Online Appendix to Government Debt Management: The Long and the Short of It*

Faraglia, Elisa † Marcet, Albert‡ Oikonomou, Rigas § Scott, Andrew ¶

September 26, 2018

*Faraglia gratefully acknowledges support from the Cambridge Endowment of Research in Finance (CERF), Marcet from Plan Nacional (Spanish Ministry of Science), Monfispol, AGAUR (Generalitat de Catalunya), the Axa Foundation, the Excellence Program of Banco de España, European Research Council under the EU 7th Framework Programme (FP/2007-2013) Advanced ERC Grant Agreement n. 324048 - APMPAL and the Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2015-0563) and Scott from the ESRCs World Economy and Finance program. We are indebted to Hafedh Bouakez, Francis Breedon, Vasco Carvalho, Martin Ellison, George Hall, Esther Hauk, Andrea Lanteri, Ricardo Nunes, Will Roberds and Ken Singleton. Special thanks are due to George Hall for generously providing us with the data enabling us to construct valuation data on market debt and the editor for suggesting how to deal with repurchases. We also benefited from the comments of participants in seminars in UQAM, NYU, Exeter, Cambridge, HEC-EP Lausanne, UC Louvain, at the Atelier de macroeconomie CIRPEE-DEEP-TSE in Toulouse, at the REDg Macroeconomics Workshop in U. Aut'onoma de Barcelona, at Atlanta Fed, at the Conference Government Debt: Constraints and Choices at Becker-Friedman Institute, University of Chicago, at the 13th Conference in Economic Theory and Econometrics in Naxos, at the Economic Policy after the Financial Crisis Worshop in EUI, Florence and the 8th Shanghai Macroeconomics Workshop. We thank Athan Zafirov and Tongbin Zhang for excellent research assistance. Corresponding Author: Rigas Oikonomou, IRES, Université Catholique de Louvain, Coll'ege L. H. Dupriez, 3 Place Montesquieu 1348 Louvain la Neuve, Belgium.

†University of Cambridge and CEPR.

‡Institut d'Anàlisi Econòmica-CSIC, ICREA, Barcelona GSE, UAB, MOVE and CEPR.

§Université Catholique de Louvain. Email: Rigas.Oikonomou@uclouvain.be

¶London Business School and CEPR.

This appendix has two sections. Section A contains some analytic results that have been left out of the text and are given here for completeness. We first give some additional details about the complete markets model under buyback and no buyback. Then we set up the Lagrangeans that have not been included in the main text, namely for the model with optimal repurchases and transaction costs of section 6.3, the model with coupons of section 7.1 and the 'callable bonds' model of section 7.3. We derive the first order optimality conditions for each model.

Section B contains the Numerical Appendix. It discusses in detail the implementation of the 'Forward States' and 'Condensed PEA' algorithms, and several practical issues on solving portfolio models with incomplete markets with the PEA. We discuss how we selected the state variables of the core vector, $\mathbf{X}_t^{core}$ and of the 'out'-vector, $\mathbf{X}_t^{out}$, as well as the order of the polynomials of the states that were used. Moreover, we report how many linear combinations of state variables were added to the approximations. Finally, we discuss how we constructed approximations for the shadow cost calculation presented in Section 6.2 and we report on the accuracy of the simulated models.

A Some Theoretical Results

For simplicity we take the case $S = 1$ throughout this section.

A.1 Complete Markets and Buyback

We describe in this section the debt management strategy under complete financial market assuming buyback. This provides more details for the calculations in section 3.4.1 in the main text.

Let $\mathbf{g}^t = (g_0, g_1, ..., g_t)$ be the history of government spending shocks up to date t. As in ABN and as in the main text of this paper z represents the present discounted value of the government surplus contingent on $\mathbf{g}^{t-1}$ and the current realization of spending g_t. Substituting for equilibrium taxes this is:

$$z_t(\mathbf{g}^{t-1}, g_t) = E_t \sum_{i=0}^{\infty} \frac{\beta^i}{u_{c,t}} \left[(u_{c,t+i} - v_{x,t+i})(c_{t+i} + g_{t+i}) - g_{t+i} u_{c,t+i} \right]. \tag{1}$$

We assume for simplicity that government expenditure follows a two step Markov process taking values $g^H > g^L$ with probabilities μ_{HH} and μ_{LL} of remaining in the same state. The government debt is given initially by b^1_{-1} and b^N_{-i} for $i = 1, ..., N$, at $t = 0$.

Following standard arguments as in Chari and Kehoe (1999) the equilibrium conditions if there are complete markets for Arrow securities is given by the implementability constraint at date 0

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[(u_{c,t} - v_{x,t})(c_t + g_t) - u_{c,t} g_t \right] = E_0 \sum_{i \in \{1,N\}} \sum_{t=0}^{i-1} \beta^t u_{c,t} b^i_{t-i}.$$

The corresponding Lagrangean that gives the Ramsey equilibrium under complete markets is

$$\mathcal{L}_{CM} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(T - c_t - g_t) \right. \right. \tag{2}$$

$$\left. + \Lambda((u_{c,t} - v_{x,t})(c_t + g_t) - u_{c,t} g_t)] - \Lambda \sum_{i \in \{1,N\}}^{N} \sum_{t=0}^{i-1} \beta^t u_{c,t} b_{t-i}^i \right\}$$

where Λ is the Lagrange multiplier of the implementability constraint.

The first order conditions for an optimum are given by:

$$u_{c,t} - v_{x,t} - \Lambda[u_{cc,t} c_t + u_{c,t} - v_{x,t} + v_{xx,t}(c_t + g_t)] = 0 \quad \text{for} \quad t \geq N \tag{3}$$

$$u_{c,t} - v_{x,t} + \Lambda[u_{cc,t} c_t + u_{c,t} - v_{x,t} + v_{xx,t}(c_t + g_t)] - \Lambda u_{cc,t} \sum_{i \in \{1,N\}} b_{t-i}^i = 0 \ \text{ for } \quad t \leq N - 1. \tag{4}$$

For $t \geq N$ the first order conditions above are time-invariant so that $c_t(\mathbf{g}^t) = c_t^{CM}$ takes two possible values $c_t(\mathbf{g}^{t-1}, g^i) = c^i$ for $i = H, L$. Therefore, given the Markov assumption on g, there are two possible values for $z_t(\mathbf{g}^{t-1}, g^i) = z^i$ and $p^{N-1}(\mathbf{g}^{t-1}, g^i) = p_i^{N-1}$ for $i = H, L$ and for all $t \geq N$.

It is clear that the first order conditions (14)-(15) in the main text coincide with (3) in this appendix for constant $\lambda_t = \Lambda$. Therefore c_t^{CM} satisfies the first order conditions of the incomplete markets model. All that is left to show is that the budget constraints under incomplete markets are satisfied.

As shown, for example, in Angeletos (2002) a necessary and sufficient condition for the period-t budget constraints to hold is that (17) in the main text holds. Therefore all we need to show is that if we implement $\left\{c_t^{CM}\right\}$ then z takes only values z^H and z^L an optimal portfolio needs to satisfy

$$b_{t-1}^1\left(\mathbf{g}^{t-1}\right) + p_i^{N-1}\left(\mathbf{g}^{t-1}\right) b_{t-1}^N\left(\mathbf{g}^{t-1}\right) = z^i \ \text{ for } i = H, L \ \ \forall t. \tag{5}$$

So that

$$\begin{pmatrix} 1 & p_H^{N-1} \\ 1 & p_L^{N-1} \end{pmatrix} \begin{pmatrix} b_{t-1}^1\left(\mathbf{g}^{t-1}\right) \\ b_{t-1}^N\left(\mathbf{g}^{t-1}\right) \end{pmatrix} = \begin{pmatrix} z^H \\ z^L \end{pmatrix}. \tag{6}$$

yielding

$$\begin{pmatrix} b_{t-1}^1\left(\mathbf{g}^{t-1}\right) \\ b_{t-1}^N\left(\mathbf{g}^{t-1}\right) \end{pmatrix} = \begin{pmatrix} \frac{p_H^{N-1} z^L - p_L^{N-1} z^H}{p_H^{N-1} - p_L^{N-1}} \\ \frac{z^H - z^L}{p_H^{N-1} - p_L^{N-1}} \end{pmatrix} = \begin{pmatrix} B_1^{BB} \\ B_N^{BB} \end{pmatrix}. \tag{7}$$

To see that $B_N^{BB} > 0$ note first that clearly surpluses are higher when g^L occurs, therefore $z_H < z_L$.

We now argue that generically $p_H^{N-1} - p_L^{N-1} < 0$. Assume a symmetric g process such that $\mu = \mu_{HH} = \mu_{LL}$ and CRRA utility we have $p_H^{N-1} = \beta^{N-1}\left[\mu_{N-1} + (1 - \mu_{N-1})\left(\frac{c^L}{c^H}\right)^{\gamma_c}\right]$ for risk aversion $-\gamma_c$, where $\mu_j = \Pr ob(g_{t+j} = g^i \mid g_t = g^i)$. Hence

$$p_H^{N-1} - p_L^{N-1} = \beta^{N-1}(1 - \mu_{N-1})\left(\left(\frac{c^L}{c^H}\right)^{\gamma_c} - \left(\frac{c^H}{c^L}\right)^{\gamma_c}\right).$$

As long as consumption is a normal good $c^H < c^L$ hence $p_H^{N-1} - p_L^{N-1} < 0$ for $\gamma_c < 0$. This shows that, generically, $B_N^{BB} > 0$.

To see that $B_1^{BB} < 0$ note that since $B_1^{BB} = z^H - p_H^{N-1} B_N^{BB}$ and $B_N^{BB} > 0$, as long as initial debt is close to zero, z^H is close to zero and B_1^{BB} is negative.

As long as debt limits are sufficiently loose to contain B_1^{BB}, B_N^{BB} this gives the equilibrium under incomplete markets.

A.2 Complete Markets and No Buyback

Under no buyback the period t budget constraint of the consumer is given by equation (3) in the main text. In this case the government issues two kinds of bonds, but it really holds N kinds of bonds every period: in addition to the bonds that mature and produce income at t, namely $(b_{t-1}^1 + b_{t-N}^N)$, the government also holds long bonds that have not yet matured: namely, $b_{t-N+1}^N, ..., b_{t-1}^N$. Even though these non-maturing bonds do not show up in equation (3) they do appear in equation (20), stating that total wealth equals discounted surpluses z. We now give more details to show that equation (20), along with equilibrium prices, is necessary and sufficient for a competitive equilibrium under incomplete markets and no buyback.

To prove that (20) is necessary, proceed as suggested in the paragraph preceding equation (20). Let s_t be the government primary surplus as in the main text, define total wealth as $W_t = b_{t-1}^1 + \sum_{j=1}^N p_t^{N-j} b_{t-j}^N$ and using equilibrium prices we can show

$$s_t + \beta E_t \left[\frac{u_{c,t+1}}{u_{c,t}} W_{t+1} \right] = W_t. \tag{8}$$

Iterating forward and assuming no Ponzi games in the value of bonds W_t yields (20) in the main text for all $t = 0, 1, ...$ hence this is a necessary condition.

To show that this is a sufficient condition the previous steps can be reversed to show that any portfolio satisfying (20) and a no Ponzi game condition in W_t also satisfies (3).

