Online Appendix

Quantitative economic geography meets history: Questions, answers and challenges

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This Online Appendix consists of five sections. In Section A.1, I derive the equilibrium conditions of the model of Section 2.1.1. In Section A.2, I provide the proofs of Theorems 1 to 3. In Section A.3, I show the isomorphisms presented in Section 2.1.3. In Section A.4, I present the special case of the model that I use for the quantitative application of Section 2.2. Finally, in Section A.5, I present the strategy of taking this model to Hungarian historical data.

A.1 Derivation of the model's equilibrium conditions

Denote total spending on tradables (by workers and landlords) at location r by $X(r)$. Note that this also equals the nominal income of workers residing at r. This is because their income is ultimately spent on tradables, either directly or indirectly (through spending on housing, which is spent on tradables by landlords). As a result, spending on housing can be expressed as

$$P_H(r) H(r) = (1-\nu) X(r) \tag{1}$$

where $P_H(r)$ denotes the price of housing, and $H(r)$ denotes the quantity of housing at r. Note that $H(r)$ is exogenously given in equilibrium, as the supply of housing is fixed at each location. Rearranging (1) yields the equilibrium price of housing:

$$P_H(r) = (1-\nu) \frac{X(r)}{H(r)} \tag{2}$$

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Using (2) and the Fréchet distribution of idiosyncratic location tastes, one can express the number of workers choosing to reside at r and work at s as

$$L(r,s) = \frac{\bar{a}(r,s)\,N(r)^{-\lambda}\left[w(s)\,/\left(P(r)^{\nu}\,X(r)^{1-\nu}\,H(r)^{\nu-1}\right)\right]^{1/\eta}\kappa(r,s)^{-1/\eta}}{\sum_{u=1}^{S}\sum_{v=1}^{S}\bar{a}(u,v)\,N(u)^{-\lambda}\left[w(v)\,/\left(P(u)^{\nu}\,X(u)^{1-\nu}\,H(u)^{\nu-1}\right)\right]^{1/\eta}\kappa(u,v)^{-1/\eta}}\bar{L} \tag{3}$$

where $w(s)$ denotes the wage at s, and $P(r)$ denotes the ideal price index of tradables at r. Similarly, the expected utility of a worker can be expressed as

$$\bar{U} = \delta\left[\sum_{r=1}^{S}\sum_{s=1}^{S}\bar{a}(r,s)\,N(r)^{-\lambda}\left[w(s)\,/\left(P(r)^{\nu}\,X(r)^{1-\nu}\,H(r)^{\nu-1}\right)\right]^{1/\eta}\kappa(r,s)^{-1/\eta}\right]^{\eta} \tag{4}$$

where $\delta = \Gamma(1-\eta)$, such that $\Gamma(\cdot)$ denotes the gamma function.

On the production side, perfect competition among firms at location s implies that the local price of tradable good s, $p(s,s)$, equals its marginal cost of production:

$$p(s,s) = \bar{A}(s)^{-1}\,L(s)^{-\alpha}\,w(s)$$

At any other location r, no arbitrage guarantees that the price of the good equals the marginal cost of production and trade:

$$p(s,r) = \bar{A}(s)^{-1}\,L(s)^{-\alpha}\,w(s)\,\tau(s,r) \tag{5}$$

In equilibrium, the ideal price index at location r is given by the equation

$$P(r)^{1-\sigma} = \sum_{s=1}^{S} p(s,r)^{1-\sigma}$$

due to CES preferences for tradables. Using (5), one can rewrite this equation as

$$P(r)^{1-\sigma} = \sum_{s=1}^{S}\bar{A}(s)^{\sigma-1}\,L(s)^{\alpha(\sigma-1)}\,w(s)^{1-\sigma}\,\tau(s,r)^{1-\sigma}\,. \tag{6}$$

Market clearing for tradable good s implies

$$w(s)\,L(s) = \sum_{r=1}^{S}\pi(s,r)\,X(r)$$

where $\pi(s,r)$ denotes the share of good s in total spending on tradables at r. Due to CES preferences for tradables, this share can be obtained as

$$\pi(s,r) = p(s,r)^{1-\sigma}\,P(r)^{\sigma-1} = \bar{A}(s)^{\sigma-1}\,L(s)^{\alpha(\sigma-1)}\,w(s)^{1-\sigma}\,P(r)^{\sigma-1}\,\tau(s,r)^{1-\sigma}$$

where I used equation (5). This allows me to rewrite the market clearing condition as

$$\bar{A}(s)^{1-\sigma} w(s)^{\sigma} L(s)^{1-\alpha(\sigma-1)} = \sum_{r=1}^{S} P(r)^{\sigma-1} X(r) \tau(s,r)^{1-\sigma}. \tag{7}$$

