Risk Aversion, Labor Risk, and Aggregate Risk Sharing with Financial Frictions[*]

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Abstract

If agents in workhorse business cycle models with financial frictions can index contracts to observable aggregates, they share aggregate financial risk (almost) perfectly, eliminating the financial accelerator mechanism. I show that in the standard specification of the Bernanke, Gertler, and Gilchrist (1999) framework with TFP shocks this happens because: *i)* borrowers and lenders are implicitly assumed to have identical, logarithmic utility, and *ii)* the representative lender's human wealth comoves closely with aggregate financial wealth. Non-state-contingent borrowing rates can arise optimally if *i)* lenders' risk aversion is increased to plausible degrees, or *ii)* at identical preferences, lenders face uninsurable countercyclical idiosyncratic risk in labor productivity.

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1 Introduction

Recent developments in the literature on macro-financial DSGE models have brought to light the fact that in several conventional frameworks, the relevance of financial frictions in aggregate fluctuations is eliminated if agents are allowed to share aggregate risk embedded in returns to risky assets optimally. One such workhorse model is that set up by Bernanke et al. (1999), henceforth BGG, building on earlier work by Bernanke and Gertler (1989) and Carlstrom and Fuerst (1997). It features capital-managing entrepreneurs and households who provide funding to the former. A costly state verification (CSV) friction emanates from idiosyncratic shocks to entrepreneurs' held capital, as formalized by Townsend (1979). The presence of such a financial friction gives rise to a financial accelerator mechanism which generates amplification and added persistence of aggregate shocks, compared to a frictionless real business cycle framework. A crucial nuance in this mechanism is the exogenously imposed constraint that the lenders must, on aggregate, receive a predetermined return. That is, the lenders' return does not respond to realizations of aggregate shocks, observed at the time of repayment, and the borrowing entrepreneurs bear all aggregate financial risk.

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This assumption regarding the predeterminacy of returns has drawn criticism, for example by Chari (2003). There is no explicitly modelled reason why in the presence of a risk-averse lender and a borrower with time-varying investment opportunities the counterparties cannot engage in mutual insurance against aggregate risk. This could be achieved by agreeing on a lender's return which is indexed to observable outcomes to be realized in the macroeconomy. Carlstrom et al. (2016) (henceforth, CFP) formalize this idea in the BGG framework. They show that, in the privately optimal one period contract, the *ex post* return to the lenders is indexed one-for-one to the return on entrepreneurial capital, adjusted for fluctuations in the borrower's and lender's marginal valuations of wealth.

The fact that borrowing entrepreneurs' *ex post* liabilities adjust to capital return shocks significantly dampens financial accelerator dynamics. In the non-state-contingent lender return case employed by BGG, a positive aggregate shock to the return on capital, for example due to increased productivity, leads to a significant increase in entrepreneurs' net worth because of their predetermined liabilities. The relatively higher net worth decreases entrepreneurial leverage, increases their ability to hold assets, boosting asset prices which feed into further net worth and investment increases – the financial accelerator mechanism.

However, if lender returns are indexed to capital returns, and possibly other observables, there is sharing of aggregate financial risk between the borrower and lender. A positive shock leads to an increase in *ex post* entrepreneurial liabilities, a smaller increase in net worth, and a dampened drop, if any, in leverage. This is exactly the mechanism at work in the treatment of CFP. Similar ideas have been presented by Krishnamurthy (2003) in a stylized three period model with borrowing contraints in the spirit of Kiyotaki and Moore (1997), and by Di Tella (2017) in the infinite horizon framework studied by Brunnermeier and Sannikov (2014). A key implication for the dynamics of borrowers'/entrepreneurs' balance sheets in all these frameworks, compared to a contract with predetermined lender returns, is that fluctuations in entrepreneurial leverage should be minimal. Or equivalently, the magnitude of relative fluctuations in entrepreneurial net worth should be close to equal to those in held assets, and not amplified by leverage.

A descriptive summary of aggregate time-series data on US firms' balance sheets demonstrates a considerable degree of volatility.[1] For a simple illustration, Figure 1 displays quarterly aggregate nonfinancial business balance sheet data from the Federal Reserve Board Flow of Funds Accounts. It graphs the HP-filtered cyclical components of nonfinancial firms' net worth and leverage, alongside that of gross value added (GVA) in the nonfarm business sector – all in logs, for the period 1976Q1–2015Q3.[2] As is evident, aggregate non-financial sector leverage exhibits non-negligible countercyclicality over the business cycle. And strikingly, since the 1980s, the unconditional second moments of the cyclical components of the balance sheet variables have seemed to drift farther from the implications of privately optimal aggregate risk sharing discussed above. Increased volatility in US firms' balance sheet variables has been pointed out in earlier work by Fuentes-Albero (2016). This finding becomes the more intriguing if one were to expect that the rapid development of financial markets and instruments during this period should have made aggregate state dependent borrowing contracts and privately optimal risk sharing more easily

[1] To be more precise in the labeling between model and data, one could think of entrepreneurial wealth in the model as inside equity, and the entrepreneurs' external financing as the sum of outside equity and debt financing. The degree of external finance return indexation to the return on assets then mirrors the relative magnitudes of outside equity and debt. In the model's solution, to a first order, this degree of indexation is constant over time, implying a constant ratio of outside equity to external finance. As long as this ratio implied by the model is less than 1 (i.e. external financing is not fully in the form of outside equity), one can easily establish a positive relation between $\textit{leverage in model} \equiv \frac{\textit{assets}}{\textit{inside equity}}$ and $\textit{leverage in data} \equiv \frac{\textit{assets}}{\textit{equity}}$.

[2] To capture nonfinancial activities, I measure assets as *Nonfinancial Assets* and debt as the sum of *Loans* and *Debt Securities* – all measured at market values. The conclusions are virtually unchanged when simply using *Total Assets* and *Total Liabilities* instead of nonfinancial assets and debt. All of these statements follow also for considering only the nonfinancial corporate business sector.

implementable. For example, regarding the management of interest rate risks, the market for interest rate swaps emerged in the early 1980s and grew rapidly during the decade (Saunders, 1999). And all this happened during a time of lower volatility in the real economy, well-documented as the Great Moderation and evident in the fluctuations of GVA.

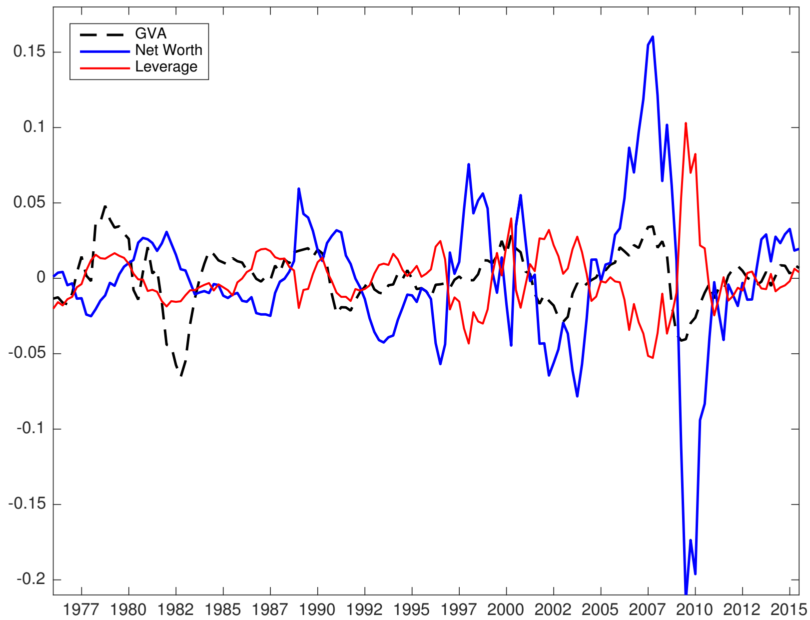


Figure 1: Log-deviations from HP-trend of gross value added in the US non-farm business sector, non-financial corporate net worth and leverage. HP-filter smoothing parameter 1,600. Data: FOFA, NIPA-BEA.

Assets measured as Nonfinancial Assets (FOFA Tables B.103 and B.104, line 2), debt as the sum of Loans (FOFA Tables B.103 and B.104, lines 30 and 27, respectively) and Debt Securities (FOFA Table.103, line 26) – all measured at market values. *Net worth* = *Assets* − *Debt*, *Leverage* = *Assets*/*Net worth*. GVA measure from NIPA-BEA Table 1.3.5 line 3. All variables deflated by the implicit price index for the nonfarm business sector (NIPA-BEA Table 1.3.4, line 3).

Explaining the observed changes in balance sheet dynamics is beyond the scope of this paper. It is nontheless clear that significant fluctuations in the net worth and leverage of non-financial firms are a prevalent phenomenon in the US economy. Also, these fluctuations are synchronized with the business cycle, exactly like the basic financial accelerator mechanism under non-state-contingent debt would predict.[3] And as demonstrated for example in the work by Giroud and Mueller (2017), the health of non-financial firms' balance sheets had significant relevance for real activity during the Great Recession.

In this paper, I pursue the idea that non-aggregate-state-contingent lender returns, and the implied countercyclical fluctuations in entrepreneurial leverage might be the outcome of privately optimal aggregate risk sharing between households and entrepreneurs in the BGG framework. A slight reformulation of the borrowing entrepreneurs' problem and preferences allows to establish that, to a first order, the conventional BGG-CFP assumption of individual entrepreneurs with linear utility consuming a constant fraction of wealth is equivalent to assuming that there is a representative entrepreneur who receives logarithmic utility from consumption and owns a continuum of *firms*, each running individual projects subject to limited liability and firm-specific risk.[4]

[3]The correlation between the above graphed GVA and leverage cyclical components is -0.48 in the sample post-1984, a commonly estimated structural break date for the Great Moderation, and -0.33 when only considering data up to 2007Q4 to eliminate the effect of the Great Recession.

[4]Similarly to CFP, Dmitriev and Hoddenbagh (2017) analyze optimal contracting in the BGG framework and consider

This establishes that by assuming logarithmic utility from consumption for the representative lender, as BGG and CFP do, one is effectively studying a risk-sharing problem between two agents with identical preferences over consumption. While being a valid theoretical benchmark, it also demonstrates that if one were to instead consider the conventional BGG specification with households that do not have logarithmic utility, the high degree of sharing aggregate financial risk found by CFP might not necessarily follow. Relatively more aggregate risk would trivially be taken on by agents with lower aversion to fluctuations in consumption. To shed light on this issue, I consider households with conventional Epstein and Zin (1989) preferences and compute under which values of risk aversion and intertemporal elasticity of substitution does the implied optimal one-period financial contract yield non-contingent borrowing rates in response to persistent yet stationary total factor productivity shocks. Under the calibration employed by CFP, this happens with a household risk aversion parameter of 13.2 and intertemporal elasticity parameter of 1.0, or CRRA utility with risk aversion of 5.92, for example.[5] Under a close to unit root TFP process, these numbers can be significantly smaller.

In addition to the agents' preferences affecting their optimal sharing of aggregate financial risk, also their exposure to aggregate risk through other sources of wealth matters. In the specification used by CFP, entrepreneurs' total wealth equals their financial wealth while the representative household is also endowed with human wealth. With logarithmic utility requiring optimal consumption to be a constant fraction of one's total wealth, optimal risk sharing between households and entrepreneurs effectively requires sharing financial returns in a way that works to neutralize fluctuations in human wealth. That is, for any positive shock to households' human wealth, they should cede more of their financial returns to the entrepreneurs. With relative fluctuations in aggregate financial wealth and human wealth comoving closely in response to TFP shocks, the agents end up sharing realized financial returns close to equally.

Given that idiosyncratic shocks to human wealth are conceivably less diversifiable than the idiosyncratic risk embedded in owning individual assets or financing entrepreneurial projects, I also consider households' countercyclical uninsurable idiosyncratic risk as an effective source of increased risk aversion. I do so by introducing uninsurable household risk in the model, emanating from shocks to individual labor productivity. In the asset pricing literature, the introduction of countercyclical idiosyncratic risk is a common way of increasing agents' effective risk aversion towards aggregate fluctuations.[6] This risk will separate individual households' consumption from aggregate human wealth fluctuations and generate a force towards less financial risk sharing in the model, even when borrowers and lenders have identical expected utility preferences over consumption.

To study the relevance of idiosyncratic lender risk, I build on the approach by Constantinides and Duffie (1996) and construct a tractable no-trade equilibrium in which households' consumption exhibits rich heterogeneity brought about by shocks to their idiosyncratic labor productivity. By specifying that the variance of these idiosyncratic shocks is time-varying and moves in response to any aggregate shocks affecting the economy, one can study the cyclical properties of idiosyncratic lender risk required to ensure the optimality of financial contracts in which the lender does not take on any aggregate financial risk brought about by the shocks. While the framework allows to generate non-state-contingency of lender returns in response to various aggregate shocks and at varying degrees of persistence in idiosyncratic risk, my main analysis focuses on a real economy with aggregate TFP shocks. I find that when the variance of idiosyncratic shocks is assumed to be persistent, then in response to a 1% drop in aggregate TFP the

varying degrees of entrepreneurial risk aversion. However, in their formulation, entrepreneurs are assumed to only consume when they die, which cannot be optimal for an agent that is not risk neutral. Thus, their underlying entrepreneurial preference structure is different from the one in this paper.

[5] To be precise, one must be careful with definitions of risk aversion when agents can vary labor supply in response to shocks to wealth, as is the case in the BGG model. See Swanson (2015) and Section 3.1 for more.

[6] The work of Mankiw (1986), Constantinides and Duffie (1996), Krusell and Smith (1997), Storesletten et al. (2007), Schmidt (2016), Constantinides and Ghosh (2017) are a few prominent examples.

cross-sectional standard deviation in quarterly labor productivity growth would have to increase by about 7.5% – fluctuations conformable with empirical findings on the cyclicality of cross-sectional heterogeneity in labor earnings growth, e.g. see Storesletten et al. (2004). In an extension with nominal rigidities, I in addition study monetary policy and capital quality shocks and find that again, the countercyclical variation in labor risk required for the BGG contract to be optimal is of realistic magnitudes.

The rest of the paper is organized as follows. Section 2 describes the environment with a representative household and entrepreneur, defines the competitive equilibrium and discusses some properties of optimal risk sharing. In Section 3, I calibrate the model and analyze optimal risk sharing for various household preferences in a real business cycle framework with TFP shocks. Section 4 extends the framework to allow for household heterogeneity and labor productivity risk and defines the corresponding competitive no-trade equilibrium. In Section 5, I calibrate the idiosyncratic risk features of this framework and analyze the implications for aggregate risk sharing and aggregate dynamics, also considering an extension with nominal rigidities. Section 6 concludes.

2 The Benchmark Representative Household Model

2.1 The Environment

For comparability with earlier work in the literature, the framework of the model environment closely follows the treatment of BGG and CFP in most parts. Time in the model is discrete and infinite. The model features two central types of agents, called households and entrepreneurs – a unit mass of each. For the purposes of Sections 2 and 3, there is no heterogeneity inside either of the two groups of agents, so I will focus on a representative household and a representative entrepreneur. There is also a unit mass of "firms" indexed by $j \in [0, 1]$, new capital producers, a representative financial intermediary and a representative final goods producer, all discussed below.

The representative household is infinitely-lived, has time discount factor β and labor-augmented Epstein-Zin preferences. It consumes the final good and sells labor in competitive markets. The household saves in period t by depositing savings in a financial intermediary. These deposits yield gross real returns R^d_{t+1} in $t + 1$. The returns are not predetermined in t and are realized at $t + 1$, possibly depending on aggregate shocks.

As in BGG and CFP, the representative financial intermediary accepts deposits from households and extends loans, between t and $t + 1$, to the continuum of firms. The intermediary is effectively a pass-through entity that diversifies all idiosyncratic risk arising from lending to firms hit with individual shocks. Yet aggregate risk on each extended loan, and on the whole loan portfolio remains. As CFP, I assume that there is free-entry into the financial intermediation market and gross returns to the depositors cannot be negative. This implies that in equilibrium, the gross real returns on households' deposits, R^d_{t+1} will equal the returns on the intermediary's loan portfolio.[7]

The representative entrepreneur is also infinitely lived, with time discount factor β_e. In the spirit of BGG, it is the only agent assumed to participate in the market for direct ownership (inside equity) of firms. The entrepreneur consumes dividends paid by the firms and it is restricted from participating in any other financial markets. Otherwise it might be able to undo the financial frictions faced by the firms it owns. As assumed by CFP, the representative entrepreneur is not endowed with any labor.

[7] Alternatively, one can assume that the households own the financial intermediary and provide frictionless equity financing to arrive at identical equilibrium conditions.

Firm j maximizes its value to shareholders by investing in productive capital. Firms are assumed to be the only entities who can hold capital between periods t and $t+1$. At the end of each period, they purchase physical capital, financed by their accumulated wealth, referred to as *net worth*, and external financing provided by the financial intermediary. At the beginning of period $t+1$ each firm's capital holdings K^j_{t+1} are scaled by an idiosyncratic shock ω^j_{t+1} which is observed by the firm, but by the lender only if a monitoring cost is incurred. This idiosyncratic shock is i.i.d across time and firms and independent of any aggregate realizations, with density $f(\omega)$, cumulative distribution $F(\omega)$ and a mean of one. Let R^k_{t+1} denote the aggregate gross return to a unit of the final good invested in capital, meaning the average return in the cross-section of firms. R^k_{t+1} is perfectly observed by all agents in the economy. Then, the total return to a unit of the final good invested in firm j's capital project at time t is $\omega^j_{t+1}R^k_{t+1}$.[8] To be more precise:

$$R^k_{t+1} \equiv \frac{r_{t+1} + (1-\delta)Q_{t+1}}{Q_t} \tag{1}$$

where r_{t+1} is the rental rate on capital, δ the depreciation rate and Q_{t+1} the relative price of capital in $t+1$.

