Security Design with Ratings*

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Abstract

We investigate the effect of public information, such as ratings, on the security design problem of a privately informed issuer. We show that the presence of ratings has important implications for both the form of security designed and the amount of inefficient retention. The model predicts that issuers will design informationally sensitive securities (i.e., levered equity) when ratings are sufficiently informative relative to the gains from trade. Otherwise, issuers opt for a standard debt contract. In either case, informative ratings increase market liquidity by decreasing the reliance on inefficient retention to convey high quality, and perhaps counterintuitively, decrease price informativeness.

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1 Introduction

In this paper, we introduce public information into the security design problem of a privately informed issuer. A liquidity-constrained issuer has existing assets that generate a random future cash flow X. To raise capital, the issuer can design and issue a security F, backed by her asset's cash flows, to a competitive market of risk-neutral investors.[1] The issuer has private information about the quality of her assets (high or low), which may hinder her ability to raise funds in the market. To address this problem, the seller can design a security in attempt to signal information to investors. In addition, after the security is designed, investors observe an informative (but imperfect) signal, R, which we refer as a *rating*. Thus, there are two potential sources of information that investors receive: (i) the choice of security, F, and (ii) the rating, R. After observing F and R, investors bid for the security and the market clearing price is determined.

As a benchmark, we first consider the model without ratings. In this case, an issuer with high-quality assets chooses to perfectly signal her "type" to investors by choosing to issue a debt contract, $F = \min\{d, X\}$, and to retain the reminder of cash flows. Hence, the issuer retains a levered equity claim. The issued debt level d is determined by the minimum amount of cash flow retention needed to separate from the low type, who issues a claim to all of her cash flows, $F = X$. Because asset quality is perfectly revealed by the choice of security, prices reflect all available information. This information transmission, however, is not free because retention of cash flows is costly for the high-type issuer.

We then analyze the model with ratings. We show that there exists a unique equilibrium satisfying standard refinements and provide a full characterization of the equilibrium as it depends on the informativeness of ratings. We then focus on how ratings affect: (1) the *form* of security issued, (2) the *level* of cash flows retained by the issuer, and (3) the informativeness of prices.

How do ratings affect the form of security issued? When ratings are not very informative, the form of security issued remains standard debt. Intuitively, when there is little information

[1] For example, the issuer could be a firm with profitable investment opportunities that raises capital by selling claims to cash flows generated by existing assets, or a bank selling asset-backed securities in order to make more loans.

conveyed by ratings, the most credible signal of high quality is the seller's willingness to retain the portion of the (random) cash flow that is most sensitive to her privately known asset quality. By issuing debt, the issuer retains a levered equity claim—the most informationally sensitive component of the cash flow. However, once ratings are sufficiently informative, the opposite is true: the most credible signal of high asset quality is to issue the most informationally sensitive portion of the cash flow. Doing so creates exposure to "ratings risk," signaling confidence that the issuer expects the rating to authenticate her high quality. We characterize the precise condition on ratings informativeness at which the issuer switches from issuing a debt security (and retaining levered equity) to issuing a levered equity security (and retaining debt), and refer to this condition as *α-informativeness*.

How do ratings affect inefficient retention? When ratings are not very informative, the types separate by choice of retention levels. For sufficiently informative ratings, however, the high-type issuer starts to rely (at least in part) on the rating to convey information to investors. Doing so requires some degree of pooling—if all information is revealed by choice of security, there is nothing left for the ratings to convey. Therefore, the high-type issuer retains a smaller portion of the residual cash flows, thereby reducing her amount of inefficient retention. On the other hand, because the low-type issuer also chooses this level of retention (with at least some probability) as opposed to selling everything, her inefficient retention increases. We precisely characterize the condition on ratings informativeness at which retention levels switch from separating to (at least some degree of) pooling. We refer to this condition as *β-informativeness*, and show that it is strictly weaker than α-informativeness.

How do ratings affect the informativeness of prices? When ratings are β-informative, some degree of pooling occurs in equilibrium and hence the rating conveys meaningful, but not complete, information to investors. However, the amount of information transmitted to investors in equilibrium is strictly less than without ratings—because, without ratings, equilibrium play is completely revealing. Thus, perhaps counterintuitively, the introduction of informative ratings decreases the total amount of information transmitted to investors and prices become less informative.

In the empirical literature, it is well established that information asymmetries between firms and outsiders, as well as the presence of ratings, affect firms' financing and investment decisions (Bharath et al., 2008; Graham and Harvey, 2001; Kisgen, 2006). Our model is motivated by this fact, and its main predictions align with existing evidence.

First, more informative public information leads the firm to issue more informally sensitive securities. Chang et al. (2006) find that as the number of analysts that follow a firm increases, firms are more likely to issue equity as opposed to debt. Moreover, Chang et al. (2009) find that as the quality of firms' auditors increases, companies issue more equity. Derrien and Kecskés (2013) show that an exogenous decrease in the number of analysts who follow a firm reduces the firm's external funding and subsequent investments. In addition, they find that the issuance of equity and risky debt falls by more than that of short-term, safer, debt.

Second, by reducing the incentive to signal through retention, informative ratings improves firms' access to external finance. Faulkender and Petersen (2005) document that firms with a debt rating have significantly more leverage. In addition, Sufi (2007) finds that the presence of third-party certification increases firm debt issuances and subsequent investments.[2]

Naturally, the "rating" in our model could be interpreted more broadly. For example, R could represent information contained in mandated disclosures from the issuer.[3] Expanding further, in Section 7 we show that our main findings extend to a setting in which investors receive dispersed private information instead of (or in addition to) public information. What is essential for our results is that a privately informed issuer is potentially exposed to fluctuations in the price of her security due to the presence of noisy information observed by investors, which naturally captures many market and informational settings.

1.1 Related Literature

Following Myers and Majluf (1984), an extensive literature studies security design in the presence of adverse selection. However, its two common modeling approaches eliminate the possibility that the *form* of the security design can signal the seller's private information.

[2]To the best of our knowledge, our model's prediction that introducing ratings can lower price informativeness has not been empirically tested.

[3]In order to implement Section 942(b) of the Dodd-Frank act, the SEC introduced rules that require ABS issuers to provide standardized, asset-level information to potential investors prior to the offering.

In the approach of Nachman and Noe (1994), the issuer needs to raise a fixed amount of funds, which leads all seller types to offer the same security. In the approach of DeMarzo and Duffie (1999), securities are designed before the arrival of private information (ex ante), and signaling occurs ex post through the choice of quantity of the designed security (see also DeMarzo, 2005; Biais and Mariotti, 2005). Under both approaches, the equilibrium prediction is the issuance of standard debt securities, as debt minimizes the security's sensitivity to the issuer's private information.

To allow for the possibility that the form of the security could convey the seller's private information, we analyze the ex post security design problem. More importantly, we characterize how equilibrium security design changes with the introduction of relevant public information. DeMarzo (2005) discusses debt as the equilibrium security design in an ex post setting without ratings, although the proof is found in DeMarzo et al. (2015).[4]

The optimality of debt is less robust when investors have private information, as shown by Axelson (2007). He studies a setting where only investors have private information, and shows that the seller may choose to issue an informationally sensitive security when there are sufficiently many investors in order to minimize the rents extracted by these investors.

The use of informationally sensitive securities may also be desirable in order to extract information from investors or to induce them to acquire information. Some examples include Boot and Thakor (1993), Fulghieri and Lukin (2001), Yang (2015) and Yang and Zeng (2015). Relatedly, Dang et al. (2010) argue that debt promotes market liquidity as it discourages information acquisition. Our paper contributes to this literature by providing a new rationale for the issuance of informationally sensitive claims.

Our model builds on the framework developed in Daley and Green (2014) who study how the presence of *grades* affects equilibrium behavior in signaling games, such as in the canonical models of Spence (1973) and Leland and Pyle (1977).[5] In their model, the signal space is one dimensional. Whereas, the signal space is an (infinite-dimensional) set of functions from realized cash flows to security payoffs in this paper.

[4]An ex post security design problem is also analyzed in Asriyan and Vanasco (2020), but in a nonexclusive screening.

[5]How ratings may affect equilibrium behavior is also explored in Boot et al. (2006), who show that ratings can work as a coordination mechanism in situations where multiple equilibria would otherwise arise.

2 Model

There are two periods. A risk-neutral agent owns an asset that generates a stochastic cash flow X in period 2. The agent has an incentive to raise cash by issuing a claim to some portion of the cash flow due to, for example, credit constraints or capital requirements. We refer to this agent as the *seller* or *issuer* and capture her incentive in reduced form by assuming that she discounts period-2 payoffs at $\delta \in (0,1)$, whereas, there is a competitive market of risk-neutral investors, whose common discount factor is 1.[6]

At the beginning of the first period, the seller privately observes a signal $t \in \{L, H\}$, also referred to as her *type*. We denote the distribution and density functions of X conditional on t by Π_t and π_t, respectively, where $\pi_t(x) > 0$ over a common support $x \in [0, \overline{x}]$. The densities satisfy the monotone likelihood ratio property (MLRP): $\frac{\pi_H(x)}{\pi_L(x)}$ is increasing in x.

After observing her type, the seller issues *security*, $F = \phi(X)$, where $\phi : [0, \overline{x}] \to [0, \overline{x}]$.[7] Specifically, for any realization of the cash flow x, $\phi(x)$ is the amount paid to the purchaser of the security and $x - \phi(x)$ is the amount retained by the seller. Following the literature, we restrict attention to securities in which both $\phi(x)$ and $x - \phi(x)$ are nondecreasing in x, which is typically justified on the grounds of moral hazard. Denote the set of all such securities by $\mathcal{F}$. After the seller designs the security, the security receives a *rating*, which is a public signal correlated with t.

Investor's share a common prior $\mu_0 \equiv \Pr(t = H) \in (0,1)$. Based upon the security offered for sale, F, and the realized rating, r, investors update their prior to a final belief $\mu_f(F, r) \equiv \Pr(t = H|F, r)$. Since the market is competitive, the price paid for the security is

$$P(F|\mu_f) = E^{\mu_f}[F] = \mu_f E[F|H] + (1 - \mu_f) E[F|L]. \tag{1}$$

The seller's realized payoff is $U(F, P, x) \equiv P + \delta(x - \phi(x))$.

Notice that because the seller values cash today more than investors do, the uniquely efficient outcome is to sell the entire cash flow (i.e., $F = X$).

[6] This approach is also used in DeMarzo and Duffie (1999), Biais and Mariotti (2005), DeMarzo (2005), and Holmström and Tirole (2011), among others.

[7] That is, $\phi(\cdot)$ is a function, whereas the security F is the random variable $\phi(X)$.

Solution Concept

To handle the common problems posed by the freedom of off-equilibrium-path beliefs in signaling games, our solution concept is perfect Bayesian equilibrium satisfying the D1 refinement (Banks and Sobel, 1987; Cho and Kreps, 1987), hereafter simply referred to as *equilibrium*. Essentially, D1 requires investors to attribute the offer of an unexpected security to the type who is "more likely" to gain from the offer compared to her equilibrium payoff (a formal description of the refinement is found in the Online Appendix.).

Throughout the paper, we provide both the argument and the intuition for each main result. Formal details of the proofs are relegated to the Online Appendix.

Debt and Levered Equity

Before beginning the analysis, it will be useful to develop notation for two particular forms of securities.

Definition 1. *A* ***debt*** *security, F_d^D, is characterized by its* ***face value****, $d \in [0, \overline{x}]$, as $F_d^D = \min\{d, X\}$. Let $\mathcal{F}^D \equiv (F_d^D)_{d \in [0,\overline{x}]}$ be the set of all debt securities.*

If the seller issues a debt security with face value d, she retains a *levered equity* claim: $X - F_d^D = \max\{0, X - d\}$. Conversely, if the seller issues a levered equity security, she retains a debt claim.

Definition 2. *A* ***levered equity*** *security, F_a^A, is characterized by the strike cash flow, $a \in [0, \overline{x}]$, as $F_a^A = \max\{0, X - a\}$. Let $\mathcal{F}^A \equiv (F_a^A)_{a \in [0,\overline{x}]}$ be the set of all levered equity securities.*

Of course, $F_{\overline{x}}^D = F_0^A = X$. That is, selling the entire cash flow is a special case of both forms of securities.

3 The No-Ratings Benchmark

As a benchmark, consider the model without ratings (or, equivalently, one in which ratings are completely uninformative). We obtain the following result.