Finally, labor market clearing implies that the nominal income of residents of location r equals

$$X(r) = \sum_{s=1}^{S} w(s) L(r,s)$$

while the population of this location equals

$$N(r) = \sum_{s=1}^{S} L(r,s)$$

and the employment of location s equals

$$L(s) = \sum_{r=1}^{S} L(r,s).$$

Using (3) and (4), these last three equations can be rewritten as

$$H(r)^{-\frac{1-\nu}{\eta}} P(r)^{\frac{\nu}{\eta}} X(r)^{1+\frac{1-\nu}{\eta}} N(r)^{\lambda} = \left(\frac{\bar{U}}{\delta}\right)^{-1/\eta} \bar{L} \sum_{s=1}^{S} \bar{a}(r,s) w(s)^{1+\frac{1}{\eta}} \kappa(r,s)^{-1/\eta}, \tag{8}$$

$$H(r)^{-\frac{1-\nu}{\eta}} P(r)^{\frac{\nu}{\eta}} X(r)^{\frac{1-\nu}{\eta}} N(r)^{1+\lambda} = \left(\frac{\bar{U}}{\delta}\right)^{-1/\eta} \bar{L} \sum_{s=1}^{S} \bar{a}(r,s) w(s)^{1/\eta} \kappa(r,s)^{-1/\eta} \tag{9}$$

and

$$w(s)^{-1/\eta} L(s) = \left(\frac{\bar{U}}{\delta}\right)^{-1/\eta} \bar{L} \sum_{r=1}^{S} \bar{a}(r,s) H(r)^{\frac{1-\nu}{\eta}} N(r)^{-\lambda} P(r)^{-\frac{\nu}{\eta}} X(r)^{-\frac{1-\nu}{\eta}} \kappa(r,s)^{-1/\eta}. \tag{10}$$

(3), (4), (6), (7), (8), (9) and (10) constitute a system of $5S + S^2 + 1$ equations and $5S + S^2 + 1$ unknowns: locations' residential populations $N(r)$, employment levels $L(s)$, wage levels $w(s)$, total spending on tradables $X(r)$ and ideal price index $P(r)$, cross-location commuting flows $L(r,s)$, as well as the economy-wide level of workers' expected utility, $\bar{U}$. These are the equilibrium conditions I use to prove Theorems 1 to 3 in Section A.2 and show the isomorphisms with alternative models in Section A.3.

A.2 Proofs of theorems

Proof of Theorem 1. (6), (7), (8), (9) and (10) and (4) consitute a system of $5S+1$ equations and $5S+1$ unknowns: $P(r)$, $w(s)$, $X(r)$, $N(r)$, $L(s)$ and $\bar{U}$. This system is a special case of the systems considered in Allen et al. (2020):[1]

$$\prod_{h=1}^{H} x_h(r)^{\gamma_{kh}} = \sum_{s=1}^{S} K_k(r,s) \prod_{h=1}^{H} x_k(r)^{\kappa_{kh}} x_h(s)^{\beta_{kh}} \qquad k=1,2,...,H \tag{11}$$

such that $H=5$, $x_1(r)=P(r)$, $x_2(r)=w(r)$, $x_3(r)=X(r)$, $x_4(r)=N(r)$, $x_5(r)=L(r)$,

$$K_1(r,s) = \bar{A}(s)^{\sigma-1} \tau(s,r)^{1-\sigma},$$

$$K_2(r,s) = \bar{A}(r)^{\sigma-1} \tau(r,s)^{1-\sigma},$$

$$K_3(r,s) = K_4(r,s) = K_5(r,s) = \left(\frac{\bar{U}}{\delta}\right)^{-1/\eta} \bar{L}\bar{a}(r,s) H(r)^{\frac{1-\nu}{\eta}} \kappa(r,s)^{-1/\eta},$$

κ_{kh} is the (k,h) entry of a 5×5 matrix $\mathbf{K}$ of zeros, γ_{kh} is the (k,h) entry of the matrix

$$\mathbf{\Gamma} = \begin{bmatrix} 1-\sigma & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 1-\alpha(\sigma-1) \\ \nu/\eta & 0 & 1+\frac{1-\nu}{\eta} & \lambda & 0 \\ \nu/\eta & 0 & \frac{1-\nu}{\eta} & 1+\lambda & 0 \\ 0 & -1/\eta & 0 & 0 & 1 \end{bmatrix}$$

and β_{kh} is the (k,h) entry of the matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 1-\sigma & 0 & 0 & \alpha(\sigma-1) \\ \sigma-1 & 0 & 1 & 0 & 0 \\ 0 & 1+\frac{1}{\eta} & 0 & 0 & 0 \\ 0 & 1/\eta & 0 & 0 & 0 \\ 1-\frac{\nu}{\eta} & 0 & -\frac{1-\nu}{\eta} & -\lambda & 0 \end{bmatrix}.$$

Theorem 1 in Allen et al. (2020) shows that the solution to (11) exists and is unique under the condition that the largest eigenvalue of the matrix $|\left(\mathbf{B}(\mathbf{\Gamma}-\mathbf{K})^{-1}\right)_{kh}|$ is strictly less than one.[2] As $\mathbf{\Gamma}$ and $\mathbf{B}$ are functions of the model's structural parameters only, the condition guaranteeing existence and uniqueness only depends on these parameters. This proves Theorem 1.

Proof of Theorem 2. Theorem 1 in Allen et al. (2020) also shows that if the largest

[1] In particular, see Remark 3 in Allen et al. (2020).

[2] Remark 2 in Allen et al. (2020) addresses the issue that $\bar{U}$ is an endogenous constant, determined by equation (4).

eigenvalue of $|\left(\mathbf{B}\left(\mathbf{\Gamma}-\mathbf{K}\right)^{-1}\right)_{kh}|$ is strictly less than one, then an algorithm that consists of iteratively applying (11) is guaranteed to converge to the solution of (11). In the context of the model of Section 2.1.1, this implies that the researcher can guess any initial distribution of $P(r)$, $w(s)$, $X(r)$, $N(r)$ and $L(r)$, plug them into equations (6), (7), (8), (9) and (10), and update the distributions using the left-hand sides of these equations. Applying this procedure iteratively, the distributions of $P(r)$, $w(s)$, $X(r)$, $N(r)$ and $L(r)$ are guaranteed to converge to the equilibrium values of these variables. Once these variables are known, equations (3) and (4) provide commuting flows $L(r,s)$ and workers' expected utility $\bar{U}$ in closed form. This proves Theorem 2.