Hence an allocation is an incomplete market equilibrium if and only if (20) holds for $p_t^j = E_t \beta^j \frac{u_{c,t+j}}{u_{c,t}}$ and the corresponding debt limits. As we show in the main text, the complete market allocations do not satisfy all these requirements: if we have bond limits then (20) can not hold for all t.

A.3 Optimal Repurchases: the Ramsey Program

In the optimal repurchase (OR) model of Section 6.3 the government maximizes the utility of the household subject to the following constraints

$$\sum_{i \in \{S,N\}} p_t^i b_t^i (1 - \mathcal{T}^i(b_t^i)) = b_{t-S}^S + b_{t-N}^N - R_{t-N+1} + p_t^{N-1} R_t (1 + \mathcal{T}^R(R_t)) + g_t - \tau_t (T - x_t) \tag{9}$$

$$T - x_t = c_t + g_t + \mathcal{TC}_t \tag{10}$$

$$0 \le b_t^i \le \frac{\overline{M}_i}{\sum_{j=1}^i \beta^j}, \quad 0 \le R_t \le b_{t-1}^N \tag{11}$$

where $\mathcal{TC}_t \equiv \sum_{i \in \{S,N\}} p_t^i b_t^i \mathcal{T}^i(b_t^i) + p_t^{N-1} R_t \mathcal{T}^R(R_t)$ represents total transaction costs.

To simplify the solution of this model we assume that the government treats as exogenous the function $\mathcal{TC}_t$, in other words it does not take derivatives of $\mathcal{TC}_t$ with respect to consumption and the bonds.[1]

The Lagrangian of the OR model is:

$$\mathcal{L} = E_0 \sum_t \beta^t \Bigg[u(c_t) + v(T - c_t - g_t - \mathcal{TC}_t) + \lambda_t [\sum_{i \in \{S,N\}} b_t^i \beta^i u_{c,t+i}(1 - \mathcal{T}(b_t^i)^i) \tag{12}$$

$$-\beta^{N-1} u_{c,t+N-1} R_t (1 + \mathcal{T}^R(R_t)) - (b_{t-S}^S + b_{t-N}^N - R_{t-N+1}) u_{c,t}$$

$$- g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t)] + \sum_{i \in \{S,N\}} \xi_{U,t}^i \Big(\frac{\overline{M}_i}{\sum_{j=1}^i \beta^j} - b_t^i\Big) + \sum_{i \in \{S,N\}} \xi_{L,t}^i b_t^i + \xi_{U,t}^R (b_{t-1}^N - R_t) + \xi_{L,t}^R R_t \Bigg].$$

The FONC are given by:

$$u_{c,t} - v_{x,t} + \lambda_t(-u_{cc,t} g_t + u_{c,t} + u_{cc,t}(T - x_t) + v_{xx,t}(T - x_t) - v_{x,t}) - u_{cc,t} [B_{t-S} \lambda_t - B\lambda_{t-S}] = 0$$
$$E_t \beta(-u_{c,t+1} \lambda_{t+1} + u_{c,t+1} \lambda_t (1 - \mathcal{T}_t^1 - \mathcal{T}_{b_t^1}^1 b_t^1)) + \xi_{L,t}^1 - \xi_{U,t}^1 = 0 \quad \text{for} \quad i = S$$
$$E_t \beta^N(-u_{c,t+N} \lambda_{t+N} + u_{c,t+N} \lambda_t (1 - \mathcal{T}_t^N - \mathcal{T}_{b_t^N}^N b_t^N)) + E_t \beta(\xi_{U,t+1}^R) + \xi_{L,t}^N - \xi_{U,t}^N = 0 \quad \text{for} \quad i = N$$
$$E_t \beta^{N-1}(u_{c,t+N-1} \lambda_{t+N-1} - u_{c,t+N-1} \lambda_t (1 + \mathcal{T}_t^R + \mathcal{T}_{R_t}^R R_t)) + \xi_{L,t}^R - \xi_{U,t}^R = 0$$

where

$$\begin{aligned} B_t &\equiv b_t^S + B_{t-N+1+S}^{net} \\ B_t^{net} &\equiv b_{t-1}^N - R_t \\ B\lambda_t &\equiv \lambda_t (1 - \mathcal{T}_t^S) b_t^S + B\lambda_{t-N+1+S}^{net} \\ B\lambda_t^{net} &\equiv \lambda_{t-1}(1 - \mathcal{T}_{t-1}^N) b_{t-1}^N - \lambda_t (1 + \mathcal{T}_t^R) R_t. \end{aligned}$$

A.4 Coupon Bonds and No Buyback: the Ramsey Program

We solve the optimal policy problem under no buyback and coupons of section 7.1 assuming for simplicity that $S = 1$. As in the rest of the paper we introduce debt limits, these are parameterized as:

$$b_t^N \in \left[\frac{\underline{M}_N}{\sum_{j=1}^N \beta^j + \kappa \sum_{j=1}^N \sum_{i=1}^j \beta^i}, \frac{\overline{M}_N}{\sum_{j=1}^N \beta^j + \kappa \sum_{j=1}^N \sum_{k=1}^j \beta^k} \right] \equiv [\widetilde{\underline{M}}_N, \widetilde{\overline{M}}_N] \tag{13}$$

so that $\widetilde{\underline{M}}$'s are in terms of the value of debt.

[1] Without this assumption we would need to keep track of the fact that there is a conditional expectation in the determination of $\mathcal{TC}_t$, therefore the solution to the model would feature both current and lagged values of the multiplier on this constraint. This would add yet more state variables in the model but with minimal quantitative effects.

Note that another way to simplify the planner's program (avoid having to keep track of the resource constraint as a separate object in the Lagrangean) is to assume $\mathcal{TC}_t$ do not enter the feasibility constraint. In this case transaction costs do not impact the overall resources of the economy, this would correspond to a situation where a financial firm can charge transaction costs on bond issuances without actually spending labor resources in it. When we run the model under this assumption (which appears sometimes in the literature) we found virtually no effect on our results.

Letting $[\underline{\widetilde{M}}_1, \widetilde{\overline{M}}_1] \equiv [\frac{\underline{M}_1}{\beta}, \frac{\overline{M}_1}{\beta}]$ be the analogous constraint set for one year debt the planning problem is given by:

$$\mathcal{L} = E_0 \sum \beta^t \Big\{ u(c_t) + v(T - c_t - g_t) + \lambda_t \Big[b_t^1 \beta u_{c,t+1} + b_t^N \Big(\beta^N u_{c,t+N} + \sum_{j=1}^{N} \beta^j u_{c,t+j} \kappa \Big)$$
$$- b_{t-1}^1 u_{c,t} - b_{t-N}^N u_{c,t} - \kappa \sum_{j=1}^{N} b_{t-j}^N u_{c,t} - g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \Big]$$
$$+ \sum_{i \in \{1,N\}} \xi_{U,t}^i (\widetilde{\overline{M}}_i - b_t^i) + \sum_{i \in \{1,N\}} \xi_{L,t}^i (b_t^i - \underline{\widetilde{M}}_i) \Big\}.$$

The first order condition for consumption is:

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t})$$
$$+ u_{cc,t} \kappa \sum_{j=1}^{N} (\lambda_{t-j} - \lambda_t) b_{t-j}^N + u_{cc,t} \sum_{i \in \{1,N\}} (\lambda_{t-i} - \lambda_t) b_{t-i}^i = 0$$

and off corners the analogous conditions for b_t^1 and b_t^N are:

$$\lambda_t E_t(u_{c,t+1}) = E_t(\lambda_{t+1} u_{c,t+1}) \tag{14}$$

$$\lambda_t E_t(\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t(\kappa \sum_{j=1}^{N} \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N}). \tag{15}$$

For brevity we summarized the properties of this model in Table 4 in the main text. In Figure 1 of this appendix we show a typical sample of long and short debt (analogous to Figures 6-9 in the main text). As was claimed in the text, when bonds pay positive coupons the properties of the model remain very close to the no buyback and zero coupons case.

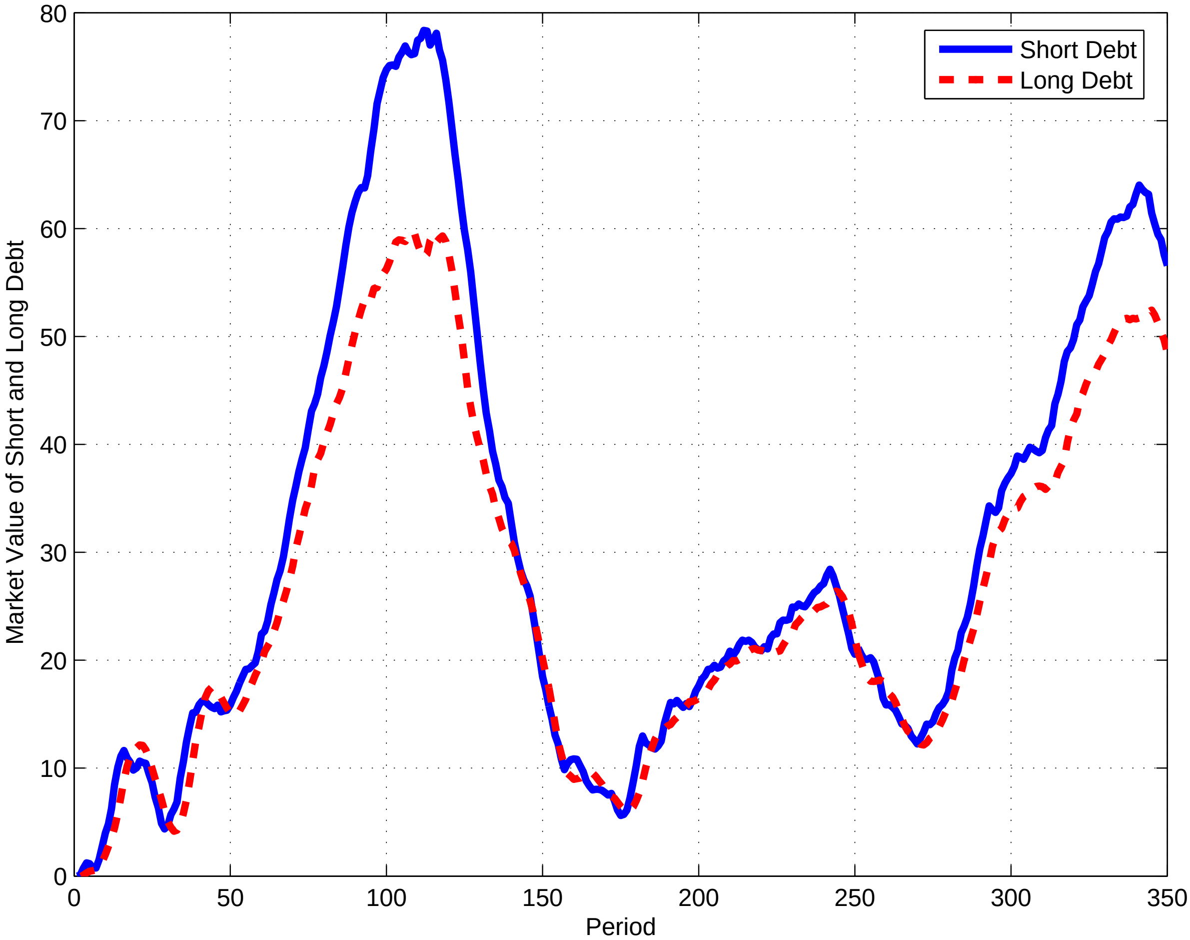
A.5 Callable Bonds: the Ramsey Program

As explained in Section 2 of the paper the US government issued callable bonds in the past. These types of securities give the issuer the option to buy them back after $m < N$ years, at every coupon date, until the bond matures. Their price at buyback is at par. We showed that historically the US government has repurchased callable bonds at the start of the call window.