Proposition 1. *Without ratings, there is a unique equilibrium and it is separating. In it, the low type issues a full claim to her cash flows, X, and the high type issues the debt security $\min\{d^{LC}, X\}$ with d^{LC} given by*

$$E[\min\{d^{LC}, X\}|H] + \delta E[X - \min\{d^{LC}, X\}|L] = \underline{u} \equiv E[X|L]. \tag{2}$$

The first component of the result is that, of all the securities available in $\mathcal{F}$, the high type will issue a debt security. Intuitively, since high cash flow realizations are more indicative of $t = H$, the high type is more willing than the low type to retain the claim that only pays off in such realizations. Hence, issuing debt is the "least costly" way to separate from the low type. A similar result is discussed in DeMarzo (2005). We offer a new and relatively simple proof of this result, which is a special case of Theorem 1(a) (see Section 5).

Given that the high type will issue some debt security, the choice of the seller becomes single-dimensional—select $d \in [0, \overline{x}]$—and the model is similar to many signaling environments that have been studied in the existing literature. Because retention is costly, the high type retains as little as possible—by setting d^{LC} as high as possible—subject to the low type weakly preferring her full-information payoff to imitating the high type's issuance, as stated in (2). Because the equilibrium is separating, the seller's information is revealed to investors and security prices accurately reflect all information.

4 Ratings and Informativeness

The rating is a random variable R, with type-dependent density function q_t on $\mathbb{R}$.[8] The informativeness of a rating realization, r, is captured by $\beta(r) \equiv \frac{q_L(r)}{q_H(r)}$.[9] Without loss, order the ratings such that β is weakly decreasing. For convenience, we assume that q_H, q_L are continuous almost everywhere, the informativeness of ratings is bounded (i.e., $\beta(\cdot) \in (0, \infty)$), and, unless otherwise stated, ratings contain non-trivial information (i.e., $\beta(r) \neq 1$ on a set of positive measure).

[8] To accommodate situations with a finite set of rating outcomes $\{y_1, y_2, \dots\}$, with probabilities $p_t(y_n)$, let $q_t(r) = p_t(y_n)$ for $r \in [n, n+1)$ and $q_t(r) = 0$ for all other r.

[9] If $q_H(r) = q_L(r) = 0$, we adopt the convention that $\beta(r) = 1$.

While $\beta(r)$ measures the informativeness of a particular ratings realization, r, the informativeness of the rating system, $\{q_L, q_H\}$, will be the critical determinant of the model's predictions. Blackwell (1951) and Lehmann (1988) provide the two predominant notions for what it means for one system to be unambiguously more informative than another, which endow only partial orderings of systems. We will show that there are two critical measures of informativeness for our analysis, each of which is based on the differing expectations of the seller types. Both measures are strictly weaker than the notions of Blackwell and Lehmann, and each endows a complete ordering over rating systems.

Consider the market belief that $t = H$ after observing the chosen security, F, but prior to the realization of the rating. We refer to this as the *interim* belief. An arbitrary interim belief is denoted μ, and $\mu(F)$ indicates the interim belief conditional on the seller's chosen security F. For an interim belief μ, the final market belief given $R = r$ is given by Bayes rule

$$\mu_f(\mu, r) = \frac{\mu q_H(r)}{\mu q_H(r) + (1-\mu) q_L(r)} = \frac{\mu}{\mu + (1-\mu)\beta(r)}. \tag{3}$$

Let $\alpha_t(\mu)$ denote the expected posterior belief from the type t's perspective:

$$\alpha_t(\mu) \equiv E_R[\mu_f(\mu, R)|t]. \tag{4}$$

Immediately, $\alpha_H(\mu) \geq \alpha_L(\mu)$ with the inequality being strict if and only if $\mu \notin \{0, 1\}$. Also, the difference between them, denoted by $\alpha(\mu) \equiv \alpha_H(\mu) - \alpha_L(\mu)$, can be shown to be continuous and single-peaked.

In determining the equilibrium security design, the relevant measure of rating system informativeness is the maximum difference in expected final market belief between the seller types: $\hat{\alpha} \equiv \max_\mu \alpha(\mu)$. The key will be how this measure of informativeness compares to the gains from trade.

Definition 3. *Ratings are* $\boldsymbol{\alpha}$***-informative*** *if* $\hat{\alpha} > \delta$.

To ease exposition, we assume that $\hat{\alpha} \neq \delta$ unless otherwise stated.

5 Equilibrium Security Design

We begin with the main result.

Theorem 1 (Effect of Ratings on Security Design)**.** *In the unique equilibrium,*

(a) If ratings are α-informative, then both types issue levered-equity securities.

(b) If ratings are not α-informative, then both types issue debt securities.

The intuition behind Theorem 1 is as follows. With no or relatively uninformative ratings, the most credible way for the seller to signal high value is to demonstrate willingness to retain the most informationally sensitive portion of the cash flow, which entails issuing debt. However, if ratings are α-informative, the opposite is true: the most credible way to signal high value is to create exposure to "ratings risk" by offering up for sale the most informationally sensitive portion of the cash flow (i.e., issue levered equity). Doing so demonstrates confidence that rating will be a positive indicator of asset quality.

To arrive at this result, first recall that the final update from the interim to final belief is a straightforward application of Bayes rule. We can therefore use (1) and (4) to write the seller's expected payoff given any security F and interim belief μ as

$$\begin{aligned} u_t(F,\mu) &= E[P(F)|t,\mu] + \delta\left(E[X-F|t]\right) \\ &= \alpha_t(\mu)E[F|H] + (1-\alpha_t(\mu))E[F|L] + \delta\left(E[X-F|t]\right). \end{aligned} \tag{5}$$

A key part of proving Theorem 1 is characterizing the solution to the following maximization problem.

$$\begin{aligned} &\max_{F,\mu} u_H(F,\mu) \\ &s.t.\ u_L(F,\mu) = k. \end{aligned} \tag{M(k)}$$

The following lemma explains why the solution to M(k) is important.

Lemma 1. *In any equilibrium, if the low type's payoff is $u_L = k$, then the high type issues a security $F^*(k)$, which results in an interim belief $\mu(F^*(k))$, such that the pair $\{F^*(k), \mu(F^*(k))\}$ solves* M(k).

Intuitively, if the high type does not select a security that solves $\mathrm{M}(k)$, then by D1, the off-path issuance of a security that does solve $\mathrm{M}(k)$ will be attributed to the high type since she stands to gain more than does the low type from this deviation. This attribution makes the deviation profitable, breaking the equilibrium.

In $\mathrm{M}(k)$, $u_L = k$ by the constraint. Hence, we can replace the objective in $\mathrm{M}(k)$ with $u_H(F,\mu) - u_L(F,\mu)$ without changing the set of solutions. Next, we have from (5) that

$$u_H(F,\mu) - u_L(F,\mu) = (\alpha(\mu) - \delta)\left(E[F|H] - E[F|L]\right) + \delta\left(E[X|H] - E[X|L]\right).$$

Finally, $\delta\left(E[X|H] - E[X|L]\right)$ is a constant unaffected by F and μ. So, the solutions to $\mathrm{M}(k)$ are identical to the solutions to the following:

$$\begin{aligned} &\max_{F,\mu}\, (\alpha(\mu) - \delta)\left(E[F|H] - E[F|L]\right) \\ &s.t.\ u_L(F,\mu) = k. \end{aligned} \qquad \mathrm{M}'(k)$$

This equivalence sheds light on the importance of α-informativeness. Notice that $E[F|H] - E[F|L] \geq 0$ for all $F \in \mathcal{F}$. Therefore, if ratings are not α-informative, the objective in $\mathrm{M}'(k)$ is negative, and the high type seeks to minimize the sensitivity of the security's expected payment to her private information about the quality of the underlying asset. This is accomplished by issuing debt, which is said to be minimally *information sensitive* among securities in $\mathcal{F}$.

On the other hand, if ratings are α-informative, the high type wants to select a security that instead maximizes the sensitivity to the private information, so long as it leads to a belief $\mu(F)$ such that $\alpha(\mu(F)) > \delta$.[10] Intuitively, if ratings are sufficiently informative, the way for the high type to maximize her payoff is by designing a security that is most sensitive to the true quality of the underlying asset.

Lemma 2. *For all $k \in [\underline{u}, \overline{u})$, the solution to* $\mathrm{M}(k)$ *is unique. If ratings are α-informative, then $F^*(k)$ is a levered equity security. Otherwise, $F^*(k)$ is a debt security.*

[10] For sufficiently high priors, it will not be feasible to achieve $\alpha(\mu(F)) > \delta$. In this case, both types sell their entire cash flows (see Proposition 3).

6 Equilibrium Retention and Characterization

The space of feasible securities that could be issued, $\mathcal{F}$, is infinite dimensional. Theorem 1 reduces the dimensionality of the relevant space; either the face value of debt (if ratings are not α-informative) or the strike cash flow of a levered equity claim (if ratings are α-informative).

Once the signaling space has been reduced to a single dimension, we can apply the framework developed in Daley and Green (2014) to fully characterize the equilibrium. They show that the measure of informativeness that determines whether the equilibrium will be separating or involve pooling depends on the expected likelihood ratios of the seller types and payoff parameters. Given the utility function of the seller in our model, the relevant condition is as follows.

Definition 4. *Ratings are* ***β-informative*** *if* $\frac{E[\beta(R)|L]}{E[\beta(R)|H]} - 1 > \left(\frac{\delta}{1-\delta}\right)\left(\frac{\Pi_L(d^{LC})-\Pi_H(d^{LC})}{1-\Pi_H(d^{LC})}\right)$.

The following lemma establishes the relative strength of the informativeness measures.

Lemma 3. *α-informativeness implies β-informativeness. The converse is not true.*

The following proposition describes equilibrium retention.

Proposition 2 (Effect on Retention)**.**

(a) If ratings are β-informative, in the unique equilibrium the high type retains less than is required for separation. Hence, there is at least some degree of pooling on the security issued.

(b) If ratings are <u>not</u> β-informative, then the unique equilibrium is least-cost separating as in Proposition 1.

Recall that in the no-ratings benchmark, the seller issues debt because it is the most informationally insensitive security. In addition, equilibrium play fully reveals the seller's type to the market since the high type (inefficiently) retains enough to accomplish separation. Our results demonstrate that both of these features are upended with sufficiently informative ratings, with the retention effect kicking in at a strictly lower level of ratings informativeness than the security form effect (see Lemma 3).

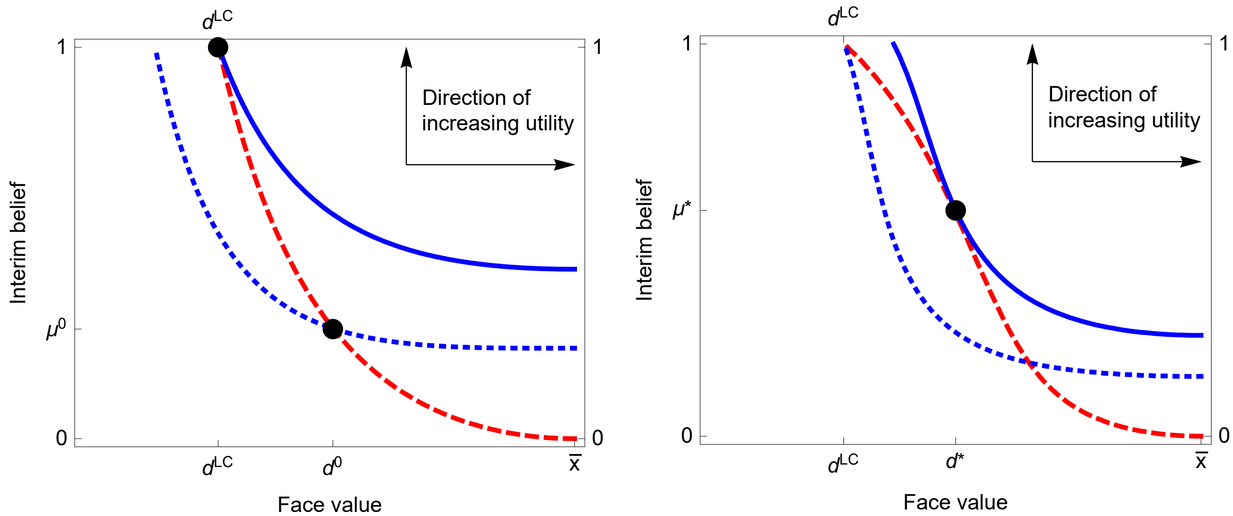
Perhaps counterintuitive at first pass is that ratings decrease the total information transmitted to investors in equilibrium. Consequently, if ratings alter retention, securities are mispriced in that they do not reflect the total information available to market participants, unlike in the no-ratings environment. Naturally, in the limit as ratings become perfectly informative, so too are prices.

