynamic of consumption. If none of the constraints are ever binding (i.e. $\nu_b=\nu_l=0$), it becomes

$$\mathbb{E}\left[\frac{1}{u'(c(\theta',x',b'))}\bigg|\theta\right]\leq\frac{1}{u'(c(\theta,x,b))},$$

with strict inequality if $\eta<1$, in our case. We therefore obtain a positive martingale, which by the supermartingale theorem, converges almost surely to $-\infty$. This is what the literature has called immiseration.

Thus, with $\eta<1$, when none of the constraints are binding, consumption decreases. However, this reduction cannot go on indefinitely given the sovereign's limited enforcement constraint. This constraint puts a lower bound to the supermartingale and therefore acts as a stopper for immiseration. Conversely, the lender's constraint puts an upper bound to the supermartingale which prevents consumption to increase indefinitely. As a result, in a contract with tow-sided limited enforcement constraints and impatient borrower, risk sharing is only partial. The contract cannot converge to the first-best allocation characterised by constant consumption over time.

Having determined the inverse Euler Equation, we can now show existence. To ensure the existence of the Lagrange multipliers — and therefore of the above contract, we need to the following technical assumption (Marcet and Marimon, 2019).

**Assumption B.1** (Interiority). *There is an $\epsilon>0$, such that, for all $\theta\in\Theta$, there is a sequence $\{\ddot{c}(\theta^t),\ddot{n}(\theta^t)\}$ satisfying equations (16) and (17) in which each outside option is replaced by $V^{af}(\theta_t)+\epsilon$ and $Z+\epsilon$, respectively.*

This assumption ensures the uniform boundedness of the Lagrange multipliers. For equations (16) and (17), it requires that, in spite of the enforcement constraints, there are strictly positive rents to be shared among the contracting parties. Otherwise there may not exist a constrained-efficient risk-sharing agreement. Given this, we can show that, under general conditions, a Fund contract exists.

**Proposition B.2** (Existence of Fund Contract)**.** *For every $\theta \in \Theta$ there is a $\underline{b}(\theta) < 0$ such that if $b_0(\theta) \geq \underline{b}(\theta)$, then there exist a Fund contract with initial condition $(\theta, b_0(\theta))$. Furthermore, there is a $\underline{t}(\theta, \underline{b}(\theta))$ such that for $t > \underline{t}(\theta, \underline{b}(\theta))$ the Fund contract is at steady state.*

*Proof.* See Appendix C □

Proposition B.2 is made of two parts. First, a Fund contract exists if — among other requirements — the initial level of private indebtedness is not too high.[37] Thus, if an economy is in an initial state $(\theta, b_0(\theta))$ but $b_0(\theta) < \underline{b}(\theta)$ then the private debt will need to be restructured — i.e. to a $\ddot{b}_0(\theta) \geq \underline{b}(\theta)$ — for a Fund contract to exist. Second, the Fund contract is characterised by an ergodic distribution. Hence, in the long-run, the relative Pareto weight moves within the same set of values over and over again. The exact shape of the ergodic distribution is the purpose of the next definition and lemma.

Having shown existence of the Fund contract, we can now determine the correspondence between the Fund contract established in Section 3.1 and the decentralised Fund contract presented in section 3.3. For that purpose, we first establish the Second Welfare Theorem.

**Proposition B.3** (Second Welfare Theorem)**.** *Given initial conditions $\{\theta_0, b_0, x_0\}$, a Fund's allocation can be decentralised as a competitive equilibrium with endogenous borrowing limits.*

*Proof.* See Appendix C □

This proposition states that there is a direct correspondence between, on the one hand, $a$ and, on the other hand, $x$ given by

$$u'(c(\theta, a, b)) = \frac{1 + \nu_l(\theta, x, b)}{1 + \nu_b(\theta, x, b)} \frac{1}{x}.$$

In words, for a given $\theta$, if $a$ and $x$ satisfy the above correspondence, then $B(\theta, x, b) = B(\theta, a, b)$, $c(\theta, a, b) = c(\theta, x, b)$, $c_p(\theta, a, b) = \tau_p(\theta, x, b)$, $c_f(\theta, a, b) = \tau_f(\theta, x, b)$, $c_p(\theta, a, b) +$

[37] Note that $\underline{b}(\theta) = \min_b\{b : \theta^- Z - b \geq V^b(\theta, \underline{x}, b)\}$.

$c_f(\theta,a,b) = \tau(\theta,x,b)$ and $n(\theta,a,b) = n(\theta,x,b)$. In that same logic, we have that $W^b(\theta,a,b) = V^b(\theta,x,b)$ and $W_p(\theta,a,b) + W_f(s,a,b) = V^l(s,x,b)$. Thus, the endogenous limits (3) and (23) are exactly and uniquely binding when they are binding in the Fund contract.

Properly speaking the correspondence relates to, on the one hand, $x$ and $b$ and, on the other hand, $\omega$ as only the entire sovereign's debt matter in the Fund. The split of $\omega$ between $a$ and $b$ is irrelevant for the Fund as the sovereign defaults $\omega$ and not selectively on $a$ or $b$.

For completeness of the argument, we also show that the First Welfare Theorem holds. That is, a recursive competitive equilibrium allocation with borrowing limits implements the constrained efficient allocation of the Fund.

**Proposition B.4** (First Welfare Theorem). *Given initial conditions $\{\theta_0, b_0, a_0\}$, a competitive equilibrium with endogenous borrowing limits implements the constrained efficient allocation of the Fund.*

*Proof.* See Appendix C □

We end this subsection with a result relating to the endogenous borrowing limits. Using the intertemporal budget constraints, we can construct the asset holdings that make the consumption allocations in the Fund contract satisfy the present value of the budget. This leads to the following proposition.

**Lemma B.1** (Borrowing and Net Present Value Constraints). *At some period $t$ and $n$ with $t \neq n$, if the participation constraint of one of the contracting parties is binding, the borrowing limit for of the constrained agent in the decentralised economy is determined by*

$$\mathcal{A}_b(\theta^t) = \mathbb{E}_t \sum_{j=0}^{\infty} Q_f(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))], \tag{B.1}$$

$$\mathcal{A}_f(\theta^n) = \mathbb{E}_t \sum_{j=0}^{\infty} Q_f(\theta^{n+j}, \omega(\theta^{n+j})|\theta^n) c_l(\theta^{n+j}, a(\theta^{n+j}), b(\theta^{n+j}))), \tag{B.2}$$

*with $Y(\theta^t, x(\theta^t), b(\theta^t)) \equiv \theta(\theta_t) f(n(\theta^t, x(\theta^t), b(\theta^t)))$ for all $t$ and $\theta^t$.*

*Proof.* See Appendix C □

Given this, (9) and (21) truly represent a net present value (NPV) constraint in equilibrium. In any state, the decentralised asset portfolio between the sovereign and the Fund is a whole plan of contingent asset position to the indefinite future. The whole contingent plan of asset holdings corresponds to the whole plan of transfers $\{\tau(\theta^t)\}_{t=0}^{\infty}$, which is clearly not a one period decision. The fact that the whole plan can be determined recursively does not mean that the asset positions in $\theta^{t+1}$ — that is $\omega(\theta^{t+1})$ — refer only to a set of contingent payoffs at $t+1$. Rather, $\omega(\theta^{t+1})$ represents the NPV of all future Fund's transfers starting from $\theta^{t+1}$. Therefore when (21) binds with strictly positive probability, the Fund refuses to grant an alternative plan embedded in some other $\ddot{\omega}(\theta^{t+1})$, which would render the NPV negative. Equivalently, this means that the Fund should not lend too much at too low a price or it would end up loosing money. Hence, the lender's constraint is a present value — or more lively, a no bailout — constraint, which is conceptually distinct from the borrower's borrowing constraint, (i.e. a sovereignty constraint).

**Corollary B.1** (Welfare Equivalence)**.** *Any sequence of private bond positions $\{b(\theta^t)\}_{t=0}^{\infty}$ being sustained in a RCE with the same initial $b(\theta_0)$ leads to the same welfare.*

*Proof.* See Appendix C □

A direct corollary of Lemma B.1 is that (B.1) and (B.2) do not depend on the private bond holdings. Thus, irrespective of the sequence of private bonds $\{b(\theta^t)\}_{t=0}^{\infty}$, as long as, the different sequences have the same initial starting point, $b(\theta_0)$ and can be sustained in the Fund contract, they will lead to the same present discounted value for the lenders and the borrower. This is a *Ricardian equivalence* result applied to equations (B.1) and (B.2).

## C Proofs

### Proof of Proposition B.1

The first order condition on consumption reads $u'(c) = \frac{1+\nu_l}{1+\nu_b}\frac{1}{x}$. The law of motion of the relative Pareto weight is given by $x' = \frac{1+\nu_b}{1+\nu_l}\eta x$. Combining those two equations one obtains

$$x' = \frac{1+\nu_b(\theta,x,b)}{1+\nu_l(\theta,x,b)}\eta x = \frac{1}{u'(c(\theta',x',b'))}\frac{1+\nu_l(\theta',x',b')}{1+\nu_b(\theta',x',b')}. \tag{C.1}$$

Moreover, observe that using the above first-order condition

$$\frac{1+\nu_b(\theta,x,b)}{1+\nu_l(\theta,x,b)}\eta x = \eta\left[\frac{1}{u'(c(\theta,x,b))}\frac{1+\nu_b(\theta,x,b)}{1+\nu_l(\theta,x,b)}\frac{1+\nu_l(\theta,x,b)}{1+\nu_b(\theta,x,b)}\right] = \eta\frac{1}{u'(c(\theta,x,b))}.$$

Hence, one can rewrite (C.1) as

$$\eta\frac{1}{u'(c(\theta,x,b))} = \frac{1}{u'(c(\theta',x',b'))}\frac{1+\nu_l(\theta',x',b')}{1+\nu_b(\theta',x',b')}.$$

Taking expectations on both sides with respect to $\theta'$ leads to

$$\eta \frac{1}{u'(c(\theta,x,b))} = \mathbb{E}\left[\frac{1}{u'(c(\theta',x',b'))}\frac{1+\nu_l(\theta',x',b')}{1+\nu_b(\theta',x',b')}\middle|\theta\right].$$

This equation is the inverse Euler equation. It gives the dynamic of consumption over time and therefore the extent of insurance. If none of the constraint ever binds and $\eta = 1$, then the contract achieves full insurance. However, whenever one of those two point is no true, consumption is not constant across states. Insurance is thus only partial in our environment. $\square$

**Proof of Proposition B.2**

Consider the model in detrended form presented in Appendix A. If one has that $\{\tilde{b}(\theta^t)\}_{t=0}^{\infty} = 0$, we are back to the standard model of Ábrahám et al. (2021) without moral hazard.

To show existence, one needs to determine whether the assumptions to apply Theorem 3(i) in Marcet and Marimon (2019) are met: A1 well defined Markovian process for $\gamma$, A2 continuity in $\{c, n\}$ and measurability in $\gamma$, A3 non-empty feasible sets, A4 uniform boundedness, A5 convex technologies, A6 concavity for the lender and strict concavity for the sovereign, A7 interiority. Assumption A1, A2, A5 and A6 are trivially met as elicited in Section 2. Since $c$ and $n$ are bounded, payoffs functions are bounded as well. This combined with the fact that the outside options are finite ensure that A4 is met. Assumption B.1 ensures A7.

One is left to show that A3 is met. If one assumes that the sequence of debt is different than zero for some $t > 0$ and especially for $t = 0$, it is the initial $\tilde{b}_0$ that is crucial for existence. If $\tilde{b}_0$ is such that the following break even condition holds:

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \tilde{\tau}(\theta^t)\middle|\theta_0\right] = Z - \tilde{b}_0,$$

then a contract exists. The break even condition is a consequence of the homogeneity of degree 1 of the problem's solution (Marcet and Marimon, 2019, Lemma 1A). Whenever the break even condition holds, one obtains for all $t$ and $\theta^t$, $V^l(\gamma^t, \tilde{x}(\gamma^t), \tilde{b}(\gamma^t)) \geq Z - \tilde{b}(\theta^t)$. However, should it not be the case, the initial debt is too large to be absorbed by the Fund. The debt has to be restructured until the above break even condition holds.

The homegeneity of degree one in $\mu = (\mu_b, \mu_l)$ allows us to redefine the contracting problem using $x$ as a co-state variable. This combined with Assumption B.1 ensures that there exists a $C > 0$ such that for the Lagrange multiplier $\vartheta$, $||\vartheta|| \leq ||x||C$. Accounting for the lender's participation constraint there is a $\bar{C}$ such that $||\vartheta|| \leq \bar{C}$. We can therefore define the set of of feasible Lagrange multipliers by $L = \{\vartheta \in \mathbb{R}^2_+ : ||\vartheta|| \leq \bar{C}\}$ and the set of feasible consumption and labor by $A = \{(c, n) \in \mathbb{R}^2_+ : n \leq 1\}$.

With this, one can use Theorem 3(i) in Marcet and Marimon (2019). That is the correspondence $SP : A \times L \to A \times L$ mapping non-empty, convex, and compact sets to themselves, is non-empty, convex-valued, and upper hemicontinuous. We can therefore apply Kakutani's fixed point theorem and existence immediately follows.

Regarding the steady state, the lower bound of the ergodic set is determined by the lowest achievable relative Pareto weight in the contract. It represents the lowest value that the sovereign accepts in the contract. The upper bound represents the highest relative Pareto weight that makes the sovereign's constraint bind; therefore it is the highest weight that the lender may need to accept. This means that every time the highest productivity shock hits (i.e. $\gamma_{max}$), the sovereign climbs to the top of the ergodic set. In opposition, for a sufficiently long string of lowest productivity shock (i.e. $\gamma_{min}$), the sovereign eventually hits the bottom of the set — owing to immiseration.

To show the existence of a unique stationary equilibrium, one shows that the dynamic of the contract satisfies the conditions given by Stokey et al. (1989, Theorem 12.12). Set $\ddot{x}$ as the midpoint of $[\underline{\tilde{x}}, \tilde{\overline{x}}]$ and define the transition function $Q : [\underline{\tilde{x}}, \tilde{\overline{x}}] \times \mathcal{X}([\underline{\tilde{x}}, \tilde{\overline{x}}]) \to \mathbb{R}$ as

$$Q(x, G) = \sum_{\theta'|\theta} \pi(\theta'|\theta) \mathbb{I}\{x' \in G\}$$

We want to show is that $\ddot{x}$ is a mixing point such that for $N \geq 1$ and $\epsilon > 0$ one has that $Q(\underline{\tilde{x}}, [x, \tilde{\overline{x}}])^N \geq \epsilon$ and $Q(\tilde{\overline{x}}, [\underline{\tilde{x}}, x])^N \geq \epsilon$. Starting at $\tilde{\overline{x}}$, for a sufficiently long but finite series of $\gamma_{min}$, the relative Pareto weight transit to $\underline{\tilde{x}}$. Hence for some $N < \infty$, $Q(\tilde{\overline{x}}, [\underline{\tilde{x}}, \ddot{x}])^N \geq \pi(\gamma_{min})^N > 0$ where $\pi(\gamma_{min})$ is the stationary probability of drawing $\gamma_{min}$. Moreover, starting at $\underline{\tilde{x}}$, after drawing $N < \infty$ $\gamma_{max}$, the relative Pareto weight transit to $\tilde{\overline{x}}$ meaning that $Q(\underline{\tilde{x}}, [\ddot{x}, \tilde{\overline{x}}])^N \geq \pi(\gamma_{max})^N > 0$. Setting $\epsilon = \min\{\pi(\gamma_{min})^N, \pi(\gamma_{max})^N\}$ makes $\ddot{x}$ a mixing point and the above theorem applies. $\square$

### Proof of Lemma 1

Recall that, in the detrended version of the model, the lower bound is defined by $\underline{x} = \min_{\gamma \in \Gamma}\{x : \widetilde{V}^b(\gamma, x, b) = \widetilde{V}^{af}(\gamma)\}$, while the upper bound corresponds to $\overline{x} = \max_{\gamma \in \Gamma}\{x : \widetilde{V}^b(\gamma, x, b) = \widetilde{V}^{af}(\gamma)\}$.