As is conventional in this line of models starting with Carlstrom and Fuerst (1997), I assume that there is enough inter-period anonymity in financial markets that only one-period contracts between the firms and the intermediary are feasible. Firms derive returns to capital from capital gains in the price of capital when selling it and renting it out to a representative final goods producer, as evident in the definition of R^k_{t+1} above. These returns are then used to cover any payments previously contracted to be made to the lender. Note that because of constant returns to scale in the final goods production function, one can equivalently assume that the firms themselves have access to the production technology, hire labor and combine it with their capital to produce output. In any case, what matters is that $\omega^k_{t+1}R^k_{t+1}$ is the return to a unit of final good invested in capital by entrepreneur j.

As in BGG, I assume that monitoring costs are a proportion μ of the realized gross payoff to a given firm's capital: $\mu\omega^j_{t+1}R^k_{t+1}Q_tK^j_{t+1}$. Also, firms have limited liability in that each individual firm's project cannot make payouts in excess of the proceeds $\omega^j_{t+1}R^k_{t+1}Q_tK^j_{t+1}$. That is, even though the firm is owned by an entrepreneur who could inject equity into firm j, equity injections or dividend payments to the owner can only be made *after* the payments with the lender have been settled. This assumption renders each individual contracting problem identical to that in BGG and CFP. The firms are assumed to liquidate all their capital and all capital must be repurchased. This assumption dates back to BGG who make it to ensure that agency problems affect the entire capital stock and not just the marginal investment. Finally, as is a common assumption in the literature to prevent the firms from growing out of their financial constraints and become self-financing in the long run, I assume that $\beta_e < \beta$.

There is a representative final goods producer who runs a Cobb-Douglas production function in aggregate labor L_t and capital K_t, producing $A_tK^\alpha_tL^{1-\alpha}_t$. It rents capital from firms for rental rate r_t, and labor from the household for wage rate W_t, both in competitive markets. A_t is a TFP shock that follows a stationary AR(1) process in logs. It is the only source of aggregate uncertainty in the baseline model and its realization is publicly observed at the beginning of period t.

The household also owns competitive new capital producers who produce new capital subject to

[8]Equivalently, to stay in line with BGG and CFP's narrative of a continuum of heterogeneous entrepreneurs doing the investing in capital subject to financial frictions, a previous version of this paper instead considered a "family" of entrepreneurs $j \in [0,1]$ each operating their own projects, yet perfectly sharing the idiosyncratic risk arising from their projects. I choose the framing of a continuum of firms held by a representative owner as it is more natural in light of the literature on heterogeneous firms and does not rely on constructs such as the "family of entrepreneurs".

adjustment costs and sell it to firms. Following CFP, they take $I_t\vartheta\left(\frac{I_t}{I_{ss}}\right)$ units of the final good and transform these into I_t investment goods, i.e. gross capital investment. ϑ is convex and I_{ss} is the steady state level of gross investment. These investment goods are sold at price Q_t. I make the standard assumptions that $\vartheta(1) = 1$, $\vartheta'(1) = 0$ and $\vartheta''(1) = \phi_Q$. This normalizes the capital price in steady state to 1 and guarantees that at steady state, the elasticity of the capital price to I_t is ϕ_Q, a key calibration target. New capital producers earn possibly non-zero profits in equilibrium, paid to households, whereas steady state profits are zero.

2.2 Equilibrium

In this section, I present the agents' problems and derive their equilibrium optimality conditions.

2.2.1 Households

The representative household maximizes lifetime utility of streams of consumption C_t and hours worked L_t:

$$\mathcal{V}_t(d_t) = \max_{C_t, L_t, d_{t+1}} \left\{ (1-\beta)u(C_t, L_t) + \beta \mathbb{E}_t \left[\mathcal{V}_{t+1}(d_{t+1})^{1-\xi}\right]^{\frac{1-\frac{1}{\psi}}{1-\xi}} \right\}^{\frac{1}{1-\frac{1}{\psi}}} \tag{2}$$

subject to the budget constraint

$$C_t + d_{t+1} \leq W_t L_t + R_t^d d_t + \Pi_t^I \tag{3}$$

where d_{t+1} denotes the household's choice of deposits saved in the intermediary, and Π_t^I are profits of new capital producers. Although I consider recursive equilibria, for brevity of notation, I assume that the aggregate state is encompassed by allowing for an aggregate state contingent value function $\mathcal{V}_t$.

Following Uhlig (2010), I assume $u(C, L) = [C\Phi(L)]^{1-\frac{1}{\psi}}$, where Φ is positive, thrice differentiable, decreasing and concave. These preferences are consistent with long run growth and give flexibility in calibrating the elasticity of labor supply. Given this, the household's first order necessary conditions for labor supply and deposits are then:

$$C_t \left[-\frac{\Phi'(L_t)}{\Phi(L_t)}\right] = W_t \tag{4}$$

$$1 = \mathbb{E}_t \left[M_{t+1} R_{t+1}^d\right] \tag{5}$$

$$\text{with } M_{t+1} \equiv \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}} \left(\frac{\mathcal{V}_{t+1}}{\mathbb{E}_t\left[\mathcal{V}_{t+1}^{1-\xi}\right]^{\frac{1}{1-\xi}}}\right)^{\frac{1}{\psi}-\xi} \left(\frac{\Phi(L_{t+1})}{\Phi(L_t)}\right)^{1-\frac{1}{\psi}} \tag{6}$$

2.2.2 Entrepreneurs

The representative entrepreneur maximizes its lifetime utility over streams of consumption C_t^e:

$$\mathcal{V}_t^e\left(\left\{n_t^j\right\}_{j\in[0,1]}\right) = \max_{C_t^e, \{n_{t+1}^j\}_j} \left\{\tilde{U}(C_t^e) + \beta_e \mathbb{E}_t\left[V_{t+1}^e\left(\left\{n_{t+1}^j\right\}_j\right)\right]\right\}$$

subject to the budget constraint

$$C_t^e + \int_0^1 q_t^j n_{t+1}^j dj \leq \int_0^1 \left(q_t^j + div_t^j\right) n_t^j dj$$

where n^j_{t+1} denotes the share of firm j's net worth acquired by the entrepreneur at the end of period t, div^j_t are the dividends paid by firm j in t, and q^j_t is the time t ex-dividend price of firm j's net worth. $\tilde{U}(C)$ is a standard CRRA momentary utility function, set to the specific case of log-utility in what is to follow

The entrepreneur's first order necessary conditions which price the firms' equity are:

$$q^j_t = \mathbb{E}_t\left[M^e_{t+1}(q^j_{t+1} + div^j_{t+1})\right], \quad j \in [0,1]$$
$$\text{with } M^e_{t+1} \equiv \beta_e \frac{\tilde{U}'(C^e_{t+1})}{\tilde{U}'(C^e_t)}$$

with equality in equilibrium because the firms must be held by the entrepreneur. The key take-away is that, in equilibrium, the firms will thus use the entrepreneur's stochastic discount factor M^e_t when maximizing their value and discounting future dividend streams. Also, market clearing for firms' shares requires $n^j_t = 1, \forall t, j$, verifying that the only source of the entrepreneur's consumption are dividends paid by the firms

$$C^e_t = \int_0^1 div^j_t dj$$

2.2.3 Final Goods and New Capital Producers

The representative final goods producer's optimization yields the demand for labor and capital:

$$W_t = (1-\alpha)A_t K^\alpha_t L^{-\alpha}_t \quad (7)$$
$$r_t = \alpha A_t K^{\alpha-1}_t L^{1-\alpha}_t \quad (8)$$

New capital producers' profits are given by:

$$\Pi^I_t = Q_t I_t - I_t \vartheta\left(\frac{I_t}{I_{ss}}\right) \quad (9)$$

Their optimization with respect to I_t yields that the equilibrium capital price follows:

$$Q_t = \vartheta\left(\frac{I_t}{I_{ss}}\right) + \frac{I_t}{I_{ss}}\vartheta'\left(\frac{I_t}{I_{ss}}\right) \quad (10)$$

The law of motion for aggregate capital is:

$$K_{t+1} = I_t + (1-\delta)K_t \quad (11)$$

2.2.4 Firms and the Loan Contract

Let us denote firm j's accumulated internal wealth after paying the lender yet before paying dividends in t by E^j_t, for equity. And let the firm net worth N^j_t be the firm's internal wealth after paying dividends. This net worth is accumulated by purchasing capital K^j_t in $t-1$, earning rental returns and capital gains on $\omega^j_t K^j_t$, paying back the contracted upon payment to the lender in t, and paying dividends to the owner.

Because of the imperfect obsevability of firm j's idiosyncratic capital shock ω^j_{t+1}, the costly state verification problem arises. Firm j's investment of K^j_{t+1} units of capital yields $\omega^j_{t+1}K^j_{t+1}$ units in $t+1$ which generates an income flow of $\omega^j_{t+1}\left[r_{t+1} + (1-\delta)Q_{t+1}\right]K^j_{t+1} = \omega^j_{t+1}R^k_{t+1}Q_t K^j_{t+1}$. Following Townsend

(1979) and Williamson (1986), one can show that if payoffs are linear in the project outcome $\omega_{t+1}^{j}K_{t+1}^{j}$, and there is no random monitoring, the optimal contract is risky debt.[9] Since idiosyncratic firm risk is fully diversified in the financial intermediary's portfolio, this is true on the lender's side. As for firm j, below it will be clear that if risky debt is the optimal contract, then the firm's value function is linear, closing the logical circle, analogously as demonstrated by CFP.

By risky debt we mean that monitoring only occurs for low realizations of ω_{t+1}^{j}. More specifically, in the absence of aggregate uncertainty, i.e. when r_{t+1} and Q_{t+1} are known at the time of signing the contract, the borrower and lender agree on a cutoff $\bar{\omega}_{t+1}^{j}$ and an implied promised repayment to the lender: $\bar{\omega}_{t+1}^{j}R_{t+1}^{k}Q_{t}K_{t+1}^{j}$. If $\omega_{t+1}^{j} < \bar{\omega}_{t+1}^{j}$, the borrower does not have sufficient funds to pay the lender. He declares bankruptcy, the lender incurs the monitoring cost and gets all of the remaining funds, which yields him an income flow of $(1-\mu)\omega_{t+1}^{j}R_{t+1}^{k}Q_{t}K_{t+1}^{j}$. If $\omega_{t+1}^{j} \geq \bar{\omega}_{t+1}^{j}$, no monitoring occurs, the borrower repays the promised amount $\bar{\omega}_{t+1}^{j}R_{t+1}^{k}Q_{t+1}K_{t+1}^{j}$ and holds on to the remaining income flow of $(\omega_{t+1}^{j}-\bar{\omega}_{t+1}^{j})R_{t+1}^{k}Q_{t}K_{t+1}^{j}$. Note that $\bar{\omega}_{t+1}^{j}$ implicitly determines an interest rate $R_{t+1}^{def,j}$ earned by the lender that is subject to default risk, defined by: $R_{t+1}^{def,j}(Q_{t}K_{t+1}^{j}-N_{t}^{j}) = \bar{\omega}_{t+1}^{j}R_{t+1}^{k}Q_{t}K_{t+1}^{j}$.

In the presence of aggregate uncertainty, however, the optimal contract involves the lender and borrower agreeing upon a schedule of $\{\bar{\omega}_{t+1}^{j}\}$, with a specific value of the cutoff for each possible realization of the aggregate state. Conditional on having observed aggregate outcomes and thus knowing the implied $\bar{\omega}_{t+1}^{j}$, the optimality of risky debt, now for each realization of the aggregate state, remains. The CSV problem takes as exogenous the aggregate returns on capital and the opportunity cost of the lender.

Let $\Gamma(\bar{\omega})$ denote the expected gross share of the returns to firm j's held capital going to the lender. And let $\mu G(\bar{\omega})$ be the expected monitoring costs:

$$\begin{aligned}
\Gamma(\bar{\omega}) &\equiv \int_{0}^{\bar{\omega}} \omega f(\omega)d\omega + \bar{\omega}\int_{\bar{\omega}}^{\infty} f(\omega)d\omega = \int_{0}^{\bar{\omega}} \omega f(\omega)d\omega + \bar{\omega}[1-F(\bar{\omega})] \\
\mu G(\bar{\omega}) &\equiv \mu\int_{0}^{\bar{\omega}} \omega f(\omega)d\omega
\end{aligned}$$

Noting that

$$\begin{aligned}
\Gamma'(\bar{\omega}) &= 1-F(\bar{\omega}) > 0 \\
\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega}) &= [1-F(\bar{\omega})][1-\mu\bar{\omega}h(\bar{\omega})] > 0 \text{ if } \bar{\omega} < \bar{\omega}^{*}
\end{aligned}$$

we have that the firm's expected net share $[1-\Gamma(\bar{\omega})]$ is decreasing in $\bar{\omega}$ and that of the lender, $[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]$ increasing.[10]

Let us define and denote firm j's leverage attained in period t, going into period $t+1$ as: $\kappa_{t}^{j} \equiv \frac{Q_{t}K_{t+1}^{j}}{N_{t}^{j}}$. Then, integrating out the realization of ω_{t+1}^{j}, conditional on the aggregate realizations of (r_{t+1}, Q_{t+1}), we can write the expected $t+1$ equity of the firm and the lender's return $R_{t+1}^{l,j}$ conditional on aggregates

[9] The proof is exactly as for the conventional CSV problem without aggregate uncertainty, only applied for each realization of the aggregate state separately.

[10] In the above, $h(\omega) \equiv f(\omega)/[1-F(\omega)]$ is the hazard rate and $\bar{\omega}^{*}$ is the cutoff value at which the lender's net share is maximized. Assuming that $\frac{\partial[\omega h(\omega)]}{\partial\omega} > 0$ and $\lim_{\omega\to+\infty} \omega h(\omega) > \frac{1}{\mu}$, as will be satisfied by the log-normal distribution employed in the computations, there exists a unique such $\bar{\omega}^{*}$. At the optimum, it cannot be the case that for any realization of aggregate shocks $\bar{\omega}^{j} > \bar{\omega}^{*}$. Because then, $\bar{\omega}^{j}$ can be reduced, the borrower made better off and the participation constraint slackened. In the calibration and simulations employed, $\bar{\omega}_{t}$ will be significantly below $\bar{\omega}^{*}$.

as:

$$
\begin{aligned}
E^j_{t+1} &\equiv [1-\Gamma(\bar{\omega}^j_{t+1})]R^k_{t+1}Q_tK^j_{t+1} = [1-\Gamma(\bar{\omega}^j_{t+1})]R^k_{t+1}\kappa^j_t N^j_t \\
R^{l,j}_{t+1} &\equiv \frac{[\Gamma(\bar{\omega}^j_{t+1})-\mu G(\bar{\omega}^j_{t+1})]R^k_{t+1}Q_tK^j_{t+1}}{Q_tK^j_{t+1}-N^j_t} = [\Gamma(\bar{\omega}^j_{t+1})-\mu G(\bar{\omega}^j_{t+1})]R^k_{t+1}\frac{\kappa^j_t}{\kappa^j_t-1}
\end{aligned}
$$

And the relation between N^j_t and E^j_t is naturally:

$$N^j_t = E^j_t - div^j_t$$

Since all firms are identical, apart from their equity, the relevant idiosyncratic state variable for firm j will just be E^j_t. Let us denote the value of a firm with period t equity E^j_t, before paying dividends by $\tilde{V}_t(E^j_t)$. Given equity, the contracting problem is to choose K^j_{t+1} and the schedule $\left\{\bar{\omega}^j_{t+1}\right\}$ subject to the lender's participation constraint. Or equivalently, one can choose κ^j_t and $\left\{\bar{\omega}^j_{t+1}\right\}$. Because firm j cannot raise external financing without any net worth, dividends necessarily cannot exceed equity $div^j_t \le E^j_t$, and to continue operating a capital project, the inequality must be strict. $div^j_t < 0$ is understood as equity injections by the owner into the firm. Firm j's value function will thus satisfy the Bellman equation:

$$\tilde{V}_t(E^j_t) = \max_{\{\bar{\omega}^j_{t+1}\},\kappa^j_t,div^j_t} \left\{div^j_t + \mathbb{E}_t\left[M^e_{t+1}\tilde{V}_{t+1}(E^j_{t+1})\right]\right\} \tag{12}$$

$$\text{s.t. } \mathbb{E}_t\left[M_{t+1}R^{l,j}_{t+1}\right] = \mathbb{E}_t\left\{M_{t+1}[\Gamma(\bar{\omega}^j_{t+1})-\mu G(\bar{\omega}^j_{t+1})]R^k_{t+1}\frac{\kappa^j_t}{\kappa^j_t-1}\right\} \ge \mathbb{E}_t[M_{t+1}R^l_{t+1}] = 1$$

$$E^j_{t+1} = \max\{\omega^j_{t+1}-\bar{\omega}^j_{t+1},0\}R^k_{t+1}\kappa^k_t\left(E^j_t - div^j_t\right), \quad div^j_t \le E^j_t$$

As mentioned in section 2.2.4, in equilibrium the firms apply the entrepreneur's stochastic discount factor M^e_{t+1}. The lender's participation constraint arises as the result of the intermediary being a pass-through entity, combining with the facts that in equilibrium all contracts will offer the same expected return to the lender $R^l_t = R^{l,j}_t, \forall j$ and as elaborated above $R^l_t = R^d_t$ in equilibrium, and finally employing the household's Euler equation.

As is commonly done in computational models of firm heterogeneity, for example by Khan and Thomas (2008), one can also redefine the firm's value measured in units of the entrepreneur's marginal utility, taken as given by firm j, as $V_t(E^j_t) \equiv U'(C^e_t)\tilde{V}_t(E^j_t)$, and rewrite (13) as:

$$V_t(E^j_t) = \max_{\{\bar{\omega}^j_{t+1}\},\kappa^j_t,div^j_t} \left\{\tilde{U}'\left(C^e_t\right)div^j_t + \beta_e\mathbb{E}_t\left[V_{t+1}(E^j_{t+1})\right]\right\} \tag{13}$$

This redefinition of the value function is not directly useful here for solving the firm's equivalent problem, but defining $V_t(E^j_t)$ in such a way makes it easy to point out the close similarities between this setup and that used by CFP later on (see Section 2.3.1 and Appendix B).