To understand the argument behind Proposition 2, consider first the case where ratings are not α-informative, meaning the solution to M(k) involves issuing debt. Hence, we can restate the problem as

$$\max_{d,\mu} \left(\alpha(\mu) - \delta\right) \left(E[\min\{d, X\}|H] - E[\min\{d, X\}|L]\right)$$
$$s.t.\ u_L(F_d^D, \mu) = k.$$

Since the signaling space has been reduced to a single dimension (Theorem 1), the solution can be illustrated graphically. Let us examine how the solution to M$(\underline{u})$ depends on rating informativeness. Starting with the case of no ratings, Figure 1(a) shows the low type's indifference curve for $u_L = \underline{u}$ (in dashed-red). Therefore, the depicted (d^0, μ^0) satisfies the constraint, but it does not solve M$(\underline{u})$ since shifting to points with higher μ-values on the low type's indifference curve strictly increases u_H. In fact, without ratings, the indifference curves for the two types satisfy the single-crossing property, meaning the unique solution to M$(\underline{u})$ is the boundary solution: $(d, \mu) = (d^{LC}, 1)$. This is the property underlying Proposition 1: without ratings, $u_L = \underline{u}$, and the high type separates by choosing $F^D_{d^{LC}}$ which leads to the separating interim belief $\mu(F^D_{d^{LC}}) = 1$.

For any (d, μ), the addition of ratings decreases u_L and increases u_H. This is depicted in Figure 1(b). The low type's indifference curve is higher than without ratings (i.e., a higher interim belief is needed to offset the negative impact of the rating and keep $u_L = \underline{u}$), whereas the high type's indifference curve is lower than without ratings. Whether or not the solution to M$(\underline{u})$ is altered, then, is determined by whether the high type's curve falls below the low's at $(d^{LC}, 1)$. This is why Theorem 2 hinges on β-informativeness, which is the precise



(a) No Ratings (b) β-informative Ratings

Figure 1: Solving M($\underline{u}$). In both panels, the dashed-red indifference curve is for the low-type $u_L(F_d^D, \mu) = \underline{u}$, the solid-blue indifference curve is for the high type at the optimal value of M($\underline{u}$), and the dotted-blue indifference curve is for the high type at a suboptimal value.

condition that determines whether ratings are informative enough to move the solution to M($\underline{u}$) away from the boundary. Finally, if $(d^{LC}, 1)$ does not solve M($\underline{u}$), then separation is not possible in equilibrium since (1) separation implies $u_L = \underline{u}$, and, recall, (2) if $u_L = k$, D1 requires that the high type select a security $F^*(k)$ that leads to an the interim belief $\mu(F^*(k))$ such that $(F^*(k), \mu(F^*(k)))$ is a solution to M(k).

Intuitively, as ratings become increasingly informative, the high type wishes to rely at least partially on them in equilibrium. But reliance on ratings requires some degree of pooling—if the types separate by choice of security, there is no useful information left for the ratings to convey. Simultaneously, reliance on the rating allows the high type to reduce her degree of inefficient retention by selecting a face value of debt that is strictly higher than the d^{LC}.

Appendix A provides a complete characterization of the equilibrium. To do so, we establish the properties of the solution to M(k) that allow us to apply Proposition 3.8 of Daley and Green (2014), and thus first solve M(k) for all feasible low-type utilities to then connect the solution locus to equilibria. A brief summary of these findings is as follows. First, Proposition 2(a) already describes the equilibrium when the ratings are not β-informative. When ratings are β-informative, the unique equilibrium involves partial (full) pooling if the prior is below (above) a threshold. In partial pooling equilibria, the low type mixes between selling the entire cash flow and mimicking the high type, where the form of security issued by the high type is characterized by Theorem 1.

6.1 Eliminating Inefficient Retention

Informative ratings decrease the reliance on inefficient retention to convey high value, and more so when the market's prior belief is more favorable. It is natural to ask then if ratings can eliminate signaling via retention, which is answered in the next proposition.

Proposition 3. *If ratings are sufficiently informative to satisfy* $\frac{E[\beta(R)|L]}{E[\beta(R)|H]}-1 > \left(\frac{\delta}{1-\delta}\right)\left(\frac{\pi_H(\overline{x})-\pi_L(\overline{x})}{\pi_H(\overline{x})}\right)$*, then there exists* $\tilde{\mu} \in (0,1)$ *such that both types efficiently sell their entire cash flow (i.e., issue* $F = X$*) for all* $\mu_0 \geq \tilde{\mu}$*.*

The rating informativeness criterion in the proposition is analogous to β-informativeness except with the RHS evaluated at the upper limit of cash flows, $\overline{x}$, rather than at d^{LC}.[11] This criterion is stronger than β-informativeness, but weaker than α-informativeness (see the proof of Lemma 3).

7 Security Design with Dispersed Information

We now extend our findings to a setting in which investors receive private signals about the asset cash flows in addition to the public rating. To do so, we augment the model of Section 2 with investor-specific private signals, S_i, and specify that the security is sold through a uniform-price, k-unit auction mechanism.

Dispersed Information. In addition to the public rating with properties described in Section 4, each investor $i \in \{1, 2, ..., n\}$ observes signal S_i. Conditional on seller type t, investor's private signals are identically and independently distributed, with type-dependent density function ξ_t. We assume the density functions $\{\xi_H, \xi_L\}$ share the properties of the density functions for the rating described in Section 4, $\{q_H, q_L\}$.

[11] Because $\Pi_H(\overline{x}) = \Pi_L(\overline{x}) = 1$, the condition in the proposition is from L'Hospital's rule.

This setting incorporates two special cases: i) our baseline model in which only the public signal is informative: if $\xi_H(s) = \xi_L(s)$ for all s; and ii) a model in which only private signals dispersed among the investors are informative: if $q_H(r) = q_L(r)$ for all r.

Sales Mechanism. The seller splits security F into $k < n$ identical units and elicits simultaneous sealed bids for a unit from each of the n (potential) investors. Each unit of security F is then allocated to the k highest bidders, who pay a price set equal to the $(k+1)^{st}$ highest bid. This mechanism has been studied extensively in the auction literature (Milgrom, 1981; Milgrom and Weber, 1982; Pesendorfer and Swinkels, 1997; Kremer, 2002), with its most well-known version being the second-price auction ($k = 1$). It also is employed (and discussed) in the security design context by Axelson (2007).

As is typical, we focus on the unique symmetric equilibrium of the sales stage. Let Y_m^k be the k^{th} order statistic of out of m draws of investor signals, and let Y_{-i}^k be the k^{th} order statistic of signals other than S_i.

In any (symmetric) equilibrium of the auction, the bid of an investor who observes public rating r and private signal s is given by $b(r, s) = E[F|R = r, S_i = s, F, Y_{-i}^k = s]$.[12] The first difference with our baseline model is that now bidders condition on their private signal and on it being *pivotal*—that is, on being tied with the k^{th} highest signal among all other bidders, $Y_{n-1}^k = s$. Let μ continue to denote the interim belief upon observing security F (i.e., $\mu = Pr(t = H|F)$), and let g_t^k be the type-dependent density of Y_{n-1}^k, and thus of Y_{-i}^k. With this, we can index an investor's bid as follows

$$b(\mu, r, s) = E[F|L] + \tilde{\mu}_f(\mu, r, s)(E[F|H] - E[F|L]), \tag{6}$$

where

$$\tilde{\mu}_f(\mu, r, s) \equiv \frac{\mu}{\mu + (1 - \mu)\beta(r)\gamma^k(s)}, \tag{7}$$

and where $\gamma^k(s) = \frac{\xi_L(s)g_L^k(s)}{\xi_H(s)g_H^k(s)}$ is an adjusted likelihood ratio of private signal S that incorporates the equilibrium bidding strategies of investors, as it measures the informativeness of signal s, conditional on s also being the k^{th} order statistic of $n - 1$ draws.

The price of security F the seller receives is given by the $(k+1)^{st}$ highest bid, $b(\mu, r, Y_n^{k+1})$.

[12] As first shown in Milgrom (1981).

Thus, the seller's expected payoff given any security F and interim belief μ is

$$u_t(F,\mu) = \rho_t^k(\mu)E[F|H] + (1-\rho_t^k(\mu))E[F|L] + \delta\left(E[X-F|t]\right), \tag{8}$$

where

$$\rho_t^k(\mu) \equiv E[\tilde{\mu}_f(\mu, R, Y_n^{k+1})|t]. \tag{9}$$

The expression in (8) is analogous to (5) for the model with only ratings, where the expected posterior $\alpha_t(\mu)$ is now adjusted to $\rho_t(\mu)$ to incorporate investors' equilibrium bidding strategies in the presence of dispersed information and the auction protocol.The second difference with our baseline model, seen in (9), is that the seller is exposed to the variation in price induced by the $(k+1)^{st}$ order statistic of n draws of S, Y_n^{k+1}, in addition to the variation induced by rating R.

In our baseline model with ratings, we showed that the seller's decision to *i*) design a particular *form* of security, and *ii*) to perfectly separate or (partially) pool with the other seller type, each depended on a distinct notion of rating informativeness (α- and β-informativeness, respectively). The following two notions of signal informativeness nest the notions from the baseline model.

Definition 5 (Signals Informativeness)**.**

- *Investors' signals are* $\boldsymbol{\rho}$**-informative** *if* $\hat{\rho}^k > \delta$.
- *Investors' signals are* $\boldsymbol{\gamma}$**-informative** *if* $\frac{E\left[\beta(R)\gamma^k\left(Y_n^{k+1}\right)|L\right]}{E\left[\beta(R)\gamma^k\left(Y_n^{k+1}\right)|H\right]} - 1 \geq \left(\frac{\delta}{1-\delta}\right)\left(\frac{\Pi_L\left(d^{LC}\right)-\Pi_H\left(d^{LC}\right)}{1-\Pi_H(d^{LC})}\right)$.

The notion of ρ-informativeness generalizes the notion of α-informativeness, and γ-informativeness generalizes β-informativeness. Analogously, ρ-informativeness implies γ-informativeness, but not the converse (generalizing Lemma 3). First, we re-establish our main finding on security form.

Theorem 2 (Effect of Signals on Security Design)**.** *In the unique equilibrium,*

(a) If signals are ρ-informative, then both types issue levered-equity securities.

(b) If signals are <u>not</u> ρ-informative, then both types issue debt securities.

Next, we re-establish our main finding on the effect of signals on equilibrium retention:

Proposition 4 (Effect of Signals on Retention)**.**

(a) If signals are γ-informative, in the unique equilibrium the high type retains less than is required for separation. Hence, there is at least some degree of pooling on the security issued.

(b) If signals are not γ-informative, then the unique equilibrium is least-cost separating as in Proposition 1.

Theorem 2 and Proposition 4 show that our results do not rely on the information observed by investors being exclusively public. What matters is the degree to which the privately informed seller can rely on the price to move based on information received by the investor side of the market. If investors will receive very noisy information, its impact on price will be too low, and the seller chooses to perfectly convey her type through the issuance of debt securities. If, instead, investor information will be accurate enough to meaningfully impact prices, the seller exposes herself to this force first by pooling (according to the β or γ criterion), and second by issuing levered equity (according to the α or ρ criterion).

7.1 Example—The Effect of Dispersed Information

In this example we illustrate the impact of investors receiving common versus dispersed information on the form of the issued security. To do so, we analyze the level of signal precision required to achieve ρ-informativeness, which as we have shown shifts the security form from debt to levered equity. We will analyze two settings. In the first setting, there are two signals and each signal is publicly observed. In the second, there are two investors and each investor observes a private signal. In both cases, the signals are normally distributed with type-dependent mean and type-independent precision. Figure 2 illustrates the minimum signal precision needed to achieve ρ-informativeness as a function of the discount factor.