The key insight is to see that the sovereign's outside option is independent of the level of indebtedness, while the sovereign's value increases with the relative Pareto weight by definition. Assume now by contradiction that the lower bound $\underline{x}(\gamma, b)$ is a function of $\gamma$ and the level of debt $b$. That is for some $\ddot{b} \neq b$, $\underline{x}(\gamma, b) \neq \underline{x}(\gamma, \ddot{b})$. This implies that either $\widetilde{V}^b(\gamma, \underline{x}(\gamma, b), b) > \widetilde{V}^b(\gamma, \underline{x}(\gamma, \ddot{b}), \ddot{b})$ or $\widetilde{V}^b(\gamma, \underline{x}(\gamma, b), b) < \widetilde{V}^b(\gamma, \underline{x}(\gamma, \ddot{b}), \ddot{b})$ depending on which of the two relative Pareto weight is the largest. The former case leads to the fact that $\widetilde{V}^b(\gamma, \underline{x}(\gamma, b), b) > \widetilde{V}^{af}(\gamma)$, while the latter case leads to $\widetilde{V}^b(\gamma, \underline{x}(\gamma, b), b) < \widetilde{V}^{af}(\gamma)$. Both cases contradict the fact that $\underline{x}(\gamma, b)$ is the relative Pareto weight for which the sovereign's constraint binds. It must therefore be that for all $\ddot{b} \neq b$, $\underline{x}(\gamma) = \underline{x}(\gamma, b) = \underline{x}(\gamma, \ddot{b})$. The same reasoning applies to the upper bound. □

**Proof of Proposition 1**

We conduct a proof by contradiction. The present proof only considers the economy in equilibrium. It might be that default occurs off equilibrium path. This situation is, however, outside the scope of the proposition. The proof follows the argument of Thomas and Worrall (1994) and Zhang (1997). The participation constraint of the borrower ensures that the value of the borrower is at most equal to its outside option. Hence, the borrower is at most indifferent between defaulting or not. □

**Proof of Proposition B.3**

Following Alvarez and Jermann (2000) we prove the proposition by construction. First, define the Fund's asset price as

$$q_f(\theta', x', b'|\theta) = \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', x'', b''|\theta')\right] \max\left\{\frac{u'(c(\theta', x', b'))}{u'(c(\theta, x, b))}\eta, 1\right\}.$$

Second, as shown in Lemma B.1, iterating over the budget constraint of the sovereign gives

$$\begin{aligned} &a(\theta^t) + b(\theta^t) = \\ &\mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j}))], \end{aligned} \tag{C.2}$$

where, $Y(\theta^t, x(\theta^t), b(\theta^t)) = \theta(\theta_t) f(n(\theta^t, x(\theta^t), b(\theta^t)))$ for all $t$ and $\theta^t$. Similarly, iterating over the consolidated budget constraint of the two lenders leads to

$$\begin{aligned} a_l(\theta^t) + b_l(\theta^t) &= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})|\theta^t) c_l(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})) \qquad \text{(C.3)} \\ &= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})|\theta^t)[Y(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})) \end{aligned}$$

$$
\begin{aligned}
&- c(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j}))] \\
=\ & - a(\theta^t) - b(\theta^t).
\end{aligned}
$$

The market clearing conditions in the Fund and the private bond market implies that $a_l(\theta^t) + a(\theta^t) = 0$ and $b(\theta^t) + b_l(\theta^t) = 0$, respectively, for all $t$ and $\theta^t$.

We now need to establish the correspondence between the initial conditions, $(x_0, b_0)$, in the Fund contract and the initial conditions in the recursive competitive equilibrium, $(a_0, a_{l,0}, b_0, b_{l,0})$. Given (C.2) and (C.3) evaluated at $t = 0$, one can determine $\bar{a}(\theta_0, a_0, b_0)$ using the budget constraint

$$
\begin{aligned}
c(\theta_0, a_0, b_0) + q_f(\theta_0, \omega_1)(\bar{a}' - \delta a_0) + \sum_{\theta_1|\theta_0} q_f(\theta_1, \omega_1(\theta_1)|\theta_0)\hat{a}'(\theta_1) + q_p(\theta_0, \omega_1)(b' - \delta b_0) \\
\leq \theta_0 f(n) + (1 - \delta + \delta\kappa)(a_0 + b_0).
\end{aligned}
$$

and the fact that $\sum_{\theta_1|\theta_0} q_f(\theta_1, \omega_1(\theta_1)|\theta_0)\hat{a}'(\theta_1) = 0$. Once, $\bar{a}(\theta_0, a_0, b_0)$ is determined, one can find the holdings of Arrow-type securities $\hat{a}'(\theta', \theta_0, a_0, b_0)$ for all $\theta' \in \Theta$. We can then retrieve the entire portfolio recursively for $t > 0$.

Third, define the endogenous borrowing limits such that

$$
\begin{aligned}
\mathcal{A}_b(\theta) &= a(\theta, \underline{x}(\theta, b), b) + b(\theta, \underline{x}(\theta, b), b), \\
\mathcal{A}_l(\theta, b) &= a_l(\theta, \overline{x}(\theta, b), b) + b_l(\theta, \overline{x}(\theta, b), b), \\
\mathcal{B}_l(\theta, a) &= a_l(\theta, \overline{x}(\theta, b), b) + b_l(\theta, \overline{x}(\theta, b), b).
\end{aligned}
$$

This definition implies that $a'(\theta', \theta, a, b) + b' \geq \mathcal{A}_b(\theta')$, $a_l'(\theta', \theta, a, b) + b_l' \geq \mathcal{A}_l(\theta', b')$ and $a_l'(\theta', \theta, a, b) + b_l' \geq \mathcal{B}_l(\theta', a'(\theta'))$. Hence, the constructed asset holdings satisfy the competitive equilibrium constraints for both the lenders and the sovereign.

Fourth, defining $I(\theta, a, b)$ as the Lagrange multiplier attached to the sovereign's budget constraint, one ensures optimality of the policy functions by setting

$$
I(\theta, a, b) = \frac{1 + \nu_l(\theta, x, b)}{1 + \nu_b(\theta, x, b)} \frac{1}{x}.
$$

Hence, since $c(\theta, x, b)$ and $n(\theta, x, b)$ satisfy the optimality conditions in the Fund, $c(\theta, a, b)$ and $n(\theta, a, b)$ are also optimally determined in the competitive equilibrium. For the lenders, consumption is optimal if the asset portfolio is optimally determined. For this observe that

$$
q_f(\theta', \omega'(\theta')|\theta) = \frac{1}{1+r}\pi(\theta'|\theta)\frac{u'(c(\theta', a'(\theta'), b'))}{u'(c(\theta, a, b))}\eta\left[(1 - \delta + \delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right]
$$

$$\geq \frac{1}{1+r}\pi(\theta'|\theta)\left[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'}q_f(\theta'',\omega''(\theta'')|\theta')\right]$$

$$\text{if } a'(\theta',\theta,a,b)+b' > \mathcal{A}_b(\theta'),$$

$$q_f(\theta',\omega'(\theta')|\theta) = \frac{1}{1+r}\pi(\theta'|\theta)\left[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'}q_f(\theta'',\omega''(\theta'')|\theta')\right]$$

$$\geq \frac{1}{1+r}\pi(\theta'|\theta)\frac{u'(c(\theta',a'(\theta'),b'))}{u'(c(\theta,a,b))}\eta\left[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'}q_f(\theta'',\omega''(\theta'')|\theta')\right]$$

$$\text{if } a_l'(\theta',\theta,a,b)+b_l' > \mathcal{A}_l(\theta',b')$$

Similarly for the private bond,

$$q_p(\theta,\omega') = \frac{1}{1+r}\sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{u'(c(\theta',a'(\theta'),b'))}{u'(c(\theta,a,b))}\eta[(1-\delta+\delta\kappa)+\delta q_p(\theta',\omega'')]$$

$$\geq \frac{1}{1+r}\sum_{\theta'|\theta}\pi(\theta'|\theta)[(1-\delta+\delta\kappa)+\delta q_p(\theta',\omega'')]$$

$$\text{if } a'(\theta',\theta,a,b)+b' > \mathcal{A}_b(\theta') \text{ for at least one } \theta'\in\Theta,$$

$$q_p(\theta,\omega') = \frac{1}{1+r}\sum_{\theta'|\theta}\pi(\theta'|\theta)[(1-\delta+\delta\kappa)+\delta q_p(\theta',\omega'')]$$

$$\geq \frac{1}{1+r}\sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{u'(c(\theta',a'(\theta'),b'))}{u'(c(\theta,a,b))}\eta[(1-\delta+\delta\kappa)+\delta q_p(\theta',\omega'')]$$

$$\text{if } a_l'(\theta',\theta,a,b)+b_l' > \mathcal{B}_l(\theta',a'(\theta')) \text{ for at least one } \theta'\in\Theta.$$

Hence the portfolio is optimally determined. It then directly follows that $W^b(\theta,a,b) = V^b(\theta,x,b)$ and $W_p(\theta,a,b)+W_f(s,a,b) = V^l(s,x,b)$.

We therefore obtain a one-to-one map between $(x,b)$ and $\omega = a+b$ for a given $\theta$. More precisely, $B(\theta,x,b) = B(\theta,a,b)$, $c(\theta,a,b) = c(\theta,x,b)$, $c_p(\theta,a,b) = \tau_p(\theta,x,b)$, $c_f(\theta,a,b) = \tau_f(\theta,x,b)$, $c_p(\theta,a,b)+c_f(\theta,a,b) = \tau(\theta,x,b)$ and $n(\theta,a,b) = n(\theta,x,b)$. Moreover the endogenous limits of the sovereign and the lenders bind uniquely and exclusively when the participation constraints of the sovereign and the lenders bind, respectively. □

### Proof of Proposition B.4

Following Alvarez and Jermann (2000) we prove the proposition by construction. As for the proof of Proposition B.3, one establishes a one-to-one mapping from $(x,b)$ to $\omega = a+b$. The key equation linking those two objects is

$$I(\theta,a,b) = u'(c(\theta,a,b)) = u'(c(\theta,x,b)) = \frac{1+\nu_l(\theta,x,b)}{1+\nu_b(\theta,x,b)}\frac{1}{x},$$

where $I(\theta, a, b)$ is the Lagrange multiplier attached to the sovereign's budget constraint in the competitive problem. Given the initial bond holdings $(a_0, a_{l,0}, b_0, b_{l,0})$, the above condition enables to identify $x(\theta_0)$ if none of the enforcement constraint binds. We can subsequently determine consumption and labor.

For the participations constraints (16) and (17), one lets $\nu_l(\theta, x, b) = 0$ if $a'_l(\theta', \theta, a, b) + b'_l > \mathcal{A}_l(\theta', b')$ and $\nu_b(\theta, x, b) = 0$ if $a'(\theta', \theta, a, b) + b' > \mathcal{A}_b(\theta')$. Otherwise, $\nu_l(\theta, x, b)$ and $\nu_b(\theta, x, b)$ are determined by the above condition. We later show that in equilibrium, $\mathcal{A}_l(\theta', b') = \mathcal{B}_l(\theta', a'(\theta'))$ for all $(\theta', a'(\theta'), b')$. Hence, the two participations constraints are satisfied. Furthermore, given that the sovereign's and lenders' intertemporal budget constraints are satisfied, the resource feasibility constraints are also satisfied. $\square$

**Proof of Proposition 2**

We conduct a proof by contradiction. Recall the timing of actions: the sovereign first borrows in the private bond market before going to the Fund. This implies that $\omega_l(\theta')$ is the total credit line in state $\theta'$ and $a'_l(\theta') = \omega_l(\theta') - b'_l$ is the residual provided by the Fund after the private borrowing. Given the above timing, in state $(s', a'_l(\theta'), b'_l)$, for the following to be true $W^f(s', a'_l(\theta'), b'_l) + W^p(\theta', a'_l(\theta'), b'_l) = \theta Z + b'_l$, one needs that $W^f(s', a'_l(\theta'), b'_l) = \theta Z$ and $W^p(\theta', a'_l(\theta'), b'_l) = b'_l$, which then implies that $\mathcal{A}_f(\theta', b'_l) = \mathcal{B}_l(\theta', a'_l(\theta'))$.

Assume now by contradiction that $\mathcal{A}_f(\theta', b'_l) > \mathcal{B}_l(\theta', a'_l(\theta'))$ for a given state $(\theta', a'_l(\theta'), b'_l)$. More precisely, one has that $a'_l(\theta') + b'_l > \mathcal{A}_f(\theta', b'_l)$, and $a'_l(\theta') + b'_l = \mathcal{B}_l(\theta', a'_l)$, which implies that by definition, $W_f(s', a'_l, b'_l) > \theta Z$, and $W_p(\theta', a'_l, b'_l) = b'_l$. Observe that given equation (14) and (25), the Fund-provided asset price is equal to the risk-free price, while the private bond price is above it. Thus, the sovereign strictly prefers to accumulate debt in the private bond market than in the Fund. If the private lenders would accept to lend such that $\ddot{b}'_l > b'_l$, the Fund would then provide $\ddot{a}'_l(\theta') = \omega_l(\theta') - \ddot{b}'_l < a'_l(\theta')$ keeping the total level of indebtedness in state, $\theta'$, $\omega_l(\theta')$, constant. With this one gets

$$\omega_l(\theta') = \ddot{a}'_l(\theta') + \ddot{b}'_l > \mathcal{A}_f(\theta', \ddot{b}'_l) > \mathcal{A}_f(\theta', b'_l),$$
$$\omega_l(\theta') = \ddot{a}'_l(\theta') + \ddot{b}'_l = \mathcal{B}_l(\theta', a'_l) > \mathcal{B}_l(\theta', \ddot{a}'_l),$$

and the negative spread disappears. Moreover, with this new level of lending, the private lenders are better off as

$$W_p(\theta', \ddot{a}'_l, \ddot{b}'_l) > \ddot{b}'_l > b'_l = W_p(\theta', a'_l, b'_l),$$

where the first inequality comes from the fact that $\omega_l(\theta') > \mathcal{B}_l(\theta', \ddot{a}'_l)$. Hence, when the

Fund's constraint does not bind, the private lenders' constraint does not as well. Otherwise, the private lenders unnecessarily restrict their lending capacity.