In proposing the model of section 7.3, our intention is not to motivate the empirical observations of why the Treasury chooses to buyback at the first call date. Rather we seek to establish that removing debt from the market before, but close to, the maturity date is akin to the model of no buyback and that our findings about the importance of short term debt and the positive comovement between short and long debt still hold.

This is why we assume that the buyback of the N-year bond occurs automatically m years after issuance. We keep our calibration $N = 10$ and we set the recall date $m = 5$. Notice that a lower m makes the model closer to the buyback section, since buyback is equivalent with $m = 1$. If we were to find that even for a low m the model behaves similar to no buyback, higher m are likely to be

Figure 1: **Market Value of Short and Long Debt under no Buyback+Coupons**



Notes: The Figure plots a typical sample path from the no buyback model with positive coupons. As explained in text the value of the coupon κ is calibrated so that bonds trade on average at par. The upper bound on b_t^N is given in (13). The lower bound is zero. The value of short term debt in a given period t in the Figure, is constructed by adding the coupon payments and principals which are to mature in $t+1$ to the market value of one year bonds issued in t.

be even closer to no-buyback. The call window in the data for 10 year bonds starts 2 years before maturity, suggesting a buyback period of $m = 8$. Our model choice of a much lower $m = 5$ means that if we find that the role for short bonds is close to no buyback this suggests that the same is likely to happen in practice.

The budget constraint of the government is:

$$\sum_{i \in \{1,N\}} p_t^i b_t^i = b_{t-1}^1 + p_t^{N-m} b_{t-m}^N + g_t - \tau_t (T - x_t).$$

The ad hoc debt constraints for the N year bond are

$$b_t^N \in \left[\frac{\underline{M}_N}{\sum_{j=0}^{m-1} \beta^{N-j}}, \frac{\overline{M}_N}{\sum_{j=0}^{m-1} \beta^{N-j}} \right] \equiv [\widetilde{\underline{M}}_N, \widetilde{\overline{M}}_N].$$

Letting $[\widetilde{\underline{M}}_1, \widetilde{\overline{M}}_1] = [\frac{\underline{M}_1}{\beta}, \frac{\overline{M}_1}{\beta}]$ be the analogous constraints for one year debt and substituting the equilibrium expressions for the tax rate and the bond prices we represent the planning problem as follows:

$$\mathcal{L} = E_0 \sum_t \beta^t \Big\{ u(c_t) + v(T - c_t - g_t) + \lambda_t \Big[\sum_{i \in \{1,N\}} b_t^i \beta^i u_{c,t+i} - b_{t-1}^1 u_{c,t} - b_{t-m}^N \beta^{N-m} u_{c,t+N-m}$$

$$-g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \Big]$$

$$+ \sum_{i \in \{1,N\}} \xi_{U,t}^i (\widetilde{\overline{M}}_i - b_t^i) + \sum_{i \in \{1,N\}} \xi_{L,t}^i (b_t^i - \widetilde{\underline{M}}_i) \Big\}.$$

The first order conditions for the optimum are given by:

$$u_{c,t} - v_{x,t} + \lambda_t \Big(u_{cc,t} c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t} \Big) + u_{cc,t} \Big[(\lambda_{t-1} - \lambda_t) b_{t-1}^1 + (\lambda_{t-N} - \lambda_{t-N+m}) b_{t-N}^N \Big]$$

$$\beta E_t (u_{c,t+1} \lambda_t - u_{c,t+1} \lambda_{t+1}) + \xi_{L,t}^1 - \xi_{U,t}^1 = 0$$

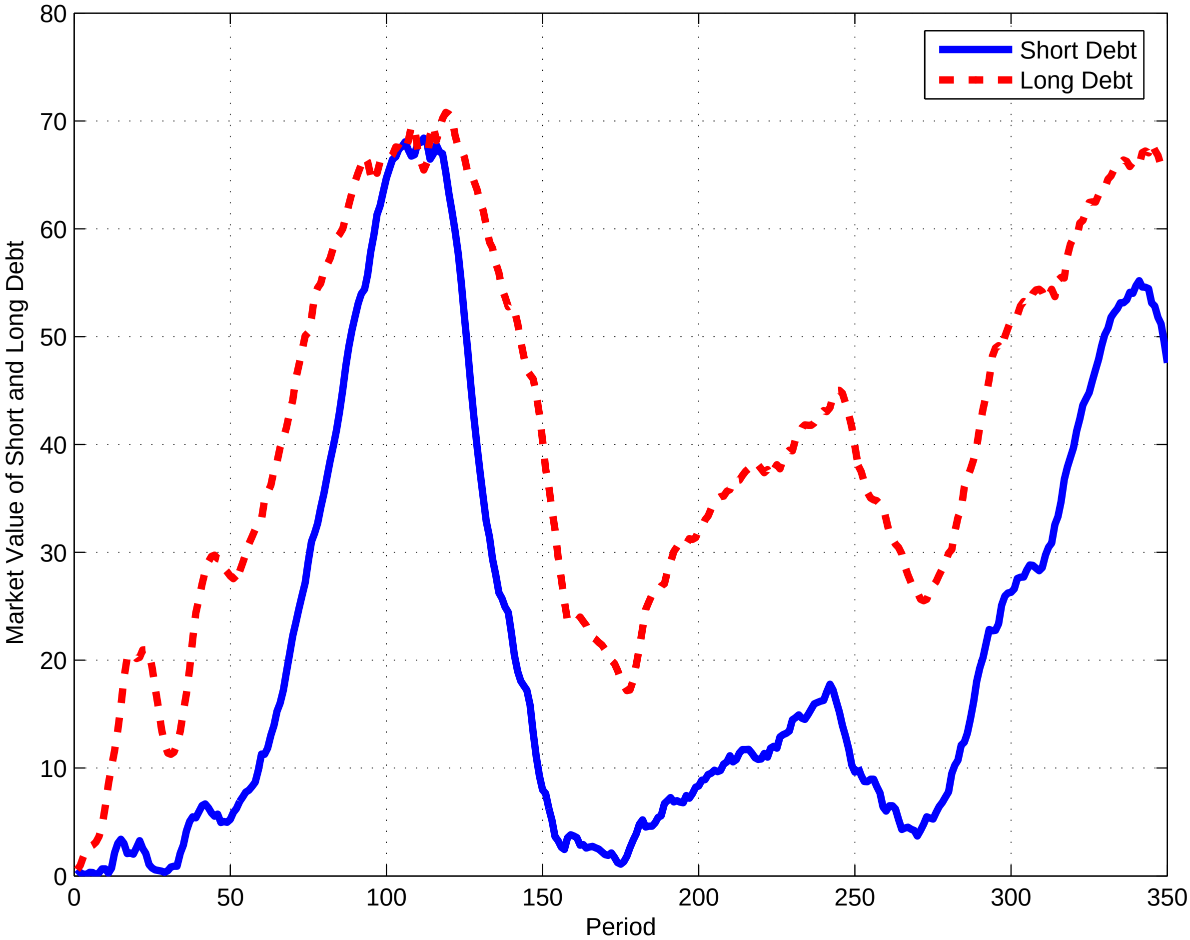
$$\beta^N E_t (u_{c,t+N} \lambda_t - u_{c,t+N} \lambda_{t+m}) + \xi_{L,t}^N - \xi_{U,t}^N = 0.$$

We assume $\widetilde{\underline{M}}_1 = \widetilde{\underline{M}}_N = 0$. In Figure 2 we plot a typical sample of the market value of short and long debt. Notice that assuming that the government repurchases debt from the market $m = 5$ years after issuance, does indeed reduce the share of short bonds in the portfolio compared to a model with no buyback. The portfolio is somewhere between the 'full buyback model' studied in the paper (i.e. when $m = 1$) and the no buyback model where $m = N$. In terms of the moments reported in Table 4 in the paper the model of this section gives us the following: $\overline{\mathcal{S}}_t = 18.5\%$, $\sigma_{\mathcal{S}_t} = 10.6\%$, $\rho_{\mathcal{S}_t, \mathcal{S}_{t-1}} = 0.88$ $\rho_{\widetilde{b}_t^S, \widetilde{b}_t^N} = 0.84$ $\%_{\mathcal{S}_t = 0} = 0.35\%$.

We view these results as encouraging because they confirm the hypothesis that even if the government buys back some of the debt before maturity there is still a role for short bonds. First, because the share of short debt is very rarely zero in simulations (e.g. $\%_{\mathcal{S}_t = 0} = 0.35\%$ versus the analogous figure in the buyback no lending model in the paper of 13%). As we mentioned, the choice of $m = 5$ is quite conservative. The data suggest that $m = 8$ would be more appropriate. With $m = 8$ we expect the model to generate results very close to the no buyback ones.

This is only a partial study of callable bonds. Clearly, the modelling of callable bonds can be made closer to the data by introducing that they can be repurchased at par or that transaction costs are involved in their recall. A model taking all these features into account is beyond the scope of this paper. However the message that there is still a role for short bonds comes out clearly from the analysis.

Figure 2: **Market Value of Short and Long Debt under 'Callable' Bonds**



Notes: The Figure plots a typical sample path from the model of Section A.5. We assume that government buybacks of 10 year bonds occur 5 years after issuance.

B Numerical Appendix

In section 4 of the paper we described the 'Condensed PEA' that deals with the high dimensionality of the state vector, and the 'Forward States PEA' that deals with the indeterminacy of the portfolio generated by the use of the PEA. This numerical appendix outlines in greater detail the two methods, their implementation and the steps we followed to approximate the conditional expectations in the different models. In particular, we report how we selected the state variables of the core vector, $\mathbf{X}_t^{core}$, the 'out'-vector, $\mathbf{X}_t^{out}$ and the order of the polynomials of the states that were used. Moreover, we report how many linear combinations of state variables were added to the approximations, and also discuss some practical features of our numerical procedure that can help the algorithm's convergence.

B.1 Implementation of "Condensed PEA" and "Forward states PEA"

B.1.1 Selection of variables in the approximation

Recall that 'Condensed PEA' divides the state vector $\mathbf{X}$ into two subvectors: the core vector, $\mathbf{X}_t^{core}$, which includes variables that (we believe a priori) are of primary importance in the approximation, and the $\mathbf{X}_t^{out}$ vector, which includes the remaining state variables and possibly higher order polynomial terms. 'Forward States PEA' resolved the portfolio indeterminacy issue through approximating $E_t u_{c,t+i}$ with $E_t(\Phi^i(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^i))$ and $E_t \lambda_{t+1} u_{c,t+i}$ with $E_t(\Psi^i(\mathbf{X}_{t+1}, \boldsymbol{\delta}^i))$. To clearly show how we implemented these two methods, we bring them now together and in what follows we outline the 'Condensed PEA' using since the first iteration (i.e. with core state variables only) 'Forward States' to solve the portfolio choice.