Notice that, regardless of the discount factor, the model with privately dispersed signals requires "more" total information (in terms of signal precision) in order for the form of the

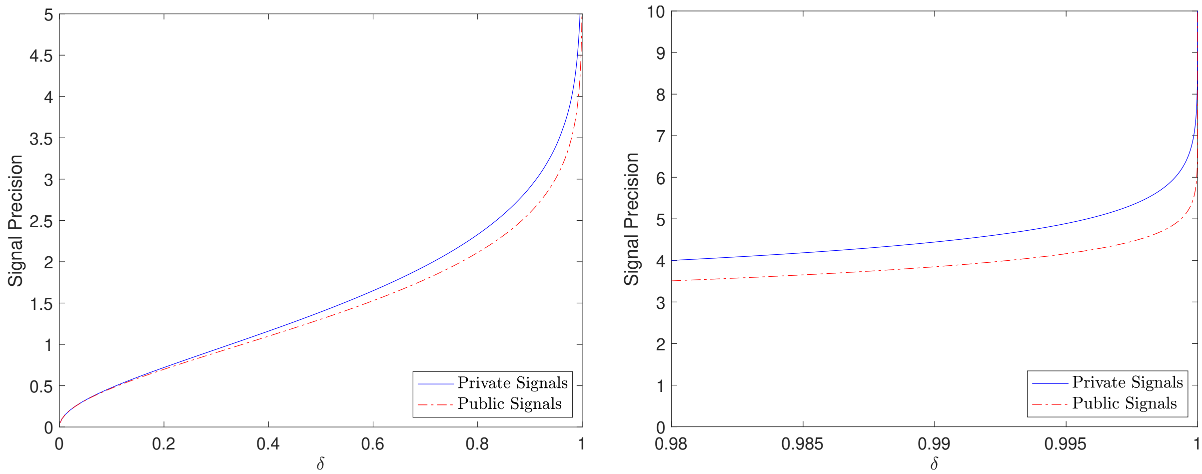


FIGURE 2: Plot of the threshold signal precision as it depends on δ and whether signals are public or private. Above the threshold, the equilibrium security form shifts from debt to levered equity. On the left-panel, the axis range is $\delta \in [0, 1]$, where on the right-panel, we focus on the more relevant range where the difference in discount rates (i.e., $1 - \delta$) ranges from zero to two percent.

security to shift from debt to levered equity. For example, when the difference between the seller and investors discount rate is 1 percentage point (i.e., $\delta = 0.99$), the seller issues levered equity when the signal precision is above 3.85 with public signals, but only when the signal precision is above 4.44 when signals are private. Because the auction does not perfectly aggregate the private information of all investors, privatizing the signals has a similar effect to lowering their precision and keeping them public.

8 Conclusion

We have analyzed the effect of ratings (or other publicly disclosed information) on a general security design problem with a privately informed seller. Sufficiently informative ratings incent high-value sellers to issue informationally sensitive securities and to decrease their inefficient retention (which is completely eliminated when the initial market belief about types is favorable). Consequently, low-value sellers are induced to pool with high-value ones with positive probability, and prices are less informative than in the (separating) equilibrium of the no-ratings environment. Our results are robust to settings in which investors receive dispersed information.

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A Detailed Equilibrium Characterization

We here provide a complete equilibrium characterization. To do so, we solve M(k) for all $k \in [\underline{u}, \overline{u})$, where $\overline{u} \equiv E[X|H]$ is a strict upper bound on the payoff the low type can achieve in equilibrium. We then connect the solutions of M(k) to equilibrium, following Daley and Green (2014).

Technically, the the payoffs in our model do not satisfy the assumptions in Daley and Green (2014), so in what follows we establish the properties of the solution to M(k) that are sufficient to apply Proposition 3.8 of Daley and Green (2014).

We begin by analyzing the case when ratings are not α-informative. The properties of the solution in such case are recorded in the following lemma and illustrated in Figure A.1. Let $\underline{d}(k)$ be the unique solution to $u_L(F^D_{\underline{d}(k)}, 1) = k$. That is, $\underline{d}(k)$ is the face value of a debt security required for $u_L = k$ given that it engenders a belief that $t = H$ with probability one (e.g., $\underline{d}(\underline{u}) = d^{LC}$).

Lemma A.1. *Suppose ratings are not α-informative. Then,*

(a) The solution to M(k)*, denoted $(F^*(k), \mu^*(k))$, is unique for all $k \in [\underline{u}, \overline{u})$.*

(b) $F^*(k)$ is a debt security, with face value $d^*(k)$, for all $k \in [\underline{u}, \overline{u})$.

(c) $\mu^*(k) < 1$ if and only if $\frac{E[\beta(R)|L]}{E[\beta(R)|H]} - 1 > \left(\frac{\delta}{1-\delta}\right)\left(\frac{\Pi_L(\underline{d}(k)) - \Pi_H(\underline{d}(k))}{1 - \Pi_H(\underline{d}(k))}\right)$.

(d) d^* and μ^* are continuous and strictly increasing in k (modulo boundary conditions).

As seen in Theorem 1, if we further increase the informativeness of ratings, the seller no longer issues debt, but instead its complement: levered equity. We again begin by characterizing the solutions to M(k), this time for the case when ratings are α-informative.

Lemma A.2. *Suppose ratings are α-informative. Then,*

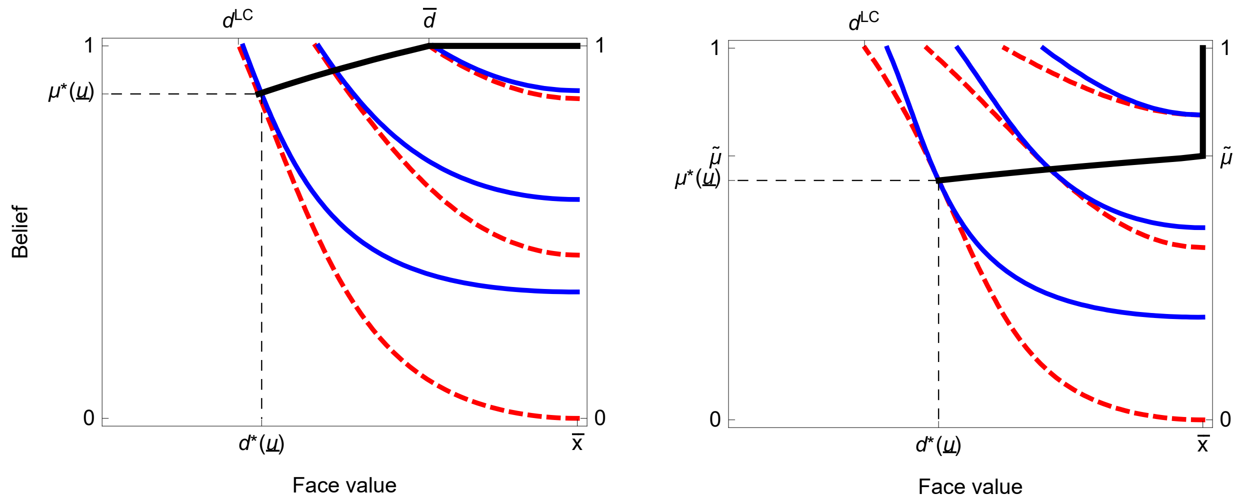
(a) The solution to M(k)*, denoted $(F^*(k), \mu^*(k))$, is unique for all $k \in [\underline{u}, \overline{u})$.*

(b) $F^*(k)$ is a levered equity security with retention level $a^*(k)$, for all $k \in [\underline{u}, \overline{u})$.

(c) $\mu^*(k) < 1$ for all $k \in [\underline{u}, \overline{u})$.

(d) a^* is strictly decreasing, and μ^* strictly increasing, in k (modulo boundary conditions).

The final component is connecting the solutions of M(k) to equilibrium. For each prior belief μ_0, only a single value for the low type's payoff is consistent with equilibrium. The result established in the following Proposition combines the results in Lemmas A.1 and A.2 with those in Proposition 3.8 in Daley and Green (2014).



(a) β-informative, but positive retention for all priors. (b) β-informative, and zero retention for high priors.

FIGURE A.1: The effect of rating informativeness on $(d^*(\cdot), \mu^*(\cdot))$, depicted as heavy-black curve. Low type's indifference curves in dashed-red, high type's in solid-blue.

Proposition A.1. *In the unique equilibrium we have that*

(a) if $\mu_0 < \mu^*(\underline{u})$, there is some degree of pooling, as the high type issues security $F^*(\underline{u})$ and the low type mixes between security $F^*(\underline{u})$ and selling the entire cash flow, X, with probability $\frac{\mu_0(1-\mu^*(\underline{u}))}{\mu^*(\underline{u})(1-\mu_0)} \in (0,1)$ and the complementary probability, respectively.

(b) if $\mu_0 \geq \mu^*(\underline{u})$, there is full pooling as both types select the unique security $F^*(k)$ such that $\mu^*(k) = \mu_0$.

Figure A.1 illustrates Proposition A.1, as well Proposition 3, for two cases in which ratings are β-informative, but not α-informative. Hence, debt is the equilibrium security design. In Panel (a), β-informativeness implies pooling, but with positive levels of retention for all priors. That is, for all μ_0, the high type issues debt with face value $d \leq \overline{d} < \overline{x}$. Panel (b) increases the informativeness of ratings such that the criterion in Proposition 3 is satisfied, and the seller efficiently sells her entire cash flow for high priors.

B Proof Details

B.1 Preliminaries and Definitions

Fact B.1. *For any $t \in \{L, H\}$ and $F \in \mathcal{F}\backslash\{0\}$,*

1. $E[F|H] > E[F|L]$.
2. $u_t(F, \mu)$ *is strictly increasing in* μ.
3. *There exists unique* $d, a \in [0, \overline{x}]$ *such that* $E[F|t] = E[F_d^D|t] = E[F_a^A|t]$.
4. *For* $\eta \in [0, 1]$*, let* $F^\eta \equiv (1 - \eta)F + \eta X$*. Then* $F^\eta \in \mathcal{F}$*, and if* $F \neq X$*, then* $E[F^\eta|t]$ *and* $u_t(F^\eta, 1)$ *are strictly increasing in* η.

Fact B.2. *In any PBE,* $u_t \in [\underline{u}, \overline{u})$ *for any* $t \in \{L, H\}$.

Lemma B.1. *The expected posterior beliefs have the following properties:*

1. $\alpha_t(\cdot)$ *is strictly increasing for any* $t \in \{H, L\}$.
2. $\alpha_H(\mu) \geq \alpha_L(\mu)$ *with the inequality being strict if and only if* $\mu \notin \{0, 1\}$.
3. $\alpha(\mu) \equiv \alpha_H(\mu) - \alpha_L(\mu)$ *is continuous and single-peaked.*
4. $\frac{d}{d\mu}\frac{\alpha'_H(\mu)}{\alpha'_L(\mu)} < 0$.

Lemma B.2. *With dispersed investor information (Section 7), for all* $k \geq 1$*, adjusted expected posteriors,* $\rho_t^k(\cdot)$*, preserve the following properties of* $\alpha_t(\cdot)$*:*

1. $\rho_t^k(\cdot)$ *is strictly increasing for any* $t \in \{H, L\}$.
2. $\rho_H^k(\mu) \geq \rho_L^k(\mu)$*, with the inequality being strict if and only if* $\mu \notin \{0, 1\}$.
3. $\rho^k(\mu) \equiv \rho_H^k(\mu) - \rho_L^k(\mu)$ *is continuous and single-peaked, with* $\hat{\rho}^k \equiv \max_\mu \rho^k(\mu)$.
4. $\frac{d}{d\mu}\frac{\rho_H^{k\prime}(\mu)}{\rho_L^{k\prime}(\mu)} < 0$.

The D1 Refinement

Fix $k \in [\underline{u}, \overline{u})$ and $F \in \mathcal{F}$, and consider the equation $u_t(F, \mu) = k$. By Fact B.1(2), there is at most one solution for μ. If it exists, denote it by $b_t(F, k)$—that is, $u_t(F, b_t(F, k)) = k$. Next, let $B_t(F, k) \equiv \{\mu : u_t(F, \mu) > k\}$. From Fact B.1(2), the connection between b_t and B_t is immediate: if $b_t(F, k)$ exists, then $B_t(F, k) = (b_t(F, k), 1]$. If $b_t(F, k)$ fails to exist, then either $B_t(F, k) = [0, 1]$ or $B_t(F, k) = \emptyset$.

In our model, the D1 refinement can be stated as follows. Fix an equilibrium endowing expected payoffs $\{u_L, u_H\}$. Consider a security F that is not in the support of either type's strategy. If $B_L(F, u_L) \subset B_H(F, u_H)$, then D1 requires that $\mu(F) = 1$ (where $\subset$ denotes strict inclusion). If $B_H(F, u_H) \subset B_L(F, u_L)$, then D1 requires that $\mu(F) = 0$.

B.2 Proofs of Lemmas

The lemmas are proved in the following order: B.1, A.1, A.2, 1, 3, B.2. Notice that Lemmas A.1 and A.2 together subsume Lemma 2.