Now assume the opposite situation — i.e. $\mathcal{A}_f(\theta', b_l') < \mathcal{B}_l(\theta', a_l'(\theta'))$. Particularly, for a given state $(\theta', a_l'(\theta'), b_l')$, one has $a_l'(\theta') + b_l' = \mathcal{A}_f(\theta', b_l')$ and $a_l'(\theta') + b_l' > \mathcal{B}_l(\theta', a_l')$. In this case, a negative spread appears on the Fund-provided assets. On distinguishes two cases. First, if $b_l' > 0$, we directly reach a contradiction as the sovereign prefers to hold debt in the Fund rather than the private bond market due to the negative spread. Second if $b_l' \leq 0$, the sovereign holds all its assets in the private bond market. If instead the private lenders would accept to borrow less such that $\ddot{b}_l' > b_l'$, the Fund would then provide $\ddot{a}_l'(\theta') = \omega_l(\theta') - \ddot{b}_l' < a_l'(\theta')$ keeping the total level of indebtedness in state, $\theta'$, $\omega_l(\theta')$, constant. We would then obtain

$$\omega_l(\theta') = \ddot{a}_l'(\theta') + \ddot{b}_l' = \mathcal{A}_f(\theta', b_l') > \mathcal{A}_f(\theta', \ddot{b}_l'),$$
$$\omega_l(\theta') = \ddot{a}_l'(\theta') + \ddot{b}_l' > \mathcal{B}_l(\theta', a_l') > \mathcal{B}_l(\theta', \ddot{a}_l'),$$

Again, the private lenders would be better off performing this lending policy instead of the other. Hence, when the private lenders' constraint does not bind, the Fund's constraint does not either.

As a result, it can only be the case that the Fund's constraint binds when the private lenders' constraint binds and vice versa. This means that in equilibrium for all $(s', a_l'(\theta'), b_l')$, $\mathcal{A}_f(\theta', b_l') = \mathcal{B}_l(\theta', a_l'(\theta'))$. □

**Proof of Corollary 1**

We conduct a proof by construction. Following Proposition 2, we do not distinguish between the Fund and the private lenders. We refer to two lending entities as the lenders. We distinguish three cases:

1. The sovereign's and lenders' participation constraints are not binding.

   The lenders' Euler equation reads
$$q_f(\theta', \omega'|\theta) = \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta')\right],$$
$$q_p(\theta, \omega') = \sum_{\theta'|\theta} \frac{\pi(\theta'|\theta)}{1+r}[(1-\delta+\delta\kappa) + \delta q_p(\theta', \omega'')],$$

   and the sovereign's Euler equations are
$$q_f(\theta', \omega'|\theta) = \beta\pi(\theta'|\theta)\frac{u'(c(\theta', \omega'))}{u'(c(\theta, \omega))}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta')\right]$$

$$q_p(\theta,\omega') = \beta \sum_{\theta'|\theta} \pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}[(1-\delta+\delta\kappa)+\delta q_p(\theta',\omega'')]$$

If none of the two constraints is ever binding,

$$\begin{aligned}
\sum_{\theta'|\theta} q_f(\theta',\omega'|\theta) &= \beta \sum_{\theta'|\theta} \pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}\left[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right] \\
&= \sum_{\theta'|\theta} \pi(\theta'|\theta)\frac{1}{1+r}\left[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right], \\
q_p(\theta,\omega') &= \beta \sum_{\theta'|\theta} \pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}[(1-\delta+\delta\kappa)+\delta q_f(\theta',\omega'')] \\
&= \sum_{\theta'|\theta} \pi(\theta'|\theta)\frac{1}{1+r}[(1-\delta+\delta\kappa)+\delta q_p(\theta',\omega'')],
\end{aligned}$$

It then follows that $Q_p(\theta,\omega') = \sum_{\theta'|\theta} Q_f(\theta',\omega'|\theta)$.

2. The sovereign's participation constraint is not binding and the lenders' participation constraint binds.

   The lenders' Euler equation reads

$$\begin{aligned}
q_f(\theta',\omega'|\theta) - \varphi_f(\theta') &= \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right], \\
q_p(\theta,\omega') - \sum_{\theta'|\theta}\varphi_p(\theta') &= \sum_{\theta'|\theta}\frac{\pi(\theta'|\theta)}{1+r}[(1-\delta+\delta\kappa)+\delta q_p(\theta',\omega'')],
\end{aligned}$$

   where $\sum_{\theta'|\theta}\varphi_p(\theta') = \sum_{\theta'|\theta}\varphi_f(\theta')$ under Proposition 2. The sovereign's Euler equations are

$$\begin{aligned}
q_f(\theta',\omega'|\theta) &= \beta\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}\left[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right], \\
q_p(\theta,\omega') &= \beta \sum_{\theta'|\theta} \pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}[(1-\delta+\delta\kappa)+\delta q_p(\theta',\omega'')].
\end{aligned}$$

   If the sovereign's participation constraint never binds,

$$\begin{aligned}
&\sum_{\theta'|\theta} q_f(\theta',\omega'|\theta) = \beta \sum_{\theta'|\theta} \pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}\left[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')\right] \quad \text{and} \\
&\sum_{\theta'|\theta} q_f(\theta',\omega'|\theta) > \sum_{\theta''|\theta'}\frac{\pi(\theta'|\theta)}{1+r}[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''|\theta')],
\end{aligned}$$

   Moreover, $Q_p(\theta,\omega') = \sum_{\theta'|\theta} Q_f(\theta',\omega'|\theta)$.

3. The sovereign's participation constraint binds and the lenders' participation constraint is not binding.

   The lenders' Euler equation is

$$q_f(\theta', \omega'|\theta) = \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta')\right],$$
$$q_p(\theta, \omega') = \sum_{\theta'|\theta} \frac{\pi(\theta'|\theta)}{1+r}[(1-\delta+\delta\kappa) + \delta q_p(\theta', \omega'')],$$

   and the sovereign's Euler equations are

$$q_f(\theta', \omega'|\theta)u'(c(\theta,\omega)) - \varphi_b(\theta') = \beta\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta')\right],$$
$$q_p(\theta,\omega')u'(c(\theta,\omega)) - \sum_{\theta'|\theta}\varphi_b(\theta') = \beta \sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}[(1-\delta+\delta\kappa) + \delta q_p(\theta', \omega'')].$$

   If the lenders' participation constraint never binds,

$$\sum_{\theta'|\theta} q_f(\theta', \omega'|\theta) > \beta \sum_{\theta'|\theta}\pi(\theta'|\theta)\frac{u'(c(\theta',\omega'))}{u'(c(\theta,\omega))}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta')\right] \quad \text{and}$$
$$\sum_{\theta'|\theta} q_f(\theta', \omega'|\theta) = \sum_{\theta''|\theta'} \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''|\theta')\right],$$

   Moreover, $Q_p(\theta,\omega') = \sum_{\theta'|\theta} Q_f(\theta',\omega'|\theta)$.

From those three cases, one can conclude that the bond price in the private market, $q_p(\theta,\omega')$, is always equal to the price in the Fund, $\sum_{\theta'|\theta} q_f(\theta'|\theta,\omega)$. As a result, the division of debt between $b'$ and $\bar{a}'$ will be indeterminate if the sovereign can freely access the Fund and the private bond market as we show in Proposition 4. □

**Proof of Proposition 3**

Recall that the endogenous limit for private lenders is defined as

$$W^p(\theta', a_l'(\theta'), \mathcal{B}_l(\theta', a_l'(\theta')) - a_l'(\theta')) = b_l',$$

where $a_l'(\theta') = A_l(\theta', \theta, a, b)$ is taken as given. This condition is obtained from applying the following transversality condition to the private lenders' value:

$$\lim_{n\to\infty} \mathbb{E}\left\{\left[\prod_{j=0}^{n} Q_p\Big(\theta^{t+j}, \omega\big(\theta^{t+j+1}\big)\Big)\right] b_l\big(\theta^{t+j+1}\big)\middle|\theta^t\right\} = 0, \quad \text{with}$$

$$Q_p(\theta^{t+j},\omega(\theta^{t+j+1})) = \frac{q_p(\theta^{t+j},\omega(\theta^{t+j+1}))}{1-\delta+\delta\kappa+\delta q_p(\theta^{t+j+1},\omega(\theta^{t+j+2}))}.$$

It means that the private lenders should never lend more than the present discounted value of the private debt. Whenever this condition is breached, private lenders run the risk of recording losses, violating the assumption of competitiveness. This arises whenever the negative spread kicks in.

Given Proposition 2 and Corollary 1, from the pricing equation (14), one sees that, when equation (23) binds, $q_p(\theta,\omega')$ settles above the risk-neutral pricing. In such circumstances, the private lenders would loose money if they continue to lend to the sovereign as they discount the future at rate $\frac{1}{1+r}$, while the discount factor is $Q_p > \frac{1}{1+r}$. This implies that

$$W^p(\theta', a_l'(\theta'), b_l') = \mathbb{E}\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c_p(\theta^t)\middle|\theta_0\right] < \mathbb{E}\left[\sum_{t=0}^{\infty}\prod_{j=0}^{t} Q_p(\theta^j,\omega(\theta^{j+1}))c_p(\theta^t)\middle|\theta_0\right] = b_l'.$$

The private lenders must therefore stop their lending activity if they do not want to run losses. However, they cannot force the sovereign to repay in advance. As a result, the private lenders can only refuse to roll-over the maturing portion of the debt — that is $b' > \delta b$.

Given this, it is clear that in the case of short term debt (i.e. $\delta = 0$), the binding constraint of the lenders directly translate to a complete shutdown of private lending. □

**Proof of Proposition 4**

We conduct a proof by construction. When (23) does not bind, the budget constraint reads

$$c + q_p(\theta,\omega')(b'-\delta b) + \sum_{\theta'|\theta} q_f(\theta',\omega'|\theta)(a'(\theta')-\delta a) = \theta f(n) + (1-\delta+\delta\kappa)(b+a).$$

Given that $\sum_{\theta'|\theta} q_f(\theta',\omega'|\theta)\hat{a}(\theta') = 0$ and Corollary 1, it can be rewritten as

$$\begin{aligned} c + q_f(\theta,\omega')(b'-\delta b) + q_f(\theta,\omega')(\bar{a}'-\delta\bar{a}) &= \theta f(n) + (1-\delta+\delta\kappa)(b+a), \\ c + q(\theta,\omega')(\bar{\omega}'-\delta(b+a)) &= \theta f(n) + (1-\delta+\delta\kappa)(b+a). \end{aligned}$$

Having the same price and being equally accessible, private and Fund-provided bonds are prefect substitute, so that the decomposition of $\bar{\omega}'$ between $b'$ and $\bar{a}'$ is indeterminate. □

**Proof of Corollary 2**

We conduct a proof by construction. Setting $\bar{a}' = 0$ implies that the sovereign exclusively accumulates debt in the private bond market, resolving the indetermination. None of the debt is located in the Fund which solely provides insurance. However, it is not always possible to set $\bar{a}' = 0$ if one does not want the constraint $W_p(\theta', a_l', b_l') \geq b_l'$ to be violated. More precisely, the maximal level of debt the private lender can absorb is given by

$$b' = \min_{\theta' \in \ddot{\Theta}} \{\theta Z - W^l(\theta', \mathcal{A}_l(\theta'))\},$$

where $W^l = W_p + W_f$ and $\mathcal{A}_l(\theta') \equiv \mathcal{A}_f(\theta', b') = \mathcal{B}_l(\theta', a')$ under Proposition 2. Moreover, $\ddot{\Theta}$ designate the set of all $\theta'$ such that $\pi(\theta'|\theta) > 0$. Then define

$$\underline{\underline{a}}(\theta, b) = \bar{\omega}'(\theta, a, b) - \min_{\theta' \in \ddot{\Theta}} \{\theta Z - W^l(\theta', \mathcal{A}_l(\theta'))\},$$

as the minimal level of debt the Fund can absorb in a given state $(\theta, b)$. Such a threshold value exists given Propositions B.2 and B.3.

Obviously, $\underline{\underline{a}}(\theta, b) \leq 0$ as $\min_{\theta' \in \ddot{\Theta}} \{\theta Z - W^l(\theta', \mathcal{A}_l(\theta'))\} \geq \bar{\omega}'(\theta, a, b)$ by definition of the lenders' participation constraint. Furthermore, $\underline{\underline{a}}(\theta, b) \geq \delta b$ given Proposition 3. □

### Proof of Lemma B.1

First, the transversality condition of the borrower is:[38]

$$\lim_{j \to \infty} \mathbb{E}_t Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[a(\theta^{t+j}) + b(\theta^{t+j})] = 0,$$

where

$$Q_f(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t) = Q_f(\theta^{t+j}, \omega(\theta^{t+j})|\theta^{t+j-1}) \cdots Q_f(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t).$$

Recall that, under Corollary 1, and define

$$q(\theta^t, \omega(\theta^{t+1})) \equiv q_p(\theta^t, \omega(\theta^t)) = \sum_{s^{t+1}|\theta^t} q_f(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t),$$
$$Q(\theta^t, \omega(\theta^{t+1})) \equiv Q_p(\theta^t, \omega(\theta^t)) = \sum_{s^{t+1}|\theta t} Q_f(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t),$$

for all $t$ and $\theta^t$. Using the borrower's budget constraint, one gets

$$(a(\theta^t) + b(\theta^t))(1 - \delta + \delta\kappa + \delta q(\theta^t, \omega(\theta^{t+1}))) =$$
$$c(\theta^t, a(\theta^t), b(\theta^t)) + q(\theta^t, \omega(\theta^{t+1}))a(\theta^{t+1}) + q(\theta^t, \omega(\theta^{t+1}))b(\theta^{t+1}) - Y(\theta^t, a(\theta^t), b(\theta^t)),$$

[38]The differentiability and strict concavity and convexity assumptions of the functional forms guarantee the local uniqueness of the policy and value functions. This in turn implies that the transversality conditions are satisfied.

where, $Y(\theta^t, a(\theta^t), b(\theta^t)) = \theta(\theta_t)f(n(\theta^t, a(\theta^t), b(\theta^t)))$ for all $t$ and $\theta^t$. Iterating forward the budget constraint and using the transversality condition as well as the equilibrium price relationship, one obtains

$$
\begin{aligned}
&a(\theta^t) + b(\theta^t) = \\
&\qquad \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))].
\end{aligned}
$$

Similarly, the transversality condition of the lender is:

$$
\lim_{t\to\infty} \mathbb{E}_t Q(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t)[a_l(\theta^{t+1}) + b_l(\theta^{t+1})] = 0.
$$

Using the consolidated budget constraint of both lenders, one gets

$$
\begin{aligned}
&(a_l(\theta^t) + b_l(\theta^t))(1 - \delta + \delta\kappa + \delta q(\theta^t, \omega(\theta^{t+1}))) = \\
&\qquad c_l(\theta^t, a(\theta^t), b(\theta^t)) + q(\theta^t, \omega(\theta^{t+1}))a_l(\theta^{t+1}) + q(\theta^t, \omega(\theta^{t+1}))b_l(\theta^{t+1}).
\end{aligned}
$$

Iterating forward the budget constraint and using the transversality condition as well as the equilibrium price relationship, one obtains

$$
\begin{aligned}
a_l(\theta^t) + b_l(\theta^t) &= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t) c_l(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) \\
&= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))] \\
&= -a(\theta^t) - b(\theta^t).
\end{aligned}
$$

The market clearing conditions in the Fund and the private bond market implies that $a_l(\theta^t) + a(\theta^t) = 0$ and $b(\theta^t) + b_l(\theta^t) = 0$, respectively, for all $t$ and $\theta^t$.