The $\mathbf{X}^{core}$ vector:

In all the models presented in the paper, but the 3 bond model, we used the core vector

$$\mathbf{X}_{t+1}^{core} = \left\{1,\ g_{t+1},\ \{b_t^i\}_{i=1,N},\ \lambda_t,\ \{(b_t^i)^2\}_{i=1,N},\ \{g_{t+1} b_t^i\}_{i=1,N}\right\} \tag{16}$$

i.e. $\mathbf{X}_{t+1}^{core}$ is composed of a constant, the level of government spending in $t+1$, the levels of date t variables (the bonds and the multipliers), the square of the bonds and the interaction term between the bonds in t and g_{t+1}.

We solve the system of FONCs after integrating out the term g_{t+1} as discussed in the text. We use the analytical formula for the conditional expectation of g_{t+1} at time t given by:

(17)

$$\int g_{t+1} f_{g_{t+1}|g_t} dg_{t+1} = \omega_t + (\underline{g} - \omega_t)\Phi\left(\frac{\underline{g} - \omega_t}{\sigma_\epsilon}\right) + (\overline{g} - \omega_t)\left[1 - \Phi\left(\frac{\overline{g} - \omega_t}{\sigma_\epsilon}\right)\right] - \sigma(\left[\phi\left(\frac{\overline{g} - \omega_t}{\sigma_\epsilon}\right) - \phi\left(\frac{\underline{g} - \omega_t}{\sigma_\epsilon}\right)\right]$$

where $\omega_t \equiv \rho_g g_t + (1 - \rho_g) g_{ss}$, Φ (ϕ) is the standard normal cdf (pdf), $\overline{g}$ and $\underline{g}$ are upper and lower bounds on government spending[2]. σ_ϵ is the standard deviation of the spending shock.[3]

We have chosen to introduce higher order terms in the core vector for three reasons. First, approximating the conditional expectations $E_t u_{c,t+i}$ means that we are approximating the bond prices. If we include only the levels of the bonds, $\{b_t^i\}_{i=1,N}$, we are imposing that bonds are close substitutes in terms of their influence on prices.[4]. This is not a property that we are likely to find in equilibrium and non-linear terms may be potentially important. Secondly, if the ad hoc debt limits are occasionally binding this is known to introduce non-linearities so that higher order terms may be important. Finally the 'Condensed PEA' inclusion criterion suggested that these non linear terms are important for the approximations. Indeed when we solved the models without higher order terms and tested whether they should be included in the linear combinations, the percentage gains in R^2 were substantial. Higher order terms therefore should be included in the polynomials, either in $\mathbf{X}_{t+1}^{core}$ or in $\mathbf{X}_{t+1}^{out}$. We ultimately chose to introduce some higher order terms in the core vector and left others for the linear combinations (see below) finding this helpful for the stability of the algorithm.

The $\mathbf{X}^{out}$ vector:

The 'out' vector is different for each of the models presented in the paper. To identify the elements of $\mathbf{X}^{out}$ we use as guidance the FONC.

[2]As discussed in the calibration section, we assume that spending fluctuates between 15 and 35 percent of steady state GDP.

[3]The expression is reached as follows:

$$\int g_{t+1} f_{g_{t+1}|g_t} dg_{t+1} = \int_{-\infty}^{\underline{g}-\omega_t} \underline{g} dF(\epsilon_{t+1}) + \int_{\overline{g}-\omega_t}^{\infty} \overline{g} dF(\epsilon_{t+1}) + \int_{\underline{g}-\omega_t}^{\overline{g}-\omega_t} (\omega_t + \epsilon_{t+1}) dF(\epsilon_{t+1})$$

$$= \omega_t + \int_{-\infty}^{\underline{g}-\omega_t} (\underline{g} - \omega_t) dF(\epsilon_{t+1}) + \int_{\overline{g}-\omega_t}^{\infty} (\overline{g} - \omega_t) dF(\epsilon_{t+1}) + \int_{\underline{g}-\omega_t}^{\overline{g}-\omega_t} \epsilon_{t+1} dF(\epsilon_{t+1})$$

where F denotes the cdf of ϵ. Standard results give:

$$\int_{-\infty}^{\underline{g}-\omega_t} (\underline{g} - \omega_t) dF(\epsilon_{t+1}) = \int_{-\infty}^{\frac{\underline{g}-\omega_t}{\sigma_\epsilon}} (\underline{g} - \omega_t) \frac{1}{\sqrt{2\pi}} e^{-\frac{z_{t+1}^2}{2}} dz_{t+1} = (\underline{g} - \omega_t) \Phi(\frac{\underline{g} - \omega_t}{\sigma_\epsilon})$$

where z is a standard normal variable and Φ is the cdf. Analogously:

$$\int_{\overline{g}-\omega_t}^{\infty} (\overline{g} - \omega_t) dF(\epsilon_{t+1}) = (\overline{g} - \omega_t)(1 - \Phi(\frac{\overline{g} - \omega_t}{\sigma_\epsilon}))$$

Finally,

$$\int_{\underline{g}-\omega_t}^{\overline{g}-\omega_t} \epsilon_{t+1} dF(\epsilon_{t+1}) = \int_{\frac{\underline{g}-\omega_t}{\sigma_\epsilon}}^{\frac{\overline{g}-\omega_t}{\sigma_\epsilon}} \sigma_\epsilon \frac{1}{\sqrt{2\pi}} z_{t+1} e^{-\frac{z_{t+1}^2}{2}} d\epsilon_{t+1} = -\sigma(\phi(\frac{\overline{g} - \omega_t}{\sigma_\epsilon}) - \phi(\frac{\underline{g} - \omega_t}{\sigma_\epsilon}))$$

Putting everything together we get (17)

[4]To see this consider the approximation of $E_t u_{c,t+N} \approx \gamma_0^N + \gamma_1^N \int g_{t+1} f(g_{t+1}|g_t) dg_{t+1} + \gamma_2^N b_t^1 + \gamma_3^N b_t^N + \gamma_4^N \lambda_t$ under linear polynomials. Clearly there are (infinitely) many pairs (b_t^1, b_t^N) that give the same bond price (holding λ_t fixed).

Notice that the optimal portfolio is nonetheless identified under linear polynomials since b_t^i, $i = 1, N$ influence all conditional expectations and enter in a nonlinear fashion in the system of FONCs (for example in the budget constraint of the government).

Consider first the *buyback model.* The first order conditions, in the case where $S = 1$, are

$$u_{c,t} - v_{x,t} + \lambda_t\Big(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t}\Big) + u_{cc,t}\sum_{i\in\{1,N\}}\Big(\lambda_{t-i} - \lambda_{t-i+1}\Big)b^i_{t-i} = 0 \tag{18}$$

$$\beta^i E_t\Big(u_{c,t+i}\lambda_t - u_{c,t+i}\lambda_{t+1}\Big) + \xi^i_{L,t} - \xi^i_{U,t} = 0 \quad \text{for} \quad i = 1, N. \tag{19}$$

When markets are incomplete, the term $\sum_{i\in\{1,N\}}\Big(\lambda_{t-i} - \lambda_{t-i+1}\Big)b^i_{t-i}$ summarises interest rate manipulation under commitment (see FMOS (2016)). Suppose that a positive spending shock arrives in period t and that $b^N_t > 0$. Since $\Big(\lambda_{t-1} - \lambda_t\Big)b^N_t$ becomes negative, the government finds optimal to promise a tax cut in $t + N - 1$ and lower the marginal utility of consumption in that period. It is then evident that the terms $\lambda_{t-1}b^N_t$ and $\lambda_t b^N_t$ are important determinants of $u_{c,t+N-1}$ and $u_{c,t+N}$ and hence they should be accounted for when we approximate the conditional expectations.[5]

Applying the above argument to determine which states potentially exert a significant influence to the expectations of date $t+1$, $t+N-1$ and $t+N$ variables in the buyback model, we include in $\mathbf{X}^{out}_{t+1}$ the following terms: $\lambda_t b^N_t$, $\lambda_t b^1_t$, $\lambda_t b^N_{t-1}$, $\lambda_{t-1}b^N_{t-1}$, $\lambda_{t-N+1}b^N_{t-N+1}$ and $\lambda_{t-N+2}b^N_{t-N+1}$.

Two more comments about this choice are necessary. Firstly, despite the fact that each of the above terms is potentially important for (some of) the conditional expectations we wish to approximate, it is unlikely that each term bears the same importance to each conditional expectation. For example, the term $\lambda_t b^N_t$ clearly exerts an influence on $u_{c,t+N}$ (through the FONC) but it is less likely to exert a significant influence on $u_{c,t+N-1}$. In this case the 'Condensed PEA' will assign a coefficient close to zero to $\lambda_t b^N_t$ in the approximation of $E_t u_{c,t+N-1}$ and a coefficient different from zero in the approximation of $E_t u_{c,t+N}$. This shows how convenient it is to include these terms in $\mathbf{X}^{out}$ where having coefficients close to zero for some state variables is not an issue, as opposed to including them in $\mathbf{X}^{core}$, in which case variables with close to zero coefficients may cause convergence problems.

Secondly, as explained before, (18) suggests that the cross terms between λ and b are potentially important for the solution. However, one may wonder whether the levels of these variables should also be included in the state vector. The FONCs show that the influence of λ_{t-N+1} on the optimal allocation in $t + 1$ is close to zero if b^N_{t-N+1} is close to zero. The effect of changes in the value

[5] In the text the implementation of 'Forward States' to the buyback model was summarized in the following equations

$$\lambda_t = \frac{E_t(\Psi^i(\mathbf{X}_{t+1}, \boldsymbol{\delta}^i))}{E_t(\Phi^i(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^i))} \qquad \text{for} \qquad i = S, N \tag{20}$$

$$\sum_{i\in\{S,N\}} b^i_t\beta^i E_t(\Phi^i(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^i)) = \sum_{i\in\{S,N\}} b^i_{t-1}\beta^{i-1}\Phi^i(\mathbf{X}_t, \boldsymbol{\gamma}^i) + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t) \tag{21}$$

Notice that in (21) we parameterize the term $E_t u_{c,t+N-1}$ as $\Phi^N(\mathbf{X}_t, \boldsymbol{\gamma}^N)$. In other words we apply the standard PEA to this term. An alternative is to define $E_t u_{c,t+N-2} = \Phi^N(\mathbf{X}_t, \boldsymbol{\gamma}^{N-1})$ and then use Forward States to get: $E_t u_{c,t+N-1} = E_t\Phi^N(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^{N-1})$. We follow the latter route in the numerical implementation. We therefore write (21) as follows

$$\begin{aligned} b^1_t\beta E_t(\Phi^1(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^1)) + b^N_t\beta^N E_t(\Phi^N(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^N)) &= b^1_{t-1}u_{c,t} + b^N_{t-1}\beta^{N-1}E_t(\Phi^{N-1}(\mathbf{X}_{t+1}, \boldsymbol{\gamma}^{N-1})) \\ &\quad + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t) \end{aligned} \tag{22}$$

i.e. when $S = 1$ and realizing that $E_t u_{c,t} = u_{c,t} = \Phi^1(\mathbf{X}_t, \boldsymbol{\gamma}^1)$.