We can guess that the continuation value function is linear, i.e. $V_{t+1}(E^j_{t+1}) = V_{t+1}E^j_{t+1}$, where, with an abuse of notation, V_{t+1} is now understood to be a *variable* that measures the marginal valuation of an additional unit of equity to the firm. Plugging in the law of motion for E^j_{t+1} and applying the law of iterated expectations to integrate out the realization of ω_{t+1}^j, the Bellman equation becomes:

$$V_t(E_t^j) = \max_{div_t^j \leq E_t^j} \left\{ div_t^j \left(\tilde{U}'(C_t^e) - \beta_e \mathbb{E}_t \left[V_{t+1}[1 - \Gamma(\bar{\omega}_{t+1}^j)] R_{t+1}^k \right] \kappa_t^j \right) \right\} +$$
$$+ E_t^j \times \max_{\{\bar{\omega}_{t+1}^j\}, \kappa_t^j} \left\{ \beta_e \mathbb{E}_t \left[V_{t+1}[1 - \Gamma(\bar{\omega}_{t+1}^j)] R_{t+1}^k \right] \kappa_t^j \right\}$$
$$\text{s.t. } \mathbb{E}_t \left\{ M_{t+1}[\Gamma(\bar{\omega}_{t+1}^j) - \mu G(\bar{\omega}_{t+1}^j)] R_{t+1}^k \right\} \kappa_t^j \geq \kappa_t^j - 1$$

In equilibrium, the constraint $div_t^j \leq E_t^j$ could not be binding as, by linearity, it would have to be binding for all firms $j \in [0, 1]$, implying no net worth were to be left for the firms and no capital K_{t+1} could be acquired.[11] The individual div_t^j are thus not pinned down, and in equilibrium it must be the case that:

$$\tilde{U}'(C_t^e) = \beta_e \mathbb{E}_t \left[V_{t+1}[1 - \Gamma(\bar{\omega}_{t+1})] R_{t+1}^k \right] \kappa_t^j$$

Since the participation constraint was already initially written independently of E_t^j, the above clearly verifies the guess that the firm's value function $V_t(E_t^j)$ is linear in equity and the problem of choosing κ_t^j and $\{\bar{\omega}_{t+1}^j\}$ is independent of firm j's equity. Thus, given that the optimal choices of κ_t^j, $\{\bar{\omega}_{t+1}^j\}$ are unique, which can be proved rigorously, each firm chooses the same leverage ratio κ_t and cutoff schedule $\{\bar{\omega}_{t+1}\}$. This implies that the Bellman equation can be written as

$$V_t = \max_{\{\bar{\omega}_{t+1}\}, \kappa_t} \left\{ \beta_e \mathbb{E}_t \left[V_{t+1}[1 - \Gamma(\bar{\omega}_{t+1})] R_{t+1}^k \right] \kappa_t \right\} \tag{14}$$
$$\text{s.t. } \mathbb{E}_t \left\{ M_{t+1}[\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] R_{t+1}^k \right\} \kappa_t \geq \kappa_t - 1 \tag{15}$$

further implying that $\tilde{U}'(C_t^e) = V_t$. Given that all firms choose the same κ_t and $\{\bar{\omega}_{t+1}\}$, the distribution of internal wealth across the firms does not matter for aggregates and we need to only track the aggregate level of firms' internal wealth. And although the distribution of dividend payments is not pinned down in equilibrium, we have established in Section 2.2.2 that it must necessarily be the case that $C_t^e = \int_0^1 div_t^j dj$, with C_t^e satisfying the Euler equation:

$$\tilde{U}'(C_t^e) = \beta_e \mathbb{E}_t \left[\tilde{U}'(C_{t+1}^e)[1 - \Gamma(\bar{\omega}_{t+1})] R_{t+1}^k \right] \kappa_t \tag{16}$$

And the aggregate net worth of firms then evolves as

$$N_t = [1 - \Gamma(\bar{\omega}_t)] R_t^k \kappa_{t-1} N_{t-1} - C_t^e \tag{17}$$
$$\text{where } \kappa_t \equiv \frac{Q_t K_{t+1}}{N_t} \tag{18}$$

Note that the leverage rate κ_t is simultaneously the inverse of the firms' (and thus the entrepreneur's) share of financial wealth in the economy. Because each firm needs a positive amount of net worth to operate its project, I assume that the entrepreneur provides transfers from other firms to any firms who default and must pay out all returns to the lender. These transfers are inconsequential as the distribution of wealth across firms is irrelevant for the aggregates.

Taking the first order conditions to the contracting problem of maximizing the firm's continuation value in (14) subject to (15) with respect to κ_t and $\{\bar{\omega}_{t+1}\}$, and using these in (14) to yield that the

[11] To be precise, one can first establish that the value function is *affine*, and given an affine value function, it must be the case that in equilibrium $\tilde{U}'(C_t^e) = \beta_e \mathbb{E}_t \left[V_{t+1}[1 - \Gamma(\bar{\omega}_{t+1})] R_{t+1}^k \right] \kappa_t$, yielding *linearity*.

Lagrange multiplier on (15) equals V_t, one can summarize the resulting optimality condition as:

$$\frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})} = \left(\beta_e \frac{\tilde{U}'(C^e_{t+1})}{\tilde{U}'(C^e_t)}\right)^{-1} M_{t+1} \quad \left(= \frac{M_{t+1}}{M^e_{t+1}}\right) \tag{19}$$

which holds state-by-state, for each realization of aggregate uncertainty in $t+1$. Of course, in the set of equilibrium conditions that determine period t realizations, this condition shows up with time indices lagged by one period compared to (19), in order to pin down the current $\bar{\omega}_t$.

The equilibrium lender return is:

$$R^l_{t+1} = [\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})] R^k_{t+1} \frac{\kappa_t}{\kappa_t - 1} \tag{20}$$

which is the most natural variable to capture the degree of aggregate risk sharing. BGG imposed that R^l_{t+1} is predetermined in t and thus constant across realizations of aggregate uncertainty, whereas CFP showed that under optimal contracting, it comoves significantly with R^k_{t+1}.

A thorough analysis of the properties of the privately optimal contract and its implications in the standard BGG framework is presented by Carlstrom et al. (2016), with all the insights extending to the set up presented above. To reiterate, the key optimality condition governing aggregate risk sharing is (19). Given the assumptions in Footnote 10, one can show that the left hand side is strictly increasing in $\bar{\omega}_{t+1}$. Therefore, naturally, whenever the household values wealth relatively more, meaning M_{t+1} is high, all else equal, also $\bar{\omega}_{t+1}$, and thus the lender's net share $[\Gamma(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})]$ and the lender's return R^l_{t+1} is high, to provide consumption insurance to the households. Conversely, when the value of firms' internal net worth captured by $\tilde{U}'(C^e_{t+1}) = V_{t+1}$ is high, the contract calls for a lower $\bar{\omega}_{t+1}$ allowing the borrowers to hold on to more net worth, all else equal. Or alternatively, one can just see this as the outcome of optimal risk sharing between the household and the entrepreneur, aiming to equalize marginal rates of substitution M_{t+1} and M^e_{t+1}, subject to the altered marginal costs of redistribution incurred due to bankruptcy costs.

2.2.5 Market Clearing and Equilibrium Definition

In equilibrium the household's deposits fund the firms' projects:

$$d_{t+1} = Q_t K_{t+1} - N_t \tag{21}$$

Combining this condition, the households' and entrepreneurs' budget constraints, the definition of leverage and the rental and labor market equilibrium conditions with new capital producers' profits, one arrives at the aggregate resource constraint:

$$C_t + C^e_t + I_t \vartheta\left(\frac{I_t}{I_{ss}}\right) + \mu G(\bar{\omega}_t) R^k_t Q_{t-1} K_t = A_t K^\alpha_t L^{1-\alpha}_t \tag{22}$$

Given that several of the equilibrium conditions were imposed in the derivations above, we can define a competitive equilibrium of the model presented as follows.

Definition 1. *A* competitive equilibrium of the representative agent model *is a collection of stochastic processes for:*

1. *a price system* $\{r_t, W_t, R^k_t, R^l_t, Q_t\}$,

2. *household's consumption, stochastic discount factor and value function realization* $\{C_t, M_t, \mathcal{V}_t\}$

3. *entrepreneurial consumption, net worth and leverage quantities and contractual cutoffs* $\{C_t^e, N_t, \kappa_t, \bar{\omega}_t\}$

4. *aggregate labor, investment and capital quantities* $\{L_t, I_t, K_{t+1}\}$

such that equations: (1), (2), (4)–(8), (10), (11), (16)–(20), (22), *with* $R_t^d = R_t^l$, *where applicable, are satisfied, given the stochastic process for* $\{A_t\}$, *and initial conditions* (K_0, E_0, A_0).

2.3 Results on Privately Optimal Risk Sharing

2.3.1 Equivalence with CFP Model

In this section I will argue that the setup with a representative entrepreneur with logarithmic utility who owns firms that are subject to idiosyncratic shocks is effectively equivalent, to a first order approximation, to the standard approach used by BGG and CFP. Further details and the precise entrepreneurs' problem in the CFP model are presented in Appendix B.

In the benchmark setup employed by BGG and CFP, a unit mass of entrepreneurs are assumed to invest in capital subject to the CSV problem directly, have linear utility from consumption and a time discount factor identical to that of the households. To be precise, let us denote this time discount factor as β_e^{CFP}. Since the entrepreneurs are financially constrained, it is optimal for them to postpone consumption indefinitely. To keep entrepreneurs saving themselves out of financial constraints, it is assumed that each faces a constant probability $1-\gamma$ of dying each period. The dying entrepreneurs are replaced by an equal of mass entering ones who get a transfer from the survivors to start operations. Entrepreneurs only consume when they die. This means that in each period, a fraction $1-\gamma$ is consumed and the remaining fraction of entrepreneurial equity is invested. Using the same notation as above:

$$C_t^e = (1-\gamma)[1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1}$$
$$N_t = \gamma[1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1}$$

And even though the entrepreneurs have linear utility, their marginal valuation of an extra unit of equity is stochastic because they face time-varying investment opportunities:

$$V_t = (1-\gamma) + \gamma\beta_e^{CFP}\mathbb{E}_t\left\{V_{t+1}[1-\Gamma(\bar{\omega}_{t+1})]R_{t+1}^k\right\}\kappa_t \tag{23}$$

with V_t the marginal valuation of a unit of equity at the beginning of t before the death shock (and consumption) is realized. The participation constraint in the contracting problem is identical across the two models. And the optimality condition for risk sharing through $\bar{\omega}_{t+1}$ in the CFP model is:

$$\frac{\Gamma'(\bar{\omega}_{t+1})}{\Gamma'(\bar{\omega}_{t+1}) - \mu G(\bar{\omega}_{t+1})} = \left(\gamma\beta_e^{CFP}\frac{V_{t+1}}{V_t-(1-\gamma)}\right)^{-1} M_{t+1} \tag{24}$$

Going back to the model set up in Section 2.1, if the entrepreneurs have logarithmic utility $\tilde{U}(C) = \log(C)$, then guessing that consumption is a constant fraction of equity, and combining (16) and (17) yields the standard optimal consumption result of an agent that has only financial wealth:

$$C_t^e = (1-\beta_e)[1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1}$$
$$N_t = \beta_e[1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1}$$

So we see right away that if $\beta_e = \gamma$, then the two models imply identical entrepreneurial consumption and net worth accumulation, conditional on all other equilibrium variables.

To fully establish identical dynamics for these models, it remains to be shown that (23) and (24) imply the same risk sharing behavior as (16) and (19) do. Appendix B establishes this to a first order approximation as $\beta_e^{CFP} \to 1$.[12] To see why this might be the case, notice the similarities between these pairs of equilibrium conditions. In both cases, the marginal valuation of an extra unit of wealth V_t, which also equals $U'(C_t^e)$ in my setup, must satisfy an Euler equation which determines how the log-deviations of V_t from steady state are related across time. And the similarities between (19) and (24) are evident. When log-linearized, the only difference is the appearance of β_e^{CFP} in both conditions for the CFP model.

2.3.2 The Relevance of Human and Financial Wealth Dynamics

An important determinant of aggregate financial risk sharing in the economy is the behavior of human and financial wealth dynamics. To make the analysis of this idea clear, let us consider the household utility specification $\xi = \psi = 1$, i.e. log-utility from consumption, and similarly log-utility for the entrepreneur. Also, given that in a first order approximation certainty equivalence applies, let us consider how the economy behaves after a TFP shock has been realized and no future shocks are expected. Under such a household utility specification, we have that the household consumes a constant fraction $(1-\beta)$ of its total wealth:[13]

$$C_t = (1-\beta)\left(R_t^l d_t + \mathcal{H}_t\right) = (1-\beta)\left\{\left[\Gamma(\bar{\omega}_t) - \mu G(\bar{\omega}_t)\right] R_t^k Q_{t-1} K_t + \mathcal{H}_t\right\} =$$
$$= (1-\beta)\left\{\left[\Gamma(\bar{\omega}_t) - \mu G(\bar{\omega}_t)\right] \mathcal{F}_t + \mathcal{H}_t\right\}$$
$$\text{where } \mathcal{F}_t \equiv [r_t + (1-\delta)Q_t]K_t$$
$$\mathcal{H}_t \equiv \sum_{j=0}^{\infty} \frac{1}{\mathcal{R}_{t,t+j}^l} W_{t+j} L_{t+j} \text{ with } \mathcal{R}_{t,t+j}^l \equiv \prod_{s=1}^{j} R_{t+s}^l, \text{ and } \mathcal{R}_{t,t}^l = 1$$

with $\mathcal{H}_t$ and $\mathcal{F}_t$ standing for the human and financial wealth in the economy, respectively. Following Section 2.3.1, the entrepreneur consumes fraction $(1-\beta_e)$ of its total (financial) wealth:

$$C_t^e = (1-\beta_e)[1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1} = (1-\beta_e)[1-\Gamma(\bar{\omega}_t)]\mathcal{F}_t$$

Suppose that the economy is shocked in period t, while previously having been in steady state. Using these optimal consumption policies in the privately optimal risk sharing condition (19) under log-utility, we have:

$$\frac{\Gamma'(\bar{\omega}_t)}{\Gamma'(\bar{\omega}_t) - \mu G'(\bar{\omega}_t)} = \frac{[1-\Gamma(\bar{\omega}_t)]\mathcal{F}_t}{\left[\Gamma(\bar{\omega}_t) - \mu G(\bar{\omega}_t)\right]\mathcal{F}_t + \mathcal{H}_t} \times \frac{1-\beta_e}{1-\beta}\frac{\beta}{\beta_e}\frac{C_{ss}}{C_{ss}^e}$$

[12] The statement must be made in a limiting sense because if $\beta_e^{CFP} = 1$, then V_t is not finite in the CFP model.

[13] To be precise, the representative household's financial wealth also contains the value of ownership of new capital producers who make profits outside of steady state. However, the profits are zero in steady state and one can verify quantitatively that the magnitude of their fluctuations is very small. For example, in response to a 1% TFP shock in the baseline calibration of this model the profits increase by an amount that is 0.07% of steady state aggregate output. Thus, for clarity, I am currently abstracting from the new capital producers' value in the household's financial wealth.

Therefore, given that the left hand side is increasing and the right hand side decreasing in $\bar{\omega}_t$, this establishes a negative relationship between $\mathcal{H}_t/\mathcal{F}_t$ and $\bar{\omega}_t$. That is, whenever the human wealth in the economy increases more than the financial wealth, the gains accrue to the household and it is thus optimal to leave a larger share of financial weatlh, implied by a lower $\bar{\omega}_t$, to the firms. For example, if $\mathcal{H}_t/\mathcal{F}_t = \mathcal{H}_{ss}/\mathcal{F}_{ss}$, then $\bar{\omega}_t = \bar{\omega}_{ss}$ and the aggregate financial risk is shared perfectly, meaning that R^l_t responds to the shock by the same relative amount as R^k_t.

Of course, $\mathcal{H}_t$ and $\mathcal{F}_t$ are themselves equilibrium objects, dependent on $\bar{\omega}_t$ itself, but this note emphasizes that it is important to keep in mind that shocks which affect human and financial wealth differently, could have markedly different implications for how the aggregate financial risk is to be shared. This motivates the discussion of the importance of TFP shock persistence in Section 3.1 and the model's extension to uninsurable household labor productivity risk in Section 4, which aims to detach a single household's consumption, at least partly, from the total human capital in the economy.

3 Quantitative Analysis of the Representative Household Model

3.1 Calibration

In the calibration of model parameters I pursue targets from earlier literature, following BGG and CFP wherever possible for comparability. One time period t is considered to be a quarter. As CFP, I set the capital share in production to be $\alpha = 0.35$, capital price elasticity with respect to investment $\phi_Q = 0.5$ and the depreciation rate $\delta = 0.025$. TFP follows $\log A_t = \rho_A \log A_{t-1} + \varepsilon^A_t$ with $\rho_A = 0.95$, as used by CFP as a benchmark, varied below. ε^A_t is i.i.d mean-zero. For computing impulse responses, the other properties of ε^A_t are irrelevant. For simulations, I assume that ε^A_t follows a normal distribution with standard deviation 0.0072, following King and Rebelo (1999).

As is common since Carlstrom and Fuerst (1997), the idiosyncratic entrepreneurial capital shock is log-normal: $\log \omega \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$. Following the discussion in Section 2.3.1 and the targets set by CFP, the parameters (μ, β_e, σ) pertaining to the entrepreneurial financial frictions are pinned down, jointly with all other parameters, to yield in steady state: (i) a spread of 200 basis points (annualized) between the lender return R^{def}_{ss} subject to default risk and the riskless lender return R^l_{ss}, both as defined in Section 2.2.4, (ii) a quarterly bankruptcy rate $F(\bar{\omega}_{ss})$ of 0.75%, (iii) a leverage ratio of $\kappa_{ss} = 2$. Exactly as in CFP, this results in $(\mu, \beta_e, \sigma) = (0.63, 0.94, 0.28)$. Based on the discussion in Section 2.3.1, I set $\tilde{U}(C) = \log(C)$.