If the participation constraint of one of the contracting parties is binding, the borrowing limit for of the constrained agent in the decentralised economy is determined by

$$
\mathcal{A}_b(\theta^t) = \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))], \tag{C.4}
$$

$$
\mathcal{A}_l(\theta^n) = \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{n+j}, \omega(\theta^{n+j})|\theta^n) c_l(\theta^{n+j}, a(\theta^{n+j}), b(\theta^{n+j}))), \tag{C.5}
$$

where $\mathcal{A}_l(\theta^n) \equiv \mathcal{A}_f(\theta^n) = \mathcal{B}_l(\theta^n)$ under Proposition 2. Further note that one distinguishes between $t$ and $n$ with $t \neq n$ as the sovereign's and the lenders' constraints cannot bind at the same time if the contract is feasible. $\square$

### Proof of Corollary B.1

We conduct a proof by construction. Consider two sequences of private bonds $\{b(\theta^t)\}_{t=0}^{\infty}$ and $\{\ddot{b}(\theta^t)\}_{t=0}^{\infty}$ satisfying the definition of a RCE with $\ddot{b}(\theta_0) = b(\theta_0)$ and $\ddot{b}(\theta^t) \neq b(\theta^t)$ for all $t > 0$ and $\theta^t \neq s_0$. Hence, at $t = 0$, the budget constraint reads

$$\begin{aligned}
a(\theta_0)(1-\delta+\delta\kappa) = &\sum_{s^1|\theta_0} q_f(\theta^1, \omega(\theta^1)|\theta_0)(a(\theta^1) - \delta a(\theta_0)) + q_p(\theta_0, \omega(\theta^1))(b(\theta^1) - \delta b(\theta_0)) + \\
&c(\theta_0, a(\theta_0), b(\theta_0)) - b(\theta_0)(1-\delta+\delta\kappa) - Y(\theta_0, a(\theta_0), b(\theta_0)), \\
\ddot{a}(\theta_0)(1-\delta+\delta\kappa) = &\sum_{s^1|\theta_0} q_f(\theta^1, \ddot{\omega}(\theta^1)|\theta_0)(\ddot{a}(\theta^1) - \delta \ddot{a}(\theta_0)) + q_p(\theta_0, \ddot{\omega}(\theta^1))(\ddot{b}(\theta^1) - \delta \ddot{b}(\theta_0)) + \\
&c(\theta_0, \ddot{a}(\theta_0), \ddot{b}(\theta_0)) - \ddot{b}(\theta_0)(1-\delta+\delta\kappa) - Y(\theta_0, \ddot{a}(\theta_0), \ddot{b}(\theta_0)).
\end{aligned}$$

Given that $\ddot{b}(\theta_0) = b(\theta_0)$ and the initial asset holdings in the Fund being $\ddot{a}(\theta_0) = a(\theta_0) = 0$, it holds that $\omega(\theta_0) = \ddot{\omega}(\theta_0)$. The two budget constraints can therefore be combined resulting to the fact that $a(\theta^1) + b(\theta^1) = \ddot{a}(\theta^1) + \ddot{b}(\theta^1)$, where we used Corollary 1. Iterating forward the same argument for $t > 0$, we obtain that $a(\theta^t) + b(\theta^t) = \ddot{a}(\theta^t) + \ddot{b}(\theta^t)$, or equivalently,

$$\begin{aligned}
&\mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))] \\
&\qquad = \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \ddot{\omega}(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, \ddot{a}(\theta^{t+j}), \ddot{b}(\theta^{t+j})) - Y(\theta^{t+j}, \ddot{a}(\theta^{t+j}), \ddot{b}(\theta^{t+j}))],
\end{aligned}$$

for all $t$ and $\theta^t$. The generalisation of the argument for any $t$ relies on the fact that the alternative private bond sequence $\ddot{b}(\theta^t) \neq b(\theta^t)$ is consistent with (21) for all $t$ and $\theta^t$.

Thus, a given sequence of private bonds $\{b(\theta^t)\}_{t=0}^{\infty}$ for which the sovereign's problem with borrowing limits $\mathcal{A}_b(\theta^t)$ and the lender's problem with NPV limits $\mathcal{A}_f(\theta^t, b(\theta^t))$ and $\mathcal{B}_l(\theta^t, a(\theta^t))$ have a solution, the alternative private bond sequence $\{\ddot{b}(\theta^t)\}_{t=0}^{\infty}$ that can be sustained as a RCE with $\ddot{b}(\theta_0) = b(\theta_0)$ and $\ddot{b}(\theta^t) \neq b(\theta^t)$ for all $t > 0$ and $\theta^t \neq s_0$ is equivalent to $\{b(\theta^t)\}_{t=0}^{\infty}$. $\square$

### Proof of Proposition 5

Given the definitions of the sovereign's endogenous borrowing limits, it holds that for all $\theta$ and for all level of private debt $b$ within the Fund's prescription $\bar{\omega}$, $V^b(\theta, \mathcal{A}_b(\theta), b) = V^{ap}(\theta, \mathcal{A}_b^{ap}(\theta)) = V^{af}(\theta)$. There is therefore no *partial* default incentive when the borrower's constraint binds and $b \geq \bar{\omega}$.

Turning now to the case in which the Fund's participation constraint binds, assume there exists a level of private debt $\bar{\bar{b}} \geq \bar{\omega}'$ such that for all $\theta$

$$V^f(\theta, \mathcal{A}_f(\theta, \bar{\bar{b}}_l) - \bar{\bar{b}}_l, \bar{\bar{b}}_l, 0) = V^f(\theta, \mathcal{A}_f(\theta, 0, 1), 0, 1),$$

which implies that for all $b < \bar{\bar{b}}$

$$V^f(\theta, \mathcal{A}_f(\theta, b) - b_l, b_l, 0) > V^f(\theta, \mathcal{A}_f(\theta, 0, 1), 0, 1),$$
$$W^b(\theta, -\mathcal{A}_f(\theta, \bar{\bar{b}}) - \bar{\bar{b}}, \bar{\bar{b}}) < V^{ap}(\theta, -\mathcal{A}_f(\theta, 0, 1)).$$

In that situation, the sovereign will gain from repudiating its private debt when the lender's participation constraint binds with $b < \bar{\bar{b}} < 0$. The private lenders anticipate this behavior. They impose a risk premium for all $b' < \bar{\bar{b}}$ whenever the lender's participation constraint binds with strictly positive probability in the next period. Even if the risk premium might be relatively small, this directly reduces the amount of debt the sovereign can raise from the private lenders. Under the assumption that the sovereign desires to accumulate no more debt than the Fund can provide, the sovereign does not accumulate more than $-\bar{\bar{b}}$ in the private bond market to avoid this risk premium and simply accumulates more debt in the Fund. This in turn implies that *partial* defaults never occur on equilibrium path as the sovereign never accumulates a sufficient level of private debt in the states in which *partial* defaults would be attractive. Conversely if for all $\theta$ and $b \geq \bar{\omega}$

$$V^f(\theta, \mathcal{A}_f(\theta, b) - b, b, 0) < V^f(\theta, \mathcal{A}_f(\theta, 0, 1), 0, 1),$$

then, there is no advantage in entering in *partial* default. $\square$

**Proof of Proposition 6**

In light of Proposition 5, the sovereign will enter in *partial* default only if it overborrowed beforehand. In what follows, one refers to the decentralised Fund contract as it enables a better exposition of the argument. Let's focus first on the sovereign's participation constraint and consider that there are three productivity states in the economy. Assume further that for a given Fund's lending policy $\bar{\omega}' = \bar{a}' + b'$,

$$\begin{aligned}
&\bar{a}' + b' + \hat{a}'(1) > \mathcal{A}_b(1) \quad \text{and} \quad \bar{a}'_l + b'_l + \hat{a}'_l(1) = \mathcal{A}_f(1, b'_l),\\
&\bar{a}' + b' + \hat{a}'(2) > \mathcal{A}_b(2) \quad \text{and} \quad \bar{a}'_l + b'_l + \hat{a}'_l(2) > \mathcal{A}_f(2, b'_l),\\
&\bar{a}' + b' + \hat{a}'(3) = \mathcal{A}_b(3) \quad \text{and} \quad \bar{a}'_l + b'_l + \hat{a}'_l(3) > \mathcal{A}_f(3, b'_l).
\end{aligned}$$

The borrower decides to overborrow the amount $\delta a + \grave{b}' < \bar{a}' + b'$ with $\delta a \geq \bar{a}'$ and $\grave{b}' < b'$. If it keeps the same level of insurance, it gets

$$
\begin{aligned}
\delta a + \grave{b}' + \hat{a}'(1) \geq \mathcal{A}_b(1) \quad &\text{and} \quad \delta a_l + \grave{b}'_l + \hat{a}'_l(1) > \mathcal{A}_f(1, b'_l), \\
\delta a + \grave{b}' + \hat{a}'(2) \geq \mathcal{A}_b(2) \quad &\text{and} \quad \delta a_l + \grave{b}'_l + \hat{a}'_l(2) > \mathcal{A}_f(2, b'_l), \\
\delta a + \grave{b}' + \hat{a}'(3) < \mathcal{A}_b(3) \quad &\text{and} \quad \delta a_l + \grave{b}'_l + \hat{a}'_l(3) > \mathcal{A}_f(3, b'_l).
\end{aligned}
$$

If the borrower decides to default on its private debt, it gets

$$
\begin{aligned}
\delta a + \hat{a}'(1) > \bar{a}' + b' + \hat{a}'(1) > \mathcal{A}_b(1) = \mathcal{A}_b^{ap}(1) \quad &\text{and} \quad \delta a_l + \hat{a}'_l(1) \geq \mathcal{A}_f^{ap}(1), \\
\delta a + \hat{a}'(2) > \bar{a}' + b' + \hat{a}'(2) > \mathcal{A}_b(2) = \mathcal{A}_b^{ap}(2) \quad &\text{and} \quad \delta a_l + \hat{a}'_l(2) > \mathcal{A}_f^{ap}(2), \\
\delta a + \hat{a}'(3) > \bar{a}' + b' + \hat{a}'(3) = \mathcal{A}_b(3) = \mathcal{A}_b^{ap}(3) \quad &\text{and} \quad \delta a_l + \hat{a}'_l(3) > \mathcal{A}_f^{ap}(3),
\end{aligned}
$$

which is clearly a better option than repaying the private debt. Thus, with this level of insurance, the borrower will default in all states. In other words, the default decision is not state contingent. Instead, the borrower can decide to reshuffle the insurance such that $\delta a + \grave{b}' + \grave{\hat{a}}'(3) = \mathcal{A}_b(3)$, meaning that the borrower would not default in the third state. For that purpose, the Arrow-type securities become

$$
\grave{\hat{a}}'(3) = \hat{a}'(3) - [(\delta a + \grave{b}') - (\bar{a}' + b')] \equiv \hat{a}'(3) - \Delta,
$$

and for all $i \in \{1, 2\}$ and a given $\theta \in \{1, 2, 3\}$,

$$
\grave{\hat{a}}'(i) = \hat{a}'(i) + \Delta \frac{\pi(3|\theta)}{\sum_{j=1}^{2} \pi(j|\theta)} < \hat{a}'(i).
$$

Basically, the borrower takes more insurance in the third state and less in the other two states. Notice that in the states in which the borrower takes less insurance, one has a double burden: more debt and less insurance. Now the question is: can the Fund sustain this reshuffle of Arrow-type securities? To answer that question, define $\ddot{\hat{a}}'_l(3)$ such that $\delta a_l + \ddot{\hat{a}}'_l(3) = \mathcal{A}_f^{ap}(3)$. In words, $\ddot{\hat{a}}'(3)$ represents the highest level of insurance the Fund can provide in state 3. Given this definition, one gets that

$$
\delta a_l + \hat{a}'_l(3) \geq \delta a_l + \ddot{\hat{a}}'_l(3) = \mathcal{A}_f^{ap}(3),
$$

leading to $\hat{a}'_l(3) \geq \ddot{\hat{a}}'_l(3)$. Using the definition of $\grave{\hat{a}}'_l(3)$,

$$
\delta a_l + \grave{\hat{a}}'_l(3) = \delta a_l + \hat{a}'_l(3) - [(\delta a_l + \grave{b}'_l) - (\bar{a}'_l + b'_l)]
$$

$$= \bar{a}'_l + \hat{a}'_l(3) - (\grave{b}'_l - b'_l)$$
$$\geq \delta a_l + \ddot{\hat{a}}'_l(3) - (\grave{b}'_l - b'_l),$$

where the inequality comes from the fact that $\hat{a}'_l(3) \geq \ddot{\hat{a}}'_l(3)$ and $\bar{a}'_l \geq \delta a_l$. Rearranging the expression leads to $(\grave{b}'_l - b'_l) \geq \ddot{\hat{a}}'_l(3) - \grave{\hat{a}}'_l(3)$. As one assumed that $\grave{b}'_l > b'_l$, for the above inequality to hold it must be that $\grave{\hat{a}}'_l(3) < \ddot{\hat{a}}'_l(3)$. This in turn implies that $\delta a_l + \grave{\hat{a}}'_l(3) < \mathcal{A}^{ap}_f(3)$. The Fund will therefore not accept this reshuffle as its participation constraint is violated in the third state if the borrower defaults on its private debt. Moreover, notice that if the reshuffling of Arrow-type securities is such that for at least one of the two states $i \in \{1, 2\}$, $\delta a + \grave{\hat{a}}'(i) < \mathcal{A}^{ap}_b(i)$, then it is not optimal for the borrower to perform the reshuffling of Arrow-type securities. Hence, the mechanism is the following. The sovereign cannot reshuffle because it is either not optimal for itself (as it would loose too much if the third state does not realize) or because the Fund refuses this reshuffle (as it would violate its constraint). As a result, being unable to insure its overaccumulation of debt, the borrower will partially default in all future states as soon as it accumulates more debt than what the Fund prescribes.