Proof of Lemma B.1. Note that

$$\alpha(\mu) = \int \frac{\mu}{\mu + (1-\mu)\beta(r)}(q_H(r) - q_L(r))dr$$

is bounded, twice continuously differentiable and meets the criteria for exchanging the order of integration and differentiation by the functional form of the integrand, which has bounded first and second partial derivatives with respect to μ. Thus, result 1 follows inmediately from $\mu_f(\mu, \cdot)$ being strictly increasing in μ, and result 2 from $\mu_f(\cdot, r)$ being weakly increasing in r together with the MLRP property that $\frac{q_H(r)}{q_L(r)}$ is weakly increasing in r. For result 3, note that $\alpha(0) = \alpha(1) = 0$, and since ratings are informative, $\alpha(\mu) > 0$ for all $\mu \in (0, 1)$. Thus, it must be that $\alpha'(0) > 0$ and that $\alpha'(1) < 0$. Finally, note that:

$$\begin{aligned}
\alpha'(\mu) &= \int \frac{(1-\beta(r))}{(\mu + (1-\mu)\beta(r))^2} q_L(r)dr \\
\alpha''(\mu) &= -2\int \frac{(1-\beta(r))^2}{(\mu + (1-\mu)\beta(r))^3} q_L(r)dr
\end{aligned}$$

with $\alpha''(\mu) < 0$ for all $\mu \in (0, 1)$. Thus, $\alpha(\mu)$ is single-peaked in μ. Finally, result 4 is established in Daley and Green,2014 (Lemma A.1) and Karlin, 1968, (Chapter 3, Proposition 5.1). □

Proof of Lemma A.1. Recall that the solutions to M(k) are identical to the solutions to M$'(k)$, which we show are characterized by (a)-(d).

Starting with (b), fix $k \in [\underline{u}, \overline{u})$ and let $\{F^*, \mu^*\}$ be a solution to M$'(k)$, where $F^* = \phi^*(X)$ and $F^* \notin \mathcal{F}^D$. By Fact B.1(3), let d be the unique solution to $E[F^*|L] = E[F_d^D|L]$, and $\phi_d(x) = \min\{d, x\}$. Since $F^* \in \mathcal{F}$, ϕ^* is non-decreasing and $\phi^*(x) \leq x$ for all x. Thus, there exists an $\tilde{x} \in (0, \overline{x})$ such that $\phi^*(x) \leq \phi_d(x)$ for all $x \leq \tilde{x}$ with strict inequality for a positive measure set of $x < \tilde{x}$, and $\phi^*(x) \geq \phi_d(x)$ for all $x \geq \tilde{x}$ with strict inequality for a positive measure set of $x > \tilde{x}$. Next,

$$\begin{aligned}
&E[F^* - F_d^D|H] - E[F^* - F_d^D|L] \\
&\quad= \int_0^{\overline{x}} (\phi^*(x) - \phi_d(x))(\pi_H(x) - \pi_L(x))dx \\
&\quad= \int_0^{\tilde{x}} (\phi^*(x) - \phi_d(x))(\pi_H(x) - \pi_L(x))dx + \int_{\tilde{x}}^{\overline{x}} (\phi^*(x) - \phi_d(x))(\pi_H(x) - \pi_L(x))dx \\
&\quad= \int_0^{\tilde{x}} (\phi^*(x) - \phi_d(x))\left(\frac{\pi_H(x)}{\pi_L(x)} - 1\right)\pi_L(x)dx + \int_{\tilde{x}}^{\overline{x}} (\phi^*(x) - \phi_d(x))\left(\frac{\pi_H(x)}{\pi_L(x)} - 1\right)\pi_L(x)dx
\end{aligned}$$

$$= \int_0^{\tilde{x}} (\phi^*(x) - \phi_d(x)) \left(\frac{\pi_H(x)}{\pi_L(x)}\right) \pi_L(x)dx + \int_{\tilde{x}}^{\overline{x}} (\phi^*(x) - \phi_d(x)) \left(\frac{\pi_H(x)}{\pi_L(x)}\right) \pi_L(x)dx$$
$$> \int_0^{\tilde{x}} (\phi^*(x) - \phi_d(x)) \left(\frac{\pi_H(\tilde{x})}{\pi_L(\tilde{x})}\right) \pi_L(x)dx + \int_{\tilde{x}}^{\overline{x}} (\phi^*(x) - \phi_d(x)) \left(\frac{\pi_H(\tilde{x})}{\pi_L(\tilde{x})}\right) \pi_L(x)dx$$
$$= \left(\frac{\pi_H(\tilde{x})}{\pi_L(\tilde{x})}\right) \int (\phi^*(x) - \phi_d(x))\pi_L(x)dx = \left(\frac{\pi_H(\tilde{x})}{\pi_L(\tilde{x})}\right) \left(E[F^*|L] - E[F_d^D|L]\right) = 0,$$

where the last inequality follows from MLRP: $\frac{\pi_H(\tilde{x})}{\pi_L(\tilde{x})} = \max_{x \leq \tilde{x}} \frac{\pi_H(x)}{\pi_L(x)}$ and that $\frac{\pi_H(\tilde{x})}{\pi_L(\tilde{x})} = \min_{x \geq \tilde{x}} \frac{\pi_H(x)}{\pi_L(x)}$. Thus, the last inequality results from maximizing the weights assigned to the negative points and minimize the weights assigned to the positive ones.

It follows that $E\left[F_d^D|H\right] - E\left[F_d^D|L\right] < E\left[F^*|H\right] - E\left[F^*|L\right]$, and

$$u_L(F_d^D, \mu^*) = \alpha_L(\mu^*) \left(E[F_d^D|H] - E[F_d^D|L]\right) + (1-\delta)E[F_d^D|L] + \delta E[X|L] < k.$$

Since $u_L(F_d^D, \mu^*)$ is continuous and increasing in d, there exists security $F_{d'}^D \in \mathcal{F}^D$, with $d' > d$, such that $u_L(F_{d'}^D, \mu^*) = k$, satisfying the constraint for $\mathrm{M}'(k)$. Further, because $E[F_{d'}^D|L] > E[F_d^D|L] = E[F^*|L]$ and $u_L(F_{d'}^D, \mu^*) = k = u_L(F^*, \mu^*)$, it must be that $E[F_{d'}^D|H] - E[F_{d'}^D|L] < E[F^*|H] - E[F^*|L]$. But then the objective in $\mathrm{M}'(k)$ attains a higher value at $\{F_{d'}^D, \mu^*\}$ than at $\{F^*, \mu^*\}$ since $\alpha(\mu^*) - \delta < 0$. This contradicts that $\{F^*, \mu^*\}$ solves $\mathrm{M}'(k)$. Hence, any solution to $\mathrm{M}'(k)$ must be a debt security.

To establish (a), we show that there is a unique solution to the following re-statement of $\mathrm{M}'(k)$:

$$\max_{d,\mu} \ v^D(d,\mu) \quad s.t. \ \ h^D(d,\mu) = k$$

where

$$v^D(d,\mu) \equiv (\alpha(\mu) - \delta)\left(E[\min\{d,X\}|H] - E[\min\{d,X\}|L]\right)$$
$$h^D(d,\mu) \equiv \alpha_L(\mu)\left(E[\min\{d,X\}|H] - E[\min\{d,X\}|L]\right) + (1-\delta)E[\min\{d,X\}|L] + \delta E[X|L].$$

Define $\mu_\ell(k)$ to be the unique solution to $u_L(X, \mu_\ell(k)) = k$. Since $X = F_{\overline{x}}^D$ and $u_L(F_d^D, \mu)$ is increasing in both d and μ, any $\{d, \mu\}$ that satisfies the constraint in $\mathrm{M}'(k)$ must have $\mu \in [\mu_\ell(k), 1]$.

Let us look first for a solution, $\{d^*, \mu^*, \lambda^*\}$, with interior $\mu^* \in (\mu_\ell(k), 1)$, where λ denotes the multiplier on the constraint. Such a solution is characterized by the first-order conditions (second-order conditions are verified at the end of this proof):

$$\left(\alpha'(\mu) - \lambda\alpha_L'(\mu)\right)\left(E[\min\{d,X\}|H] - E[\min\{d,X\}|L]\right) = 0 \qquad \text{(FOC-}\mu\text{)}$$
$$\left(\alpha(\mu) - \delta - \lambda\alpha_L(\mu)\right)\int_d^{\overline{x}} \left(\pi_H(x) - \pi_L(x)\right)dx - \lambda(1-\delta)\int_d^{\overline{x}} \pi_L(x)\,dx = 0 \qquad \text{(FOC-}d\text{)}$$

In any solution, $d^* > 0$, as $u_L(F_0^D, \mu) = \delta E[X|L] < \underline{u} \leq k$, in violation of the constraint. From

Fact B.1(1), the second term in the LHS of FOC-μ is positive, and thus $\lambda^* = \frac{\alpha'(\mu^*)}{\alpha'_L(\mu^*)}$. The second condition, FOC-d, requires $\lambda^* < 0$, since $\alpha(\mu) - \delta < 0$ for all μ. Thus, $\alpha'(\mu^*) < 0$ which implies $\mu^* \in [\hat{\mu}, 1]$ (Lemma B.1(3)). Combining these two conditions, we obtain the system of equations that determines $\{d^*, \mu^*\}$:

$$\frac{1 - \Pi_H(d^*)}{1 - \Pi_L(d^*)} - 1 = \frac{1-\delta}{(\alpha(\mu^*) - \delta)\frac{\alpha'_L(\mu^*)}{\alpha'(\mu^*)} - \alpha_L(\mu^*)} \tag{B1}$$

$$h^D(d^*, \mu^*) = k. \tag{B2}$$

Let $d_1 : \mu \mapsto d_1(\mu)$ denote the mapping from beliefs to debt levels implied by (B1) and $d_2^k : \mu \mapsto d_2^k(\mu)$ denote the mapping implied by (B2). First, note that $d_1(\cdot)$ is continuous and strictly increasing in μ since the LHS of (B1) is strictly increasing in d due to hazard rate dominance (implied by MLRP):

$$\frac{d}{dd}LHS = \left[\frac{\pi_L(d)}{1 - \Pi_L(d)} - \frac{\pi_H(d)}{1 - \Pi_H(d)}\right]\frac{1 - \Pi_H(d)}{1 - \Pi_L(d)} > 0,$$

and the RHS of (B1) is strictly increasing in μ when $\alpha(\mu) - \delta < 0$:

$$\frac{d}{d\mu}RHS = \frac{1-\delta}{\left((\alpha(\mu) - \delta)\frac{\alpha'_L(\mu)}{\alpha'(\mu)} - \alpha_L(\mu)\right)^2}\left(\frac{\alpha(\mu) - \delta}{\alpha'(\mu)^2}\left(\alpha''_H(\mu)\alpha'_L(\mu) - \alpha'_H(\mu)\alpha''_L(\mu)\right)\right) > 0,$$

since $\frac{d}{d\mu}\frac{\alpha'_H(\mu)}{\alpha'_L(\mu)} < 0$ from Lemma B.1(4).

Second, $d_2^k(\cdot)$ is continuous and strictly decreasing in μ for all k since $\alpha'_L(\cdot) > 0$ (Lemma B.1(1)). Therefore, there is at most one pair $\{d^*(k), \mu^*(k)\}$ such that $d_1(\mu^*(k)) = d_2^k(\mu^*(k)) = d^*(k)$ (i.e., solves (B1) and (B2)). If this pair exits, it is then the unique solution to M$'(k)$. If it fails to exists, the unique solution to M$'(k)$ is a boundary solution. In this case: (i) if $d_1(1) < d_2^k(1)$ the unique solution is $\{d^*(k), \mu^*(k)\} = \{d_2^k(1) = \underline{d}(k), 1\}$; and ($ii$) if instead $d_1(1) > d_2^k(1)$, then given there is no intersection, $d_1(\mu_\ell(k)) > d_2^k(\mu_\ell(k))$ as well, and the unique solution is $\{d^*(k), \mu^*(k)\} = \{\overline{x}, \mu_\ell(k)\}$.

Next, (c) is a matter of direct calculation. For any $k \in [\underline{u}, \overline{u})$, $\mu^*(k) < 1$ if and only if $d_1(1) > d_2^k(1) = \underline{d}(k)$. This holds if and only if

$$\frac{1 - \Pi_H(\underline{d}(k))}{1 - \Pi_L(\underline{d}(k))} - 1 < \frac{\left(\frac{\alpha'(1)}{\alpha'_L(1)}\right)(1-\delta)}{\alpha(1) - \delta - \left(\frac{\alpha'(1)}{\alpha'_L(1)}\right)\alpha_L(1)}. \tag{B3}$$

By straightforward calculations $\alpha(1) = 0$, $\alpha_L(1) = 1$, $\alpha'_t(1) = E[\beta(R)|t]$, and $E[\beta(R)|H] = 1$. So (B3) becomes

$$E[\beta(R)|L] > \frac{(1 - \Pi_H(\underline{d}(k))) - \delta(1 - \Pi_L(\underline{d}(k)))}{1 - \Pi_H(\underline{d}(k)) - \delta(1 - \Pi_H(\underline{d}(k)))} = \frac{\delta(\Pi_L(\underline{d}(k)) - \Pi_H(\underline{d}(k)))}{(1-\delta)(1 - \Pi_H(\underline{d}(k)))} + 1.$$

Finally, for (d), note that changes in k do not impact the mapping $d_1(\cdot)$. For any two $k, k' > 0$ such that $k' > k$ we have that $d_2^{k'}(\mu^*(k)) > d_2^k(\mu^*(k)) = d_1(\mu^*(k))$. Since we have shown that $d_1(\cdot)$ is strictly increasing, it must be that (modulo boundary solutions) $\mu^*(k') > \mu^*(k)$ and thus $d^*(k') > d^*(k)$. Finally, both $\mu^*(k)$ and $d^*(k)$ are continuous in k since d_1 and d_2^k are continuous in μ and k.