Proof of Theorem 3. Using wages $w(s)$ and the matrix of commuting flows $L(r,s)$ (including commuting from a location to itself), the researcher can recover each location's residential population, employment and spending on tradables as

$$N(r)=\sum_{s=1}^{S}L(r,s),$$

$$L(s)=\sum_{r=1}^{S}L(r,s)$$

and

$$X(r)=\sum_{s=1}^{S}w(s)\,L(r,s)$$

respectively. Plugging $L(s)$ and $X(r)$ into equations (6) and (7), $2S$ unknowns remain in these $2S$ equations: namely, the ideal price index $P(r)$ and the fundamental productivity $\bar{A}(r)$ of each location r. This system is again a special case of the systems considered in Allen et al. (2020), such that $H=2$, $x_1(r)=P(r)$, $x_2(r)=\bar{A}(r)$,

$$K_1(r,s)=L(s)^{\alpha(\sigma-1)}\,w(s)^{1-\sigma}\,\tau(s,r)^{1-\sigma},$$

$$K_2(r,s)=w(r)^{-\sigma}\,L(s)^{\alpha(\sigma-1)-1}\,X(s)\,\tau(r,s)^{1-\sigma},$$

$\mathbf{K}$ is a 2×2 matrix full of zeros,

$$\mathbf{\Gamma}=\begin{bmatrix}1-\sigma & 0\\ 0 & 1-\sigma\end{bmatrix}$$

and

$$\mathbf{B}=\begin{bmatrix}0 & \sigma-1\\ \sigma-1 & 0\end{bmatrix}.$$

It is straightforward to show that the largest eigenvalue of $|\left(\mathbf{B}\left(\mathbf{\Gamma}-\mathbf{K}\right)^{-1}\right)_{kh}|$ equals one. In this case, Theorem 1 in Allen et al. (2020) implies that the solution to the system exists and is unique up to scale, although iterating on the system is not guaranteed to converge to the solution. Thus, the set of fundamental productivities $\bar{A}(s)$ rationalizing the observed data is unique (up to scale). Next, plugging the price indices $P(r)$ just recovered, as well as the observed $L(r,s)$, $w(s)$ and $N(r)$, into equation (3), one can obtain the fundamental amenity level of any location pair (r,s) up to scale by simply rearranging this equation. As price indices are uniquely identified by the previous step, these fundamental amenity levels are also unique (up to scale). This proves Theorem 3.

A.3 Proofs of isomorphisms

Land use in production. The representative firm's first order conditions imply that land rents are given by

$$R(s) = \frac{1-\mu}{\mu}\frac{w(s)\,L(s)}{\Lambda(s)}$$

where $\Lambda(s)$ equals the exogenous supply of production land at s in equilibrium. As a result, the firm's marginal cost of production equals

$$\begin{aligned} &\mu^{-\mu}(1-\mu)^{-(1-\mu)}\,\bar{A}(s)^{-1}\,L(s)^{-\alpha}\,w(s)^{\mu}\,R(s)^{1-\mu} \\ &= \left(\mu\bar{A}(s)\,\Lambda(s)^{1-\mu}\right)^{-1} L(s)^{-[\alpha-(1-\mu)]}\,w(s) \end{aligned} \tag{12}$$

which has the same form as the marginal cost of production in the model of Section 2.1.1 once fundamental productivities and the agglomeration parameter are redefined as

$$\tilde{\bar{A}}(s) = \mu\bar{A}(s)\,\Lambda(s)^{1-\mu}$$

and

$$\tilde{\alpha} = \alpha - (1-\mu)\,.$$

Finally, the income of an employee at s, $y(s)$, equals her wage income plus a $1/L(s)$ fraction of rents,

$$y(s) = w(s) + \frac{R(s)\,\Lambda(s)}{L(s)} = \mu^{-1}w(s) \tag{13}$$

and total spending on tradables at residential location r is given by

$$X(r) = \sum_{s=1}^{S} y(s)\,L(s) = \mu^{-1}\sum_{s=1}^{S} w(s)\,L(s)\,. \tag{14}$$

Redefining $\tilde{X}(r) = \mu X(r)$, the equilibrium conditions yield equations (3), (4), (6), (7), (8), (9) and (10) in the redefined variables, except that workers' expected utility is multiplied by a constant $\mu^{-\nu}$. Thus, this model is isomorphic to the model of Section 2.1.1.