Following CFP, I set β to 0.99. As stated above, momentary household utility has the form $u(C, L) = [C\Phi(L)]^{1-\frac{1}{\psi}}$. Like BGG and CFP, I solve the model by log-linearization around the deterministic steady state. Because of this, the properties of the function Φ only affect the equilibrium conditions through $\Phi(L_{ss})$, $\Phi'(L_{ss})$ and $\Phi''(L_{ss})$. More specifically, one needs to determine $\nu_l \equiv -\frac{\Phi'(L_{ss})L_{ss}}{\Phi(L_{ss})} > 0$ which captures the effect of L_t on M_t and $\mathcal{V}_t$, and $-\frac{\Phi'(L_{ss})}{\Phi(L_{ss})}$, which pins down L_{ss}. And finally, the determination of labor supply elasticity can be seen by log-linearizing (4) to get:

$$c_t + \underbrace{\left[\frac{\Phi''(L_{ss})L_{ss}}{\Phi'(L_{ss})} + \nu_l\right]}_{\equiv 1/\eta_l} l_t = w_t$$

with lowercase letters denoting the corresponding log-deviations from steady state values. By setting $\eta_l = 3$, one can exactly replicate the labor supply condition as used by CFP. The value of ν_l is determined independently of Φ in steady state. To see this, rewrite the labor market equilibrium condition in steady state as:

$$-\frac{\Phi'(L_{ss})}{\Phi(L_{ss})}L_{ss} = (1-\alpha)\left(\frac{K_{ss}}{L_{ss}}\right)^{\alpha}\frac{L_{ss}}{C_{ss}}$$

Similarly as in a conventional RBC model, $\frac{K_{ss}}{L_{ss}}$ and $\frac{C_{ss}}{L_{ss}}$ are pinned down by the remaining system of equilibrium conditions. This results in $\nu_l \approx 0.958$, close to the recommendation by Uhlig (2010) made based on the fact that $(1-\alpha)\left(\frac{K_{ss}}{L_{ss}}\right)^{\alpha}\frac{L_{ss}}{C_{ss}} = (1-\alpha)\frac{Y_{ss}}{C_{ss}} \approx 1$, with $Y_{ss}/C_{ss} \approx 3/2$. I then choose $-\frac{\Phi'(L_{ss})}{\Phi(L_{ss})}$ to normalize L_{ss} to 1.

As for the parameters ψ and ξ governing the household's intertemporal elasticity of substitution (IES) and risk aversion, respectively, I consider various values below. In the benchmark case of $\psi = \xi = 1$, the model's first order approximation matches the log-utility specification used by CFP. As discussed in detail by Swanson (2015), the ability of households to adjust their labor supply in response to shocks affects their attitude towards risk and thus measures of relative risk aversion, as defined by Arrow (1965) and Pratt (1964), do not exactly equal ξ in this case. Nontheless, ξ equals the coefficient of relative risk aversion when labor were to be held exogenously fixed. And ξ is larger than the *consumption-wealth coefficient of relative risk aversion* with adjusting labor, as defined and shown by Swanson (2015) to be the most adequate measure in explaining equity premia in an RBC model. This means that allowing for a variable labor margin tilts the outcomes *against* less risk sharing between the household and the entrepreneurs as the household's effective risk aversion is less than ξ.

Finally, note that unlike most applications in which the introduction of Epstein-Zin utility with ξ differing from $1/\psi$ has no effect in a first order approximation solution, it does here. Because of certainty equivalence imposed by the solution method and the fact that in standard DSGE model equilibrium conditions in period t, the stochastic discount factor M_{t+j} shows up inside expectation terms with $j > 0$, the term $\left(\frac{\mathcal{V}_{t+1}}{\mathbb{E}_t[\mathcal{V}_{t+1}^{1-\xi}]^{\frac{1}{1-\xi}}}\right)^{\frac{1}{\psi}-\xi}$ effectively equals 1 everywhere *ex ante*. However in the current case, the agents' stochastic discount factors explicitly appear *ex post*, as M_t in t, in condition (19) to determine how any realized risk is shared. And the implied realization of $\bar{\omega}_t$ then has direct first order effects on the agents' wealth distribution and equilibrium dynamics.

3.2 Second Moments and Impulse Responses

As the analysis of the representative household model focuses on how changes in the household's preferences affect privately optimal aggregate risk sharing in this baseline economy, I concentrate the quantitative analysis on presenting standard deviations and impulse response functions of key balance sheet and real variables for various preference calibrations.

Table 1 below presents results from different combinations of ψ and ξ. The benchmark is $\psi = \xi = 1$, corresponding to the infinitely patient entrepreneur limit of CFP's model. Increasing either ξ or decreasing ψ makes the household less willing to take on aggregate financial risk generated by TFP shocks. Equilibrium increases in household consumption after positive productivity shocks generate larger drops in its stochastic discount factor due to decreased elasticity of intertemporal substitution and increased risk aversion. I consider two main exercises. Firstly, keeping $\psi = 1$, I determine the ξ necessary to yield an optimal contract that implies a non-state-contingent lender's return. Secondly, I do the same while setting $\psi = 1/\xi$, i.e. employ expected utility preferences with momentary utility $\frac{[C\Phi(L)]^{1-\xi}}{1-\xi}$.

The first column of output in Table 1, denoted as $\left(\frac{\partial r_t^l}{\partial \varepsilon_t^A}\right) / \left(\frac{\partial r_t^k}{\partial \varepsilon_t^A}\right)$, refers to the TFP shock responsiveness of the net lender return, $\log(R_t^l)$, relative to that on the borrowers' assets, $\log(R_t^k)$ at impact.

For brevity, I use this as the measure of the degree of financial risk sharing. If this entry is 1.0, then there is perfect aggregate financial risk sharing between the households and entrepreneurs. This is meant in the sense that in response to an unexpected innovation in TFP at t, $\bar{\omega}_t$ and the households' share in the capital project returns do not respond, making the return on households' financial assets move one for one with that on the entrepreneurs'. If this entry is 0.0, then $\bar{\omega}_t$ responds to eliminate any effects of R_t^k on R_t^l, implying a non-state-contingent lender return – the contract imposed by BGG. And if this entry happens to be negative, it is the entrepreneurs who are providing consumption insurance to the households, increasing payouts in recessions, and vice versa. The following two columns present the standard deviation of log entrepreneurial net worth and leverage relative to that of log output y. The standard deviations of the latter and log investment i are in the last two columns, respectively. I compute the second moments based on a simulation of 10^6 quarters.

Table 1: Relative impulse response of lender return, relative standard deviations of log entrepreneurial net worth and leverage, absolute standard deviation of log output and investment (in percentages), in representative household model; simulation of 10^6 quarters.

ρ_A	ξ, ψ	$\left(\frac{\partial r_t^l}{\partial \varepsilon_t^A}\right) \Big/ \left(\frac{\partial r_t^k}{\partial \varepsilon_t^A}\right)$	$\frac{\text{std}(n)}{\text{std}(y)}$	$\frac{\text{std}(\hat{\kappa})}{\text{std}(y)}$	std(y), %	std(i), %
0.95	1.0, 1.0	0.825	0.947	0.050	2.626	2.884
	13.2, 1.0	0.000	1.498	0.495	2.875	4.040
	5.92, $1/\xi$	0.000	1.597	0.424	3.403	5.473
0.99	1.0, 1.0	0.752	0.982	0.029	6.170	4.932
	4.34, 1.0	0.000	1.100	0.206	6.311	5.534
	2.57, $1/\xi$	0.000	1.099	0.156	6.493	5.701

First of all, as demonstrated by CFP, we see that with logarithmic utility and TFP shock persistence of 0.95, there is a considerable degree of aggregate risk sharing, although not exactly close to 1.0, with the measure of risk sharing at approximately 0.82. It is still enough to generate small leverage fluctuations of about 5% of that of output, and net worth volatility at the same magnitude of numeraire output – evidence of a significant dampening of the financial accelerator mechanism.

Increasing ξ to 13.2 yields a high enough aversion to risk for the household not to be willing to take on aggregate financial risk and the contract imposed by BGG becomes the optimal private contract. An unexpected increase in household consumption in response to a positive TFP shock increases the household's utility $\mathcal{V}_t$, making the stochastic discount factor M_t drop significantly. The relative volatility of entrepreneurial net worth is about 1.5 times higher and that of leverage almost 10 times higher than under logarithmic household utility. There is also slight amplification of output fluctuations, with its standard deviation increasing about 10%.

It is worth emphasizing that because the underlying framework is a simple RBC model, the current specification of the model does not contain forces that would cause entrepreneur net worth and leverage volatility to lead to significant output volatility amplification. The only two channels that lead to more volatile output due to less financial risk sharing are the wealth effect in household labor supply, which leads to larger positive responses of labor supply when households take on a smaller wealth increase after a positive shock, and the increased capital accumulation, which arises from the firms' improved ability to invest. Neither force is quantitatively strong in the current calibration. Thus, in the current analysis, a more valid measure of real amplification caused by the financial accelerator mechanism is the volatility of investment, which *is* directly affected by the financial conditions of firms. As one can see, going from logarithmic household utility to $\xi = 13.2$, the standard deviation of investment increases about 40%. Adding nominal rigidities or working capital constraints on hiring labor to the model have the potential to introduce more output amplification due to financial frictions.

One can also can arrive at less risk sharing with lower household's IES. Setting ξ to 5.92 and $\psi = 1/5.92$, again leads to non-state-contingent lender returns. The increase in the relative volatility of net worth is slightly larger, and smaller for leverage. The implied increase in output volatility is significantly larger, almost 30%. The causes of these differences become clearer from the impulse responses below. Although the aim of the analysis here is not to match the simulated moments to the data, it is worthwhile noting that if one HP-filters the simulated series for comparability, the relative net worth and leverage volatilities generated under non-state-contingent lender returns are remarkably close to those of US nonfinancial firms during 1976Q1–2015Q3. These results are shown in Table 4 in Appendix A.

To illustrate the model's dynamics in more detail, Figure 2 below presents impulse responses to a 1% positive TFP shock over 28 quarters for the three utility calibrations under $\rho_A = 0.95$. The lower left panel shows the relative response of the quarterly lender return r_t^l. As seen above, the benchmark case exhibits a non-trivial degree of risk sharing with more than 80% of the innovation in capital returns, seen in the middle right panel, paid out to the households. And the engineered non-state-contingency of the lender returns are seen for the two other calibrations. Since in the latter two cases the firms hold on

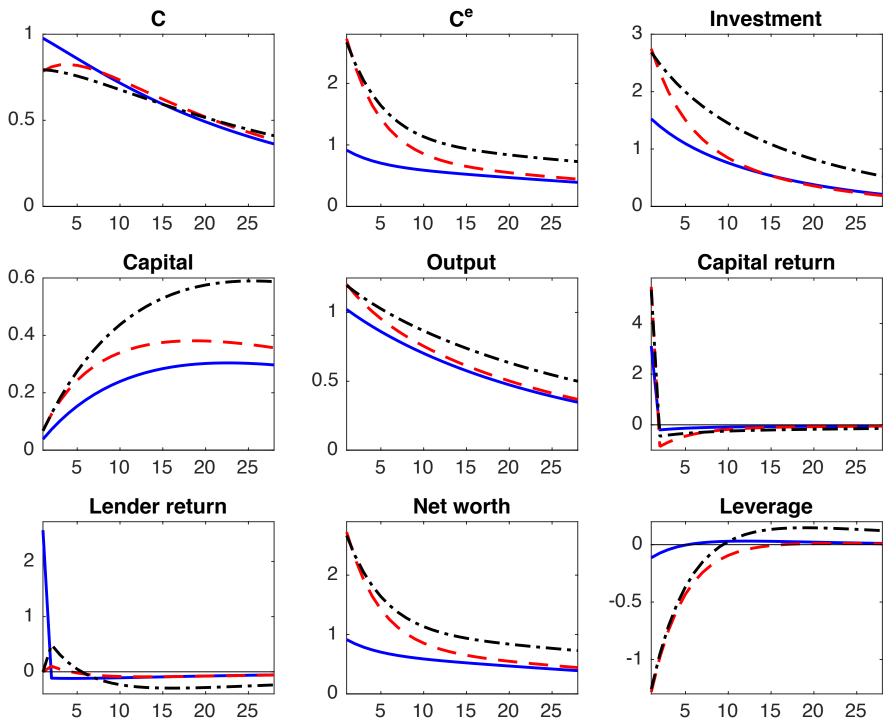


Figure 2: Impulse responses to 1% positive TFP shock in representative household model, with $\rho_A = 0.95$, (100×) log-deviations from steady state, returns annualized; Horizontal axis – quarters; Blue solid – log-utility, red dashed – $\xi = 13.2$, black dash-dotted – $\xi = 1/\psi = 5.92$.

to relatively more wealth, their net worth increases and leverage decreases significantly more at impact. Higher net worth facilitates investment, which increases the price of capital, in turn increasing returns to capital and net worth – the financial accelerator mechanism. And because of logarithmic utility for the entrepreneur, the response of C^e follows that of net worth. With less risk sharing, the household's wealth increases less, leading to lower consumption and a weaker positive wealth effect, increasing labor supply and output.

The key difference between the two parametrizations with non-state-contingent lender returns are the transitional dynamics of investment, net worth and output. In both cases, at the time of the shock, there is significant amplification. Yet for the case with high risk aversion and unitary IES, investment and net worth fall significantly faster, leading to smaller capital accumulation and a faster reversion in output. The reason is that with a low IES, the household prefers a flatter consumption profile, inducing it to save more throughout the first few periods after the shock. The extra savings flow through firms into investment, boosting entrepreneurial net worth. The fact that higher household savings should increase their wealth share, and thus entrepreneurial leverage, is counteracted by the fact that the high investment increases capital prices and entrepreneurial net worth through capital returns. Yet we do see that over time, leverage recovers faster under low IES, reflecting the household's larger accumulated wealth share. The sustained high investment yields larger capital accumulation and higher output throughout the transition path.

Note that the appearance of labor growth – which is negative along the transition path – in the household's stochastic discount factor when $\psi < 1$ makes it willing to take on a steeper consumption profile and dampens this added stimulus to the financial accelerator mechanism from low IES. If one were to exogenously set $\nu_l = 0$ for the sake of the argument, the desire for a flat profile would be stronger, the initial household savings even larger and the amplification in the low IES parametrization more significant. A $\nu_l > 0$ also explains how negative consumption growth can appear alongside lender returns above steady state values in the first few quarters after the shock.

When the persistence of the producvitity shocks is increased, less of the aggregate financial risk associated with the TFP shocks is taken on by the households, evident for $\xi = \psi = 1$. This reflects the importance of human and financial wealth dynamics for aggregate financial risk sharing in the economy, as discussed in Section 2.3.2. An increase in productivity shock persistence may increase the household's human capital responsiveness as the gains or losses from the discounted labor income throughout all future periods are accrued. This effect on human wealth is counteracted by the fact that with a more persistent shock, the transition path of household consumption is flatter, implying a smaller drop in the lender return used for discounting human wealth income after a positive productivity shock.

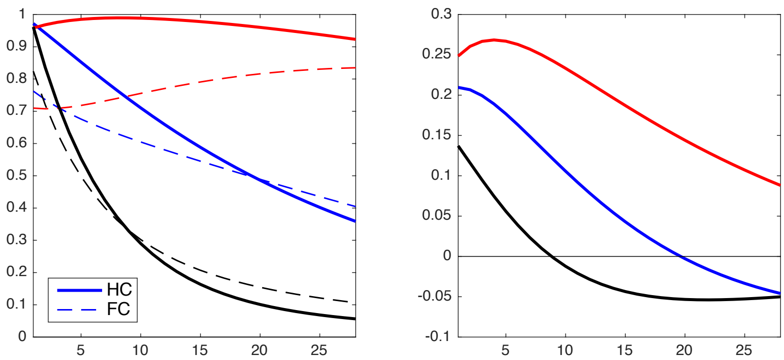


Figure 3: Left panel: Impulse responses of human (HC) and financial wealth (FC) to 1% positive TFP shock, percentage deviations from steady state, baseline calibration. Right panel: difference in HC and FC percentage deviations; Horizontal axis – quarters; Red – $\rho_A = 0.99$, blue – $\rho_A = 0.95$, black – $\rho_A = 0.85$.

At the same time, a more persistent positive TFP shock does not necessarily lead to a larger innovation in financial wealth. When shocks have *less* persistence, consumption smoothing motivates agents to save more of the initial windfall, boosting investment. As the price of capital is directly tied to investment, this channel decreases financial wealth responsiveness when TFP shocks become more persistent. The left panel of Figure 3 depicts the impulse responses of human and financial wealth, as defined in Section 2.3.2, for three different degrees of TFP shock persistence in the $\xi = \psi = 1$ calibration. All three specifications yield almost identical initial responses of human wealth, implying that for less persistent shocks a larger drop in the discount rate compensates for the lower duration of higher labor income. On the other hand, increased shock persistence does lead to lower initial responsiveness of financial wealth, resulting in larger differences in human and financial capital responses, seen in the right panel of the figure.

Because of the increased relative volatility of human wealth under $\rho_A = 0.99$, smaller changes in ξ and ψ are required to yield non-state-contingent lender returns, as households' exposure to human wealth fluctuations becomes more severe, in relative terms. One either needs $\xi = 4.34$ and $\psi = 1$, or $\xi = 1/\psi = 2.57$. Yet there is also an extra effect arising from Epstein-Zin preferences whenever $\xi \neq 1/\psi$. Persistently higher consumption brings about large innovations in household lifetime utility $\mathcal{V}_t$, making the household additionally averse to taking on the financial risk related to persistent TFP shocks. This discussion of course implies that for *less* persistent productivity shocks, ξ and ψ would have to deviate significantly from unity in order for non-state-contingent lender returns to arise. For example, for $\rho_A = 0.90$, the financial accelerator mechanism implies that financial wealth responds relatively more than human wealth on impact, and $\xi \approx 27$ while $\psi = 1$ is necessary for non-state-contingent lender return optimality.