Verifying Second-Order Conditions. We now verify that the solution given by the first-order conditions (B1)-(B2) is, in fact, a solution to $\mathrm{M}'(k)$. We verify that the determinant of the Bordered Hessian is negative at our interior critical point:

$$BH = \begin{bmatrix} 0 & h_d^D & h_\mu^D \\ h_d^D & L_{dd} & L_{d\mu} \\ h_\mu^D & L_{\mu d} & L_{\mu\mu} \end{bmatrix}$$

where $L(d,\mu) = v^D(d,\mu) - \lambda\left(h^D(d,\mu) - k\right)$.

$$\begin{aligned}
h_d^D &= \alpha_L(\mu^*)\int_{d^*}^{\overline{x}}(\pi_H(x) - \pi_L(x))\,dx + (1-\delta)\int_{d^*}^{\overline{x}}\pi_L(x)\,dx && > 0\\
d_\mu^D &= \alpha_L'(\mu^*)\left[E[\min\{d^*, X\}|H] - E[\min\{d^*, X\}|L]\right] && > 0\\
L_{dd} &= -[\left(\alpha(\mu^*) - \delta - \lambda^*\alpha_L(\mu^*)\right)(\pi_H(d^*) - \pi_L(d^*)) - \lambda^*(1-\delta)\pi_L(d^*)] && < 0\\
L_{\mu\mu} &= \left(\alpha''(\mu^*) - \lambda^*\alpha_L''(\mu^*)\right)\left[E[\min\{d^*, X\}|H] - E[\min\{d^*, X\}|L]\right] && < 0\\
L_{d\mu} = L_{\mu d} &= \left(\alpha'(\mu^*) - \lambda^*\alpha_L'(\mu^*)\right)\int_{d^*}^{\overline{x}}(\pi_H(x) - \pi_L(x))\,dx && = 0
\end{aligned}$$

Where $L_{dd} < 0$ since hazard rate dominance implies $\frac{1-\Pi_H(d)}{1-\Pi_L(d)} > \frac{\pi_H(d)}{\pi_L(d)}$ which combined with the FOC implies:

$$\frac{\pi_H(d^*)}{\pi_L(d^*)} - 1 < \frac{\lambda^*(1-\delta)}{\alpha(\mu^*) - \delta - \lambda^*\alpha_L(\mu^*)}$$

$$(\alpha(\mu^*) - \delta - \lambda^*\alpha_L(\mu^*))(\pi_H(d^*) - \pi_L(d^*)) - \pi_L(d^*)\lambda^*(1-\delta) > 0$$

where the inequality changes sign since $(\alpha(\mu^*) - \delta - \lambda^*\alpha_L(\mu^*)) < 0$. Finally, $L_{\mu\mu}(\mu^*, d^*, \lambda^*) < 0$ since $\frac{d}{d\mu}\left(\frac{\alpha_H'(\mu)}{\alpha_L'(\mu)}\right) < 0$. A sufficient condition for our solution to be a local maximum is that the bordered Hessian is negative definite. That is, $|BH_1| < 0$ and $|BH_2| > 0$. It is easy to see that $|BH_1| = -(h_d^D)^2 < 0$ and that $|BH_2| = -\left(h_d^D\right)^2 L_{\mu\mu} - \left(h_\mu^D\right)^2 L_{dd} > 0$. □

Proof of Lemma A.2. Recall that the solutions to $\mathrm{M}(k)$ are identical to the solutions to $\mathrm{M}'(k)$, which we show are characterized by (a)-(d). Lemma B.1 and α-informativeness imply that there are exactly two solutions to $\alpha(\mu) = \delta$, which we denote $\underline{\mu}, \overline{\mu}$, and that $\underline{\mu} < \hat{\mu} < \overline{\mu}$. For all $\mu \in (\underline{\mu}, \overline{\mu})$, $\alpha(\mu) > \delta$, and for all $\mu \notin [\underline{\mu}, \overline{\mu}]$, $\alpha(\mu) < \delta$. As in the proof of Lemma A.1, define $\mu_\ell(k)$ to be the unique solution to $u_L(X, \mu_\ell(k)) = k$. Because, for any μ and $F \neq X$, $u_L(X,\mu) > u_L(F,\mu)$, and u_L is increasing μ, any $\{F, \mu\}$ that satisfies the constraint in $\mathrm{M}'(k)$ must have $\mu \in [\mu_\ell(k), 1]$.

Case 1: $\mu_\ell(k) \in [0, \overline{\mu})$.

First, in any solution it must be that $\mu^*(k) \in (\underline{\mu}, \overline{\mu})$. To see this, recall that $E[F|H] - E[F|L] \geq 0$ for any $F \in \mathcal{F}$ (Fact B.1(1)). Hence, if $\mu \notin (\underline{\mu}, \overline{\mu})$, then the objective in M$'(k)$ is weakly negative. However, the objective can attain positive value. For example, select arbitrary $\mu \in (\mu_\ell(k), \overline{\mu})$ and let $\nu > 0$ solve $u_L(\nu X, \mu) = k$ (it is straightforward to show such a ν always exists, and is positive). Then,

$$\underbrace{(\alpha(\mu) - \delta)}_{>0}\underbrace{(E[(\nu X|H] - E[\nu X|L])}_{>0} > 0.$$

Because any solution must do at least this well, $\mu^*(k) \in (\underline{\mu}, \overline{\mu})$, and $\alpha(\mu^*(k)) - \delta > 0$. Notice that this establishes claim (c) of the lemma.

To establish (b), fix $k \in [\underline{u}, \overline{u})$ and let $\{F^*, \mu^*\}$ be a solution to M$'(k)$, where $F^* = \phi^*(X)$ and $F^* \notin \mathcal{F}^A$. By Fact B.1(3), let a be the unique solution to $E\left[F^*|L\right] = E[F_a^A|L]$, and $\phi_a(x) = \max\{0, x - a\}$. Since $F^* \in \mathcal{F}$, $\phi^*(x) - x$ is non-decreasing and $\phi^*(x) \geq 0$ for all x. Thus, there exists an $\tilde{x} \in (0, \overline{x})$ such that $\phi^*(x) \geq \phi_a(x)$ for all $x \leq \tilde{x}$ with strict inequality for a positive measure set of $x < \tilde{x}$, and $\phi^*(x) \leq \phi_a(x)$ for all $x \geq \tilde{x}$ with strict inequality for a positive measure set of $x > \tilde{x}$. From here the calculations run analogously to those in the proof of Lemma A.1(b), to show that $E\left[F_a^A|H\right] - E\left[F_a^A|L\right] > E\left[F^*|H\right] - E\left[F^*|L\right]$, and

$$u_L(F_a^A, \mu^*) = \alpha_L(\mu^*)\left(E[F_a^A|H] - E[F_a^A|L]\right) + (1-\delta)E[F_a^A|L] + \delta E[X|L] > k.$$

Since $u_L(F_a^A, \mu^*)$ is continuous and decreasing in a, there exists $F_{a'}^A \in \mathcal{F}^A$, with $a' > a$, such that $u_L(F_{a'}^A, \mu^*) = k$, satisfying the constraint for M$'(k)$. Further, because $E[F_{a'}^A|L] < E[F_a^A|L] = E[F^*|L]$ and $u_L(F_{a'}^A, \mu^*) = k = u_L(F^*, \mu^*)$, it must be that $E[F_{a'}^A|H] - E[F_{a'}^A|L] > E[F^*|H] - E[F^*|L]$. But then the objective in M$'(k)$ attains a higher value at $\{F_{a'}^A, \mu^*\}$ than at $\{F^*, \mu^*\}$ since $\alpha(\mu^*) - \delta > 0$. This contradicts that $\{F^*, \mu^*\}$ solves M$'(k)$. Hence, any solution to M$'(k)$ must be a levered equity security.

To establish (a), we show that there is a unique solution to the following re-statement of M$'(k)$:

$$\max_{a,\mu}\ v^A(a, \mu) \quad s.t.\ \ h^A(a, \mu) = k,$$

where

$$\begin{aligned} v^A(a,\mu) &\equiv (\alpha(\mu) - \delta)\left(E[\max\{0, X-a\}|H] - E[\max\{0, X-a\}|L]\right), \\ h^A(a,\mu) &\equiv \alpha_L(\mu)\left(E[\max\{0, X-a\}|H] - E[\max\{0, X-a\}|L]\right) \\ &\qquad + (1-\delta)E[\max\{0, X-a\}|L] + \delta E[X|L]. \end{aligned}$$

Again, the constraint implies that in any solution $\mu^* \geq \mu_\ell(k)$, and we have already established that $\mu^* \in (\underline{\mu}, \overline{\mu})$.

Let us look first for a solution, $\{a^*, \mu^*, \lambda^*\}$, with interior $\mu^* \in (\mu_\ell(k), 1)$, where λ denotes the multiplier on the constraint. Such a solution is characterized by the following first-order conditions (second-order conditions are verified at the end of this proof):

$$\left(\alpha'\left(\mu\right)-\lambda\alpha_L'\left(\mu\right)\right)\left(E[\max\{0,X-a\}|H]-E[\max\{0,X-a\}|L]\right)=0 \qquad \text{(FOC-}\mu\text{)}$$

$$\left(\alpha\left(\mu\right)-\delta-\lambda\alpha_L\left(\mu\right)\right)\int_a^{\overline{x}}\left(\pi_L\left(x\right)-\pi_H\left(x\right)\right)dx+\lambda\left(1-\delta\right)\int_a^{\overline{x}}\pi_L\left(x\right)dx=0. \qquad \text{(FOC-}a\text{)}$$

In any solution, $a^* < \overline{x}$, as $u_L(0,\mu) = \delta E[X|L] < \underline{u} \leq k$, in violation of the constraint. From Fact B.1(1) then, the second term in the LHS of FOC-μ is positive, and $\lambda^* = \frac{\alpha'(\mu^*)}{\alpha_L'(\mu^*)}$. The second condition, FOC-a, requires $\lambda^* > 0$ because we have already established that $\mu^* \in (\underline{\mu}, \overline{\mu})$, meaning $\alpha(\mu^*) - \delta > 0$. Thus, $\alpha'(\mu^*) > 0$ which implies $\mu^* < \hat{\mu}$. Combining the two conditions, we obtain the following system of equations that determines $\{a^*, \mu^*\}$:

$$\frac{1-\Pi_H\left(a^*\right)}{1-\Pi_L\left(a^*\right)}-1=\frac{1-\delta}{\left(\alpha\left(\mu^*\right)-\delta\right)\frac{\alpha_L'(\mu^*)}{\alpha'(\mu^*)}-\alpha_L\left(\mu^*\right)} \qquad \text{(B4)}$$

$$h^A(a^*,\mu^*)=k. \qquad \text{(B5)}$$

Let $a_1 : \mu \mapsto a_1(\mu)$ denote the mapping from beliefs to residual debt levels implied by (B4) and $a_2^k : \mu \mapsto a_2^k(\mu)$ denote the mapping implied by (B5). First, note that $a_1(\cdot)$ is continuous and strictly decreasing in $\mu \in (\underline{\mu}, \overline{\mu})$ since the LHS of (B5) is strictly increasing in a due to hazard rate dominance (implied by MLRP), whereas the RHS of (B5) is strictly decreasing in μ when $\alpha(\mu) - \delta > 0$, as shown in the proof of Lemma A.1(a). Second, $a_2^k(\cdot)$ is strictly increasing in μ for all k since $\alpha_L'(\cdot) > 0$. Therefore, there is at most one pair $\{a^*(k), \mu^*(k)\}$ such that $a_1(\mu^*(k)) = a_2^k(\mu^*(k)) = a^*(k)$ (i.e., solves (B4) and (B5)). If this pair exits, it is then the unique solution to M$'(k)$. If it fails to exists, the unique solution to M$'(k)$ is a boundary solution: $\mu^*(k) \in \{\mu_\ell(k), 1\}$. Since we established at the outset that $\mu^*(k) \in (\underline{\mu}, \overline{\mu})$, if the solution is boundary it must be that $\mu^*(k) = \mu_\ell(k)$ and (by definition of $\mu_\ell(k)$) $a^*(k) = 0$ (i.e., $F^*(k) = X$).

Finally, The argument for (d) is analogous to that provided for Lemma A.1(d).