The previous case was focusing on the sovereign's participation constraint. We now pass to the states in which the Fund's participation constraint binds. As before consider there exists a level of private debt $\bar{\bar{b}}' \geq \bar{\omega}'$ such that $\mathcal{A}_f(\theta', \bar{\bar{b}}') = \mathcal{A}^{ap}_f(\theta')$. It then holds for all $\grave{b}' < \bar{\omega}' \leq \bar{\bar{b}}'$ and for all $\theta'$, $\mathcal{A}_f(\theta', \grave{b}') > \mathcal{A}^{ap}_f(\theta')$. If this is not the case, this means that there is an arbitrage opportunity. Consider that among all $\theta'$, there exists a single $\ddot{\theta}'$ for which

$$\hat{a}'_l(\theta') + \bar{a}' = \mathcal{A}^{ap}_f(\theta') < \mathcal{A}_f(\theta', \grave{b}') \quad \forall \theta' \in \Theta \setminus \ddot{\theta}',$$
$$\hat{a}'_l(\ddot{\theta}') + \bar{a}' = \mathcal{A}^{ap}_f(\ddot{\theta}') = \mathcal{A}_f(\ddot{\theta}', \grave{b}').$$

The Fund can then reshuffle the Arrow-type securities. More precisely, it can sufficiently increase $\hat{a}'_l(s')$ by $\epsilon > 0$ such that $\hat{a}'_l(s') + \bar{a}' + \epsilon < \mathcal{A}_f(\grave{s}', \grave{b}')$ for all $\theta' \in S \setminus \ddot{\theta}'$. Given this increase, it can now slightly decrease $\hat{a}'_l(\ddot{\theta}')$. As a result,

$$\hat{a}'_l(s') + \bar{a}' + \epsilon < \mathcal{A}_f(s', \grave{b}') \quad \forall \theta' \in \Theta \setminus \ddot{\theta}',$$
$$\hat{a}'_l(\ddot{\theta}') + \bar{a}' - \frac{\sum_{\theta' \in S \setminus \ddot{\theta}'} \pi(\theta'|\theta)\epsilon}{\pi(\ddot{\theta}'|\theta)} < \mathcal{A}_f(\ddot{\theta}', \grave{b}'),$$

contradicting our initial assumption. To complete the argument, note that the reshuffling is such that

$$\sum_{\theta' \in \Theta \setminus \ddot{\theta}'} \pi(\theta'|\theta)(\hat{a}'_l(\theta') + \epsilon) + \pi(\ddot{\theta}'|\theta)\left(\hat{a}'_l(\ddot{\theta}') - \frac{\sum_{\theta' \in S \setminus \ddot{\theta}'} \pi(\theta'|\theta)\epsilon}{\pi(\ddot{\theta}'|\theta)}\right) = \sum_{\theta'|\theta} \pi(\theta'|\theta)\hat{a}'_l(\theta').$$

Given this, one has that for all $\grave{b}' < \bar{\omega}' \leq \bar{\bar{b}}$ and for all $\theta' \in \Theta$,

$$W^b(\theta', -\mathcal{A}_f(\theta', \grave{b}') - \grave{b}', \grave{b}') < V^{ap}(\theta', -\mathcal{A}_f^{ap}(\theta')).$$

In words, as soon as the sovereign overborrows and the Fund's participation constraint binds, it will enter in *partial* default. Now, if for all $b' \geq \bar{\omega}'$ and for all $\theta' \in S$, $\mathcal{A}_f(\theta', b') < \mathcal{A}_f^{ap}(\theta')$, then the sovereign simply never overborrows. □

**Proof of Corollary 3**

Given Proposition 5, it holds that for all $\theta$, and $a$ and $b$ such that $a + b \geq \bar{\omega}$, $D_p(\theta, a, b) = 0$, and under Proposition 1, $D_f(\theta, a, b) = 0$. Moreover, given Proposition 6, for all $\theta$ and for all $a$ and $b$ such that $a + b < \bar{\omega}$, $D_p(\theta, a, b) = 1$ and $D_f(\theta, a, b) = 0$, which implies that for all $\theta$ and for all $\omega' < \bar{\omega}'$, $q_p(\theta, \omega') = 0$. □

## D Additional Details of the Calibration

### D.1 Data Sources and Measurement

We calibrate the model for Italy. The main data sources and definitions of data variables are listed in Table D.1. The data frequency is quarterly, and the time periods are from 1992Q1 to 2019Q4, avoiding the interruption caused by COVID-19. Whenever the data souces contain the seasonally adjusted series for the relevant data variables, we use the them directly; otherwise, we seasonally adjust the data series using X11 algorithm with R package `seasonal`. For debt service and average maturity, we use annual series since quarterly ones are unavailable meanwhile we only need the sample avearge for our calibration.

To map the data to the model, we construct model consistent data measures as below.

**Labor input** For the aggregate labor input $n_t$, we use two series, the aggregate working hours $H_t$ and the total employment $E_t$. We calculate the normalized labor input as $n_t = H_t/(E_t \times 5200)$, assuming 100 hours of allocatable time per worker per week. However, for second order data moment computations, we use $H_t$ directly, since the per worker annual working hours do not show a significant cyclical pattern and both the level and the trend do not affect the computation of the moments.

**Fiscal position and private consumption** We hold the premise of fitting the *observed* fiscal behavior of Italy, so that we use directly the *data measures* of primary surplus to calibrate the model, and correspondingly, define the model consistent measure of consumption as the difference between output and primary surplus, since in the model, primary surplus

Table D.1: Data Sources and Definitions

| Series | Sources | Unit |
|---|---|---|
| Output | ECB[a] | 1 million 2010 constant euro |
| Total working hours | ECB[b] | 1 thousand hours |
| Employment | Eurostat[c] | 1000 persons |
| Government debt | Eurostat[d] | end-of-quarter percentage |
| Debt service | AMECO[e] | end-of-year percentage of GDP, annual |
| Fiscal surplus | Eurostat, Bank of Italy[f] | million euro |
| Long-term bond yields | Eurostat[g] | percentage, nominal |
| Debt maturity | OECD, EuroStat, ESM[h] | years, annual |
| Labor share | AMECO[i] | percentage, annual |

[a] Real GDP, chain linked volume; data in 1991Q1–2014Q2 under ESA95, and data in 2014Q3–2019Q4 under ESA10, with the latter series adjusted to match the former in the overlapping periods 1995Q1–2014Q2.

[b] Hours for total employment; same adjustment to data under ESA95 and ESA10 as for output.

[c] Total employment (Eurostat label `lfsi_emp_q_h`).

[d] General government consolidated gross debt (Eurostat label `gov_10q_ggdebt`); quarterly series available for 2000Q1 onwards, and for 1992Q1-1999Q4, interpolate annual series instead; measured as end-of-quarter debt stock to total GDP of previous 4 quarters.

[e] AMECO (label `UYIGE`) for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1992–1995.

[f] Eurostat (net lending, label `gov_10q_ggnfa`) 1999Q1–2019Q4; Bank of Italy (financing of the gross borrowing requirement, including privatization receipts) 1992Q1–1998Q4.

[g] EMU convergence criterion bond yields (label `irt_lt_mcby_q`).

[h] See text below; ESM data are obtained from private correspondance.

[i] Compensation of employees (UWCD) plus gross operating surplus (UOGD) minus gross operating surplus adjusted for imputed compensation of self-employed (UQGD), then divided by nominal GDP (UVGD).

$ps$ is equal to output $y$ minus consumption $c$. We have raw data on quarterly fiscal *surplus* instead of *primary surplus*. To arrive the latter from the former, we add back interest payment of the government to fiscal surplus. To be more precise, we first calculate fiscal suplus to GDP ratio (nominal quarterly GDP obtained from CEIC for Italy). Second, we obtain quarterly interest payment to GDP ratio from Eurostat (label `gov_10q_ggnfa`) for 1999Q1 onwards, and use the end-of-year annual value (obtained from AMECO and European Commission *General Government Data*) for each quarter in the year as a proxy for

1992Q1–1998Q4. Third, we add fiscal surplus to GDP and interest payment to GDP to arrive at primary surplus to GDP, and conduct seasonal adjustment to the series. And finally, we obtain the level of quarterly (*real*) primary surplus by multiplying the seasonally adjusted primary surplus to GDP ratio to (*real*) output in the same quarter.

**Government debt, spread, and maturity** Following Bocola et al. (2019) and Ábrahám et al. (2021), we calibrate the model to match the total public debt of Italy.

For the nominal risk free rate, we use the annualized short-term (3M) interest rates in the Euro money market (obtaied from EuroStat with label `irt_st_q`) for 1999Q1–2019Q4, and the annulized short-term (3M) bond return of Germany (obtained from EuroStat with label `irt_h_mr3_q`) for 1992Q1–1998Q4, before the start of Euro. To convert the nominal risk-free rate into real rate, we subtract GDP deflator of Germany from the former series. To arrive at a meaningful measure of the *real* spread, i.e., a spread unaffected by expected inflation hence rightly reflecting credit risk, we split the sample into to two parts. After the introduction of Euro, we can directly use the spread between the long-term nominal bond yields and the nominal risk-free rate, since all rates are denominated in euro and thus subject to the same inflation expectation. For the period before Euro, we follow Ábrahám et al. (2021) and use spot and forward exchange rates (retrieved from Datastream) to convert the German nominal risk free rate into Italy's local currency, hence deriving a synthetic local currency risk free rate, and finally take the difference between the local nominal long-term bond yield with the synthetic risk free rate.

The information on the maturity structure of the government debt for Italy is not comprehensive. We manage to obtain government debt maturity data over 1990–2015 for Italy from all sources listed in Table D.1.

### D.2 Estimation Results

Panel (a) of Figure D.1 plots the sample productivity series for Italy used for our calibration of the productivity shock process. It is clear that the during the 2008 Global Financial Crisis, there was prominent negative growth in productivities. This distinctive feature in the productivity dyanmics is also the main motivation for the use of Markov regime switching model (27) to calibrate the productivity shock. Correspondingly, Panel (b) shows that a 2-regime specification capture the crisis dynamics very well, with the smoothed regime probabilities reach almost 1 during the sudden drop periods observed in Panel (a).

The final estimation results are summarized in Table D.2. Note that we identify regime

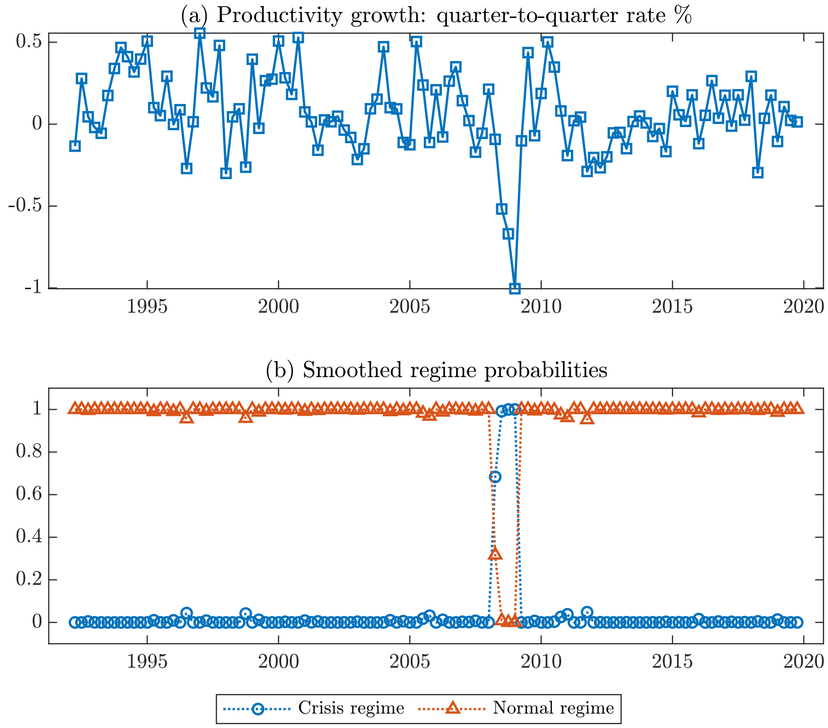



Figure D.1: Data sample and the estimated smoothed regime probabilities

1 as the crisis regime, and regime 2 as the normal regime. To overcome the local maximum problem of the highly nonlinear likelihood function, we randomize initializations of the EM algorithm of 1,000 times.

Table D.2: Parameters of the regime switching productivity process

| | $\mu(\varsigma)$ | $\rho(\varsigma)$ | $\sigma(\varsigma)$ | $P$ | $\varsigma' = 1$ | $\varsigma' = 2$ | invariant dist. |
|---|---|---|---|---|---|---|---|
| $\varsigma = 1$ | $-0.0336$ | 0.9018 | 0.0009 | $\varsigma = 1$ | 0.6633 | 0.3367 | 0.0372 |
| $\varsigma = 2$ | 0.0009 | 0.2167 | 0.0020 | $\varsigma = 2$ | 0.0130 | 0.9870 | 0.9628 |

*Notes*: $\varsigma$ denotes the current regime of productivity shock, and $\varsigma'$ denotes that of the next period.

## E Welfare Calculations

This section describes how the welfare gains depicted in Table 3 are computed. Similar to Ábrahám et al. (2021), define value of the sovereign for a sequence $\{c(\theta^t), n(\theta^t)\}$ starting from an initial state at $t = 0$ as

$$V^b(\{c(\theta^t), n(\theta^t)\}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c(\theta^t), n(\theta^t)) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[ \log(c(\theta^t)) + \gamma \frac{(1 - n(\theta^t))^{\sigma_n} - 1}{1 - \sigma_n} \right],$$

where the last equality is obtained from the functional form considered in Section 5. We denote the sovereign's allocations with the Fund by $\{c^f(\theta^t), n^f(\theta^t)\}$ and the allocations without the Fund by $\{c^i(\theta^t), n^i(\theta^t)\}$. The value for the borrower with and without the Fund is given by $W^{bf}(\theta, \omega) = W^{bf}(\{c^f(\theta^t), n^f(\theta^t)\})$ and $V^{bi}(\theta, b) = V^{bi}(\{c(\theta^t), n(\theta^t)\})$, respectively. To properly compare the two economies, we consider the point where $\omega = b =: o$. Thus $(\theta, o)$ represents the initial state for both economies. Now define $V^{bi}(\theta, o; \chi) = V^{bi}(\{(1+\chi)c(\theta^t), n(\theta^t)\})$, where $\chi(\theta, o)$ represents the consumption-equivalent welfare gain of the Fund's intervention. It then directly follows that the welfare gain is computed in the following way $V^{bi}(\theta, o; \chi) = W^{bf}(\theta, o)$. Given the above functional form, we have that $\frac{\log(1+\chi)}{1-\beta} + V^{bi}(\theta, o) = V^{bf}(\theta, o)$. The welfare gain therefore boils down to $\chi(\theta, o) = \exp\left[(V^{bf}(\theta, o) - V^{bi}(\theta, o))(1 - \beta)\right] - 1$. We concentrate our analysis to the case in which $o = 0$.