4 The Heterogeneous Households Extension

To study the relevance of uninsurable idiosyncratic lender risk for the sharing of aggregate risk, I introduce time-varying idiosyncratic risk to households' labor productivity. By employing a special case of the households' preferences introduced in Section 2.2.1, and by assuming a specific stochastic structure for idiosyncratic labor productivity shocks, building on the approach by Constantinides and Duffie (1996), a no-trade equilibrium provides tractability and a clean exposition of the question at hand. The resulting model with heterogeneous households is in its essence a slight variation of the representative household model presented above, only differing in that the stochastic discount factor used by households to price assets will now also mirror the time-varying uninsurable idiosyncratic risk which they face.

4.1 The Environment

Let there be a unit mass of *ex ante* identical infinitely-lived households, indexed by $i \in [0, 1]$. Following the exposition in Section 2.2.1, let us consider the specification of households' preferences in which $\xi = \psi = 1$, and $\Phi(L) \equiv (1 - L)^\phi$, with $\phi > 0$ a parameter. This means that we are simply considering households with expected utility preferences and a momentary utility flow from consumption and leisure given by:

$$u(C, L) = \log C + \phi \log(1 - L)$$

In addition to holding deposits $d_{i,t}$ in the financial intermediary, the households can freely trade risk-free bonds in zero net supply with each other. Yet in the equilibrium considered below, all households i will hold an identical portfolio of financial assets, and thus the holdings of such bonds are zero across $i \in [0, 1]$. This is the sense in which the equilibrium considered is a "no-trade" one, while there is active borrowing and lending between the households and the continuum of firms through the intermediary.[14]

[14]More generally, the households could be allowed to trade any set of assets in zero net supply whose returns are contingent only on aggregate realizations, e.g. a set of Arrow-Debreu securities, with one corresponding to each realization of the aggregate state. Again, in the no-trade equilibrium, the holdings of these would be zero across all households.

Also because of this, for brevity, I will abstract from trading in new capital producer equity in what is to follow.

Let us now denote the household's time discount factor as $\tilde{\beta}$. The reason for distinguishing the notation for the heterogeneous households' discount factor from the β of the representative household used prior will become clear in the calibration below. All in all, the recursive problem of household i can thus be written as:

$$\mathcal{U}_t\left(d_{i,t}, b^f_{i,t}, \eta_{i,t}\right) = \max_{c_{i,t}, l_{i,t}, d_{i,t+1}, b^f_{i,t+1}} \left\{\log(c_{i,t}) + \phi\log(1-l_{i,t}) + \tilde{\beta}\mathbb{E}^i_t\left[\mathcal{U}_{t+1}\left(d_{i,t+1}, b^f_{i,t+1}, \eta_{i,t+1}\right)\right]\right\}$$
$$\text{s.t. } c_{i,t} + d_{i,t+1} + q^f_t b^f_{i,t+1} = W_t\eta_{i,t}l_{i,t} + R^d_t d_{i,t} + b^f_{i,t} + \Pi^I_t$$

where $\eta_{i,t}$ is the productivity of a unit of household i's labor $l_{i,t}$ in period t, and $\mathcal{U}_t \equiv \frac{[\mathcal{V}_t]^{1-\xi}}{(1-\beta)(1-\xi)}$ is simply a monotonic transformation of the household value function used in Section 2. q^f_t is the period t price of a risk-free bond paying off a unit of the final good in $t+1$ and $b^f_{i,t+1}$ are household i's holdings of such a bond. $\mathbb{E}^i_t[\cdot]$ refers to the conditional expectation operator with respect to both aggregate and idiosyncratic uncertainty in $t+1$.

I assume that household i's labor productivity $\eta_{i,t}$ follows a stochastic process such that in a no-trade equilibrium, $c_{i,t} = \theta_{i,t}C_t$, where $C_t \equiv \int_0^1 c_{i,t}di$ and $\theta_{i,t}$ follows:

$$\frac{\theta_{i,t}}{\theta_{i,t-1}} = \exp\left\{\sigma_{\theta,t}\varepsilon_{i,t} - \frac{\sigma^2_{\theta,t}}{2}\right\} \tag{25}$$

where $\varepsilon_{i,t} \sim N(0,1)$, i.i.d. across households i and time t, and where $\sigma^2_{\theta,t}$ is itself a stochastic process, governing the time-varying volatility of idiosyncratic risk. Put differently, the equilibrium (log-)consumption of household i consists of an idiosyncratic component which follows a random walk with drift and time-varying volatility of innovations, and an aggregate component, with the dynamics of the latter determined in general equilibrium:

$$\log(c_{i,t}) = \log(\theta_{i,t}) + \log(C_t)$$
$$\text{with } \log(\theta_{i,t}) = -\frac{\sigma^2_{\theta,t}}{2} + \log(\theta_{i,t-1}) + \sigma_{\theta,t}\varepsilon_{i,t}$$

Solving for the households' optimal behavior given any process for $\eta_{i,t}$ then allows us to characterize the specific properties of $\eta_{i,t}$ required for this to be the case. In what is to follow, I will be referring to the variance of the (log-)growth of the idiosyncratic consumption component, $\sigma^2_{\theta,t}$ interchangeably as "idiosyncratic labor risk". This is justified by the fact that the growth of a household's idiosyncratic consumption component and the implied idiosyncratic labor productivity growth follow very similar stochastic processes, as discussed in detail below and in Appendix C.

The remaining components of the model pertaining to the representative entrepreneur, the continuum of firms, the representative financial intermediary, the final goods producer and new capital producers are unchanged from Section 2.2.

4.1.1 Discussion of Assumptions

The conceptual approach that I am applying to constructing a no-trade equilibrium is exactly that of Constantinides and Duffie (1996): we are looking for an equilibrium in which individuals' consumption behaves in a way that provides simple tractability and aggregation. Since Constantinides and Duffie

(1996) studied an endowment economy in which households' income from human wealth was completely exogenous, it was enough to make the necessary assumptions directly on a household's labor income process.[15] However, because in the current model labor supply is endogenous, an analogous assumption must be made on the households' labor *productivity* process, taking into account how the households then choose their labor supply given this process. Having solved for the households' necessary optimality conditions, we can then elaborate upon the required idiosyncratic labor productivity process that is consistent with the conjectured consumption behavior in a no-trade equilibrium.

For brevity, below I will exhibit the implied $\eta_{i,t}$ written as a function of aggregate equilibrium outcomes and the individual's $\theta_{i,t}$. But of course, these aggregate equilibrium objects are themselves functions of initial conditions and current and past realizations of any exogenous stochastic processes affecting the economy. That is, in general, if we were to collapse the initial conditions and the history of past and current aggregate shocks up to time t in a single variable $\mathcal{S}^t$, then equilibrium aggregate consumption would satisfy $C_t = \varphi_{c,t}(\mathcal{S}^t)$ for some function $\varphi_{c,t}(\mathcal{S}^t)$ – with consumption an example of an aggregate equilibrium outcome. So, for example, when below we write that $\eta_{i,t}$ must satisfy some condition $\eta_{i,t} = \lambda_{\eta,t}(\theta_{i,t}, C_t)$, how this should be interpreted is that there is an implied function $\varphi_{\eta,t}$, the composite of $\lambda_{\eta,t}$ and $\varphi_{c,t}$, such that $\eta_{i,t} = \varphi_{\eta,t}(\theta_{i,t}, \mathcal{S}^t)$. And $\varphi_{\eta,t}$, alongside the stochastic processes for the aggregate exogenous shocks and (25), thus defines the required fundamental stochastic process governing $\eta_{i,t}$. Taking the implied stochastic process for $\eta_{i,t}$ and the aggregate exogenous stochastic processes as given, each household makes their decisions optimally, in turn validating the conjectured equilibrium.

Recent empirical evidence by Guvenen et al. (2014) has suggested that it is not the the countercyclical *variance* of innovations in individual labor income that is causing idiosyncratic labor risk to be countercyclical, but rather the *left-skewness* of these innovations is countercyclical. That is, large drops in earnings become more likely during recessions. For computational ease and expositional clarity, I will nontheless examine the potential of the countercyclical *variance* of idiosyncratic labor income growth in delivering no financial risk sharing in the CFP model. In this case, the quantitative exercise is to simply characterize the required dynamics of *one* distributional parameter: the idiosyncratic shock variance $\sigma^2_{\theta,t}$. One would expect the introduction of countercyclical left-skewness in the growth process of $\theta_{i,t}$ to yield conceptually similar results, although with a specification that introduces more degrees of freedom than simply determining movements in the second moment for the underlying stochastic process of $\theta_{i,t}$. And this would obscure the precise quantitative exercise that I set out to conduct.

4.2 Equilibrium, Aggregation and No-Trade

Given that only the household sector's problem has changed, I will focus on deriving their equilibrium optimality conditions and establishing the existence of a no-trade equilibrium. A household's first order necessary conditions for labor supply, deposits and risk-free bonds are now as follows:

$$c_{i,t}^{-1} W_t \eta_{i,t} = \phi(1 - l_{i,t})^{-1} \tag{26}$$

$$1 = \mathbb{E}_t^i \left[M_{i,t+1} R_{t+1}^d \right] \tag{27}$$

$$q_t^f = \mathbb{E}_t^i \left[M_{i,t+1} \right] \tag{28}$$

$$\text{with } M_{i,t+1} \equiv \tilde{\beta} \left(\frac{c_{i,t+1}}{c_{i,t}} \right)^{-1} \tag{29}$$

[15] See Constantinides and Duffie (1996), equation (8).

By rewriting (26) and integrating across $i \in [0,1]$, we can derive an aggregate labor supply condition:

$$\begin{aligned} l_{i,t}\eta_{i,t} &= \eta_{i,t} - \phi \frac{c_{i,t}}{W_t} \\ \Rightarrow L_t &= \bar{\eta}_t - \phi \frac{C_t}{W_t} \end{aligned} \tag{30}$$

where $L_t \equiv \int_0^1 l_{i,t}\eta_{i,t} di$ is the aggregate effective labor supply and $\bar{\eta}_{i,t} \equiv \int_0^1 \eta_{i,t} di$ the average labor productivity in t. The latter is exogenous to the economy, and could itself follow some stochastic process, e.g. equivalent to introducing shocks to the household's labor productivity in a representative household business cycle model. I will simply normalize $\bar{\eta}_t = 1, \forall t$ in the computations below.

For household i, we can also use the labor supply condition in the budget constraint and substitute out individual labor supply to yield:

$$(1+\phi)c_{i,t} + d_{i,t+1} + q_t^f b_{i,t+1}^f = W_t \eta_{i,t} + R_t^d d_{i,t} + b_{i,t}^f + \Pi_t^I \tag{31}$$

Finally, we can impose the consumption and portfolio choices of household i which correspond to a no-trade equilibrium of the form we are looking for, i.e. $c_{i,t} = \theta_{i,t} C_t$, $b_{i,t+1}^f = 0$, $d_{i,t+1} = d_{t+1}$, and find that for such policies to be consistent with equilibrium it must be that $\eta_{i,t}$ satisfies:

$$\eta_{i,t} = \frac{1}{W_t}\left\{(1+\phi)\theta_{i,t}C_t + d_{t+1} - R_t^d d_t - \Pi_t^I\right\} \tag{32}$$

Following the discussion in Section 4.1.1 above, (32) thus determines the stochastic process for individual labor productivity consistent with a no-trade in equilibrium in which consumption follows $c_{i,t} = \theta_{i,t} C_t$, with $\theta_{i,t}$ satisfying (25). In Appendix C, I elaborate upon the properties of this implied labor productivity process and discuss issues such as how to ensure stationarity of the cross-sectional distribution by assuming that households die and get replaced at some exogenously given rate, as done by Constantinides and Duffie (1996). Among other things, I point out that since $\log(\theta_{i,t}/\theta_{i,t-1})$ is i.i.d. normal, i.e. $\theta_{i,t}$ is a geometric random walk with log-normal innovations, and the term $(d_{t+1} - R_t^d d_t - \Pi_t^I)$ is very small in the calibration considered, also the growth of $\log(\eta_{i,t}/\eta_{i,t-1})$ conditional on aggregate shock realizations in t is *approximately* normal.[16] Thus, the idiosyncratic labor productivity process I employ closely follows labor income processes often used in quantitative studies of Bewley-Huggett-Aiyagari models, e.g. by Kaplan and Violante (2010), with the added time-varying volatility of innovations, as for example by Storesletten et al. (2007). Moreover, the implied conditional variance of idiosyncratic labor productivity growth $\log\left(\frac{\eta_{i,t}}{\eta_{i,t-1}}\right)$ closely follows the conditional variance of $\log\left(\frac{\theta_{i,t}}{\theta_{i,t-1}}\right)$, i.e. $\sigma_{\theta,t}^2$.

The final piece of establishing validity of the no-trade equilibrium is verifying that all households are willing to hold the required portfolios of deposits and risk-free bonds. Given that in the conjectured equilibrium, $c_{i,t} = \theta_{i,t} C_t$, we can focus on Euler equation (27) for deposits, impose (25) and elaborate:

$$\begin{aligned} 1 =& \mathbb{E}_t^i \left[\tilde{\beta}\left(\frac{c_{i,t+1}}{c_{i,t}}\right)^{-1} R_{t+1}^d\right] = \\ =& \mathbb{E}_t^i \left[\tilde{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{-1} \exp\left\{-\sigma_{\theta,t+1}\varepsilon_{i,t+1} + \frac{\sigma_{\theta,t+1}^2}{2}\right\} R_{t+1}^d\right] \end{aligned}$$

Let us denote the information set which contains all information up to, and including, $t-1$ plus all information about aggregates up to t as $\mathcal{I}_t^a$. By the assumption that $\varepsilon_{i,t} \sim N(0,1)$, i.i.d. across

[16] More precisely, conditional on aggregate realizations in t, $\eta_{i,t}/\eta_{i,t-1}$ follows what is sometimes referred to as a *three-parameter lognormal* random variable.

households i and time t, we have that conditional on $\mathcal{I}_{t+1}^a$:

$$-\sigma_{\theta,t+1}\varepsilon_{i,t+1} + \frac{\sigma_{\theta,t+1}^2}{2} \sim N\left(\frac{\sigma_{\theta,t+1}^2}{2}, \sigma_{\theta,t+1}^2\right)$$

So we can apply the law of iterated expectations to integrate over $\varepsilon_{i,t+1}$ and rewrite the above Euler equation as:

$$1 = \mathbb{E}_t\left[\tilde{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{-1}\exp\left\{\sigma_{\theta,t+1}^2\right\}R_{t+1}^d\right] \tag{33}$$

One can apply the same steps for the optimality condition regarding risk-free bonds (28) and arrive at:

$$q_t^f = \mathbb{E}_t\left[\tilde{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{-1}\exp\left\{\sigma_{\theta,t+1}^2\right\}\right] \tag{34}$$

Thus, we have established that if (33) and (34) hold, then at the conjectured no-trade allocations, all households are behaving optimally by satisfying their Euler equations and that the allocation is indeed an equilibrium.[17]

We can then define the effective "representative stochastic discount factor" as:

$$M_{t+1} \equiv \tilde{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{-1}\exp\left\{\sigma_{\theta,t+1}^2\right\} \tag{35}$$

And let us define a competitive no-trade equilibrium of the heterogeneous household model by utilizing the unchanged equilibrium conditions from Section 2, as follows.

Definition 2. *A* competitive no-trade equilibrium of the heterogeneous household model *is a collection of stochastic processes for:*

1. *a price system* $\{r_t, W_t, R_t^k, R_t^l, Q_t\}$,
2. *aggregate households' consumption, deposits, and representative stochastic discount factor* $\{C_t, d_{t+1}, M_t\}$
3. *entrepreneurial consumption, net worth and leverage quantities and contractual cutoffs* $\{C_t^e, N_t, \kappa_t, \bar{\omega}_t\}$
4. *aggregate labor, investment and capital quantities, and new capital producers' profits* $\{L_t, I_t, K_{t+1}, \Pi_t^I\}$
5. *individual household consumption and labor supply* $\{c_{i,t}, l_{i,t}\}_{i\in[0,1]}$

such that equations: (1), (5), (7)–(11), (16)–(22), (26), (30), (31), (35), *with* $R_t^d = R_t^l$, $d_{i,t} = d_t$, *and* $b_{i,t}^f = 0$, *where applicable, are satisfied, given the definitions of* C_t *and* L_t, *the stochastic processes for* $\left\{A_t, \sigma_{\theta,t}^2, \eta_{i,t}\right\}$, *and initial conditions* $\left(K_0, E_0, A_0, \sigma_{\theta,0}^2, \{\eta_{i,0}\}_{i\in[0,1]}\right)$.

[17] To fully establish optimality, we must also verify that a transversality condition for the deposits holds, by again applying the law of iterated expectations and independence of $\varepsilon_{i,t}$ across time and from the aggregate realizations:

$$0 = \lim_{T\to\infty}\mathbb{E}_t^i\left[\tilde{\beta}^{T-t}u_c(c_{i,T}, l_{i,T})d_{T+1}\right] = \lim_{T\to\infty}\mathbb{E}_t\left[\left(\prod_{s=t+1}^{T}\tilde{\beta}\exp\left\{\sigma_{\theta,s}^2\right\}\right)C_T^{-1}d_{T+1}\right]$$

which is satisfied if the aggregate economy is stationary and $\tilde{\beta}\exp\left\{\sigma_{\theta,s}^2\right\} < 1$ in the long run, as in the calibration applied below.

5 Quantitative Analysis of the Heterogeneous Household Model

5.1 Calibration

Compared to the representative household model's calibration in Section 3.1, I follow the same set of calibration targets as closely as possible. The only equilibrium conditions determining aggregates which have changed are the stochastic discount factor employed in representing the households' preferences over the lending contracts, as now determined in (35), and the aggregate labor supply condition, now given by (30). As for the former, I will leave the nonstochastic steady state level of idiosyncratic risk $\sigma^2_{\theta,ss}$ and the time discount factor of the heterogeneous households unspecified for now, and instead suppose that:[18]

$$\tilde{\beta} \exp\left\{\sigma^2_{\theta,ss}\right\} = \beta = 0.99$$

This implies that the steady state risk-free interest rate in the heterogeneous household model is unchanged compared to the representative household case in Section 2 and the effective representative stochastic discount factor M_{t+1} can be rewritten as:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-1} \exp\left\{\sigma^2_{\theta,t+1} - \sigma^2_{\theta,ss}\right\}$$

The dynamics of the exogenous cyclical component of idiosyncratic risk captured by $\sigma^2_{\theta,t+1} - \sigma^2_{\theta,ss}$, under various specifications, will be the main object of interest in the quantitative analysis below.