Endogenous specialization, driven by comparative advantage. Following the same steps as in Eaton and Kortum (2002), one can show that the share of location-s goods in total spending on tradables at r equals

$$\pi(s,r) = \frac{\bar{A}(s)^{\theta} L(s)^{\alpha\theta} w(s)^{-\theta} \tau(s,r)^{-\theta}}{\sum_{u=1}^{S} \bar{A}(u)^{\theta} L(u)^{\alpha\theta} w(u)^{-\theta} \tau(u,r)^{-\theta}}$$

and the ideal price index at r equals

$$P(r) = \Gamma\left(1 - \frac{\sigma-1}{\theta}\right)^{\frac{1}{1-\sigma}} \left[\sum_{s=1}^{S} \bar{A}(s)^{\theta} L(s)^{\alpha\theta} w(s)^{-\theta} \tau(s,r)^{-\theta}\right]^{-1/\theta}.$$

Redefining $\tilde{P}(r) = \Gamma\left(1 - \frac{\sigma-1}{\theta}\right)^{\frac{1}{\sigma-1}} P(r)$ and rearranging, we obtain the equation

$$\tilde{P}(r)^{-\theta} = \sum_{s=1}^{S} \bar{A}(s)^{\theta} L(s)^{\alpha\theta} w(s)^{-\theta} \tau(s,r)^{-\theta} \tag{15}$$

while goods market clearing yields

$$\bar{A}(s)^{-\theta} w(s)^{1+\theta} L(s)^{1-\alpha\theta} = \sum_{r=1}^{S} \tilde{P}(r)^{\theta} X(r) \tau(s,r)^{-\theta}. \tag{16}$$

Equations (15) and (16) are identical to (6) and (7), except that they include the redefined price index variable, and θ everywhere instead of $\sigma - 1$. The remaining equilibrium conditions are unchanged, except that workers' expected utility is multiplied by a constant $\Gamma\left(1 - \frac{\sigma-1}{\theta}\right)^{\frac{\nu}{\sigma-1}}$. Thus, this model is isomorphic to the model of Section 2.1.1.

Endogenous specialization, driven by increasing returns. Due to monopolistic competition with a linear production technology, CES preferences and iceberg trade costs, firms in the model with increasing returns set a price that is a constant markup over their marginal cost:

$$p_{\omega}(s,r) = p(s,r) = \frac{\sigma}{\sigma-1} \bar{A}(s)^{-1} L(s)^{-\alpha} w(s) \tau(s,r) \tag{17}$$

Plugging this price into firms' profits, one can show that the profit-maximizing output of a firm equals

$$q_{\omega}(s) = q(s) = (\sigma-1) f \bar{A}(s) L(s)^{\alpha} \tag{18}$$

while the firm's employment equals

$$\ell_{\omega}(s) = \ell(s) = \frac{q(s)}{\bar{A}(s) L(s)^{\alpha}} + f = \sigma f.$$

Plugging this into the local labor market clearing condition

$$L(s) = \int_{\omega \text{ produced at } s} \ell_\omega(s)\, d\omega$$

pins down the mass of tradables produced at s as

$$N(s) = \frac{L(s)}{\ell(s)} = \frac{L(s)}{\sigma f}. \tag{19}$$

In equilibrium, the ideal price index at location r is given by the equation

$$P(r)^{1-\sigma} = \int_0^N p_\omega(r)^{1-\sigma}\, d\omega = \sum_{s=1}^{S} N(s)\, p(s,r)^{1-\sigma}.$$

Combining this with equations (17) and (19) yields

$$P(r)^{1-\sigma} = \sigma^{-\sigma}(\sigma-1)^{\sigma-1} f^{-1} \sum_{s=1}^{S} \bar{A}(s)^{\sigma-1} L(s)^{1+\alpha(\sigma-1)} w(s)^{1-\sigma} \tau(s,r)^{1-\sigma}. \tag{20}$$

By CES preferences, market clearing for a good produced at s implies

$$q(s) = \sum_{r=1}^{S} p(s,r)^{-\sigma} P(r)^{\sigma-1} X(r).$$

Combining this with equations (17) and (18) and rearranging yields

$$\bar{A}(s)^{1-\sigma} w(s)^{\sigma} L(s)^{-\alpha(\sigma-1)} = \sigma^{-\sigma}(\sigma-1)^{\sigma-1} f^{-1} \sum_{r=1}^{S} P(r)^{\sigma-1} X(r)\, \tau(s,r)^{1-\sigma}. \tag{21}$$

Normalizing the fixed cost of production to $f = \sigma^{-\sigma}(\sigma-1)^{\sigma-1}$ and redefining the agglomeration parameter of the model of Section 2.1.1 as

$$\tilde{\alpha} = \alpha + \frac{1}{\sigma-1},$$

equations (20) and (21) become identical to equations (6) and (7). The remaining equilibrium conditions are unchanged. This shows the isomorphism between this model and the model of Section 2.1.1.

A.4 Special case of the model used in the quantitative application of Section 2.2

I consider the special case of the model of Section 2.1.1 with

- no commuting, i.e., $\kappa(r,s)=\infty$ for any $r\neq s$ and $\kappa(r,r)=1$;
- no congestion externalities, i.e., $\lambda=0$;
- no differences in fundamental productivity across locations, i.e., $\bar{A}(s)=1$ for all s.

As this is a special case of the model of Section 2.1.1, the results on equilibrium existence and uniqueness (Theorem 1) and the simple algorithm that can be used to solve the equilibrium (Theorem 2) hold in this model as well. In fact, as I show below, the equilibrium conditions substantially simplify in this framework, so much so that the distribution of population can be characterized by S equations that do not involve other endogenous variables. This simple model structure is useful on two fronts. First, it implies that the condition guaranteeing equilibrium uniqueness reduces to an inequality, as opposed to the complex eigenvalue condition of the general model presented in the proof of Theorem 1 (Section A.2 of the Online Appendix). Second, it implies that the equlibrium population distribution can be obtained from a simplified version of the iterative algorithm laid out in the proof of Theorem 2 (Section A.2 of the Online Appendix).