**Welfare decomposition**

Following Ábrahám et al. (2021), we can decompose the welfare gains into four main components. As the Fund avoids default, it avoids the output penalty and the market exclusions. Those are the first two sources of welfare gains. In addition, as one can see from the two last columns of Table 3, the Fund enlarges the debt capacity of the sovereign. Finally, the Fund provides state-contingent transfer, whereas the economy without the Fund only has access to non-contingent bonds. Table E.3, presents the decomposition of the welfare gains for each of the depicted growth states and zero initial debt. As one can see, the main source of welfare gains is the larger debt capacity followed by the state contingency and the circumvention of output penalty. Note that debt capacity and state contingency are closely linked one another. Without state-contingent transfers, the sovereign could not sustain a larger indebtedness.

## F Interest Rate-Growth Differential

Given the importance of the interest rate-growth differential highlighted in our study, we add to the benchmark model a shock to the risk-free rate $r$. This enables an analysis of the insurance component related to the direct change in $r^p$ and $\gamma$. We consider a two-state Markov process for the risk-free rate. More precisely, $r \in \{r_H, r_L\}$ with probability $\pi_r(r|r_-)$. We set $r_H = 0.0132$ as in the benchmark calibration and $r_L = 10^{-4}$ with $\pi_r(r_H|r_H) = 0.995$

Table E.3: Welfare Decomposition at Zero Initial Debt

| State | No penalty (%) | Immediate return to market (%) | Greater debt capacity (%) | State-contingent insurance (%) |
|---|---|---|---|---|
| $\gamma = \gamma_{min}$ | 8.49 | 2.62 | 80.02 | 8.87 |
| $\gamma = \gamma_{med}$ | 8.79 | 2.33 | 81.66 | 7.22 |
| $\gamma = \gamma_{max}$ | 8.41 | 1.88 | 78.49 | 11.22 |

and $\pi_r(r_L|r_L) = 0.985$.

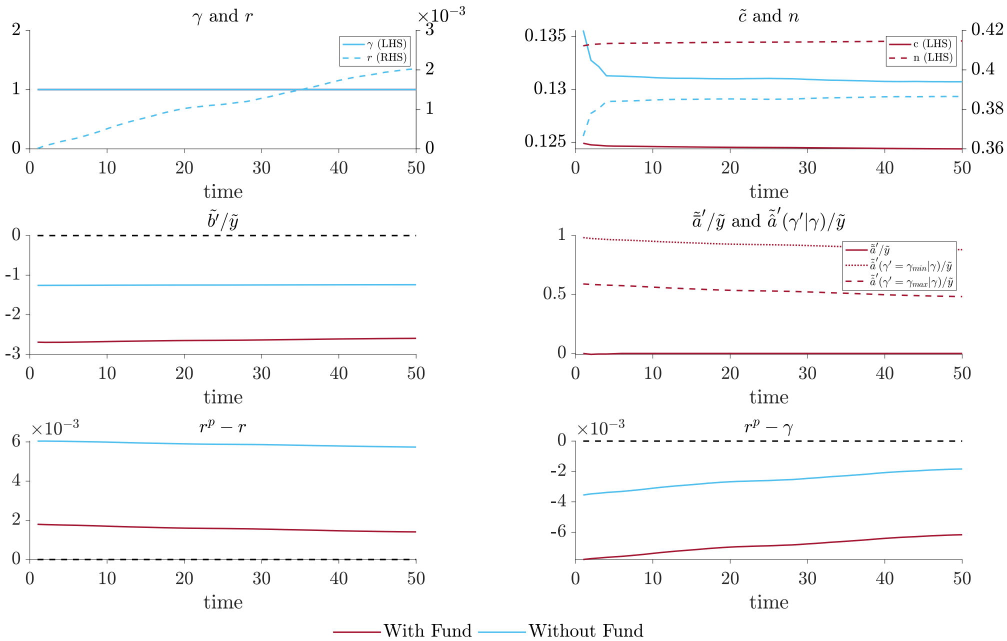
The stochastic risk-free rate directly affect the bond price — and therefore $r^p$ — as the lender discount the future differently. When $r$ reduces, $q^p$ increases as the lender gives less importance to future outcomes. In what follows we analyze the main difference between the economy with and without the Fund in steady state.

Figure F.2 depicts the impulse response function following a negative and positive interest rate shock.[39] The construction of the impulse responses follows the exact same step as highlighted previously. As one can see, the negative $r$ shock reduces consumption in the economy without the Fund. At a lower $r$, the price of debt is larger enabling a greater consumption per unit of issued debt. The effect is however very short-lived. Moreover, consumption in the economy with the Fund moves very little. One observes a slight increase in the debt held in the Fund as the lender's participation constraint might bind in some states. The opposite happens in the case of a positive interest rate shock. In the economy without the Fund, consumption is reduced as the price of debt is low. However, it quickly recovers to its steady state level. Again, the level of consumption remains very stable in the economy with the Fund. This avoids the large shift in the economy without the Fund.

Table F.4 presents the welfare gains in consumption equivalent between the economy with and without the Fund. The welfare computation is the same as in section 6.3 and is exposed in Appendix E. Again, welfare gains are important. This is due to the large jumps in consumption and labor that the stochastic $r$ generates in the economy without the Fund. Thus, even though consumption can be larger and labor can be lower in the economy without the Fund, jumps in those variables are very costly in terms of consumption smoothing. One sees that welfare gains are the highest when the risk-free rate is low. This is because debt is much cheaper to accumulate in the Fund in this situation.

[39]Figures G.13 and G.14 in Appendix G present the impulse response function for all relevant variables.

(a) Negative $r$ Shock




(b) Positive $r$ Shock

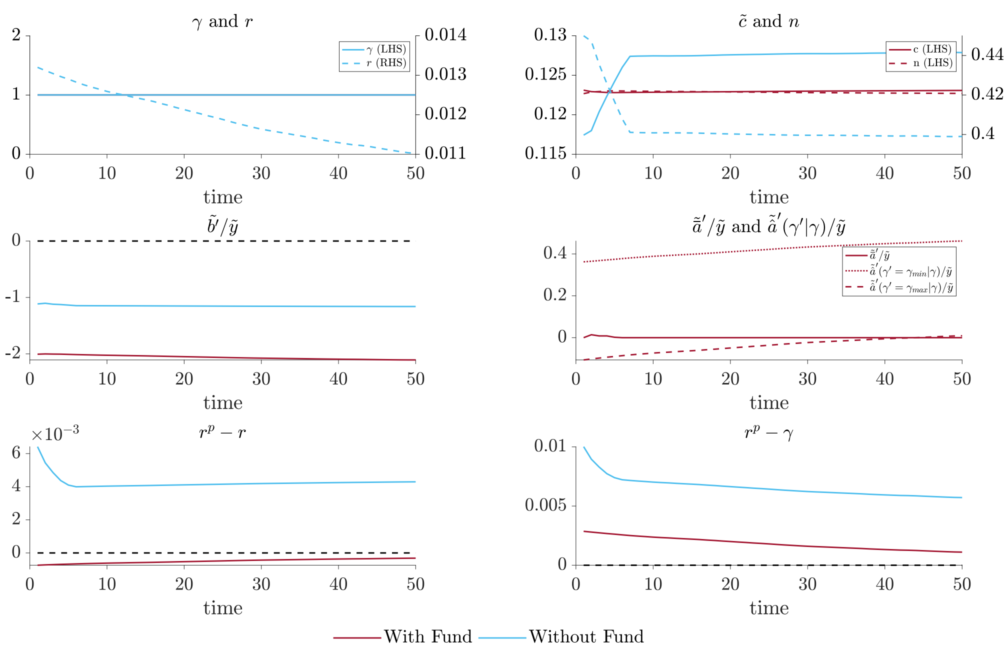



Figure F.2: Impulse Response Functions

Table F.5 depicts the decomposition of welfare gains. As before, most of the welfare gains are concentrated towards the greater debt capacity. The state contingency is also at the source of a large part of the welfare gains especially when the risk-free rate is low. This should not come as a surprise. As we noted in Figure F.2, consumption largely oscillates in

Table F.4: Welfare Comparison at Zero Initial Debt

| State | Welfare Gains (%) | Maximal Debt Absoption (% of GDP) | |
|---|---|---|---|
| | | With Fund | Without Fund |
| $(\gamma, r) = (\gamma_{min}, r_H)$ | 7.17 | 398 | 177 |
| $(\gamma, r) = (\gamma_{med}, r_H)$ | 6.53 | 195 | 111 |
| $(\gamma, r) = (\gamma_{max}, r_H)$ | 6.76 | 198 | 113 |
| $(\gamma, r) = (\gamma_{min}, r_L)$ | 22.54 | 588 | 224 |
| $(\gamma, r) = (\gamma_{med}, r_L)$ | 21.31 | 275 | 124 |
| $(\gamma, r) = (\gamma_{max}, r_L)$ | 21.44 | 277 | 127 |
| Average | 10.28 | | |

Table F.5: Welfare Decomposition at Zero Initial Debt

| State | No penalty (%) | Immediate return to market (%) | Greater debt capacity (%) | State-contingent insurance (%) |
|---|---|---|---|---|
| $(\gamma, r) = (\gamma_{min}, r_H)$ | 6.35 | 3.33 | 75.08 | 15.23 |
| $(\gamma, r) = (\gamma_{med}, r_H)$ | 6.38 | 3.28 | 76.34 | 14.00 |
| $(\gamma, r) = (\gamma_{max}, r_H)$ | 6.29 | 3.36 | 76.51 | 13.84 |
| $(\gamma, r) = (\gamma_{min}, r_L)$ | 4.42 | 2.33 | 59.64 | 33.60 |
| $(\gamma, r) = (\gamma_{med}, r_L)$ | 4.32 | 2.30 | 60.80 | 32.57 |
| $(\gamma, r) = (\gamma_{max}, r_L)$ | 4.32 | 2.31 | 60.97 | 32.39 |

the economy without the Fund in such case.

## G Additional Tables and Figures

The relative Pareto weight is the key to the dynamics of the model economy. Figure G.3 displays its law of motion. The dark grey region represents the ergodic set given in Definition 1. It is delimited by a lower bound of $\underline{\tilde{x}} = 0.09$ and an upper bound of $\tilde{\overline{x}} = 0.145$. The light grey region represents the basin of attraction of the ergodic set. As one can clearly see the upper and lower bounds of the set do not coincide. Thus, we are in the case of an imperfect risk sharing steady state. As noted earlier, the line characterizing the first best in our economy is below the 45° line as the sovereign is relatively more impatient than the lenders. This means that whenever none of the constraints is binding, the relative Pareto weight decreases. It continues to do so until it hits the value at which the sovereign's participation constraint is binding avoiding immiseration. This is different than the case of equally patient agents where the relative Pareto weight remains constant when none of the constraints is binding.

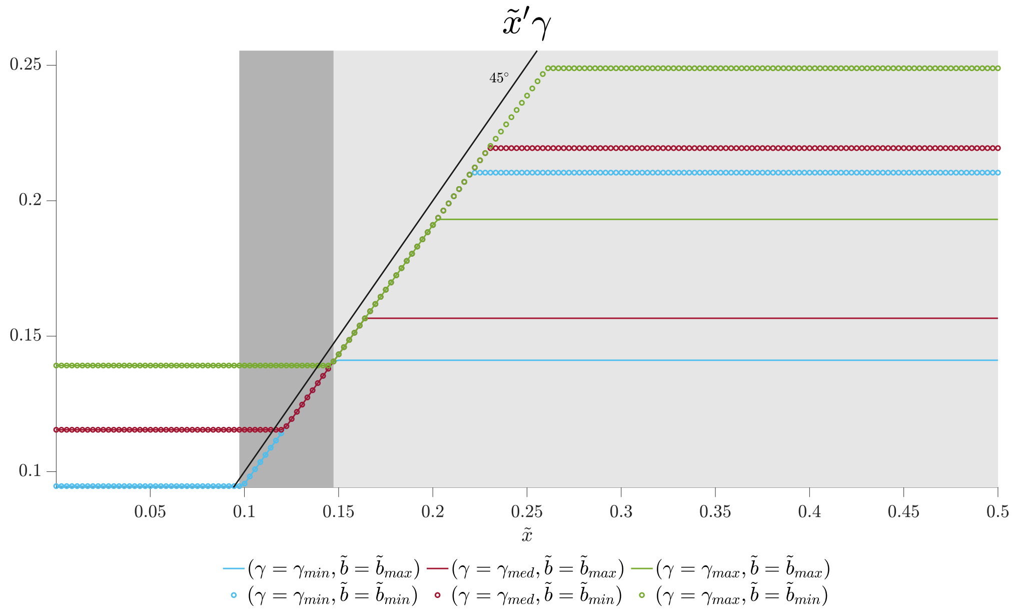



Figure G.3: Evolution of the Relative Pareto Weight in Steady State as a Function of $(\gamma, \tilde{b}, \tilde{x})$

We can illustrate the movement of the relative Pareto weights in the ergodic set with the following example. Suppose we start in the ergodic set on the first best line of the median shock (red non-horizontal dots) with a relative Pareto weight of say $\underline{\tilde{x}} = 0.13$ and $\tilde{b} = 0$. There, neither of the two participation constraints binds. If the economy remains in this state with that level of private debt, the relative Pareto weight decreases until it reaches the sovereign's binding constraint at around $\underline{\tilde{x}} = 0.12$. At this point, consider that the economy moves to the highest growth state. There, the value $\underline{\tilde{x}} = 0.12$ is now too low for the sovereign — its participation constraint therefore binds irrespective of its indebtedness level. The Planner will then increase the relative weight and set it to the minimum level to make the sovereign indifferent between reneging the contract or not — that is $\underline{\tilde{x}} = 0.145$. As long as the growth state does not change, the economy remains there.

Figures G.4 and G.5 depict the main policy functions and financial variables as a function of $(\gamma, \tilde{\omega})$ for zero debt and different levels of debt, respectively. More precisely, they both present the aforementioned statistics for the largest, the median and the lowest growth shocks. The dynamic is fairly similar to what we have highlighted in Section 6. This is because there is a direct correspondence between $\tilde{\omega}$ and $(\tilde{x}, \tilde{b})$ as discussed in Appendix B.

Figure G.7 depicts the main policy functions and financial variables as a function of $(\gamma, \tilde{b})$. Most notably, it present the aforementioned statistics for the largest and lowest growth shocks $\gamma_{max}$ and $\gamma_{min}$, as well as, the largest and lowest relative Pareto weights $\tilde{z}_{max}$

and $\tilde{z}_{min}$, respectively.

Figure G.8 presents the default set of the economy with and without the Fund's intervention. The former is depicted on the right hand side and the latter on the left hand side of the figure. Without the Fund's intervention, the sovereign defaults at different levels of labor productivity and different levels of debt depending on the labor productivity regime. In regimes of greater average growth, the sovereign defaults on relatively higher debt levels or even decides not to default. With the Fund's intervention, the sovereign never defaults consistent with Proposition 1.