The two ways of solving the model are obviously conceptually equivalent.

of the multiplier is felt more when government debt is high. This nonlinear influence seems to be (sufficiently) well captured in our specification by the cross terms and not by the levels since, as we verify in section 7.4 of the main text, we pass accuracy tests.[6].

We apply the above selection criterion to the other models. Consider the *no buyback model* and its first order conditions and budget constraint:

$$u_{c,t} - v_{x,t} + \lambda_t\Big(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t+g_t) - v_{x,t}\Big) + u_{cc,t}\sum_{i\in\{1,N\}}\Big(\lambda_{t-i}-\lambda_t\Big)b^i_{t-i} = 0$$

$$\beta^i E_t\Big(u_{c,t+i}\lambda_t - u_{c,t+i}\lambda_{t+i}\Big) + \xi^i_{L,t} - \xi^i_{U,t} = 0 \quad \text{for} \quad i = 1, N$$

$$\sum_{i\in\{1,N\}} b^i_t\beta^i E_t u_{c,t+i} = g_t u_{c,t} + u_{c,t}\sum_{1,N} b^i_{t-i} - (u_{c,t}-v_{x,t})(c_t+g_t).$$

We include in the $\mathbf{X}^{out}$ vector: $\lambda_t b^N_t$, $\lambda_t b^1_t$, $\lambda_{t-N+1}b^N_{t-N+1}$, and b^N_{t-N+1}, as these appear directly on the FONC.

Next, consider the *no buyback model with coupons*. To solve the coupon model we need to approximate the term $\sum_{j=1}^N \beta^j E_t u_{c,t+j}\kappa + \beta^N u_{c,t+N}$ and the term $\sum_{j=1}^N \beta^j E_t u_{c,t+j}\lambda_{t+j}\kappa + \beta^N u_{c,t+N}\lambda_{t+N}$. From the FONC of consumption and the government budget constraint (omitted for brevity), it is easy to show that all the lags of b^N_{t-j} and $\lambda_{t-j}b^N_{t-j}$, for $j = 1, 2, .., N-1$ should be introduced in the out vector. The X^{out} vector is therefore composed by: $\{\lambda_{t-j}b^N_{t-j}\}_{j=0}^{N-1}$, $\lambda_t b^1_t$, $\{b^N_{t-j}\}_{j=1}^{N-1}$.

Similarly, when we consider the *callable bond* model the X^{out} vector includes:

$\{b^i_t\lambda_t\}_{i=1,N}$, $\lambda_t b^N_{t-N+m}$, $\lambda_{t-N+1}b^N_{t-N+1}$, $\lambda_{t-N+m}b^N_{t-N+1}$ and $b^N_{t-N+m+1}$, where m is the repurchase date.

Finally, for each of the above models we include in $\mathbf{X}^{out}$ other higher order terms of date t variables that have not been included in $\mathbf{X}^{core}$. In each approximation we add in $\mathbf{X}^{out}$ the following terms: λ_t^2, $(b^N_t)^3$, $(b^1_t)^3$, $b^N_t b^1_t$.

We now consider the *optimal repurchases model* of section 6.3 in the main text (see a previous subsection of this online appendix for the FONC of this model). The following expectations need to be approximated with PEA in this case:

$$E_t\xi^R_{U,t+1} \quad \text{and} \quad E_t u_{c,t+i}, \quad E_t\lambda_{t+i}u_{c,t+i}, \quad i = 1, N, N-1$$

where $\xi^R_{U,t}$ is the Langrange multiplier on the constraint $R_t \leq b^N_{t-1}$.

As discussed in the text, one way to reduce the total number of state variables in this model is to rewrite the state vector as:

$$\mathbf{X}_{t+1} = \Big\{g_{t+1},\ \ B_t,\ \ B\lambda_t,\ \ \{B^{net}_{t+1-i}, B\lambda^{net}_{t+1-i}\}_{i=1}^N,\ \ \lambda_{t-N},\ \ b^N_{t-N}\Big\} \tag{23}$$

[6] Recall that $\mathbf{X}^{core}$ includes the variables λ_t,b^N_t and $b_{1,t}$ in levels. These first order terms, help us to identify the portfolio, but combined with their squares, cubes and so on can (practically speaking), explain part of the variability of some of the cross terms in $\mathbf{X}^{out}$. To avoid having residuals close to zero from the regressions of $\mathbf{X}^{out}$ on $\mathbf{X}^{core}$ when we compute linear combinations, we use an additional selection criterion that we describe in the next subsection.

where

$$B_t^{net} \equiv b_{t-1}^N - R_t$$
$$B_t \equiv b_t^S + B_{t-N+2}^{net}$$
$$B\lambda_t^{net} \equiv \lambda_{t-1}(1-\mathcal{T}^N)b_{t-1}^N + \lambda_t(1+\mathcal{T}^R)R_t$$
$$B\lambda_t \equiv \lambda_t(1-\mathcal{T}^1)b_t^1 + B\lambda_{t-N+2}^{net}.$$

As we did for the previous models we chose $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$ in the optimal repurchase model to include the state variables which appear in the FONC and which therefore exert a direct influence on the conditional expectations. We specified $\mathbf{X}^{core}$ as in (16) and $\mathbf{X}^{out}$ as follows:

$$\mathbf{X}_{t+1}^{out} = \left\{ \{b_t^i \lambda_t\}_{i=1,N},\ b_{t-N+1}^N - R_{t-N+2},\ (b_{t-N+1}^N - R_{t-N+2})\lambda_{t-N+1},\ R_{t-N+2}\lambda_{t-N+2} \right\}. \tag{24}$$

Given the specification of $\mathbf{X}_{t+1}^{out}$ (and that of $\mathbf{X}_{t+1}^{core}$) the terms $B_t, B\lambda_t, \left(B_{t+1-i}^{net}, B\lambda_{t+1-i}^{net}\right)_{i=1}^N$ appear in the approximations. For example $B_t = b_t^1 + b_{t-N+1}^N - R_{t-N+2}$ and $B_{t-N+2}^{net} = b_{t-N+1}^N - R_{t-N+2}$ are part of the state vector, but we have chosen to separate the terms b_t^1 and $b_{t-N+1}^N - R_{t-N+2}$ in the approximations assigning b_t^1 to the core and $b_{t-N+1}^N - R_{t-N+2}$ to the out vector. We did this for convenience and most importantly to be able to use as an initial guess for our approximation the solution of the no buyback model.

Moreover, notice that though in principle we could introduce R_t as a variable in $\mathbf{X}^{core}$,[7] this is not necessary to identify the optimal path of R_t. Since this is a model where the government can repurchase only after one period and we assume positive transaction costs, we do not need R_t in the core states to determine the portfolio.[8]

Finally notice that in $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$, the bond and repurchases variables are not multiplied by transaction costs. Since these variables (mostly) enter separately in the approximations and since the costs $\mathcal{T}$ are small, this does not influence the properties of the solution.

Let's now turn to the *model with three bonds*. When the government issues debt in three maturities $(1 < M < N)$ under no buyback the FONC are given by:

$$u_{c,t} - v_{x,t} + \lambda_t\Big(u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t+g_t) - v_{x,t}\Big) + u_{cc,t}\sum_{i\in\{1,M,N\}}\Big(\lambda_{t-i} - \lambda_t\Big)b_{t-i}^i = 0$$
$$\beta^i E_t\Big(u_{c,t+i}\lambda_t - u_{c,t+i}\lambda_{t+i}\Big) + \xi_{L,t}^i - \xi_{U,t}^i = 0 \quad \text{for} \quad i = 1, M, N$$
$$\sum_{i\in\{1,M,N\}} b_t^i \beta^i E_t u_{c,t+i} = g_t u_{c,t} + u_{c,t}\sum_{1,M,N} b_{t-i}^i - (u_{c,t} - v_{x,t})(c_t+g_t).$$

We need to approximate now 6 conditional expectations. We specify the 'core' and 'out' vectors as

[7] From (23) we know that $b_{t-1}^N - R_t$ is a state variable. However, this will not appear in the FONC in periods $t+1, t+N, t+N-1$ and for this reason we dropped it from the core state vector and from the out vector (24).

[8] In other words R_t can still be identified through the budget constraint or through the nonlinear transaction costs. Had we allowed the government to repurchase more than once and if the transaction costs were assumed independent of (the vector in the case of many repurchases) R we would need the control variables R to be in $\mathbf{X}^{core}$ in order to solve the model.

Moreover, since R_t is always close to zero introducing it as an independent variable in the core vector leads to convergence problems. We discuss this further below.

Table 1: **Variables used in approximations**

	$\mathbf{X}^{core}$	$\mathbf{X}^{out}$		**total**
		common var.	**ad hoc**	
BB			$\{b_t^i\lambda_t\}_{i=1,N}, \lambda_t b_{t-1}^N,$ $\lambda_{t-1}b_{t-1}^N, \lambda_{t-N+1}b_{t-N+1}^N,$ $\lambda_{t-N+2}b_{t-N+1}^N$	19
NBB			$\{b_t^i\lambda_t\}_{i=1,N},$ $\lambda_{t-N+1}b_{t-N+1}^N, b_{t-N+1}^N$	17
coupons	$1, g_{t+1}, \{b_t^i\}_{i=1,N},$ $\lambda_t, \{(b_t^i)^2\}_{i=1,N},$ $\{g_{t+1}b_t^i\}_{i=1,N}$	$\{(b_t^i)^3\}_{i=1,N},$ $\lambda_t^2, b_t^N b_t^1$	$\{\lambda_{t-i}b_{t-i}^N\}_{i=0}^{N-1},$ $\lambda_t b_t^1, \{b_{t-i}^N\}_{i=1}^{N-1}$	30
callables			$\{b_t^i\lambda_t\}_{i=1,N}, \lambda_t b_{t-N+m}^N,$ $\lambda_{t-N+1}b_{t-N+1}^N, \lambda_{t-N+m}b_{t-N+1}^N$ $b_{t-N+m+1}^N$	19
repurchases			$\{b_t^i\lambda_t\}_{i=1,N}, b_{t-N+1}^N - R_{t-N+2}$ $(b_{t-N+1}^N - R_{t-N+2})\lambda_{t-N+1}$ $R_{t-N+2}\lambda_{t-N+2}$	18
3 bonds	$1, g_{t+1}, \{b_t^i\}_{i=1,M,N},$ $\lambda_t, \{(b_t^i)^2\}_{i=1,M,N},$ $\{g_{t+1}b_t^i\}_{i=1,M,N}$		$\{b_t^i\lambda_t\}_{i=1,M,N}, \lambda_t^2,$ $\left(b_t^M\right)^3, b_t^1 b_t^M, b_t^M b_t^N$ $\{b_{t-i+1}^i\}_{i=M,N}, \{b_{t-i+1}^i\lambda_{t-i+1}\}_{i=M,N}$	26

follows:

$$\mathbf{X}_{t+1}^{core} = \left\{1, g_{t+1}, \lambda_t, \{b_t^i\}_{i=1,M,N}, \{(b_t^i)^2\}_{i=1,M,N}, \{g_{t+1}b_t^i\}_{i=1,M,N}\right\}$$

$$\mathbf{X}_{t+1}^{out} = \left\{\{b_t^i\lambda_t\}_{i=1,M,N}, \{(b_t^i)^3\}_{i=1,M,N}, \{b_t^i b_t^k\}_{i,k\in\{1,M,N\},k\neq i}, \lambda_t^2, \{b_{t-i+1}^i\}_{i=M,N}, \{b_{t-i+1}^i\lambda_{t-i+1}\}_{i=M,N}\right\}$$

Therefore we have 12 variables in the core vector and 14 variables in the out vector.