Case 2: $\mu_\ell(k) \in [\overline{\mu}, 1)$.
To begin, let $\mu_\ell(k) = \overline{\mu}$. We claim that $\{F^*, \mu^*\} = \{X, \mu_\ell(k)\}$ is the unique solution to M$'(k)$. To see this, note that it is feasible (by definition of $\mu_\ell(k)$) and produces a value of 0 for the objective since $\alpha(\overline{\mu}) = \delta$. Consider now any other candidate $\{F, \mu\}$. First, if $\mu = \mu_\ell(k)$ but $F \neq X$, then $u_L(F, \mu_\ell(k)) < u_L(X, \mu_\ell(k)) = k$, in violation of the problem's constraint. Second, if $F = 0$, then for any μ, $u_L(0, \mu) = \delta E[X|L] < \underline{u} \leq k$, also in violation of the constraint. The only remaining possibility is that $F \neq 0$ and $\mu \neq \mu_\ell(k)$. In order to satisfy the constraint, it must be that $\mu \in (\mu_\ell(k), 1] = (\overline{\mu}, 1]$. But then the objective attains a negative value, establishing the claim. Notice that $X \in \mathcal{F}^A \cap \mathcal{F}^D$.

Now let $\mu_\ell(k)$ be arbitrary in $[\overline{\mu}, 1]$, and consider the restricted version of M(k) in which only *debt* securities can be offered:

$$\begin{aligned} &\max_{d,\mu} u_H(F_d^D, \mu) \\ &s.t.\ u_L(F_d^D, \mu) = k \end{aligned} \tag{$\mathrm{M}_d(k)$}$$

For this problem, claims (a), (c), and (d) of Lemma A.1 remain true (whereas (b) is simply assumed). Because $X = F_{\overline{x}}^D \in \mathcal{F}^D$, for $\tilde{k}$ such that $\mu_\ell(\tilde{k}) = \overline{\mu}$ the unrestricted solution is feasible in the restricted problem, so it must remain the solution in the restricted problem. Lemma A.1(d) then implies that, in $\mathrm{M}_d(k)$, $d^*(k) = \overline{x}$ for all $k > \tilde{k}$ as well. Note that $k > \tilde{k}$ is equivalent to $\mu_\ell(k) > \overline{\mu}$. Thus we have that if $\mu_\ell(k) > \overline{\mu}$, the optimal debt security to offer has face value $d^* = \overline{x}$.

Turing back now to the unrestricted problem $\mathrm{M}'(k)$, for any $\mu_\ell(k) > \overline{\mu}$, since any feasible μ is in $[\mu_\ell(k), 1]$, $\alpha(\mu) - \delta < 0$ for all feasible μ. The same argument given for Lemma A.1 implies that any solution must be a debt security. So restricting to debt securities is without loss, and the solution to $\mathrm{M}'(k)$ is the same as the solution to $\mathrm{M}_d(k)$, which is $\{F^*(k), \mu^*(k)\} = \{X, \mu_\ell(k)\}$. Claims (a)-(d) follow immediately.

Verifying Second-Order Conditions. We now verify that the solution given by the first-order conditions (B4)-(B5) is, in fact, a solution to $\mathrm{M}'(k)$. We verify that the determinant of the Bordered Hessian is negative at our interior critical point:

$$BH = \begin{bmatrix} 0 & h_a^A & h_\mu^A \\ h_a^A & L_{aa} & L_{a\mu} \\ h_\mu^A & L_{\mu a} & L_{\mu\mu} \end{bmatrix}$$

where $L(a, \mu) = v^A(a, \mu) - \lambda \left(h^A(a, \mu) - k\right)$.

$$\begin{aligned}
h_a^A &= -\alpha_L(\mu^*) \int_{a^*}^{\infty} (\pi_H(x) - \pi_L(x))\, dx - (1-\delta) \int_{a^*}^{\infty} \pi_L(x)\, dx && < 0 \\
h_\mu^A &= \alpha_L'(\mu^*) \left[E[\max\{0, X - a^*\}\,|H] - E[\max\{0, X - a^*\}\,|L]\right] && > 0 \\
L_{aa} &= \left(\alpha(\mu^*) - \delta - \lambda^* \alpha_L(\mu^*)\right)(\pi_H(a^*) - \pi_L(a^*)) + \lambda^*(1-\delta)\pi_L(a^*) && < 0 \\
L_{\mu\mu} &= \left(\alpha''(\mu^*) - \lambda^* \alpha_L''(\mu^*)\right)\left[E[[\max\{0, X - a^*\}\,|H] - E[[\max\{0, X - a^*\}\,|L]\right] && < 0 \\
L_{a\mu} &= L_{\mu a} = -\left(\alpha'(\mu^*) - \lambda^* \alpha_L'(\mu^*)\right) \int_{a^*}^{\infty} (\pi_H(x) - \pi_L(x))\, dx && = 0
\end{aligned}$$

Where $L_{aa} < 0$ since hazard rate dominance implies $\frac{1-\Pi_H(d)}{1-\Pi_L(d)} > \frac{\pi_H(a)}{\pi_L(a)}$ which combined with the FOC implies:

$$\frac{\pi_H(a^*)}{\pi_L(a^*)} - 1 < \frac{\lambda^*(1-\delta)}{\alpha(\mu^*) - \delta - \lambda^* \alpha_L(\mu^*)}$$

$$(\alpha(\mu^*) - \delta - \lambda^* \alpha_L(\mu^*))(\pi_H(a^*) - \pi_L(a^*)) - \pi_L(d^*)\lambda^*(1-\delta) < 0$$

Finally, $L_{\mu\mu}(\mu^*, a^*, \lambda^*) < 0$ since $\frac{d}{d\mu}\left(\frac{\alpha_H'(\mu)}{\alpha_L'(\mu)}\right) < 0$. A sufficient condition for our solution to be a local maximum is that the bordered Hessian is negative definite. That is, $|BH_1| < 0$ and $|BH_2| > 0$. It is easy to see that $|BH_1| = -(h_a^A)^2 < 0$ and that $|BH_2| = -\left(h_a^A\right)^2 L_{\mu\mu} - \left(h_\mu^A\right)^2 L_{aa} > 0$. □

Proof of Lemma 1. Fix an equilibrium with arbitrary $u_H = \hat{u}_H$ and $u_L = k \in [\underline{u}, \overline{u})$. Since the low type has the option to choose the same security as the high type, $u_L(F, \mu(F)) \leq k$ for all $F \in S_H$. Fix now $F \in S_H$ and suppose that $u_L(F, \mu(F)) < k$. Then $F \notin S_L$, so $\mu(F) = 1 = b_H(F, \hat{u}_H)$ and $B_L(F, k) = \emptyset$. Further, it must be that $F \neq X$ since $u_L(X, 1) = \overline{u} > k$. Then for $\eta \in (0,1)$ small enough $b_H(F^\eta, \hat{u}_H) \in (0,1)$ and $B_L(F^\eta, k) = \emptyset$. Therefore, $F^\eta \notin S_L$ and $\mu(F^\eta) = 1$ (by belief consistency if $F^\eta \in S_H$, by D1 if not). Since $u_H(F^\eta, 1) > u_H(F, 1) = \hat{u}_H$, the high type would gain by deviating to F^η, breaking the equilibrium. Therefore, $u_L(F, \mu(F)) = k$, or equivalently $\mu(F) = b_L(F, k)$, for all $F \in S_H$.

Suppose now there exists $F \in S_H$ such that $F \neq F^*(k)$. Then

$$u_H(F, \mu(F)) = u_H(F, b_L(F, k)) < u_H(F^*(k), \mu^*(k)) = u_H(F^*(k), b_L(F^*(k), k)),$$

and thus $b_H(F^*(k), \hat{u}_H) < \mu^*(k) = b_L(F^*(k), k)$. D1 then implies that $\mu(F^*(k)) = 1$, meaning that deviating to $F^*(k)$ is profitable for the high type and breaking the equilibrium. Hence, if the low type's equilibrium payoff is k, then $S_H = \{F^*(k)\}$ and $\mu(F^*(k)) = \mu^*(k)$. □

Proof of Lemma 3. First, generalize the notion of β-informativeness to "β-informativeness at x" by replacing d^{LC} with arbitrary $x \in [0, \overline{x}]$ in Definition 4. See now that β-informativeness at x implies β-informativeness at all $x' \leq x$, but not the converse. The statement holds because $\frac{\Pi_L(x) - \Pi_H(x)}{1 - \Pi_H(x)}$ is nondecreasing in x, as taking the derivative yields

$$\begin{aligned}
&\frac{(1 - \Pi_H(x))\pi_L(x) - (1 - \Pi_L(x))\pi_H(x)}{(\Pi_H(x) - 1)^2} \geq 0 \\
&\iff \quad (1 - \Pi_H(x))\pi_L(x) - (1 - \Pi_L(x))\pi_H(x) \geq 0 \iff \quad \frac{\pi_L(x)}{1 - \Pi_L(x)} \geq \frac{\pi_H(x)}{1 - \Pi_H(x)},
\end{aligned}$$

where the last inequality is the definition of hazard rate dominance, which holds due to MLRP.

Now suppose that ratings are α-informative. Then, from the proof of Lemma A.2, there exists $\tilde{k}$ such that $F^*(k) = X$ and $\mu^*(k) = \mu_\ell(k) < 1$ for all $k \geq \tilde{k}$. Consider now the restricted version of M(k) in which only debt securities can be offered:

$$\begin{aligned}
&\max_{d,\mu} u_H(F_d^D, \mu) \\
&s.t.\ u_L(F_d^D, \mu) = k.
\end{aligned} \tag{$\mathrm{M}_d(k)$}$$

For this problem, claims (a), (c), and (d) of Lemma A.1 remain true (whereas (b) is simply assumed). Because $X \in \mathcal{F}^D$, for all $k \geq \tilde{k}$ the unrestricted solution is feasible in the restricted problem, so it must remain the solution in the restricted problem. From Lemma A.1(d), then, in $\mathrm{M}_d(k)$, $\mu^*(k) \leq \mu_\ell(\tilde{k}) < 1$ for all $k \leq \tilde{k}$. Lemma A.1(d) then implies that ratings are β-informative for any value of $d^{LC} = x \in [0, \overline{x}]$.

That the converse does not hold can be shown with a counterexample in which ratings are β-informative at all $x \in [0, \overline{x}]$, but are not α-informative. Let ratings be binary and symmetric (see footnote 8): $R \in \{l, h\}$ and $\Pr(R = h|t = H) = \Pr(R = l|t = L) \equiv p \in (\frac{1}{2}, 1)$. A sufficient condition for β-informativeness at any x is

$$\frac{E[\beta(R)|L]}{E[\beta(R)|H]} - 1 > \frac{\delta}{1-\delta}$$

which, using $E[\beta(R)|H] = 1$, is equivalent to

$$\frac{E[\beta(R)|L] - 1}{E[\beta(R)|L]} = \frac{(1-2p)^2}{1-3p(1-p)} > \delta.$$

Next, α-informativeness requires $\alpha(\hat{\mu}) > \delta$. For binary-symmetric ratings, $\hat{\mu} = \frac{1}{2}$ for all p, and the requirement is $\alpha(\frac{1}{2}) = (1-2p)^2 > \delta$. Since, for all $p \in (0, 1)$,

$$0 < (1-2p)^2 < \frac{(1-2p)^2}{1-3p(1-p)} < 1,$$

β-informativeness holds for all x, while α-informativeness fails, when $\delta \in \left((1-2p)^2, \frac{(1-2p)^2}{1-3p(1-p)}\right)$, producing the counterexample. □

Proof of Lemma B.2. As $g_t^k(s)$ is the density of the k-th order statistic among $m = n-1$ remaining bidders, we have that

$$g_t^k(s) = \frac{m!}{(k-1)!(m-k)!}\xi_t(s)\Xi_t(s)^{k-1}(1-\Xi_t(s))^{m-k},$$

where Ξ_t is the c.d.f. with corresponding p.d.f. ξ_t. Thus, the adjusted likelihood ratio is given by

$$B^k(r, s) \equiv \beta(r)\gamma^k(s) = \frac{q_L(r)\xi_L(s)g_L^k(s)}{q_H(r)\xi_H(s)g_H^k(s)} = \frac{q_L(r)\xi_L(s)^2\Xi_L(s)^{k-1}(1-\Xi_L(s))^{m-k}}{q_H(r)\xi_H(s)^2\Xi_H(s)^{k-1}(1-\Xi_H(s))^{m-k}}.$$

We also need the likelihood ratio of the $k+1$-th order statistic of n draws, because this is the distribution over which the seller will compute the expected posterior. Let $h_t^k(s)$ denote the density of the $k+1$ order statistic conditional on t. And define

$$\tilde{B}^k(r, s) \equiv \beta(r)\frac{h_L^k(s)}{h_H^k(s)} = \frac{q_L(r)\xi_L(s)\Xi_L(s)^{m-k}(1-\Xi_L(s))^k}{q_H(r)\xi_H(s)\Xi_H(s)^{m-k}(1-\Xi_H(s))^k}$$

We now show that adjusted likelihood ratios B^k and $\tilde{B}^k$ preserve the properties of the baseline model likelihood ratio, β: mainly, that they are both weakly decreasing in r and in s.