Figure G.9 presents the holdings of Arrow-type securities. This figure is key in explaining the insurance mechanism provided by the Fund. First, we clearly see that the sovereign goes long in the transition between a relatively high growth state to a relatively low growth state. The opposite is true for short positions. Hence, Arrow-type securities prevent large drops in consumption when growth suddenly decreases. That is, the holding of Arrow-type securities is procyclical. In other words, the prospective insurance is large when the current growth state is high. Second, one observes that the insurance taken when $\gamma' = \gamma_{min}$ decreases when the lender's participation constraint binds, while the repayment (i.e. negative holdings) when $\gamma' = \gamma_{max}$ largely increases. This is due to the negative spread.

Figure G.10 presents the transfers from the Fund and the private lenders. The Fund's primary surplus, $\tilde{\tau}_f$, represents the net savings of the sovereign in the Fund. As the relative Pareto weight increases towards the value at which the lenders' participation constraint binds, the surplus becomes negative. The opposite is true when the relative Pareto weight is decreasing. Thus, the surplus is procyclical or if one prefers the deficit is countercyclical. As already mentioned, this procyclicality is the key mechanism preventing default. Next to the net savings in the Fund, one has the net savings in the private bond economy, $\tilde{\tau}_p$. The pattern here is the opposite of the one observed before, reflecting the hedging property of the Fund. The last panel of Figure G.10 depicts the total net savings, $\tilde{\tau}_f + \tilde{\tau}_p = \tilde{\tau}$. It follows the same pattern as $\tilde{\tau}_f$. The total surplus is therefore procyclical (or countercyclical if one refers to primary deficits) as well. Furthermore, it remains modest compared to $\tilde{\tau}_f$ or $\tilde{\tau}_p$, reflecting the fact that positions in the private bond market are counterbalanced by positions in the Fund.

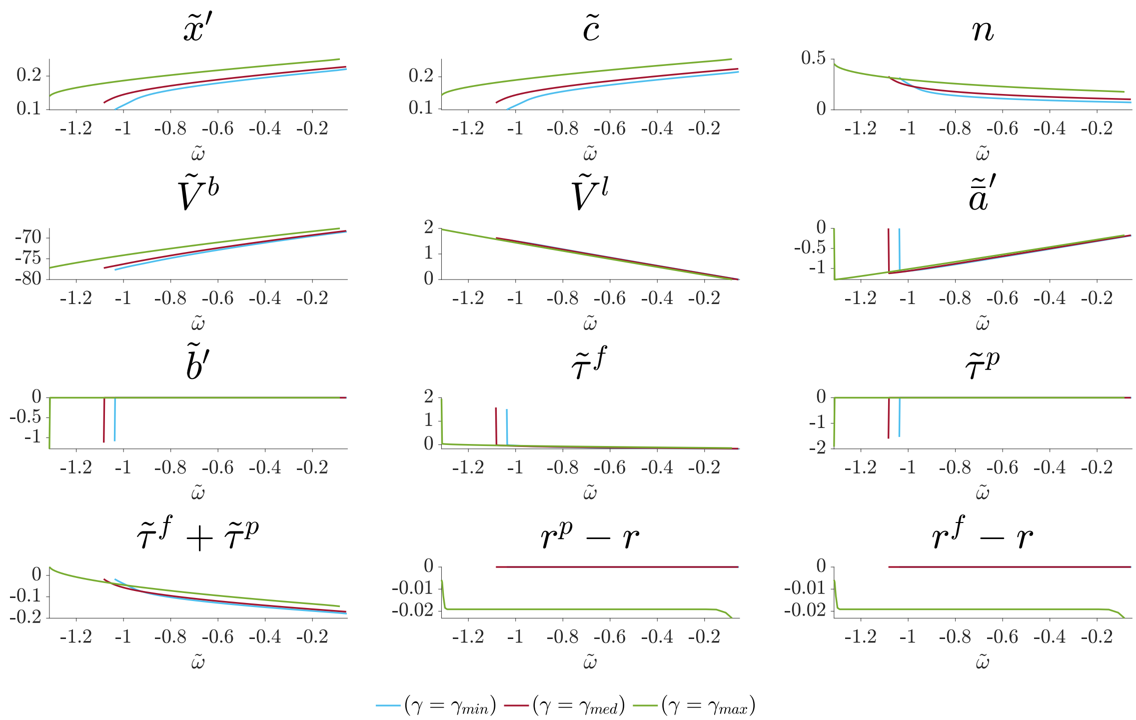



Figure G.4: Optimal Policies with Zero Private Debt as Function of $(\gamma, \tilde{\omega})$

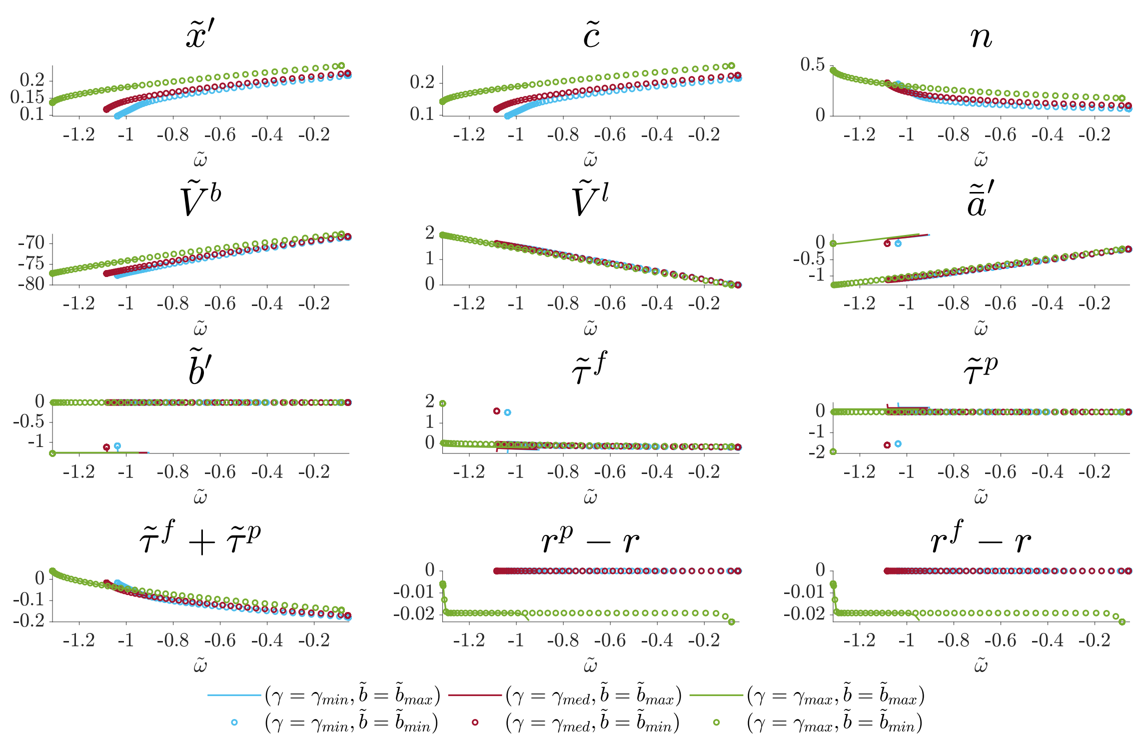



Figure G.5: Optimal Policies for Different Levels of Private Debt as Function of $(\gamma, \tilde{\omega})$

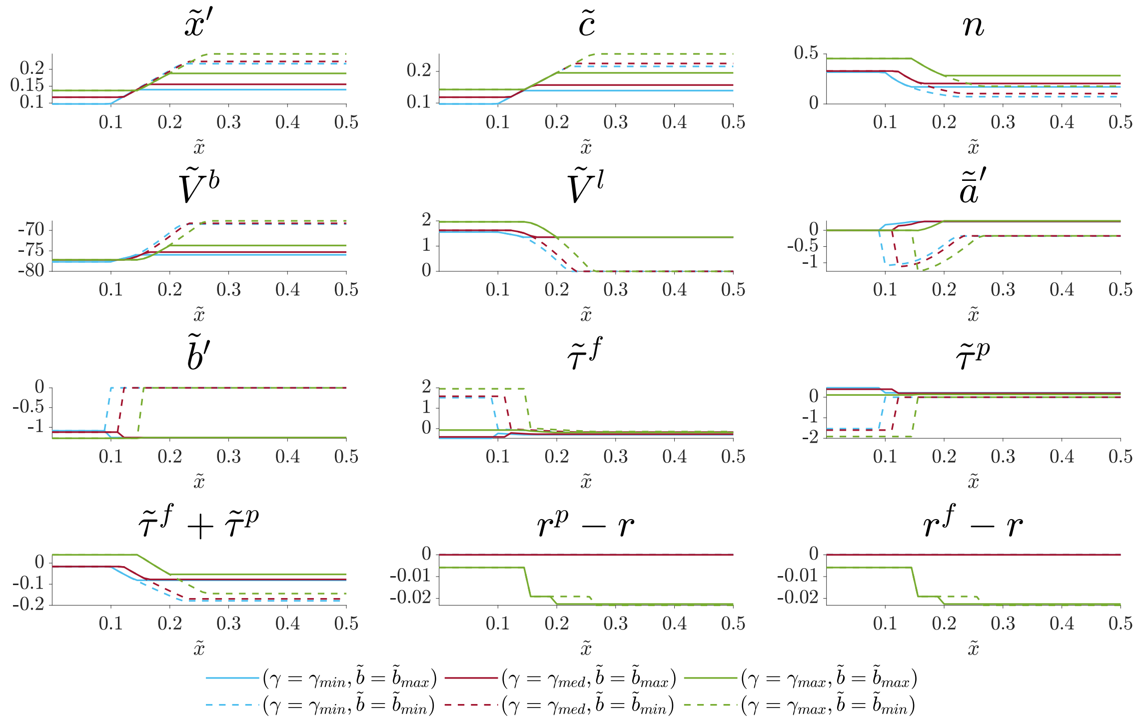



Figure G.6: Optimal Policies for Different Levels of Private Debt as Function of $(\gamma, \tilde{z})$

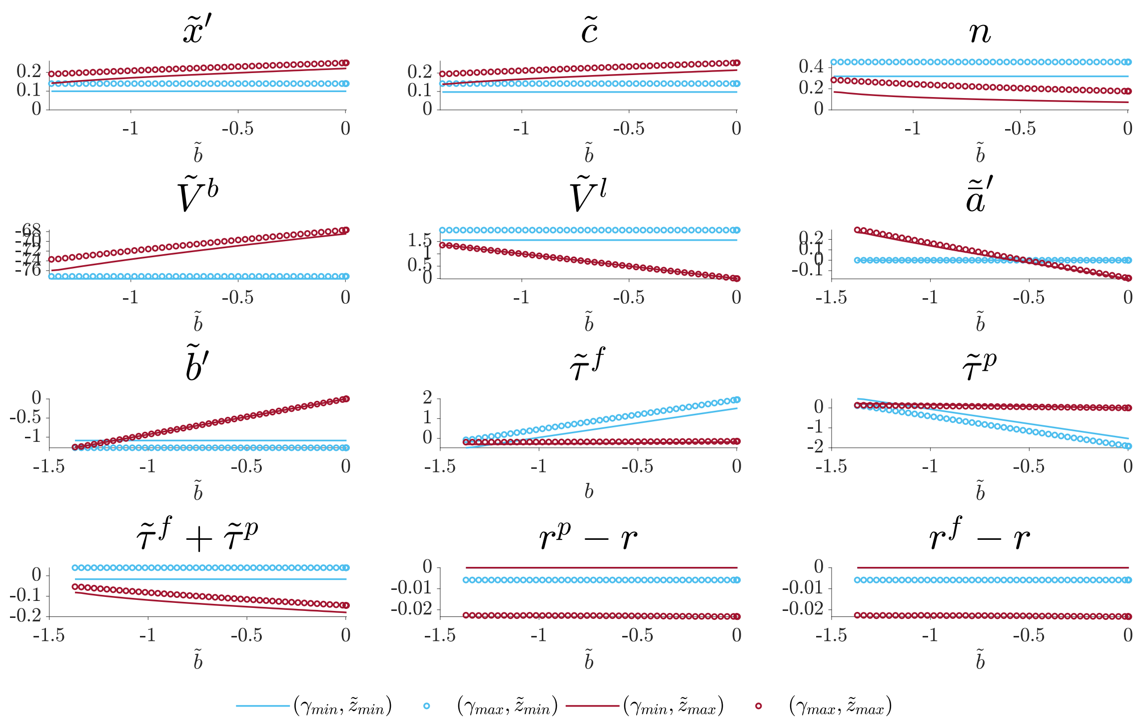



Figure G.7: Optimal Policies as Function of $(\gamma, \tilde{b})$

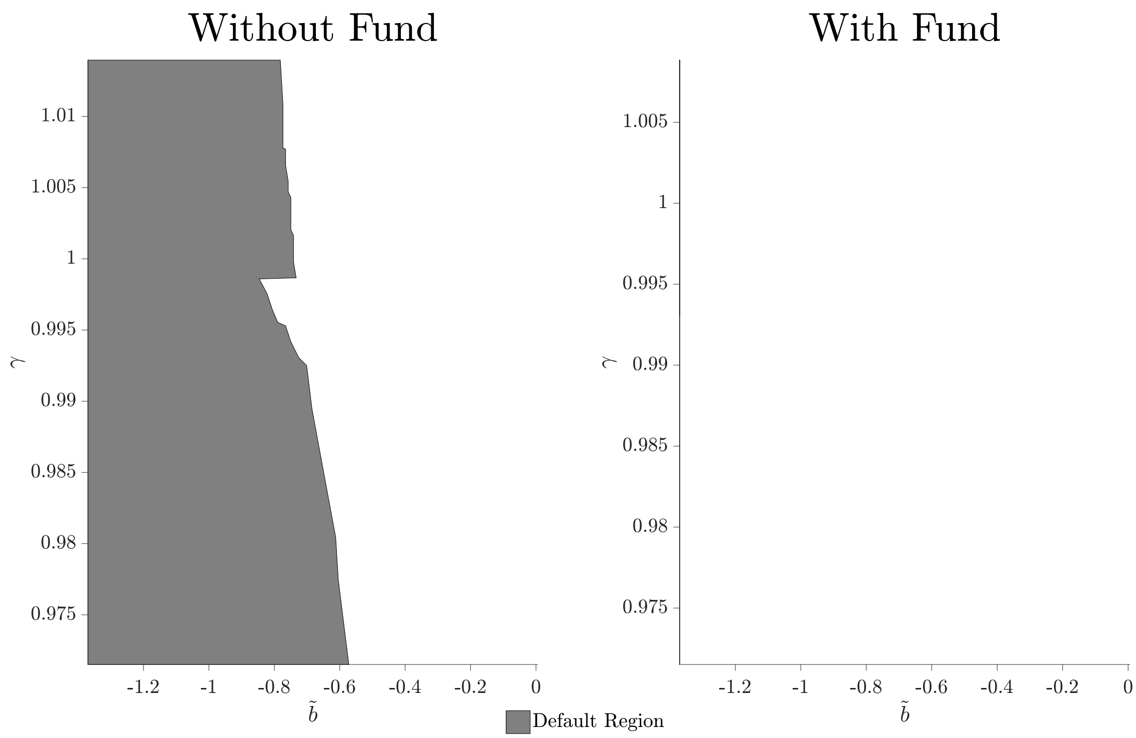



Figure G.8: Default Set as a Function of $(\gamma, \tilde{b})$

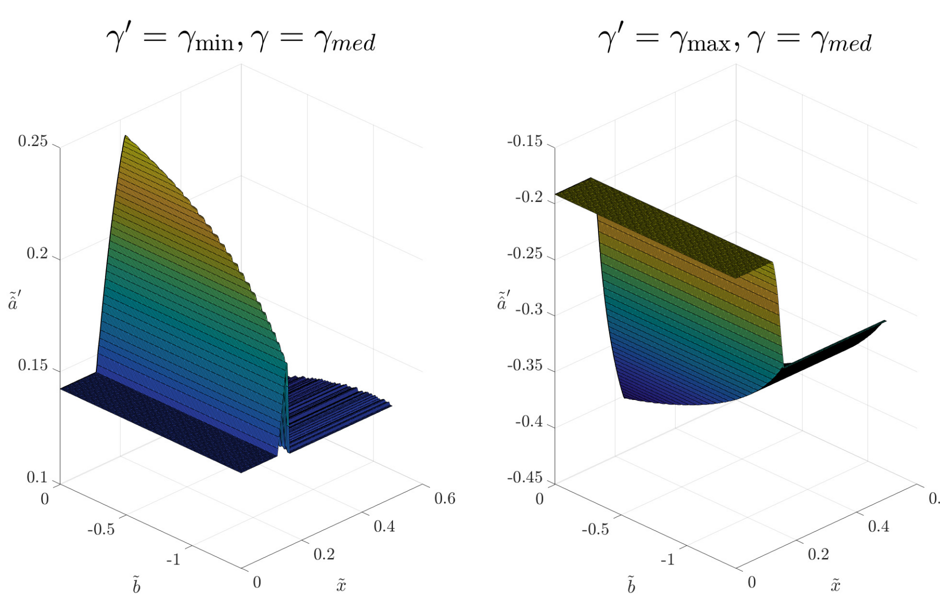



Figure G.9: Arrow-type Securities with Zero Private Debt as Function of $(\gamma, \tilde{x})$