Table 1 summarises the previous discussion on our choices for $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$.

B.1.2 An R^2 selection criterion for the elements of $\mathbf{X}^{out}$

Once we have chosen the composition of $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$, we apply the following procedure:

1. We first regress each variable X_j^{out} on $\mathbf{X}_{-j}^{out}$ and $\mathbf{X}^{core}$ and compute the R-square of the regression, R_j^2.

2. We find the variable k with the highest R_k^2, that is $k = \arg\max_{j\in\{1,2,\ldots,\text{length}(\mathbf{X}^{out})\}}\{R_j^2\}$. If $R_k^2 > 0.995$ we set the coefficient $\alpha_k^1 = 0$. In other words, we set the coefficient of this variable in the first linear combination (and in all approximations) equal to zero.

3. We repeat Steps 1 and 2 removing the excluded variables from $\mathbf{X}^{out}$ until $R_k^2 < 0.995$.

4. We apply the 'Condensed PEA' to find the coefficients $\overline{\boldsymbol{\gamma}}^{i,f}$ and $\overline{\boldsymbol{\delta}}^{i,f}$, i.e. the new fixed point in the model, with the first linear combination of the elements of $\mathbf{X}^{out}$ which 'survive' Steps 1-3.

5. When we recover $\overline{\boldsymbol{\gamma}}^{i,f}$ and $\overline{\boldsymbol{\delta}}^{i,f}$, we repeat steps 1-4 to determine which of the variables in $\mathbf{X}^{out}$ have a non-zero coefficient in the second linear combination. We apply this procedure to all linear combinations we include to the model.

To understand why the above criterion is useful notice that when $R_j^2 > 0.995$, most of the variability of X_j^{out} is either explained by the core state variables and/or X_j^{out} is highly correlated with other variables in $\mathbf{X}^{out}$. In the first case the residuals of the regression of X_j^{out} on $\mathbf{X}^{core}$ (required to estimate the linear combination) will be close to zero so that the variable does not add almost anything to the approximation. In the second case, the residuals will be highly correlated with the residuals of other $\mathbf{X}^{out}$ variables. In both cases estimating the coefficients α becomes problematic and the convergence of the model with linear combinations becomes more difficult. Since a high R_k^2 denotes that the $k-$th variable is redundant, it helps the algorithm to converge if its coefficient is set to zero beforehand.

Number of linear combinations used in the approximation

Tables 2 to 4 summarize the number of linear combinations we add to the approximations of some of the models considered in the paper. Consider first Table 2 which reports the results for the buyback models (under 'no lending', top panel and under 'lending', bottom panel).

As described in the text, a new linear combination is added when it reduces significantly the residual sum of squares obtained from the regression of $u_{c,t+i}$, (for instance) on $\mathbf{X}^{core}$ and the linear combinations which were added in the approximation in previous rounds. Our criterion is based on the percentage gain in the coefficient of variation R^2 we get when we add the new linear combination.

The rows in the table summarise the gains in R^2 for each linear combination. R_{aug}^2 is the value of the coefficient of variation we obtain when we include an additional linear combination to the model. R_{old}^2 the coefficient of variation without the additional linear combination. The row labeled LC_1 corresponds to the 'Condensed PEA' test when we solve the model only with the $\mathbf{X}^{core}$ variables and test the inclusion of the first linear combination. LC_2 tests the significance of the second linear combination and so on.

We add a further linear combination to an approximation when

$$R_{diff}^2 = \frac{R_{aug}^2 - R_{old}^2}{R_{old}^2} * 100 > 0.05,$$

in other words when the gain in R^2 is greater than 0.05 percent.

As Table 2 shows the buyback model under 'no lending' requires one linear combination. The approximations of $E_t u_{c,t+1}$ and $E_t u_{c,t+N-1}$ include a linear combination in the first round and the approximation of $E_t u_{c,t+1}\lambda_{t+1}$ includes one in the second round. In the buyback 'lending' model the importance of the $\mathbf{X}^{out}$ variables is limited and so this model does not require any linear combinations.

Table 3 reports the analogous findings in the no buyback models and Table 4 for the case of coupons. Each of these models is solved with linear combinations.

Table 2: **Linear Combinations: Buyback Model**

		BuyBack 'no Lending'				
		$u_{c,t+1}$	$u_{c,t+N}$	$u_{c,t+N-1}$	$u_{c,t+1}\lambda_{t+1}$	$u_{c,t+N}\lambda_{t+1}$
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	**0.0757**	0.0169	**0.1677**	0.0441	0.0258
	LC_2	0.0026	0.0228	0.0043	**0.0547**	0.0417
	LC_3	0.0259	0.0232	0.0234	0.0060	0.0308
	Total	1	0	1	1	0
		BuyBack 'Lending'				
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	0.0081	0.0451	0.0403	0.0385	0.0322
	Total	0	0	0	0	0

Note: The table shows the number of linear combinations in the buyback models ('no lending', top panel and 'lending, bottom panel). The columns list the conditional expectations we approximate in these models. The rows report the percentage gains in R^2 from adding a further linear combination to the model. Hence row LC_1 shows the gains when we compare the regressions with $\mathbf{X}^{core}$ only (R^2_{old}) to the regressions with $\mathbf{X}^{core}$ and one linear combination (R^2_{aug}.) In row LC_2 R^2_{old} derives from a regression on $\mathbf{X}^{core}$ and the first linear combination and R^2_{aug} adds a second linear combination and so on.
We denote in bold values of $\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$ which exceed the 0.05 threshold (above which we introduce an additional linear combination to the model).

Table 3: **Linear Combinations: No-Buyback model**

		$u_{c,t+1}$	$u_{c,t+N}$	$u_{c,t+1}\lambda_{t+1}$	$u_{c,t+N}\lambda_{t+N}$
		No BuyBack 'No Lending'			
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	0.0173	0.0166	**0.0561**	**0.0646**
	LC_2	**0.0578**	0.0112	0.0109	0.0035
	LC_3	0.0002	0.0194	0.0058	0.0001
	Total	1	0	1	1
		No BuyBack 'Lending'			
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	0.0194	**0.0590**	0.0342	**0.1448**
	LC_2	0.0167	0.0007	0.0140	0.0006
	Total	1	0	0	1

Note: The table shows the number of linear combinations in the no buyback models ('no lending', top panel and 'lending, bottom panel). See Table 3 for details.

Table 4: **Linear Combinations: No-Buyback Model with Coupons**

		$u_{c,t+1}$	$q_{c,t}$	$u_{c,t+1}\lambda_{t+1}$	$q_{\lambda c,t}$
		No BuyBack Coupons			
$\frac{R^2_{aug}-R^2_{old}}{R^2_{old}} * 100$	LC_1	0.0067	0.0154	0.0213	**0.0654**
	LC_2	0.0048	0.0108	0.0077	0.0011
	Total	0	0	0	1

Note: The table shows the number of linear combinations in the no buyback model with coupons. See Table 3 for details. We define $q_{c,t} \equiv \sum_{j=1}^{N} \beta^j u_{c,t+j}\kappa + \beta^N u_{c,t+N}$ and $q_{\lambda c,t} \equiv \sum_{j=1}^{N} \beta^j u_{c,t+j}\lambda_{t+j}\kappa + \beta^N u_{c,t+N}\lambda_{t+N}$.

The picture is similar when we consider the models not included in the tables: the callable bonds, three maturities and the model with optimal repurchases. The callable bond model needs one linear combination to be added to get accurate solutions. For the three maturities model we find that $\mathbf{X}^{core}$ is sufficient and therefore we do not include any linear combinations. The optimal repurchase model requires two linear combinations to be accurately solved. The first linear combination is introduced to the approximations of $E_t u_{c,t+i}\lambda_{t+i}$, $i = 1, N, N-1$ and $E_t \xi^R_{U,t+1}$. The second linear combination is introduced to the approximation of $E_t u_{c,t+1}$.

B.2 Some practical features of the numerical implementation

B.2.1 Dealing with occasionally binding constraints on debt

As explained in the main text, we impose an upper and lower bound on the issuance of short and long bonds. In a two bond model we have in total four constraints. These constraints are only occasionally binding and in theory we could use the approach explained, for example, in Marcet and Singleton (1999) to deal with them. Suppose that the government can issue only one bond, whatever the maturity. Marcet and Singleton suggest for every period t first to solve the unconstrained problem and check whether one of the debt constraints is violated. If it is violated, the value of the bond is set equal to the value of the debt limit and c_t and λ_t are recalculated accordingly.

Unfortunately, this cannot be easily applied in the case of more than one maturity because of the number of constraints involved. If one of the constraints is violated when solving the unconstrained problem, we need to verify that forcing the constraint is not going to generate a violation of one of the constraints on the other bond. This problem presents too many cases to be checked one by one and the computational burden increases considerably when an additional maturity is introduced to the model. For this reason we impose the following (quadratic) costs when the bonds violate the limits in the buyback model:

$$C(b^i_t) = \begin{cases} \frac{\phi_1}{2}\left(b^i_t - \frac{\overline{M}_i}{\beta^i}\right)^2 & b^i_t > \frac{\overline{M}_i}{\beta^i} \\ \frac{\phi_1}{2}\left(\frac{\underline{M}_i}{\beta^i} - b^i_t\right)^2 & b^i_t < \frac{\underline{M}_i}{\beta^i} \\ 0 & otherwise \end{cases}$$

for $i = 1, N$. ϕ_1 governs the penalty from deviating from the debt limits $\frac{\underline{M}_i}{\beta^i}$ and $\frac{\overline{M}_i}{\beta^i}$. We choose a value of ϕ_1 equal to unity. Analogous cost functions are used in the no buyback and coupons models, the debt limits have to be adjusted in these cases as described in text.

In the optimal repurchase model we have an additional constraint on the level of repurchases: $0 \leq R_t \leq b^N_{t-1}$. In this case we continue to impose $C(b^i_t)$ for $i = 1, N$ however we use Marcet and Singleton's approach to deal with this extra constraint. When R violates a limit (either because $R_t < 0$ or $R_t > b^N_{t-1}$) we fix the value of R_t to the constraint and solve the FONC to determine the optimal portfolio and the value of the multipliers, $\xi^R_{L,t}$ and $\xi^R_{U,t}$.

B.2.2 Initial conditions and sample size

In order to generate a more precise approximation of the policy functions over the debt space we use PEA with oversampling. We choose 25 different initial conditions for the debt levels b_{-1}^1 and b_{-j}^N, where $j = 1, 2, ..N-1$ uniformly distributed in the interval $\left[\frac{\underline{M}_i}{\beta^i}, \frac{\overline{M}_i}{\beta^i}\right]$ (e.g. in the buyback model). We draw 10 samples of 500 periods for each initial condition. The total number of observations is then 125000.