Unlike Theorems 1 and 2, I can no longer use the general result on model invertibility (Theorem 3) for two reasons. First, Theorem 3 relies on observing data that are unobserved in the particular historical setting of Section 2.2, such as wages at the location level. Second, Theorem 3 assumes a non-degenerate distribution of fundamental productivities $\bar{A}(s)$ that are backed out from the data, while I assume no productivity differences across locations in this special case of the model. Nonetheless, I show below that the model can be inverted to recover the only unobserved location fundamental (amenities). In particular, I show that the system of S equations characterizing the equilibrium population distribution can be solved uniquely for location amenities (up to scale) as a function of the observed population distribution, housing supply, and trade costs.

To obtain these results, first note that, due to the assumptions made above, $L(r,s)=0$ for any $r\neq s$ and $L(r,r)=N(r)=L(r)$ for any r in this model. As a consequence, spending on tradables by the residents of location r reduces to

$$X(r)=w(r)L(r).$$

Substituting these results, as well as the assumptions made on $\kappa(r,s)$, λ and $\bar{A}(s)$, into equilibrium conditions (3), (4), (6) and (7) yields

$$L(r)=\frac{\bar{a}(r,r)\left[w(r)^{\nu}/\left(P(r)^{\nu}L(r)^{1-\nu}H(r)^{\nu-1}\right)\right]^{1/\eta}}{\sum_{s=1}^{S}\bar{a}(s,s)\left[w(s)^{\nu}/\left(P(s)^{\nu}L(s)^{1-\nu}H(s)^{\nu-1}\right)\right]^{1/\eta}}\bar{L} \tag{22}$$

$$\bar{U}=\delta\left[\sum_{r=1}^{S}\bar{a}(r,r)\left[w(r)^{\nu}/\left(P(r)^{\nu}L(r)^{1-\nu}H(r)^{\nu-1}\right)\right]^{1/\eta}\right]^{\eta} \tag{23}$$

$$P(r)^{1-\sigma} = \sum_{s=1}^{S} L(s)^{\alpha(\sigma-1)} w(s)^{1-\sigma} \tau(s,r)^{1-\sigma} \tag{24}$$

and

$$w(s)^{\sigma} L(s)^{1-\alpha(\sigma-1)} = \sum_{r=1}^{S} P(r)^{\sigma-1} w(r) L(r) \tau(s,r)^{1-\sigma}. \tag{25}$$

This is a system of $3S+1$ equations and $3S+1$ unknowns: locations' populations $L(r)$, wage levels $w(r)$, ideal price indices $P(r)$, as well as the economy-wide level of workers' expected utility, $\bar{U}$.

To further simplify this system of equilibrium conditions, one can combine (22) and (23) to express the price index as

$$P(r) = \bar{L}^{\frac{\eta}{\nu}} \delta^{1/\nu} \bar{U}^{-1/\nu} \bar{a}(r,r)^{\frac{\eta}{\nu}} H(r)^{\frac{1-\nu}{\nu}} w(r) L(r)^{-\frac{1-\nu+\eta}{\nu}}. \tag{26}$$

Substituting this into equations (24) and (25) yields

$$\begin{aligned} &\bar{a}(r,r)^{-\frac{\eta(\sigma-1)}{\nu}} H(r)^{-\frac{1-\nu}{\nu}(\sigma-1)} w(r)^{1-\sigma} L(r)^{\frac{1-\nu+\eta}{\nu}(\sigma-1)} \\ &= \bar{L}^{\frac{\eta(\sigma-1)}{\nu}} \delta^{\frac{\sigma-1}{\nu}} \bar{U}^{-\frac{\sigma-1}{\nu}} \sum_{s=1}^{S} L(s)^{\alpha(\sigma-1)} w(s)^{1-\sigma} \tau(s,r)^{1-\sigma} \end{aligned} \tag{27}$$

and

$$\begin{aligned} &w(s)^{\sigma} L(s)^{1-\alpha(\sigma-1)} \\ &= \bar{L}^{\frac{\eta(\sigma-1)}{\nu}} \delta^{\frac{\sigma-1}{\nu}} \bar{U}^{-\frac{\sigma-1}{\nu}} \sum_{r=1}^{S} \bar{a}(r,r)^{\frac{\eta(\sigma-1)}{\nu}} H(r)^{\frac{1-\nu}{\nu}(\sigma-1)} w(r)^{\sigma} L(r)^{1-\frac{1-\nu+\eta}{\nu}(\sigma-1)} \tau(s,r)^{1-\sigma}. \end{aligned} \tag{28}$$

One can also show that (27) and (28) reduce to the same equation as long as wages take the form

$$w(r) = \bar{w} \cdot \bar{a}(r,r)^{-\frac{\eta}{\nu}\frac{\sigma-1}{2\sigma-1}} H(r)^{-\frac{1-\nu}{\nu}\frac{\sigma-1}{2\sigma-1}} L(r)^{\left(\alpha+\frac{1-\nu+\eta}{\nu}-\frac{1}{\sigma-1}\right)\frac{\sigma-1}{2\sigma-1}}.\text{[3]} \tag{29}$$