As $\beta(r)$ is decreasing in r, so are B^k and $\tilde{B}^k$, so it rests to show that $\frac{h_L^k(s)}{h_H^k(s)}$ decreases in s, which we show in what follows in two steps.

First, note that $\frac{\Xi_L(s)}{\Xi_H(s)}$ is weakly decreasing in s. To see this, note that

$$sign\left(\frac{d}{ds}\left(\frac{\Xi_L(s)}{\Xi_H(s)}\right)\right) = sign\left(\xi_L\Xi_H - \xi_H\Xi_L\right).$$

So it suffices to show that $\frac{\xi_L(s)}{\xi_H(s)} \geq \frac{\Xi_L(s)}{\Xi_H(s)}$. Note that we can write $\Xi_L(s) = \int_0^s \frac{\xi_L(x)}{\xi_H(x)}\xi_H(x)dx$ and since $\frac{\xi_L}{\xi_H}$ is monotonic, we have that

$$\Xi_L(s) = \int_0^s \frac{\xi_L(x)}{\xi_H(x)}\xi_H(x)dx \leq \frac{\xi_L(s)}{\xi_H(s)} \int \xi_H(x)dx = \frac{\xi_L(s)}{\xi_H(s)}\Xi_H(x).$$

Second, we have that $\frac{1-\Xi_L(s)}{1-\Xi_H(s)}$ is weakly decreasing in s. To see this, $sign\left(\frac{d}{ds}\left(\frac{1-\Xi_L(s)}{1-\Xi_H(s)}\right)\right) = sign\left(-\xi_L(1-\Xi_H)+\xi_H(1-\Xi_L)\right)$ so it suffices to show that

$$\frac{\xi_L(s)}{\xi_H(s)} \leq \frac{1-\Xi_L(s)}{1-\Xi_H(s)}$$

Then, note that

$$\frac{1-\Xi_L(s)}{1-\Xi_H(s)} = \frac{\int_s^1 \frac{\xi_L(x)}{\xi_H(x)}\xi_H(x)dx}{\int_s^1 \xi_H(x)dx} \geq \frac{\frac{\xi_L(s)}{\xi_H(s)}\int_s^1 \xi_H(x)dx}{\int_s^1 \xi_H(x)dx} = \frac{\xi_L(s)}{\xi_H(s)}.$$

With this, it follows that so are B^k and $\tilde{B}^k$ are both weakly decreasing in r and in s. With this result, we can proceed to the proof of results 1. to 4.

First, result 1 is immediate as

$$\rho_t^{k\prime}(\mu) = \int\int \frac{\beta(r)\gamma^k(s)}{\left(\mu+(1-\mu)\beta(r)\gamma^k(s)\right)^2} h_t^k(s)\, q_t(r)dsdr > 0 \tag{B6}$$

Second, as $\mu_f(\mu, r, s)$ is weakly increasing in s and in r and $\frac{h_H^k(s)}{h_L^k(s)}$ is weakly inceasing in s while $\frac{q_H(r)}{q_L(r)}$ is weakly increasing in r, result 2 follows.

Third, as signals are informative, and $\rho^k(0) = \rho^k(1) = 0$ and $\rho^k \in (0,1)$ for all $\mu \in (0,1)$, it must be that $\rho^{k\prime}(0) > 0$ and $\rho^{k\prime}(1) < 0$. In addition, note that for all $\mu \in (0,1)$

$$\rho^{k\prime\prime}(\mu) = -2\int\int \frac{\beta(r)\gamma^k(s)\left(1-\beta(r)\gamma^k(s)\right)}{\left(\mu+(1-\mu)\beta(r)\gamma^k(s)\right)^3}\left(h_H^k(s)\, q_H(r) - h_L^k(s)\, q_L(r)\right)dsdr \tag{B7}$$

$$= -2\int\int \frac{\left(1-\beta(r)\gamma^k(s)\right)^2}{\left(\mu+(1-\mu)\beta(r)\gamma^k(s)\right)^3} h_L^k(s)\, q_L(r)dsdr < 0 \tag{B8}$$

Thus, result 3 follows. Finally, result 4 is established in Lemma A.1 in Daley and Green,2014 (and also in Karlin, 1968, (Chapter 3, Proposition 5.1)). For the Daley and Green (2014) proof to go through, we need it to be the case that for any two private signals, s, s', $\gamma^k(s) \geq \gamma^k(s') \iff \tilde{B}^k(\cdot, s) \geq \tilde{B}^k(\cdot, s')$. So, as both are decreasing in s, the result follows. □

B.3 Proofs of Propositions

Proof of Proposition 1. Having no ratings means that ratings are not β-informative at any x. Hence, by Lemma A.1(c), $\mu^*(k) = 1$ and $d^*(k) = \underline{d}(k)$ for all $k \in [\underline{u}, \overline{u})$. The proposition then follows directly from Proposition A.1. $\square$

Proof of Proposition A.1. From Lemmas A.1 and A.2 we have that $F^*(k)$ and $\mu^*(k)$ are unique for all $k \in [\underline{u}, \overline{u})$. Let S_t be the support of the type t's strategy. In the proposed unique equilibrium, the high type plays a pure strategy, denoted it F_H, so $S_H = \{F_H\}$, and $S_L \subseteq \{X, F_H\}$. For completeness, we must specify the off-path beliefs: $\mu(F) = 0$ for all $F \neq F_H$.

Verifying that the proposed profile is a PBE is straightforward. To see that it satisfies D1, fix a μ_0 and consider the proposition's unique equilibrium candidate. Denote the high type's equilibrium payoff $\hat{u}_H$ and low type's equilibrium payoff k, so $F_H = F^*(k)$. Let F be an arbitrary security in $\mathcal{F}$ such that $F \neq F^*(k)$. First, if $B_L(F,k) = [0,1]$, then the low type could deviate to F and obtain a payoff strictly greater than k, regardless of $\mu(F)$, breaking the PBE. Hence, either $b_L(F,k) \in [0,1]$ exits or $u_L(F,1) < k$. If $b_L(F,k)$ exits, then since $\{F^*(k), \mu^*(k)\}$ is the unique solution to M(k), $u_H(F, b_L(F,k)) < u_H(F^*(k), \mu^*(k)) = \hat{u}_H$. By Fact B.1(2) then, $b_H(F, \hat{u}_H) > b_L(F,k)$ (or $B_H(F, \hat{u}_H) = \emptyset$) implying $B_H(F, \hat{u}_H) \subseteq B_L(F,k)$. So, $\mu(F) = 0$ is consistent with D1. If instead $u_L(F,1) < k$ (so $B_L(F,k) = \emptyset$), then there exists unique $\eta \in (0,1)$ such that $u_L(F^\eta, 1) = k$. Since $\{F^*(k), \mu^*(k)\}$ solves M(k), $u_H(F^*(k), \mu^*(k)) \geq u_H(F^\eta, 1) > u_H(F,1)$. Hence, $B_H(F, \hat{u}_H) = \emptyset$ as well, and D1 places no restriction on $\mu(F)$.

We now establish uniqueness. By Lemma 1, if the low type's equilibrium payoff is k, then $S_H = \{F^*(k)\}$ and $\mu(F^*(k)) = \mu^*(k)$. Further, if the low type selects $F \notin S_H \cup \{X\}$, then $\mu(F) = 0$, and $u_L(F,0) < u_L(X,0) \leq u_L(X, \mu(X))$ for any value of $\mu(X)$. She could therefore profitably deviate to X. Hence, $S_L \subseteq S_H \cup \{X\}$.

The final step is to characterize which values of $u_L = k$ are consistent with equilibrium, which depends on the prior, μ_0. Recall from Lemmas A.1(d) and A.2(d) that μ^* is continuous and strictly increasing in k. First, let $\mu_0 < \mu^*(\underline{u})$, and let $u_L = k$. Therefore, $S_H = \{F^*(k)\}$ and $\mu(F^*(k)) = \mu^*(k) > \mu_0$. For this belief to be consistent with seller strategies, $S_L \neq \{F^*(k)\}$. Hence, $S_L = \{F^*(k), X\}$ and $k = \underline{u}$. The precise mixing probabilities given in the proposition are required for the Bayesian consistency: $\mu(F^*(\underline{u})) = \mu^*(\underline{u})$.

Second, let $\mu_0 \geq \mu^*(\underline{u})$. Hence, there exists unique $k_0 \in [\underline{u}, \overline{u})$ such that $\mu^*(k_0) = \mu_0$. Suppose that $u_L = k > k_0$. Then $S_H = \{F^*(k)\}$ and $\mu(F^*(k)) = \mu^*(k) > \mu^*(k_0) = \mu_0$. But then for this belief to be consistent with seller strategies, $S_L \neq \{F^*(k)\}$. Hence, $S_L = \{F^*(k), X\}$ and $k = \underline{u}$, which contradicts $k > k_0$. Suppose instead that $u_L = k < k_0$. Then $S_H = \{F^*(k)\}$ and $\mu(F^*(k)) = \mu^*(k) < \mu^*(k_0) = \mu_0$. But then $\mu(F) < \mu_0$ for all F on the equilibrium path, which violates belief consistency. Hence, $k = k_0$, and $S_H = S_L = \{F^*(k_0)\}$, exactly as given in the proposition. $\square$

Proof of Proposition 2. The result is a direct implications of the properties established in Lemmas A.1 and A.2 and Proposition A.1. □

Proof of Proposition 3. First, consider the case when ratings are β- but not α-informative. Using the proof of Lemma A.1(a) and the equilibrium characterization in Proposition A.1, it is sufficient to show that $d_1(\mu_\ell(k)) > d_2^k(\mu_\ell(k))$ for all k such that $\mu_\ell(k)$ is sufficiently close to 1. Recalling that $d_2^k(\mu_\ell(k))$ satisfies $u_L(F^D_{d_2^k(\mu_\ell(k))}, \mu_\ell(k)) = k$, it follows that $d_2^k(\mu_\ell(k)) = \overline{x}$ by definition of $\mu_\ell(k)$. For any $k \in [\underline{u}, \overline{u})$, $d_1(\mu_\ell(k)) > \overline{x}$ if and only if

$$\frac{1 - \Pi_H(\overline{x})}{1 - \Pi_L(\overline{x})} - 1 < \frac{\left(\frac{\alpha'(\mu_\ell(k))}{\alpha'_L(\mu_\ell(k))}\right)(1-\delta)}{\alpha(\mu_\ell(k)) - \delta - \left(\frac{\alpha'(\mu_\ell(k))}{\alpha'_L(\mu_\ell(k))}\right)\alpha_L(\mu_\ell(k))}.$$

Because the RHS is continuous in μ, we can take the limit as $\mu_\ell(k) \to 1$, at which point the calculations are analogous to those in the proof of Lemma A.1(c), establishing the result.

Finally, consider the case when ratings are α-informative. First, for any $k \in [\underline{u}, \overline{u})$, in order to satisfy the constraint in M(k), $a^*(k) = 0$ if and only if $\mu^*(k) = \mu_\ell(k)$. Second, the proof of Lemma A.2 establishes that, for all $k \in [\underline{u}, \overline{u})$, if $\mu^*(k) \neq \mu_\ell(k)$ then $\mu^*(k) < \hat{\mu}$ and that a^* is continuous and decreasing in k. Hence, there exists $\tilde{k}$ such that $\{a^*(k), \mu^*(k)\} = \{0, \mu_\ell(k)\}$ for all $k \geq \tilde{k}$, and that $\mu_\ell(\tilde{k}) \leq \hat{\mu}$. The proposition then follows from the equilibrium characterization in Proposition A.1. □

Proof of Proposition 4. Is a direct implications of the properties established in Lemma B.2 and Proposition 2.
BD: Using Lemma B.2 in place of Lemma B.1, the proof follows the same steps used to establish Proposition 2. □

B.4 Proofs of Theorems

Proof of Theorem 1. The result is a direct implications of the properties established in Lemmas A.1 and A.2 and Proposition A.1. □

Proof of Theorem 2. The result is a direct implications of the properties established in Lemma B.2 and Theorem 1.
BD: Using Lemma B.2 in place of Lemma B.1, the proof follows the same steps used to establish Theorem 1. □