Given the initial conditions for the portfolio, we also need to specify some initial values for the λ's. For this purpose we recover initial values $\lambda_{-N} = ..., \lambda_{-1}$ that would be consistent with the deterministic steady state. As is well known in steady state the debt level in these models is indeterminate and so we can obtain a different λ (consistent with a different c) for each bond vector. Under no buyback we obviously need to set $b_{-1}^N = b_{-2}^N = ... = b_{-N+1}^N$ to be in steady state.[9]

B.2.3 Rescaling

To improve the stability of the algorithm, we rescale the variables which enter in $\mathbf{X}^{core}$ and $\mathbf{X}^{out}$. For example we use $\frac{b_t^i}{\frac{\overline{M}_i}{\beta^i}}$ and $\frac{\lambda_t - \lambda^s}{\lambda^s}$ instead of b_t^i and λ_t. This is applied to every lag of the independent variables used in the approximation. We also rescaled the dependent variables in the PEA regressions by their steady state values such that their means are close to one in the approximations. For example, we regress $\frac{u_{c,t+1}}{u_c^s}$ and $\frac{u_{c,t+1}\lambda_{t+1}}{u_c^s \lambda^*}$ on $\mathbf{X}^{core}$ and the linear combinations, to obtain the approximations of $E_t\left(\frac{u_{c,t+1}}{u_c^s}\right)$ and $E_t\left(\frac{u_{c,t+1}\lambda_{t+1}}{u_c^s \lambda^*}\right)$ respectively. The same is done for the other expectations.

Rescaling is useful because some of the coefficients could be very small without it. For example, consider the buy back no lending model; b_t^N can fluctuate in simulations between 0 and $\frac{\overline{M}_i}{\beta^i} \approx 117$ and its square between 0 and 117^2. It is obvious that the estimated coefficients of these terms may be close to zero. This makes it difficult to find a reliable convergence criterion for the model.[10] Through rescaling the state variables fluctuate between 0 and 1. This improves significantly the stability of our algorithm (see also Judd et al (2011)).

B.2.4 Convergence of PEA - Finding Good initial conditions for the coefficients

Den Haan and Marcet (1990) show that PEA does not guarantee convergence. Convergence is more likely if we use good initial conditions for the coefficients. This is even more necessary in the context

[9]Notice that since sample sizes are sufficiently long (500 observations) our results do not change when we set $\lambda_{-1} = \lambda_{-2} = ... = 0$.

[10]To see this, denote the coefficient of variable b_t^N in the approximation of $E_t u_{c,t+i}$ by γ_3^i. Let $\gamma_{3,1}^i$ be the update of this coefficient and $\gamma_{3,0}^i$ the initial value. If we use a stopping rule

$$\text{Converge if} \quad \frac{|\gamma_{3,1}^i - \gamma_{3,0}^i|}{|\gamma_{3,0}^i|} < \epsilon \tag{25}$$

and $\gamma_{3,1}^i, \gamma_{3,0}^i \approx 0$, then the behavior of (25) will be very erratic (both very high and very low values are possible, and this does not tell us much about convergence of the model's quantities). Analogously, if we use the convention

$$\text{Converge if} \quad \frac{|\gamma_{3,1}^i - \gamma_{3,0}^i|}{1 + |\gamma_{3,0}^i|} < \epsilon \tag{26}$$

for some ϵ, then the algorithm may (wrongly) converge after a few iterations.

In our codes we employ the criterion (26), but since the variables are rescalled, we are sure that coefficients which are small in values, do not matter much for the optimal policy.

of the optimal portfolio problem under incomplete markets. If the initial coefficients constitute a very poor guess of the equilibrium of the model, then the algorithm may circle for a long time and subsequently diverge.

Good initial conditions for portfolio choice models can be obtained as follows:

1. Solve portfolio models with positive transaction costs.

   For example consider solving the Ramsey problem under buyback subject to the following government budget constraint:

   $$\sum_{i=1,N} p_t^i b_t^i = \sum_{i=1,N} p_t^{i-1} b_{t-1}^i + g_t - \left(1 - \frac{v_{x,t}}{u_{c,t}}\right)(c_t + g_t) + \sum_{i=1,N} \omega_i (b_t^i)^2$$

   where $\omega_i (b_t^i)^2$ is a transaction cost paid by the government at issuance. It is obvious that in this model the optimal portfolio is determinate (even with the conventional PEA). In the limit when $\omega_i \to 0$ we obtain the buyback model considered in this paper, if $\omega_i \to \infty$ there is no trade in bonds. Hence, good initial conditions can be found from solving models with positive transactions costs and gradually reducing ω_i till 0.

2. Solve models under tight debt limits and gradually loosen them.

   We found that models with tight debt constraints converge more easily than models with looser ones. Generally speaking, models with very loose debt constraints can converge to a wrong equilibrium which features for example a constant λ as in the case of complete markets. This holds in particular because running the models with samples of 500 observations may imply that the debt limits are rarely hit, if they are very loose.[11] Moreover, when the bounds are loose, poor initial conditions may make the PEA circle or diverge. Assuming tight bounds helps the algorithm converge. The converged coefficients can then be used as initial conditions for models with looser bounds and so on.

B.2.5 Calculating the sample moments

As discussed in the text, to compute the moments reported in Table 4 in the paper, we simulated the model 1000 times using as initial conditions the values of $\mathcal{S}_t$ and the market value of debt, we recovered from the data. In 1955 the share of short debt equaled 39% and the initial debt to GDP ratio was 38% in the CRSP sample.

We then computed the values b_{-1}^1 and b_{-j}^N, $j = 1, 2, .., N$ in the deterministic steady state such that the initial share and market value of debt are consistent with these targets. For example in the

[11] To see this, consider the following example: Suppose that the initial guess for the polynomials is $E_t u_{c,t+i} = \gamma_0^i + \gamma_1^i E_t g_{t+1} + \gamma_2^i b_t^1 + \gamma_3^i b_t^N$ and $E_t \lambda_{t+1} u_{c,t+i} = \lambda^* E_t u_{c,t+i}$. Then, under very loose bounds (e.g. $-\underline{M}_i = \overline{M}_i = \infty$) for every t we get $\lambda_t = \lambda^*$ as a solution to the system of FONC, as in the case of complete markets.

Under tight bounds however, it is likely that $b_t^i = \frac{\overline{M}_i}{\beta^i}$ or $b_t^i = \frac{\underline{M}_i}{\beta^i}$ for some t, and in this case the condition $\lambda_t = \frac{E_t u_{c,t+i} \lambda_{t+1}}{E_t u_{c,t+i}}$ does not hold. λ_t is recovered from the FONC of consumption and generally it will be that $\lambda_t \neq \lambda^*$. This introduces variability in λ_t.

A similar argument, showing the importance of 'moving bounds' for convergence in the PEA, was made by Maliar and Maliar (2003).

buyback model we have

$$\frac{\beta b_{-1}^1}{\beta b_{-1}^1 + \beta^N b_{-1}^N} = 0.39 \qquad \beta b_{-1}^1 + \beta^N b_{-1}^N = 0.38 * 70$$

The analogous expressions for the other models are omitted for brevity.

Given the initial conditions for the bonds, we found the initial values of c and λ to satisfy the FONC of consumption and the government budget constraint in the deterministic steady state.

We then simulated the models and computed the market value of government debt and the share of short bonds. Notice that whereas in the buyback models to construct the market values for short and long bonds it is sufficient to use the approximations of $E_t u_{c,t+1}$ and $E_t u_{c,t+N}$, in the no buyback model this is not the case. In particular we need to compute the value of non-matured debt in period t. This requires all the prices p_t^j for $j = 2, 3, ..., N-1$. Since these prices do not affect the equilibrium properties of optimal allocations, we computed the approximations through simple regressions of $u_{c,t+i}$ on $\mathbf{X}_{t+1}$ once our algorithm has converged.[12]

Finally, note that because the model is solved with quadratic costs if the debt limits are violated, as described in subsection B.2.1, the market value of government debt can become (slightly) negative in the no lending models, in some periods and samples. The statistics reported in Table 4 in the paper are calculated after dropping samples where the market value becomes negative in 'no lending models'. For the same reason in order to avoid having a negative share of short debt in simulations (if say $b_t^1 < 0, b_t^N > 0$) or greater than unity (i.e. when $b_t^N < 0, b_t^1 > 0$) , we computed the moments using $\min\{\max\{\mathcal{S}_t, 0\}, 1\}$ in the no lending models: we forced the share to be equal to zero when it was negative and 1 when it exceeded unity. This adjustment obviously was much more frequent in the buyback 'no lending model' than in the no buyback model.[13]

B.2.6 Some limitations of the PEA

We cannot claim that the numerical algorithm we propose in this paper can solve every portfolio choice problem under incomplete markets. To make this point, we describe here a few cases where the approximation of conditional expectations under 'Forward States' may not compute accurately equilibria with multiple assets.

The first noteworthy difficulty of our methodology is that as the number of assets increases the optimal allocation may be close to the complete markets' one. Recall that in this case the portfolio and the multiplier λ are constant through time. Clearly, such equilibria cannot be approximated with polynomials of the form $E_t u_{c,t+i} = \gamma_0^i + \gamma_1^i E_t g_{t+1} + \gamma_2^i b_t^1 + \gamma_3^i b_t^N + \gamma_3^i \lambda_t + ...$; if the RHS variables are roughly constant, the estimation of the polynomial coefficients will not be reliable. Our algorithms are designed to deal with cases where markets are incomplete, this involves either few assets, or tight debt constraints or both.

[12] In these regressions we used all bonds and cross terms b_{t-j}^N and $b_{t-j}^N \lambda_{t-j}$ $j = 0, 1, ..., N-1$ as independent variables. We do not use the 'Condensed PEA' to approximate the bond prices since these approximations are performed when the algorithm has converged and thus the algorithms convergence properties do not depend on them.

[13] Recall that one of the main findings of the paper is that under no buyback long and short debt levels comove strongly. This property also holds for the issuances b_t^1 and b_t^N. It is therefore rare that b_t^1 is slightly negative and $b_t^N > 0$. If this occurs in our simulations it is likely that the overall market value is slightly negative in which case the sample is dropped as described previously.

However, under buyback and no lending, we frequently have $b_t^N >> 0$ and $b_t^1 \approx 0$ so that small negative values of short debt can occur. In these cases we set the short term share to 0.

Second, even under incomplete markets if the government can trade three or more maturities we cannot rule out equilibria where some of the assets are roughly constant over time (and thus stable coefficients for the polynomials are hard to obtain). To see this assume that in the *buyback model* the government issues three different maturities: 1 year, M year and N year bonds where $1 < M < N$. To take advantage of fiscal hedging the government will likely adopt the following debt management strategy: set $b_t^1 = \frac{M_1}{\beta} < 0$, $b_t^N = \frac{\overline{M}_N}{\beta} > 0$ and use b_t^M to finance deficits and surpluses over the business cycle.[14] Therefore the values of b_t^1 and b_t^N will be roughly constant over time and thus it will be difficult to construct approximations of the conditional expectations when the polynomials contain b_t^1 and b_t^N as state variables.