In particular, both (27) and (28) reduce to

$$\begin{aligned} &\bar{a}(r,r)^{-\frac{\eta}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}} H(r)^{-\frac{1-\nu}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}} L(r)^{\left(1-\alpha(\sigma-1)+\frac{1-\nu+\eta}{\nu}\sigma\right)\frac{\sigma-1}{2\sigma-1}} \\ &= \bar{L}^{\frac{\eta(\sigma-1)}{\nu}} \delta^{\frac{\sigma-1}{\nu}} \bar{U}^{-\frac{\sigma-1}{\nu}} \sum_{s=1}^{S} \bar{a}(s,s)^{\frac{\eta}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}} H(s)^{\frac{1-\nu}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}} L(s)^{\left(1+\alpha\sigma-\frac{1-\nu+\eta}{\nu}(\sigma-1)\right)\frac{\sigma-1}{2\sigma-1}} \tau(s,r)^{1-\sigma}. \end{aligned} \tag{30}$$

(30) constitutes a system of S equations that characterize the population level $L(r)$ at each location as a function of fundamental amenities $\bar{a}(r,r)$, housing supply $H(r)$, trade costs

[3] As I have not normalized any price yet, I can choose the value of $\bar{w}$ freely. I normalize it to one: $\bar{w} = 1$.

$\tau(s,r)$, total population $\bar{L}$ and the economy-wide utility level $\bar{U}$. Does (30) pin down a unique population distribution? Theorem 4, which is the counterpart of the general model's Theorem 1, provides a simple condition under which this is the case.

Theorem 1. *The system of equations (30) has a unique solution for the equilibrium population distribution if*

$$\alpha < \frac{1-\nu+\eta}{\nu}.$$

Moreover, under this condition, the economy-wide utility level as well as locations' price indices and wages are also unique. In other words, the model features a unique equilibrium.

Proof. Under the change in variables

$$\tilde{L}(r) = \bar{U}^{\frac{1}{1-\nu+\eta-\alpha\nu}} L(r), \tag{31}$$

equation (30) can be rewritten as

$$\begin{aligned} &\bar{a}(r,r)^{-\frac{\eta}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}} H(r)^{-\frac{1-\nu}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}} \tilde{L}(r)^{\left(1-\alpha(\sigma-1)+\frac{1-\nu+\eta}{\nu}\sigma\right)\frac{\sigma-1}{2\sigma-1}} \\ &= \bar{L}^{\frac{\eta(\sigma-1)}{\nu}} \delta^{\frac{\sigma-1}{\nu}} \sum_{s=1}^{S} \bar{a}(s,s)^{\frac{\eta}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}} H(s)^{\frac{1-\nu}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}} \tilde{L}(s)^{\left(1+\alpha\sigma-\frac{1-\nu+\eta}{\nu}(\sigma-1)\right)\frac{\sigma-1}{2\sigma-1}} \tau(s,r)^{1-\sigma} \end{aligned} \tag{32}$$

which is a system of S equations and S unknowns, $\tilde{L}(r)$ for each location r. (32) is a special case of the equations considered in Allen and Arkolakis (2014):

$$x(r)^{\gamma} = \sum_{s=1}^{S} K(r,s)\, x(s)^{\beta} \tag{33}$$

where $x(r) = \tilde{L}(r)$,

$$K(r,s) = \bar{a}(r,r)^{\frac{\eta}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}} H(r)^{\frac{1-\nu}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}} \bar{L}^{\frac{\eta(\sigma-1)}{\nu}} \delta^{\frac{\sigma-1}{\nu}} \bar{a}(s,s)^{\frac{\eta}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}} H(s)^{\frac{1-\nu}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}} \tau(s,r)^{1-\sigma},$$

$\gamma = \left(1-\alpha(\sigma-1)+\frac{1-\nu+\eta}{\nu}\sigma\right)\frac{\sigma-1}{2\sigma-1}$, and $\beta = \left(1+\alpha\sigma-\frac{1-\nu+\eta}{\nu}(\sigma-1)\right)\frac{\sigma-1}{2\sigma-1}$. Allen and Arkolakis (2014) show that the solution to (33) exists and is unique under the condition $\beta/\gamma \in (-1,1)$.[4] The condition $\beta/\gamma \in (-1,1)$ implies $\alpha < \frac{1-\nu+\eta}{\nu}$, that is, the inequality condition in Theorem 4.

Once we know that the solution to (32), $\tilde{L}(r)$, exists and is unique, the economy-wide utility level is determined uniquely as

$$\bar{U} = \left(\frac{\sum_{r=1}^{S}\tilde{L}(r)}{\bar{L}}\right)^{1-\nu+\eta-\alpha\nu} \tag{34}$$

[4]In particular, see Appendix A.1.3 in Allen and Arkolakis (2014) for the proof under a discrete set of locations. The same condition follows from the more general Theorem 1 of Allen et al. (2020), which I used to prove Theorem 1 in the general model.

where I used equation (31) together with the fact that $\bar{L} = \sum_{r=1}^{S} L(r)$. Using equation (31) again, the population of any r can be uniquely expressed as

$$L(r) = \bar{U}^{-\frac{1}{1-\nu+\eta-\alpha\nu}} \tilde{L}(r). \tag{35}$$

Finally, equilibrium wages and price indices are uniquely determined by equations (29) and (26), respectively. □

The inequality condition that guarantees equilibrium uniqueness, $\alpha < \frac{1-\nu+\eta}{\nu}$, is intuitive. It captures the idea that the model's agglomeration force (the agglomeration externality in production, α) is weak relative to the model's congestion forces: housing $1-\nu$ and the dispersion of idiosyncratic location tastes η. The next theorem, which is the counterpart of the general model's Theorem 2, presents a simple iterative algorithm that can be used to solve for the equilibrium if this inequality condition holds.