As for the households' labor supply, for comparability, I will again ensure that to a first order the aggregate labor supply condition exactly replicates that of the representative agent case in Section 2, and in CFP. To see how this is possible, let us log-linearize (30), and plug in the steady state labor supply condition to write:

$$c_t + \frac{L_{ss}}{1 - L_{ss}} l_t = w_t$$

where the lowercase letters again denote log-deviations of the corresponding aggregates from their steady state values. Comparing this to the representative agent log-linearized labor supply condition seen in Section 3.1, it is clear that in order to establish equivalence, we simply need to ensure $\frac{L_{ss}}{1-L_{ss}} = \frac{1}{\eta_l}$. So given that CFP use $\eta_l = 3$, we need $L_{ss} = 1/4$, which then pins down the required value of labor disutility ϕ. Using the more common target of 1/3 of individuals' time spent engaged in market activities, as often done in the calibration of macroeconomic models, e.g. Hansen (1985), would decrease the aggregate labor supply elasticity from 3 to 2.

Based on these choices, we have thus established that if $\sigma^2_{\theta,t} = \sigma^2_{\theta,ss}, \forall t$, i.e. there is no time-varying idiosyncratic labor risk, then the aggregate behavior of the heterogeneous household model and the representative household model under logarithmic expected utility preferences is *identical* to a first order – a convenient theroretical benchmark. Finally, even though we do not need to determine a specific value of $\sigma^2_{\theta,ss}$ for the calibration of the model based on the nonstochastic steady state, this value is relevant for the model's dynamics outside of the steady state, given the log-linear specification governing $\sigma^2_{\theta,t}$ that I introduce below.[19] Using PSID data for 1968-1993, Storesletten et al. (2004) estimate an annual specification for idiosyncratic labor earnings in which the stochastic part of log labor earnings consists of a persistent (not necessarily unit root) and transitory component, and shocks to the persistent

[18] The *nonstochastic steady state* is one in which all aggregate shock innovations are set to zero while all idiosyncratic shocks still exist.

[19] If we were to instead employ an analogous *linear* specification for $\sigma^2_{\theta,t} - \sigma^2_{\theta,ss}$, one could also solve for the model's dynamics without having to specify a value for $\sigma^2_{\theta,ss}$.

component have regime-switching conditional variance. Their estimates imply an annual autocorrelation of idiosyncratic labor income of approximately 0.95 and conditional standard deviations of the persistent shocks of 0.12 in aggregate expansions and 0.21 in contractions, with the frequency-weighted average being 0.17. This average then roughly translates into a quarterly standard deviation of 0.085. Using the average as a benchmark, I will consider the implied quarterly variance of $\sigma^2_{\theta,ss} = 7.225 \times 10^{-3}$ in what is to follow.

5.2 Risk Sharing with Aggregate TFP Shocks and Flexible Prices

In the analysis of the representative household model, the goal was to determine which household preferences yield optimality of the BGG contract of no financial risk sharing in response to a standard aggregate TFP shock. For the heterogeneous household case with uninsurable idiosyncratic labor risk, I follow a similar idea and ask: how does the households' idiosyncratic labor risk captured by $\sigma^2_{\theta,t}$ have to vary in response to the aggregate TFP shock, so that households take on none of the financial risk in the optimal contract. Moreover, given the flexibility that the Constantinides and Duffie (1996)-style approach delivers, we can conduct a similar exercise for *any* conceivable aggregate shock that could hit the economy, which I will do below.

In what is to follow, I will suppose that the log-deviation of uninsurable idiosyncratic labor risk (variance) from its steady state value, denoted $\hat{\sigma}^2_{\theta,t} \equiv \log\left(\frac{\sigma^2_{\theta,t}}{\sigma^2_{\theta,ss}}\right)$, is stationary and persistent, and exposed to innovations in the exogenous aggregate processes affecting the economy:

$$\hat{\sigma}^2_{\theta,t} = \rho_\theta \hat{\sigma}^2_{\theta,t-1} + \Upsilon' \varepsilon_t \tag{36}$$

where $\rho_\theta \in [0, 1)$. ε_t is a vector of aggregate shock realizations which includes the TFP shock $\varepsilon^A_t \in \varepsilon_t$, and any other aggregate shocks that might hit the economy. And Υ is a vector of parameters that capture the exposure of idiosyncratic labor risk to each of these shocks. Considering various choices of ρ_θ, the objective of my analysis is to determine the elements of Υ such that in response to any conceivable type of shock contained in ε_t, the household takes on no financial risk in the optimal contract. In the flexible price specification, I will assume that aggregate TFP shocks are the only source of aggregate uncertainty, so $\Upsilon \equiv [\Upsilon_A]$ and $\varepsilon_t \equiv \left[\varepsilon^A_t\right]$ are of length 1. In the case with nominal rigidities below, ε_t will also contain monetary policy and capital quality shocks.

As pointed out by CFP, because of certainty equivalence in a first order approximation solution, the behavior of the BGG and CFP models differs *only* with regards to the impact effects of any realized exogenous shocks. That is, starting the two models off at *any* initial condition (K_0, E_0, A_0), in the absence of additional aggregate shocks being realized, they would yield identical equilibrium paths. This in turn means that to establish equivalence of the BGG and CFP models with heterogeneous households using time-varying labor risk $\sigma^2_{\theta,t}$, one would only need to specify how $\sigma^2_{\theta,t}$ moves at the time of the revelation of an exogenous aggregate shock in period t, and simply set $\sigma^2_{\theta,t+m} = \sigma^2_{\theta,ss}$ for $m \geq 1$. That is, we could set $\rho_\theta = 0$, and find the Υ_0 that implies optimality of the BGG contract under $\hat{\sigma}^2_{\theta,t} = \Upsilon'_0 \varepsilon_t$. However, supposing that idiosyncratic labor risk $\sigma^2_{\theta,t}$ exhibits no persistence would be extreme and counterfactual – for example, it can be seen in Storesletten et al. (2004) that the cross-sectional standard deviation of idiosyncratic labor income innovations in the PSID exhibits persistence across time, similar to the persistence in the cross-sectional *mean* of log income. Also, the calibration method employed by Schmidt (2016) results in a persistence of the monthly AR(1) process for $\sigma^2_{\theta,t}$ in excess of 0.98. Finally, zero persistence in idiosyncratic labor risk would require unrealistically high responsiveness of $\sigma^2_{\theta,t}$ to macroeconomic shocks for the BGG contract to be optimal, as I elaborate below.

As a final note, because $\sigma^2_{\theta,t}$ has a natural lower bound of 0, there is a limit to how far the representative stochastic discount factor M_t can fall due to drops in idiosyncratic labor risk, conditional on aggregate consumption C_t. Because of this, in what is to follow, we must be careful when using solutions relying on perturbation around the steady state because their precision can potentially deteriorate significantly in the case of large aggregate shocks. Most importantly, if a very large quantitative exposure of $\hat{\sigma}^2_{\theta,t}$ to TFP shocks is necessary for the BGG contract to be optimal, then for large TFP innovations the implied approximation error in the optimal risk sharing condition

$$\frac{\Gamma'(\bar{\omega}_t)}{\Gamma'(\bar{\omega}_t) - \mu G(\bar{\omega}_t)} = \frac{M_t}{M_t^e}$$

might become significant since $M_t \geq \tilde{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{-1}$, while this boundary is irrelevant for infinitesimal fluctuations around the steady state. For this reason, I will not report simulation results for the case in which $\rho_\theta = 0$ and the required movements in $\hat{\sigma}^2_{\theta,t}$ are extreme. On the other hand, given that worries of a lack of financial risk sharing usually arise under recessionary shocks which tend to imply *increases* in idiosyncratic risk, these considerations are not an issue for studying the model's behavior in crises.

Following the quantitative analysis of the representative agent model in Section 3, let us consider TFP shocks with persistence $\rho_A = 0.95$ to be the sole exogenous shock in the model. Table 2 below presents, for various values of ρ_θ, a selection of simulated second moments and the exposure of $\hat{\sigma}^2_{\theta,t}$ to ε^A_t necessary to yield an optimal lending contract in which the households take on none of the financial risk associated with investing in capital. That is, the BGG and CFP models coincide, or in the notation introduced above, $\left(\frac{\partial r^l_t}{\partial \varepsilon^A_t}\right) / \left(\frac{\partial r^k_t}{\partial \varepsilon^A_t}\right) = 0$.

If idiosyncratic labor risk exhibits no persistence, i.e. $\rho_\theta = 0$, for the BGG contract to be optimal in response to TFP shocks, one needs $\Upsilon_A = -355.0$. That is, in response to a 1% fall in TFP, $\log\left(\sigma^2_{\theta,t}\right)$ would have to increase by 3.55 points compared to its steady state value, implying an increase of $\sigma^2_{\theta,t}$ from 7.225×10^{-3} to approximately 0.251 and an increase of the standard deviation $\sigma_{\theta,t}$ from 0.085 to 0.502 – an effectively 6-fold increase in risk. Such a change in idiosyncratic risk is clearly larger than the difference between expansions and recessions estimated by Storesletten et al. (2004), and unrealistic.

On the other hand, when idiosyncratic labor risk $\sigma^2_{\theta,t}$ exhibits persistence, the exposure to innovations in aggregate TFP required for the BGG contract to be optimal drops significantly. The main reason for this is that now, the countercyclicality of households' precautionary savings motives directly affects aggregate dynamics. When the expected value of $\sigma^2_{\theta,t+1}$ drops after a positive TFP shock, all else equal, households' marginal return to saving in period t decreases since they face less uninsurable idiosyncratic labor risk going forward. This increases the cost of external funds for firms and dampens the positive response of investment and capital prices, thus reducing the implied volatility of aggregate financial wealth in the economy.[20] On top of the dampening effects on investment, the countercyclicality of the precautionary savings motive also generates a countercyclical force on labor supply due to a wealth effect: relatively larger increases in households' consumption after positive TFP shocks push down their labor supply, all else equal, leading to a less volatile aggregate output process. These effects can be seen comparing the simulated second moments of the model with $\rho_\theta = 0.8$ and $\rho_\theta = 0.95$ in Table 2. Even though the required exposure of $\sigma^2_{\theta,t}$ to ε^A_t is more than twice as small under $\rho_\theta = 0.95$ compared to $\rho_\theta = 0.8$, the higher persistence is sufficient for the dampening effects of countercyclical precautionary motives to increase with ρ_θ.

As for the implications regarding the required cyclicality of $\hat{\sigma}^2_{\theta,t}$ in response to TFP shocks when

[20]The idea that countercyclical precautionary motives tend to stabilize a real business cycle model is, for example, discussed in more depth by Challe et al. (2017).

Table 2: Exposure of idiosyncratic risk $\hat{\sigma}^2_{\theta,t}$ to TFP innovations Υ_A, relative standard deviations of log entrepreneurial net worth and leverage, absolute standard deviation of log output and investment, given various combinations of persistence parameters ρ_θ, such that $\left(\frac{\partial r^l_t}{\partial \varepsilon^A_t}\right) / \left(\frac{\partial r^k_t}{\partial \varepsilon^A_t}\right) = 0.0$, $\rho_A = 0.95$, and std $\left(\varepsilon^A_t\right) = 0.0072$; simulation of 10^6 quarters.

ρ_θ	Υ_A	$\frac{\text{std}(n)}{\text{std}(y)}$	$\frac{\text{std}(\hat{\kappa})}{\text{std}(y)}$	std(y), %	std(i), %
0.0	-355.0	–	–	–	–
0.8	-31.3	1.023	0.248	2.539	2.475
0.95	-14.3	0.820	0.310	2.244	1.441

$\rho_\theta = 0.8$, $\Upsilon_A = -31.3$ is necessary for the BGG contract to be optimal. That is, in response to a 1% fall in TFP, the variance of the growth of log $\left(\frac{\theta_{i,t}}{\theta_{i,t-1}}\right)$ must increase by a third, or more precisely, go from $\sigma^2_{\theta,ss} = 7.225 \times 10^{-3}$ to $\sigma^2_{\theta,ss} = 9.880 \times 10^{-3}$. And the implied quarterly standard deviation would increase by about 17% from 0.085 to 0.100 – when annualized, a change about three times smaller than the differences estimated by Storesletten et al. (2004) between economic expansions and recessions.

Finally, when the persistence of idiosyncratic labor risk variance equals that of the TFP process, i.e. $\rho_\theta = \rho_A = 0.95$, the required exposure of idiosyncratic labor risk to TFP shocks drops to $\Upsilon_A = -14.3$. In this case, in response to $\varepsilon^A_t = -0.01$, the variance of the growth of log $\left(\frac{\theta_{i,t}}{\theta_{i,t-1}}\right)$ must increase to approximately $\sigma^2_{\theta,t} = 8.336 \times 10^{-3}$, and the quarterly standard deviation by about 7.5% to 0.091, from the steady state value of 0.085.

Table 4 in Appendix A again compares the relative volatilities of HP-filtered model data to empirical counterparts in the U.S. for the time period 1976Q1–2015Q3. Although the heterogeneous household specification with persistent idiosyncratic labor risk is an improvement over the representative agent model with log-log utilities, the introduction of the countercyclical precautionary savings motives dampens aggregate volatility significantly. In what is to follow, I introduce nominal rigidities as done by BGG and CFP. As emphasized by Challe et al. (2017), in a New Keynesian setting, countercyclical precautionary savings motives introduce procyclicality in aggregate demand, thus generating a countervailing amplification effect and potentially improving the heterogeneous household specification's empirical performance.

5.3 Risk Sharing with Nominal Rigidities

I will introduce nominal rigidities using the standard New Keynesian approach in which retail firms purchase the production of the final good producers at "wholesale prices", transform this into differentiated products and sell them to the household subject to monopolistic competition, while facing rigidities in price setting. Since there are no nominal rigidities between the financial intermediary and the firms, the contracting problem is unchanged. Given that the derivation of the framework is standard[21], I will simply comment on the newly introduced equilibrium objects and the equilibrium conditions which are affected.

There will now be gross inflation Π_t, an aggregate markup X_t (the ratio of the price of the retail goods bundle relative to the wholesale price), and a gross nominal interest rate R^n_t. The markup will

[21] For details see, for example, Galí (2015).

appear in final goods producers demand for labor and capital, equations (7) and (8) above:

$$W_t = X_t^{-1}(1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha}$$
$$r_t = X_t^{-1}\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}$$

where W_t and r_t are the *real* wage and rental rate. There is a Fisher equation that connects nominal and real interest rates:

$$\mathbb{E}_t\left[M_{t+1}\Pi_{t+1}^{-1}R_{t+1}^n\right] = 1$$

And the central bank sets a nominal interest rate following a Taylor rule:

$$R_{t+1}^n = R_{ss}^n \left(\Pi_t/\Pi_{ss}\right)^{\phi_\pi} \exp\left\{z_t^M\right\}$$

where z_t^M captures exogenous movements in monetary policy, following $z_t^M = \rho_M z_{t-1}^M + \varepsilon_t^M$. And $R_{ss}^n = \Pi_{ss}/\beta$ is the steady state gross nominal interest rate. Finally, there is a New Keynesian Phillips curve, which I will immediately introduce in linearized form, in log-deviations from steady state:[22]

$$\pi_t = -\kappa_\pi x_t + \beta\mathbb{E}_t\pi_{t+1}$$

Following CFP, I use parameter values $\kappa_\pi = 0.025$, $\phi_\pi = 1.5$ and $\rho_M = 0.50$.

In addition to the aggregate TFP and monetary policy shocks, the last shock I consider is a mean-zero i.i.d. capital quality shock ε_t^K, as has been studied in various general equilibrium macro-finance models since the Great Recession, for example by Gertler et al. (2012). That is, I suppose that per unit of capital K_{t+1} installed by each firm at the end of period t, only a fraction $\exp\left\{\varepsilon_{t+1}^K\right\}$ survives until the beginning of period $t+1$, and is available for production and further capital accumulation.

Similarly as above in the flexible price case, I will look for the vector of exposures $\Upsilon \equiv [\Upsilon_A, \Upsilon_M, \Upsilon_K]'$ such that if idiosyncratic labor risk $\sigma_{\theta,t}^2$ follows specification (36), the households take on no (real) financial risk induced by any of the shocks in $\varepsilon_t \equiv \left[\varepsilon_t^A, \varepsilon_t^M, \varepsilon_t^K\right]'$. I will focus on the calibration with persistent idiosyncratic labor risk $\rho_\theta = 0.95$.

Table 3 presents the required exposures of $\hat{\sigma}_{\theta,t}^2$ necessary to imply an optimal contract in which the households take on none of the financial risk associated with the TFP, monetary, and capital quality shocks. For comparison with the analysis above, the table also includes selected second moments from the model simulation when ε_t^A, with a normal distribution and standard deviation of 0.0072 is the only aggregate shock in the economy. For aggregate TFP shocks, the required exposure Υ_A is higher in

Table 3: Exposure of idiosyncratic risk $\hat{\sigma}_{\theta,t}^2$ to TFP, monetary policy and capital quality innovations Υ, relative standard deviations of log entrepreneurial net worth and leverage, absolute standard deviation of log output and investment under TFP shocks, such that $\left(\frac{\partial r_t^l}{\partial \varepsilon_t^A}\right) / \left(\frac{\partial r_t^k}{\partial \varepsilon_t^A}\right) = 0.0$, $\rho_A = \rho_\theta = 0.95$, and $\text{std}\left(\varepsilon_t^A\right) = 0.0072$; simulation of 10^6 quarters.