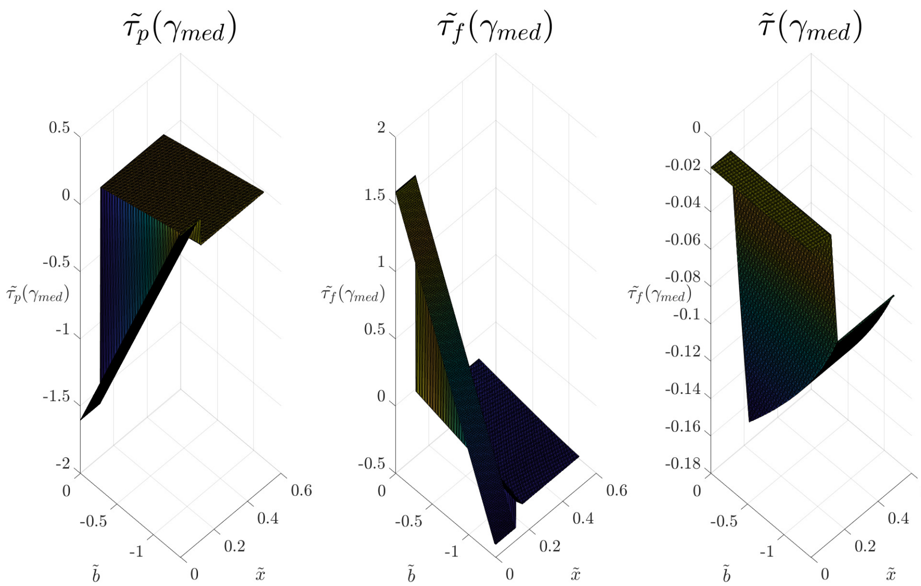



Figure G.10: Transfers as Function of $(\gamma, \tilde{b}, \tilde{x})$

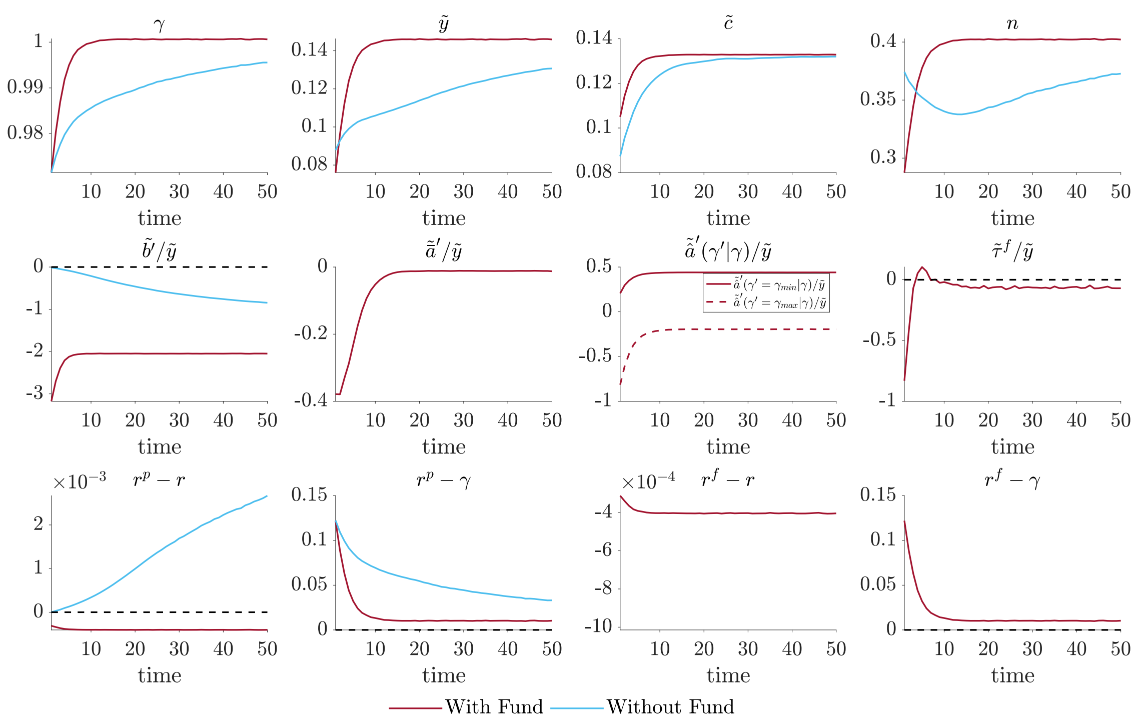



Figure G.11: Impulse Response Functions — Negative $\gamma$ Shock

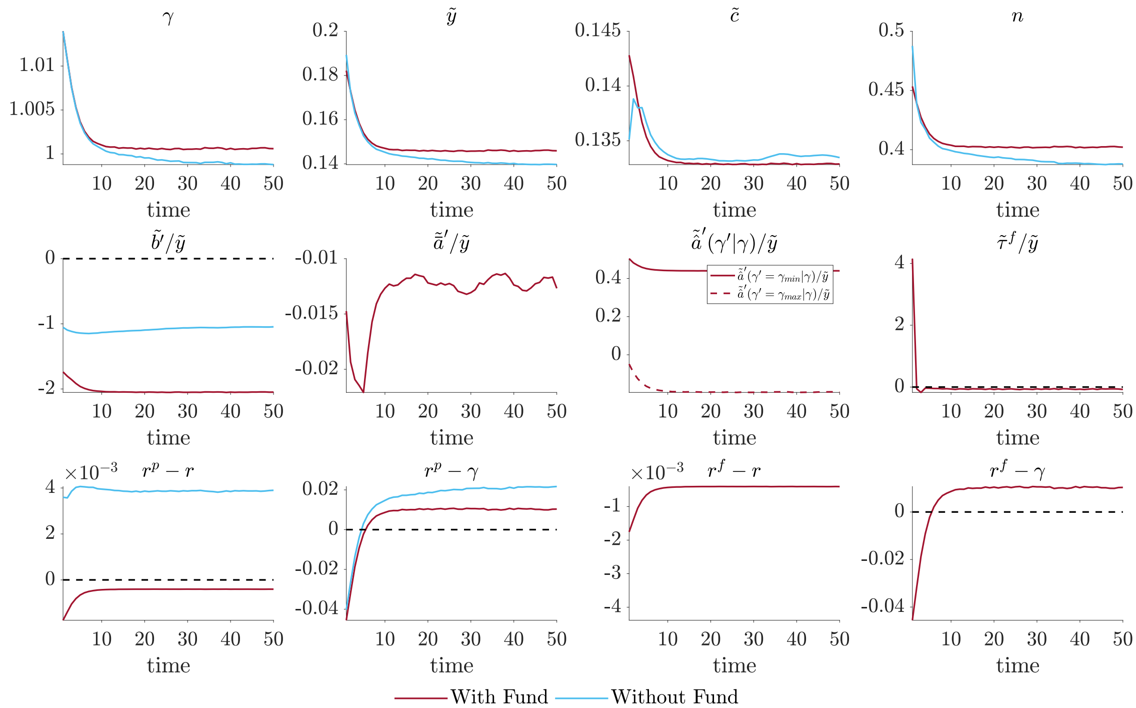



Figure G.12: Impulse Response Functions — Positive $\gamma$ Shock

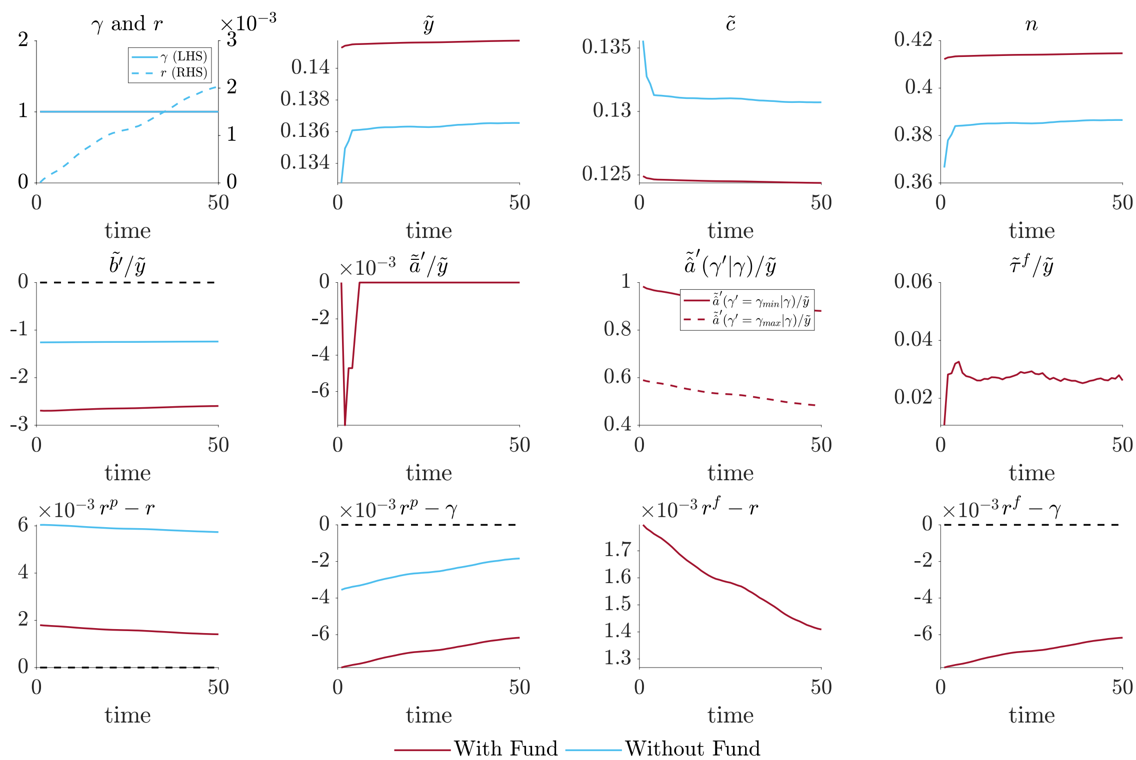



Figure G.13: Impulse Response Functions — Negative $r$ Shock

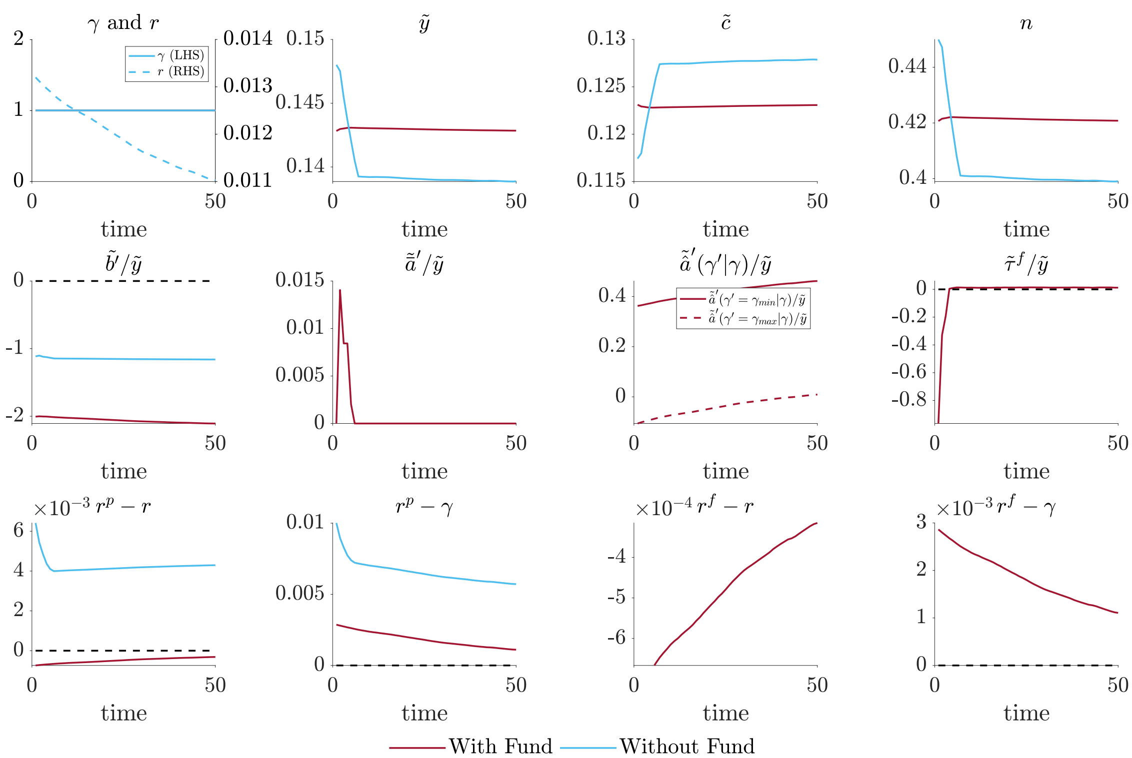



Figure G.14: Impulse Response Functions — Positive $r$ Shock