Theorem 2. *Assume $\alpha < \frac{1-\nu+\eta}{\nu}$. Then a simple iterative algorithm is guaranteed to converge to the equilibrium spatial distribution of population. In turn, workers' expected utility, price indices and wages can be obtained in closed form as a function of the former distribution.*

Proof. Allen and Arkolakis (2014) show that under the condition $\beta/\gamma \in (-1,1)$, an algorithm that consists of iteratively applying (33) is guaranteed to converge to the solution of (33). In the context of the current model, this implies that the researcher can guess any initial distribution of $\tilde{L}(r)$, plug it into equation (32), and update the distribution using the left-hand side of this equation. Applying this procedure iteratively, the distribution of $\tilde{L}(r)$ is guaranteed to converge to the equilibrium values of this variable. Once $\tilde{L}(r)$ is known, workers' expected utility $\bar{U}$ can be obtained in closed form from equation (34). Next, locations' population levels can be obtained from equation (35), while wages can be obtained from equation (29). Finally, price indices can be obtained from equation (26). □

Note that the iterative algorithm of Theorem 5 is even simpler than the one of Theorem 2 as it does not require the researcher to iterate on multiple equations and endogenous variables, only on (32) and $\tilde{L}(r)$.

Another advantage of equation (30) is that it allows the researcher to recover the distribution of fundamental amenities $\bar{a}(r,r)$ that rationalize the observed population data as an equilibrium. This result is similar in spirit to the general model's Theorem 3, even though Theorem 3 is not directly applicable here as fundamental productivity is assumed to be identical across locations. The result is stated formally as follows.

Theorem 3. *Assume that the researcher observes the values of structural parameters, locations' populations $L(r)$ and housing supply $H(r)$, as well as the matrix of trade costs $\tau(r,s)$. Then there exists a unique distribution of fundamental amenities $\bar{a}(r,r)$ (up to* *scale) that rationalize observed population levels as an equilibrium. Moreover, a simple iterative algorithm is guaranteed to converge to this distribution of amenities.*

Proof. Under the change in variables

$$\tilde{\bar{a}}(r,r) = \bar{U}^{1/\eta}\bar{a}(r,r),$$

equation (30) can be rewritten as

$$\begin{aligned} &\tilde{\bar{a}}(r,r)^{-\frac{\eta}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}} H(r)^{-\frac{1-\nu}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}} L(r)^{\left(1-\alpha(\sigma-1)+\frac{1-\nu+\eta}{\nu}\sigma\right)\frac{\sigma-1}{2\sigma-1}} \\ &= \bar{L}^{\frac{\eta(\sigma-1)}{\nu}}\delta^{\frac{\sigma-1}{\nu}}\sum_{s=1}^{S}\tilde{\bar{a}}(s,s)^{\frac{\eta}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}} H(s)^{\frac{1-\nu}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}} L(s)^{\left(1+\alpha\sigma-\frac{1-\nu+\eta}{\nu}(\sigma-1)\right)\frac{\sigma-1}{2\sigma-1}}\tau(s,r)^{1-\sigma} \end{aligned} \tag{36}$$

which is a system of S equations and S unknowns, $\tilde{\bar{a}}(r,r)$ for each location r. (36) is again a special case of equation (33), where $x(r) = \tilde{\bar{a}}(r,r)$,

$$\begin{aligned} K(r,s) = &H(r)^{\frac{1-\nu}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}} L(r)^{-\left(1-\alpha(\sigma-1)+\frac{1-\nu+\eta}{\nu}\sigma\right)\frac{\sigma-1}{2\sigma-1}}\bar{L}^{\frac{\eta(\sigma-1)}{\nu}}\delta^{\frac{\sigma-1}{\nu}} \\ &\cdot H(s)^{\frac{1-\nu}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}} L(s)^{\left(1+\alpha\sigma-\frac{1-\nu+\eta}{\nu}(\sigma-1)\right)\frac{\sigma-1}{2\sigma-1}}\tau(s,r)^{1-\sigma}, \end{aligned}$$

$\gamma = -\frac{\eta}{\nu}\frac{\sigma(\sigma-1)}{2\sigma-1}$, and $\beta = \frac{\eta}{\nu}\frac{(\sigma-1)^2}{2\sigma-1}$. Allen and Arkolakis (2014) show that the solution to (33) exists and is unique under the condition $\beta/\gamma \in (-1,1)$, which is always satisfied in the case of equation (36) as $\beta/\gamma = -\frac{\sigma-1}{\sigma} \in (-1,1)$. Thus, $\tilde{\bar{a}}(r,r)$ are uniquely pinned down by equation (36).

Allen and Arkolakis (2014) also show that whenever $\beta/\gamma \in (-1,1)$, an algorithm that consists of iteratively applying (33) is guaranteed to converge to the solution of (33). In this specific context, this means that guessing any initial distribution of $\tilde{\bar{a}}(r,r)$ and applying (36) iteratively guarantees convergence to the distribution of $\tilde{\bar{a}}(r,r)$ that rationalize the observed population data. □

Theorem 6 is even more powerful than its counterpart in the general model (Theorem 3). This is because Theorem 6 not only guarantees uniqueness of the model inversion but also offers an algorithm to conduct this procedure. Recall that it was not possible to find such an algorithm in the general model (Section 2.1.2), whereas it turns out to be possible in this special case of the model, which is used for the quantitative application.[5]

[5]The inversion procedure cannot identify location pair amenities $\bar{a}(r,s)$ for which $r \neq s$. This is intuitive: only commuting data between locations r and s would be informative about these amenities, but the special case of the model considered in this section does not feature commuting. That said, one does not need these location pair amenities to simulate the model either, as they do not enter the model's equilibrium conditions (equations 22 to 25).