Shock (s)	Υ_s	$\frac{\text{std}(n)}{\text{std}(y)}$	$\frac{\text{std}(\hat{\kappa})}{\text{std}(y)}$	std(y), %	std(i), %
ε_t^A	-29.3	0.962	0.436	2.954	1.925
ε_t^M	126.0	–	–	–	–
ε_t^K	-41.6	–	–	–	–

comparison to the flexible price case, increasing to -29.3. This is because under nominal rigities, as

[22] Given that I am employing a first-order approximation solution, the Phillips curve can be derived from assuming either Calvo (1983) or Rotemberg (1982) pricing rigidities.

foreshadowed above, the countercyclical precautionary savings motive introduces a destabilizing force through procyclical demand effects. As households consume relatively more in response to a positive TFP shock inducing a persistent fall in idiosyncratic labor risk, demand increases, markups drop, and inflation is pushed up. In fact, as seen below, the effect of this channel is strong enough to make TFP shocks *in*flationary in the current model, in contrast to them being *de*flationary in the textbook New Keynesian model, with or without the BGG-style financial frictions. All in all, the total financial wealth in the economy thus becomes more volatile, in turn requiring that households' idiosyncratic labor risk drop relatively more in expansions for it to be optimal that they do not acquire any of the unexpected financial gains. $\Upsilon_A = -29.3$ implies that in response to a 1% unexpected fall in TFP, the standard deviation of $\sigma_{\theta,t}$ would have to increase by about 15%, from 0.085 in steady state to about 0.098 at shock impact. For reference, the implied annual volatility change is roughly three times smaller than the differences estimated by Storesletten et al. (2004) between economic expansions and recessions.

Although the introduction of nominal rigidities in conjuction with the financial accelerator mechanism amplifies the relative volatility of net worth and leverage and the absolute volatility of output and investment compared to the flexible price case, the countercyclical precautionary savings motives of households still weigh heavily on dampening investment fluctuations. And this keeps the volatility of the latter below that of output. If one were to HP-filter the model simulated data, the implied volatility of the cyclical component of investment would be closer to that of output due to the higher persistence of the latter.

Figure 4 presents impulse responses to a 1% positive TFP shock over 28 quarters in the model with nominal rigidities. As delivered by the exposure of idiosyncratic labor risk $\Upsilon_A = -29.3$, one can see the (log) real lender return r_t^l not responding to the increase in productivity at impact. At the same time the annualized real return to capital increases by 2.5 percentage points, and these unexpected returns are fully acquired by the firms, pushing their net worth up and their leverage down. The main difference in comparison to the aggregate dynamics of the representative household case with flexible prices is the stabilizing effect introduced by the households' countercyclical precautionary savings motives, slightly counteracted by nominal rigidities and demand effects. The fall in precautionary savings increases firms' cost of funds, as can be seen in by the increase in the required lenders' real return going forward. This weighs on investment, allowing it to increase by slightly more than 1%, dampening the positive response in the price of capital and the returns to capital r_t^k. Also, as mentioned above, the TFP shock is inflationary in the current calibration, with the (log) aggregate markup x_t falling by 0.7 percentage points, and the annualized inflation increasing by 100 basis points causing the central bank to increase the policy rate by 150 basis points.

The required exposure of $\sigma^2_{\theta,t}$ to monetary shocks ε_t^M is $\Upsilon_M = 126.0$. Given that $\Upsilon_M/100$ measures required log-changes in $\sigma^2_{\theta,t}$ in response to a 100 basis point shock to *quarterly* nominal interest rates, we can infer that in response to a more common 25 basis point increase in the annualized nominal policy rate, idiosyncratic labor risk variance $\sigma^2_{\theta,t}$ would have to increase by slightly less than a tenth. Or more specifically, the implied standard deviation $\sigma_{\theta,t}$ would need to increase by about 4%, from 0.085 to 0.088. Given nominal rigidities, the positive nominal rate shock decreases investment, capital prices, and the financial wealth in the economy. A simultaneous increase in the households' uninsurable idiosyncratic risk makes them reluctant to take on any of this drop in financial wealth, pushing the burden fully onto the firms, eroding their net worth and setting in motion the financial accelerator mechanism.

The impulse responses to an annualized 25 basis point contractionary monetary policy shock, i.e. $\varepsilon_t^M = 0.0025/4$ can be seen in Figure 5. As is common in the textbook New-Keynesian model, output, investment and consumption drop at shock impact, all about 0.2%. Again, the exposure of idiosyncratic

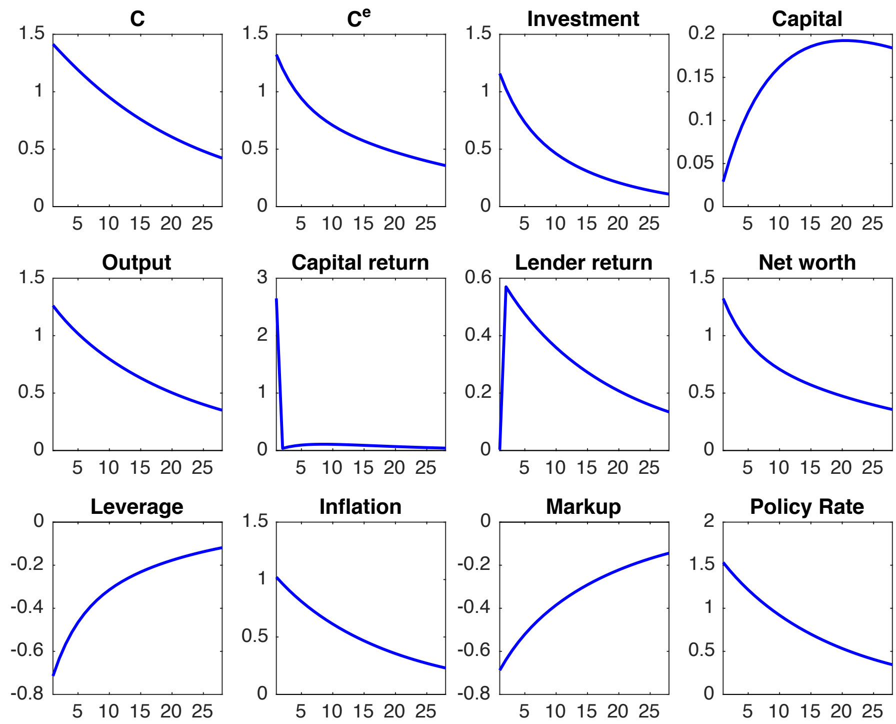


Figure 4: Impulse responses to 1% positive TFP shock in heterogeneous household model with nominal rigidities, (100×) log-deviations from steady state, returns and inflation annualized, capital and lender returns real; Horizontal axis – quarters.

labor risk $\Upsilon_M = 126.0$ implies an increase in $\sigma^2_{\theta,t}$ just enough so that the return to lenders does not drop at shock impact and the entrepreneurs suffer the full drop in the return to capital, eroding the firms' net worth. However, the households' countercyclical precautionary savings motives significantly reduce firms' borrowing costs, allowing them to rebuild net worth and aggregate investment to revert quickly, overshooting the steady state level already about a year out and exceeding it for most of the transition back to steady state. Moreover, notice that even though the monetary policy shock $\varepsilon^M_t > 0$ is contractionary, it causes inflation to drop enough that the central bank's policy rule induces them to relax monetary policy and *decrease* the policy rate by 40 basis points.

For it to be optimal that the households do not take on any financial risk associated with capital quality shocks, we need $\Upsilon_K = -41.6$. In response to a 1% drop in capital quality, i.e. $\varepsilon^K_t = -0.01$, the standard deviation $\sigma_{\theta,t}$ would have to increase by about 23%, from 0.085 to 0.105. Or, following Gertler et al. (2012) and extending this to a 5% decline in capital quality to consider a financial crisis such as the Great Recession, the standard deviation would have to more than double – a *relative* increase of approximately similar magnitude as the fluctuations between expansions and recessions that Storesletten et al. (2004) estimated given PSID data for 1968–1993. Although, in *absolute* terms, the implied increase in the annualized standard deviation from 0.17 to roughly 0.34 would be considerably larger than the empirical difference of 0.12 and 0.21 between expansions and recessions.

Figure 6 presents impulse responses to a 5% negative capital quality shock in the model with nominal rigidities. The drop in the capital stock causes consumption and output to drop by about 5% and 3%, respectively. However, because the shock starts the economy off on a transition path to steady state from a low level of capital, aggregate investment significantly exceeds its steady state value along the path.[23] Again, the increased precautionary savings motives of the households, implied by a high level

[23]Unlike Gertler et al. (2012), I have not altered any details of the model such as introducing GHH utility or habit

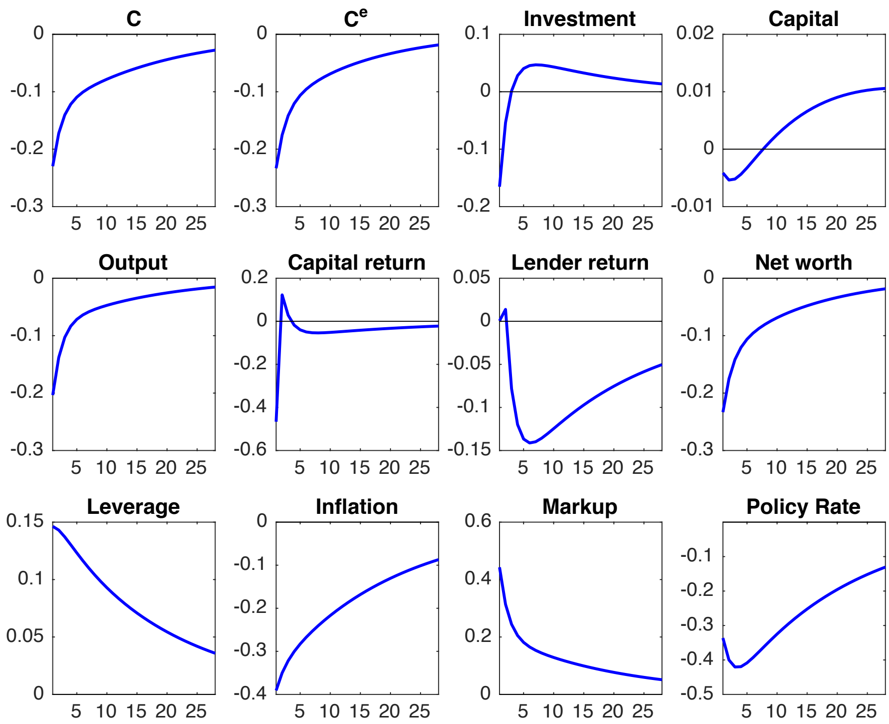


Figure 5: Impulse responses to annualized 25 basis point contractionary monetary policy shock in heterogeneous household model with nominal rigidities, (100×) log-deviations from steady state, returns and inflation annualized, capital and lender returns real; Horizontal axis – quarters.

of $\sigma^2_{\theta,t}$, help the economy rebuild the capital stock and revert to steady state faster. Since the capital quality shock is highly deflationary, the central bank responds and decreases the nominal policy rate by more than 10 percentage points. Although I have not specificed the level of the steady state inflation and thus the nominal rate, this drop would quite clearly violate the zero lower bound, which is not currently imposed in the model. Thus, a nonlinear solution which imposes the ZLB could lead to a considerably more severe crisis caused by the capital quality shock.

The last row of Table 4 in Appendix A compares the relative volatilities of HP-filtered model data if the TFP shock is the only aggregate disturbance, to empirical counterparts in the U.S. for the time period 1976Q1–2015Q3. As expected, the nominal rigidities allow the countercyclical precautionary savings motive to become an amplifying force, bringing the standard deviation of output in the model closer to that seen in the data. Yet to improve the model's fit to the data, one must necessarily consider a larger selection of aggregate disturbances than simply productivity shocks, which however, is outside the scope of the current paper.

Finally, in the scope of the exercise above, the high persistence of $\rho_\theta = 0.95$ has allowed to exemplify that at low exposures of idiosyncratic risk to various aggregate shocks, as captured by Υ, the optimal contract implies non-state-contingent returns to the households. Even though reaching the contract assumed by BGG as the privately optimal one is in itself a valid theoretical target, the dampening effects of a high ρ_θ mask the economy's ability to amplify the effects of aggregate shocks and yield slightly surprising impulse responses – such as investment contracting less than consumption in response to a contractionary monetary policy shock. To showcase how the economy with countercyclical labor risk and optimal contracting can considerably amplify the effects of aggregate shocks and exhibit dynamics

formation in households' preferences or using a lower intertemporal elasticity of substitution in order to have investment *drop* after a negative capital quality shock.

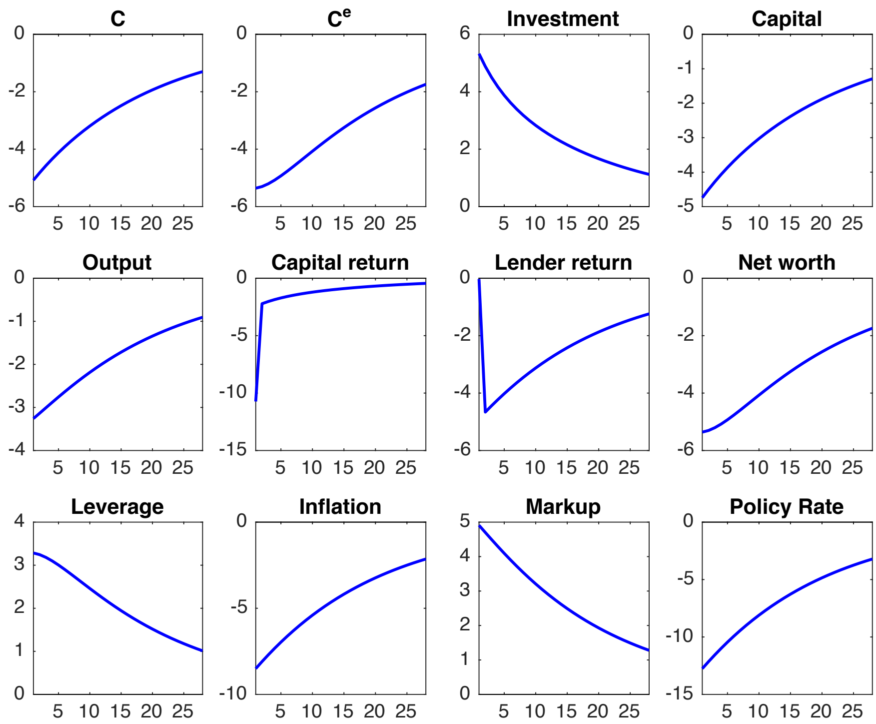


Figure 6: Impulse responses to 5% negative capital quality shock in heterogeneous household model with nominal rigidities, (100×) log-deviations from steady state, returns and inflation annualized, capital and lender returns real; Horizontal axis – quarters.

distinctive to the financial accelerator mechanism, I will consider a persistence of idiosyncratic risk equal to that of the monetary policy shock $\rho_\theta = \rho_M = 0.5$, and increase the exposure Υ_M, so that the economy's impulse responses to a monetary policy shock are similar to those of the BGG model.

For comparability, I will follow CFP and compute the economy's response to a 25 bp quarterly contractionary monetary policy shock if $\Upsilon_M = 0.0$, i.e. the economy is equivalent to the representative agent CFP model, and if $\Upsilon_M = 1100.0$. At $\Upsilon_M = 1100.0$, a 25 bp *annualized* shock would require $\sigma_{\theta,t}$ to increase from 0.085 in steady state to 0.1199 at shock impact – a considerable increase. Figure 7 below plots the results. The impulse responses in the economy with countercyclical idiosyncratic labor risk and optimal financial contracting are strikingly similar to those in the benchmark BGG model, as seen in Figure 4 in Carlstrom et al. (2016), and exhibit considerable amplification over and above the representative agent CFP model. Even though there is some financial risk sharing of the monetary policy shock, seen in the drop of lender returns in both cases, the *relative share* of the capital return innovation taken on by the household, i.e. $\left(\frac{\partial r_t^l}{\partial \varepsilon_t^M}\right) / \left(\frac{\partial r_t^k}{\partial \varepsilon_t^M}\right)$, is about 1.13 in the model without, and 0.30 in the case with countercyclical idiosyncratic risk. Given that I have introduced the fluctuations in idiosyncratic uncertainty exogenously, brought about by the monetary policy shock, one could also think of the exercise as computing the response to a joint incidence of standard monetary policy and idiosyncratic uncertainty shocks. As studied by Basu and Bundick (2017), shocks which increase uncertainty in a New Keynesian framework generate a joint drop in consumption and investment, facilitated by countercyclical markups. This explains why the economy with added countercyclical idiosyncratic risk is able to generate amplification similar to the BGG model even though some of the aggregate financial risk in the contracts between households and firms is taken on by the former, and the financial accelerator channel itself is slightly dampened in comparison to a case with no financial risk sharing.

The above therefore illustrates how a variation of the benchmark representative household framework

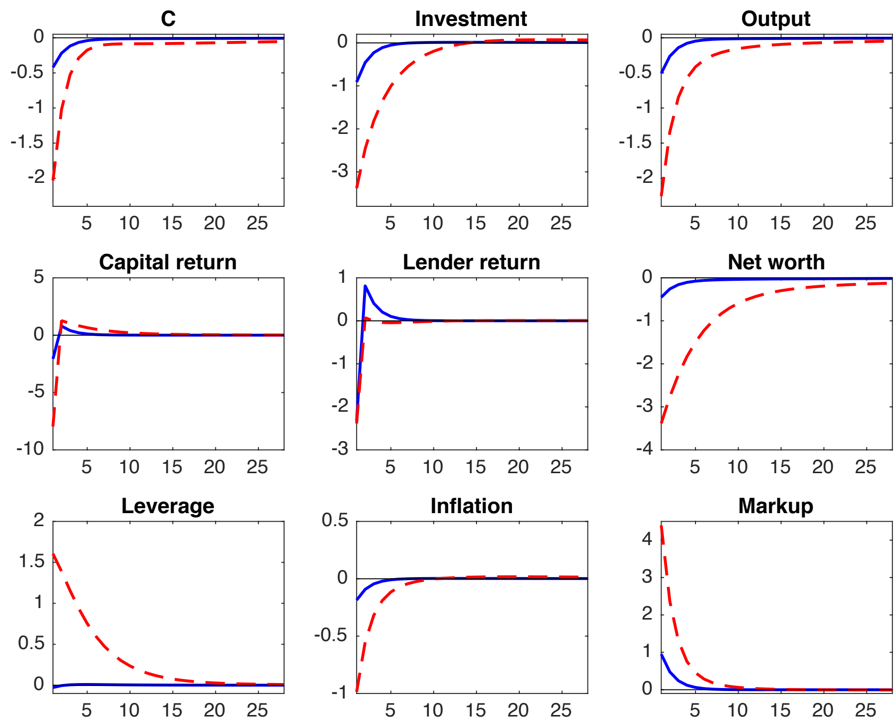


Figure 7: Impulse responses to quarterly 25 basis point contractionary monetary policy shock in heterogeneous household model with nominal rigidities and $\rho_\theta = 0.5$, $(100\times)$ log-deviations from steady state, returns and inflation annualized; Blue solid – $\Upsilon_M = 0.0$ (representative agent CFP model), red dashed – $\Upsilon_M = 1100.0$; Horizontal axis – quarters.

can further justify privately optimal contracts in which firms borrowing from households take on a significant amount of aggregate financial risk. Fully non-state-contingent lender returns and amplification of aggregate shocks are reached under logarithmic utility over consumption for both the households and the representative entrepreneur, while uninsurable idiosyncratic risk exhibits countercyclical fluctuations of realistic magnitudes.