Third, (we found that) it is not as easy to solve models where the government issues bonds of maturities close to one another (e.g. 1, 9 and 10 years or 1, 2 and 10 years) as it is to solve models where maturities are further from one another (e.g. 1, 5 and 10 years). As we have seen, no buyback models give us portfolios where all issuances have the tendency to comove strongly, however, when bonds are close substitutes, asset prices also comove strongly and so portfolios are difficult to pin down. The algorithm then tends to circle without converging.

Note that these difficulties are not relevant if the goal is to solve models with multiple assets and realistic frictions (e.g. imperfect substitutability among the assets). Small transaction costs, bond clienteles and preferences for short term (safe) assets, will give well defined demand curves for each maturity and these are realistic features of government debt markets. Our methodology is therefore broadly applicable to solve models with many assets, the limitations described in this subsection arise because our model is a simplistic one and abstracts from several realistic frictions.

B.3 Accuracy of the solutions

To check the accuracy of the solution of each model we compute the Euler Equation Errors (EEE) generated by our approximations (see for example Arouba et al (2006) for an exhaustive description of the methodology). Essentially this methodology checks that first order conditions hold with an acceptable degree of precision at many points in the state vector.

In particular, the test requires to numerically calculate each conditional expectation in the Euler equations. Ours is not a routine application of the standard accuracy test because we have expectations up to N leads, so the exact integration is not feasible. We use Monte Carlo integration to approximate the expectation integrals. In practice we draw 250 shock samples (for 25 initial conditions of public debt with 10 samples for each initial condition) of 450 periods each. We then simulate each model using our approximations. We discard the first 200 periods of each sample, and for each subsequent period [15], $\bar{t}$, we draw $k = 10000$ different shock paths of length N, the number of leads in the conditional expectations. We simulate our model for each shock path separately using our approximated policy functions and initial states given by the allocation in $\bar{t}$. We then compute the conditional expectations as the mean over the k samples. For example for the buy back and no buy

[14] If the debt limits on b_t^M are sufficiently loose, this strategy is feasible.

[15] For the 3 bond model we check the errors every 5 periods. This choice was made for computational purposes because in this model we have 125 initial conditions and 10 samples per initial condition.

back models we compute:

$$\Xi_{\bar{t},1} = E_t\Bigg(u_{c,\bar{t}+1}\Bigg) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+1}}{k}$$

$$\Xi_{\bar{t},2} = E_t\Bigg(u_{c,t+N}\Bigg) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+N}}{k}$$

$$\Xi_{\bar{t},3} = E_t\Bigg(u_{c,t+1}\lambda_{t+1}\Bigg) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+1}\lambda^i_{\bar{t}+1}}{k}$$

$$\Xi^{NBB}_{\bar{t},4} = E_t\Bigg(u_{c,t+N}\lambda_{t+N}\Bigg) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+N}\lambda^i_{\bar{t}+N}}{k} \text{ or } \Xi^{BB}_{\bar{t},4} = E_t\left(u_{c,t+N}\lambda_{t+1}\right) = \frac{\sum_{i=1}^{k} u^i_{c,\bar{t}+N}\lambda^i_{\bar{t}+1}}{k}$$

Since we have two Euler equations, we check separately each of them, calculating the value of the multiplier in period $\bar{t}$ implied by the expectations $\Xi_{\bar{t},}$ given the portfolio $b_{1,\bar{t}}, b_{N,\bar{t}}$[16]. In theory we could stop here and check the difference between the implied multiplier and the one generated by our simulation. However it is difficult to give an intuitive economic interpretation of this difference. Following the literature we then state our results in terms of of consumption deviations. To do this we calculate the implied consumption error using the FONC of c_t, given the implied multiplier.

In particular we compute the following quantities:

$$EEE^1_{\bar{t}} = \frac{\widetilde{c}^1_{\bar{t}} - c_{\bar{t}}}{\widetilde{c}^1_{\bar{t}}}$$

$$EEE^N_{\bar{t}} = \frac{\widetilde{c}^N_{\bar{t}} - c_{\bar{t}}}{\widetilde{c}^N_{\bar{t}}}$$

where $\widetilde{c}_{\bar{t}}$ is the consumption implied by the new approximation of the expectations, $\Xi_{\bar{t},}$ and $c_{\bar{t}}$ the one implied by our approximation. We compute the average EEE across all samples and initial conditions, the maximum error and the percentage of positive and negative errors. We average over 62500 errors.

As in Aruoba et al. (2006) we report the *absolute* errors using base 10 logarithms to make our findings comparable with the rest of the literature. A value of -3 means a 1$ mistake per 1000$, a value of -4 a mistake of $1 per $10000 and so on. Table 5 reports the results.

Table 5 shows that the average of the errors are between -3 and -4, that the percentage of positive errors is close to 50% and that the maximum errors are not large. Moreover, we find that it is quite unlikely that the region of the state space where the maximum error occurs is visited in simulations. These results are well within the range accepted by other authors (e.g. Aruoba et al (2006)). This suggests that the model solutions are accurate.

B.4 Shadow Cost Calculations

In section 6.2 of the paper we presented the results of an approximate "shadow cost" calculation of the loss in utility due to transaction costs. In this section we give details on how we proceeded in

[16] Since the optimal portfolio is determined through 'Forward States' it is not possible to use objects $\Xi_{\bar{t},i}$ to determined new values of $b_{1,\bar{t}}, b_{N,\bar{t}}$.

Table 5: **Euler Equation Errors**

		BB		NBB					
		lending	no lending	lending	no lending	coupons	callable	repurch.	3 bonds
EEE^1	ave	-3.97	-3.72	-3.84	-3.86	-3.89	-3.92	-3.84	-3.77
	max	-2.30	-2.28	-2.50	-2.64	-2.60	-2.51	-2.79	-2.32
	%+	0.41	0.51	0.48	0.44	0.43	0.44	0.48	0.51
EEE^N	ave	-3.18	-3.06	-3.53	-3.51	-3.30	-3.75	-3.18	-3.62
	max	-1.81	-1.93	-2.55	-2.47	-1.99	-2.49	-2.12	-2.74
	%+	0.50	0.54	0.54	0.49	0.57	0.49	0.61	0.42
EEE^{N-1}								-3.23	
								-1.94	
								0.40	
EEE^M	ave								-3.59
	max								-2.30
	%+								0.58

our calculations. As explained in the text we seek to find:

$$\chi = \frac{U^{BB} - U^{NBB}}{\mathcal{T}otal^{BB} - \mathcal{T}otal^{NBB}}$$

where $U^i = E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t^i) + v(x_t^i)\right]$ denotes the total welfare for each model $i = BB, NBB$. $\mathcal{T}otal^i = E_0 \sum_{t=0}^{\infty} \beta^t \left(\lambda_t^i u_{c,t}^i + v_{x,t}^i\right) \mathcal{T}ot_t^i$ is the total shadow transaction cost of buyback or no buyback in term of utility where $\mathcal{T}ot_t^i$ is the total transaction cost at time t for the optimal portfolio.

We then calculated numerically the four elements that determine χ using a mix of short and long run simulations in order to have a good approximation of the infinite sums. Let's take for example $U^{BB} = E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t^{BB}) + v(x_t^{BB})\right]$. In order to approximate this term we first run a long simulation of the buyback model with 100000 periods and calculated $U_{L,\bar{t}}^{BB} = \sum_{t=\bar{t}}^{\overline{T}} \beta^{t-\bar{t}} \left[u(c_t^{BB}) + v(x_t^{BB})\right]$ for every $\bar{t} = 1, ..., \overline{T}$, where $\overline{T} = 100000$. Starting from $\overline{T} = 100000$ we defined $U_{L,\overline{T}}^{BB} = \frac{\overline{U}^{BB}}{1-\beta}$ where $\overline{U}^{BB}$ is the average of $u(c^{BB}) + v(x^{BB})$ over the entire simulation. Then, iterating backwards one period we got $U_{L,\overline{T}-1}^{BB} = u(c_{\overline{T}-1}^{BB}) + v(x_{\overline{T}-1}^{BB}) + \beta U_{L,\overline{T}}^{BB}$. We continued to obtain $U_{L,\bar{t}}^{BB}$ up to $t = 1$.

After dropping the first 100 and the last 2000 periods, we regressed the generated sums on $b_{t-1}^1, b_{t-1}^N, .., b_{t-N}^N, \lambda_{t-1}, .., \lambda_{t-N}$ and g_t. This gave us an approximation $f(b_{t-1}^1, b_{t-1}^N, .., b_{t-N}^N, \lambda_{t-1}, .., \lambda_{t-N}, g_t)$ of the conditional expectation of U^{BB} in t based on the long run simulation. We used this as an 'end point' in the short run simulations.

The short run simulations were carried out as follows: We simulated our models 10000 times for 100 periods starting from the same initial condition. Continuing with the previous example of the buyback model, we calculated $U_{S,i}^{BB} = \sum_{t=1}^{100} \beta^t \left[u(c_{t,i}^{BB}) + v(x_{t,i}^{BB})\right] + \beta^{101} U_{L,i}^{BB}$ for every sample $i = 1, ..., 10000$ where $U_{L,i}^{BB} = f(b_{100,i}^1, b_{100,i}^N, .., b_{100-N,i}^N, \lambda_{100,i}, ..., \lambda_{100-N,i}, g_{100,i})$. Our approximation of U^{BB} is the average of $U_{S,i}^{BB}$ over the 10000 samples.

The above procedure was repeated for all four elements of χ, calibrating the transaction costs as explained in section 6.1 of the paper.

C Bibliography

- Aiyagari, R., Marcet, A., Sargent, T.J. and Seppälä, J. (2002) "Optimal Taxation without State-Contingent Debt" *Journal of Political Economy*, 110, 1220-1254.

- Aruoba, B., Fernandez-Villaverde, J. and Rubio-Ramírez, J-F. (2006) "Comparing Solution Methods for Dynamic Equilibrium Economies", *Journal of Economic Dynamics and Control*, 30(12), 2477-2508.

- Buera F. and Nicolini, J.P. (2004) "Optimal Maturity of Government Debt with Incomplete Markets", *Journal of Monetary Economics,* 51, 531-554.

- Chari, V.V., and Kehoe, P. (1999) "Optimal Fiscal and Monetary Policy," *Handbook of Macroeconomics*, Volume 1, Elsevier.

- den Haan, W. and Marcet, A. (1990) "Solving the stochastic growth model by parameterizing expectations" *Journal of Business and Economic Statistics*, 8, 31-34.

- Faraglia, E., Marcet, A., Oikonomou, R. and Scott, A (2016) "Long Term Government Bonds ", *mimeo LBS*.

- Judd. K., Maliar, L., and Maliar, S (2011) "Numerically Stable and Accurate Stochastic Simulation Methods for Solving Macro Models ", *Quantitative Economics* 2, 173-210.

- Judd. K., Maliar, L., and Maliar, S (2012) "Merging Simulation and Projection Aproaches to Solve High-Dimensional Problems ", *NBER* paper 18501.

- Maliar, L. and Maliar, S. (2003), "Parameterized Expectations Algorithm and the Moving Bounds," *Journal of Business and Economic Statistics,* 21, 88- 92.

- Marcet, A and Singleton, K (1999) "Equilibrium Asset Prices and Savings in a Model with Heterogeneous Agents, Incomplete Markets and Liquidity Constraints", *Macroeconomic Dynamics,* 3, June: pp 243-276.