A.5 Taking the model of Section 2.2 to the data

I apply the following strategy to quantify the impact of new bridges on the spatial distribution of population and welfare in the model. First, I invert the model to recover Hungarian settlements' fundamental amenities that rationalize observed data on these settlements' 1910 population and housing supply. This is done using Theorem 6, which guarantees a unique set of amenities (up to scale) that rationalize the data and offers an iterative algorithm to conduct the inversion procedure. When conducting the inversion, I allow for the possibility of crossing the Danube through a bridge in Pozsony, Komárom, Esztergom and Budapest, that is, at the locations of actual bridges in 1910.

Next, I simulate the model in the absence of bridges in Pozsony, Komárom and Esztergom, while I keep the possibility of crossing the Danube through a bridge in Budapest, where a bridge had already been present since 1849 (Section 2.2). When simulating the model without bridges on the upper Danube, I keep all other fundamentals (settlements' amenities and housing supply, inland trade costs, Hungary's total population and the values of structural parameters) fixed. This step is done using Theorems 4 and 5, which guarantee a unique equilibrium and offer an iterative algorithm to conduct the simulation.

Finally, I compare the 1910 equilibrium to the equilibrium without bridges on the upper Danube. First, I estimate the reduced-form effect of distance from the bridge on log settlement population, as in the data. Second, I measure the change in aggregate welfare as

$$1 - \frac{\bar{U}_{nobridges}}{\bar{U}_{1910}}$$

where $\bar{U}_{1910}$ and $\bar{U}_{nobridges}$ denote workers' expected utility in the 1910 equilibrium and in the equilibrium without bridges on the upper Danube, respectively.

As discussed in Section 2.2.2, I apply the above procedure repeatedly for a range of values of transport cost parameter ϕ. In practice, I set up a fine grid of ϕ between 0.001 and 0.1, consisting of 100 equally spaced grid points. Then, for any given value of ϕ on the grid, I invert the model using 1910 data and simulate it in the absence of bridges on the upper Danube. Given the simple nature of the iterative algorithms used for the inversion and the simulation, the process of inverting and simulating the model for all 100 grid points takes no more than a few minutes on a typical personal computer.

I choose the values of the remaining structural parameters based on the literature. It is reasonable to think that the vast majority of the cost of crossing a river on a boat, ψ, is the cost of transshipment between the boat and inland transportation. I follow the estimate of Donaldson and Hornbeck (2016) – ultimately from Fogel (1964) – that transshipment between two different modes of transportation costed 50 cents per ton in the late-19th century United States, while rail transportation costed 0.63 cents per ton-mile. Since ϕ measures the cost of inland transportation per kilometer in my context, and hence inland costs per mile equal 1.6ϕ, setting

$$\psi = \frac{50}{0.63} 1.6\phi = 126.98\phi$$

implies the same ratio of transshipment to inland (rail) costs as in Fogel's estimates.

For the dispersion parameter of idiosyncratic location tastes η, I exploit the fact that the elasticity of population to real income equals $1/\eta$ in the model (equation 22). This elasticity has been estimated at 3.3 by Monte et al. (2018) for the contemporary United States, which suggests setting $\eta = 1/3.3$. Although Hungary around 1900 differs from the current U.S. in various respects, Nagy (2020b) estimates a range of elasticities of population to real income in 1910 Hungary that are in fact close to the Monte et al. (2018) number.

To identify parameter ν, I use the fact that the share of workers' expenditure on housing equals $1 - \nu$ in the model. Lacking historical data on this share, I set it to 0.25, which is consistent with estimates of the housing expenditure share in the contemporary U.S. (Davis and Ortalo-Magné, 2011). For the elasticity of substitution across tradables σ, I follow Bernard, Eaton, Jensen and Kortum (2003) and set the value of this parameter to 4.

To choose the value of parameter α driving the strength of agglomeration forces, note that the model does not feature increasing returns as in Krugman (1991). At the same time, the economic geography literature tends to think of increasing returns as a key driver of city formation and urbanization (Fujita and Thisse, 2002). Section A.3 of the Online Appendix shows an isomorphism between the general model of Section 2.1.1 and a Krugman (1991)-style framework with increasing returns. In that isomorphic framework, the value of the agglomeration parameter equals

$$\tilde{\alpha} = \alpha + \frac{1}{\sigma - 1}$$

where α is the "true" agglomeration externality parameter, while the term $\frac{1}{\sigma-1}$ stems from the presence of increasing returns. Motivated by this, I set $\alpha = 0.06 + \frac{1}{\sigma-1} = 0.06 + \frac{1}{4-1}$, where the 0.06 number comes from agglomeration externality estimates by Ciccone and Hall (1996). Under these choices of parameters α, ν and η, the inequality condition of Theorem 4 holds and thus the model is guaranteed to feature a unique equilibrium.

I set total population $\bar{L}$ equal to the sum of Hungarian settlements' populations in 1910. Finally, I proxy the housing supply of settlement r, $H(r)$, by the land area of the settlement, as reported in the 1910 census. Just like with the other structural parameters and model fundamentals, I keep the values of $\bar{L}$ and $H(r)$ fixed between the inversion and the simulation.