6 Conclusion

Privately optimal aggregate risk sharing is significantly affected if a party is relatively more risk averse or faces uninsurable time-varying idiosyncratic risk. This paper exemplifies these ideas in a reformulation of the workhorse Bernanke et al. (1996) (BGG) model with financial frictions and analyzes their quantitative relevance for the implied aggregate dynamics. In response to aggregate total factor productivity shocks, non-state-contingent lender returns, as initially imposed by BGG, are privately optimal if the lending household has a coefficient of risk aversion parameter of 13.2 with unitary elasticity of intertemporal subsitution. Alternatively, the same outcome follows when the household has CRRA-type expected utility preferences with a risk aversion coefficient of 5.92. In a tractable no-trade equilibrium with rich consumption heterogeneity, building on the insights of Constantinides and Duffie (1996), non-state-contingent lender returns are shown to arise when households have logarithmic utility and face an uninsurable idiosyncratic labor productivity process that features persistent countercyclical risk of empirically plausible magnitudes.

Moving away from a simple two-agent setting with identical preferences, the virtual irrelevance of financial frictions for aggregate dynamics can disappear, even if agents are allowed to write privately optimal contracts to surmount these frictions. The time-varying uninsurable idiosyncratic risk faced by a party considered in this paper is only one of such plausible deviations from the benchmark. Other examples include informational asymmetries and trading frictions, such as studied by Asriyan (2015). Whether the mechanism detailed here is operative in the data is yet to be determined in further research. It is nontheless evident that the ultimate effect of financial frictions on the aggregate economy can significantly depend on other imperfections present in the environment.

Appendix

A HP-filtered Second Moments from the Model and US Data

Table 4: Relative standard deviations of entrepreneurial log net worth (n) and leverage ($\hat{\kappa}$), absolute standard deviation of log output y (in percentages), in representative and heterogeneous agent models; HP-filtered model data from simulation of 10^6 quarters; US data on non-financial business sector net worth and leverage, output as Gross Value Added of non-farm business sector; HP parameter 1,600.

		$\frac{\text{std}(n)}{\text{std}(y)}$	$\frac{\text{std}(\hat{\kappa})}{\text{std}(y)}$	$\text{std}(y), \%$
Data (76Q1–15Q3)		2.038	0.856	1.884
Representative agent baseline				
ρ_A	ξ, ψ			
0.95	1.0, 1.0	0.845	0.109	0.958
	13.2, 1.0	2.098	0.996	1.107
	5.92, $1/\xi$	2.069	1.005	1.115
0.99	1.0, 1.0	0.838	0.152	0.940
	4.34, 1.0	1.941	0.941	1.062
	2.57, $1/\xi$	1.549	0.767	0.989
Heterogeneous agent extension				
ρ_A, ρ_θ	Υ_A			
0.95, 0.80	-31.1	1.041	0.516	1.258
0.95, 0.95	-14.3	0.934	0.470	1.224
0.95, 0.95 (nom. rig.)	-29.3	1.004	0.541	1.632

Assets measured as Nonfinancial Assets (FOFA Tables B.103 and B.104, line 2), debt as the sum of Loans (FOFA Tables B.103 and B.104, lines 30 and 27, respectively) and Debt Securities (FOFA Table.103, line 26) – all measured at market values. *Net worth* = *Assets* − *Debt*, *Leverage* = *Assets*/*Net worth*. GVA measure from NIPA-BEA Table 1.3.5 line 3. All variables deflated by the implicit price index for the nonfarm business sector (NIPA-BEA Table 1.3.4, line 3).

B Entrepreneurs' Problem in the CFP Model and Equivalence to the log-Utility Representative Entrepreneur

This Appendix establishes the first order equivalence of the equilibrium in the CFP model and the model presented in Section 2 by comparing first order approximations of the equilibrium conditions which differ across the two.

B.1 CFP Model

B.1.1 Entrepreneurs' equilibrium conditions

For more details on the entrepreneurs' problem in the CFP model see Carlstrom et al. (2016). The bottom line is that in the CFP model, the optimality conditions for an entrepreneur's problem can be combined into the following Bellman equation, laws of motion and first order condition in the equilibrium variables $\{V_t, C_t^e, N_t, \bar{\omega}_t, \kappa_t\}$:

$$V_t = (1-\gamma) + \gamma\beta_e^{CFP}\mathbb{E}_t\left\{V_{t+1}[1-\Gamma(\bar{\omega}_{t+1})]R_{t+1}^k\right\}\kappa_t \tag{37}$$

$$C_t^e = (1-\gamma)[1-\Gamma(\bar{\omega}_t)]R_t^k\kappa_{t-1}N_{t-1} \tag{38}$$

$$N_t = [1-\Gamma(\bar{\omega}_t)]R_t^k\kappa_{t-1}N_{t-1} - C_t^e \tag{39}$$

$$\frac{\Gamma'(\bar{\omega}_t)}{\Gamma'(\bar{\omega}_t) - \mu G(\bar{\omega}_t)} = \left(\gamma\beta_e^{CFP}\frac{V_t}{V_{t-1}-(1-\gamma)}\right)^{-1} M_t \tag{40}$$

Plus the participation constraint (15), which effectively determines κ_t. Since these and all other equilibrium conditions are necessarily identical across the two models, I will not focus those. For brevity, let us denote the left hand side of (40) with the increasing function $\Psi(\bar{\omega}_t)$.

B.1.2 Steady state

In the non-stochastic steady state, combining (38) and (39) gives:

$$1 = \gamma[1-\Gamma(\bar{\omega})]R^k\kappa \tag{41}$$

And using this in (37) yields:

$$V = (1-\gamma) + \gamma\beta_e V[1-\Gamma(\bar{\omega})]R^k\kappa \Rightarrow V = \frac{1-\gamma}{1-\beta_e} \tag{42}$$

And (40), combined with (37) yields:

$$\Psi(\bar{\omega}) = M\left(\frac{\gamma\beta_e V}{V-(1-\gamma)}\right)^{-1} = M\left(\frac{\gamma\beta_e V}{\beta_e V}\right)^{-1} = M\gamma^{-1} \tag{43}$$

Thus, (42) separately determines V, and (41) and (43) alongside the remaining equilibrium conditions determine the rest of the steady state values.

B.1.3 First order dynamics

As mentioned in Section 2.3.1, (38) and (39) are exactly identical in the two models, given $\gamma = \beta_e$ and $\tilde{U}(C) = \log C$ in the model of Section 2.1, so their equivalence follows trivially. Also, to save on notation, I will denote $X_t^r \equiv [1-\Gamma(\bar{\omega}_t)]R_t^k\kappa_{t-1}$ in log-linearizing (37), as this product shows up in the same manner in both of the models.

Log-linearizing (37) gives, using the fact that in steady state $\gamma X^r = 1$:

$$v_t = \beta_e^{CFP}\mathbb{E}_t\left\{v_{t+1} + x_{t+1}^r\right\} \tag{44}$$

And log-linearizing (40), using the fact that in steady state $V-(1-\gamma) = \beta_e V$, yields:

$$\frac{\Psi'(\bar{\omega})\bar{\omega}}{\Psi(\bar{\omega})}\hat{\omega}_t = m_t - \left(v_t - \frac{1}{\beta_e}v_{t-1}\right) \tag{45}$$

B.2 Representative entrepreneur model

B.2.1 Entrepreneurs' equilibrium conditions

Following the analysis in Sections 2.2.4 and 2.3.1, if the entrepreneur has logarithmic utility $\tilde{U}(C) = \log(C)$, one can write the equilibrium conditions determining $\{V_t, C_t^e, N_t, \bar{\omega}_{t+1}, \kappa_t\}$ as:

$$V_t = \frac{1}{C_t^e} \tag{46}$$

$$C_t^e = (1-\beta_e)[1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1} \tag{47}$$

$$N_t = [1-\Gamma(\bar{\omega}_t)]R_t^k \kappa_{t-1} N_{t-1} - C_t^e \tag{48}$$

$$\Psi(\bar{\omega}_t) = M_t \left(\beta_e \frac{V_t}{V_{t-1}}\right)^{-1} \tag{49}$$

Plus the participation constraint (15), which again effectively pins down κ_t. As discussed in Section 2.3.1, the result that under log-utility, consumption is a constant fraction of equity can be reached by employing (48) and the entrepreneurs' Euler equation (16), with the latter now being replaced by (47). As mentioned, (47) and (48) are identical across the two models.

B.2.2 Steady state

In steady state, combining (47) and (48) implies:

$$1 = \beta_e[1-\Gamma(\bar{\omega})]R^k \kappa \tag{50}$$

And (49) implies

$$\Psi(\bar{\omega}) = M\beta_e^{-1} \tag{51}$$

which are identical to (41) and (43) whenever $\gamma = \beta_e$, so the two models have *exactly* the same non-stochastic steady states, apart from the value of V which in this case is pinned down by

$$V = \frac{1}{C^e} \tag{52}$$

B.2.3 First order dynamics

Log-linearizing (49) directly yields:

$$\frac{\Psi'(\bar{\omega})\bar{\omega}}{\Psi(\bar{\omega})}\hat{\omega}_t = m_t - (v_t - v_{t-1}) \tag{53}$$

which is equivalent to (45) whenever $\beta_e^{CFP} \to 1$.

And finally, because the Euler equation for the entrepreneur must still be satisfied by V_t, even though now redundant, it is necessarily the case that V_t satisfies

$$V_t = \beta_e \mathbb{E}_t\left\{V_{t+1}[1-\Gamma(\bar{\omega}_{t+1})]R_{t+1}^k\right\}\kappa_t$$
$$\Rightarrow v_t = \mathbb{E}_t\left\{v_{t+1} + x_{t+1}^r\right\} \tag{54}$$

which is equivalent to (45) whenever $\beta_e^{CFP} \to 1$.

So we have established the equivalence of the five equilibrium conditions relevant for determining the outcome of the entrepreneurs' problem in these two log-linearized models whenever $\beta_e^{CFP} \to 1$ and $\gamma = \beta_e$.

C Properties of the Idiosyncratic Labor Productivity Process

The analysis in Sections 4 and 5 treats the variance of the growth of the idiosyncratic consumption component $\log(\theta_{i,t})$ as the main measure of idiosyncratic labor risk, while leaving unspecified the exact stochastic properties of the actual fundamental labor productivity process $\eta_{i,t}$ that facilitates such a behavior of equilibrium household consumption. This Appendix provides further details on the implied stochastic properties of $\eta_{i,t}$ and argues that in the no-trade equilibria studied, the implied behavior of household i's labor productivity growth $g_{i,t}^{\eta} \equiv \frac{\eta_{i,t}}{\eta_{i,t-1}}$ is very similar to the growth $g_{i,t}^{\theta}$ in the idiosyncratic consumption component that it is supposed to facilitate in equilibrium, with $g_{i,t}^{\theta} \equiv \frac{\theta_{i,t}}{\theta_{i,t-1}}$ being a lognormal random variable.

To elaborate upon the implied stochastic properties of $\eta_{i,t}$, let us rewrite (32) in t by plugging in for $(1+\phi)\theta_{i,t-1}C_{t-1}$ from (32) itself in $t-1$:

$$\eta_{i,t} = \frac{1}{W_t}\left\{\frac{\theta_{i,t}}{\theta_{i,t-1}}\frac{C_t}{C_{t-1}}\left[W_{t-1}\eta_{i,t-1} - s_{t-1}\right] + s_t\right\}$$

where for brevity, $s_t \equiv d_{t+1} - R_t^d d_t - \Pi_t^I$ denotes net financial savings in period t. We can then write household i's implied labor productivity growth as:

$$\begin{aligned} g_{i,t}^{\eta} &= \Phi_{1i,t} g_{i,t}^{\theta} + \Phi_{0i,t} \\ &\text{where} \\ \Phi_{1i,t} &\equiv \frac{C_t}{C_{t-1}}\frac{W_{t-1}}{W_t}\left[1 - \frac{s_{t-1}}{W_{t-1}}\frac{1}{\eta_{i,t-1}}\right] \\ \Phi_{0i,t} &\equiv \frac{s_t}{W_t}\frac{1}{\eta_{i,t-1}} \end{aligned}$$

This means that conditional on $\eta_{i,t-1}$ and aggregate shocks up to t, $g_{i,t}^{\eta}$ has a *three-parameter lognormal* distribution, with $\Phi_{0i,t}$ being the *threshold* parameter.

Unsurprisingly, we see that the conditional distribution of labor productivity growth $g_{i,t}^{\eta} = \frac{\eta_{i,t}}{\eta_{i,t-1}}$ must depend on the prior level of productivity $\eta_{i,t-1}$, even though the growth of the idiosyncratic consumption component $g_{i,t}^{\theta}$ is i.i.d. across time. This is because we are imposing that the distribution of consumption growth is identical across households, while all households hold the *same* financial portfolio. This means that depending on the precise path of $\{\theta_{i,t}\}_t$ experienced by household i over time, the shares of human and financial wealth in its total wealth will differ. And because in each period, i's consumption equals the difference between labor income and net financial savings, a percentage change in consumption will be consistent with different percentage changes in labor income, depending on the relative size of s_t and $c_{i,t}$.[24] Because of this, any comparison of the properties of $g_{i,t}^{\eta}$ and $g_{i,t}^{\theta}$ below will necessarily be contingent on the distribution of $\eta_{i,t-1}$ considered, as reported.

If it was the case that $\Phi_{0i,t} = 0$, then $g_{i,t}^{\eta}$ would be *exactly* lognormal, conditional on $\eta_{i,t-1}$ and aggregate shocks up to t. Although this is not exactly true in the model, $\frac{s_t}{W_t}$ is actually small in the nonstochastic steady state and its proximity. This simply follows from the fact that since $\Pi_{ss}^I = 0$, we

[24]Put simply, if $c_i = w_i - s$, then the elasticity of c_i with respect to w_i depends on the magnitude of s relative to w_i.

have that $s_{ss} = (1 - R^l_{ss})d_{ss} = (1 - 1/\beta)d_{ss}$. Thus, if $\eta_{i,t-1}$ is not too small, then $\Phi_{0i,t}$ is close to zero and the distribution of $g^{\eta}_{i,t}$ conditional on aggregates and $\eta_{i,t-1}$ is barely distinguishable from that of a lognormal variable. Moreover, for the same reasons, near the steady state, $\Phi_{1i,t}$ is close to 1, implying that the conditional distribution of $g^{\eta}_{i,t}$ is actually close to that of $g^{\theta}_{i,t}$. To illustrate this, Figure 8 below compares the histograms of a sample of $g^{\theta}_{i,t}$ and the corresponding $g^{\eta}_{i,t}$ in the nonstochastic steady state of the flexible price heterogeneous household model calibration studied in Section 5.2, conditional on $\eta_{i,t-1} = 1$. More specifically, in that case, $\Phi_{0i,ss} = 1 - \Phi_{1i,ss} = -0.109$. It is clear that the two distributions are virtually indistinguishable.

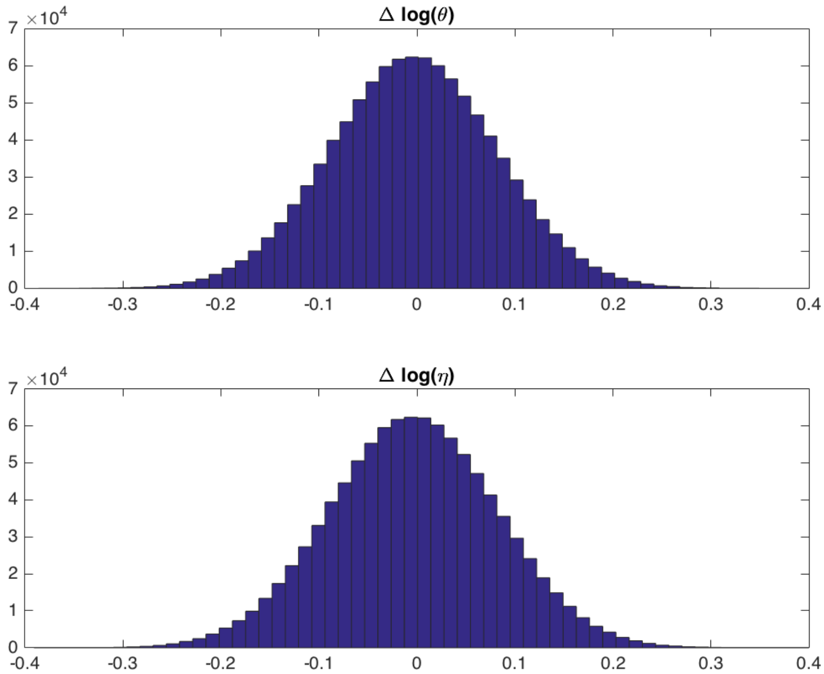
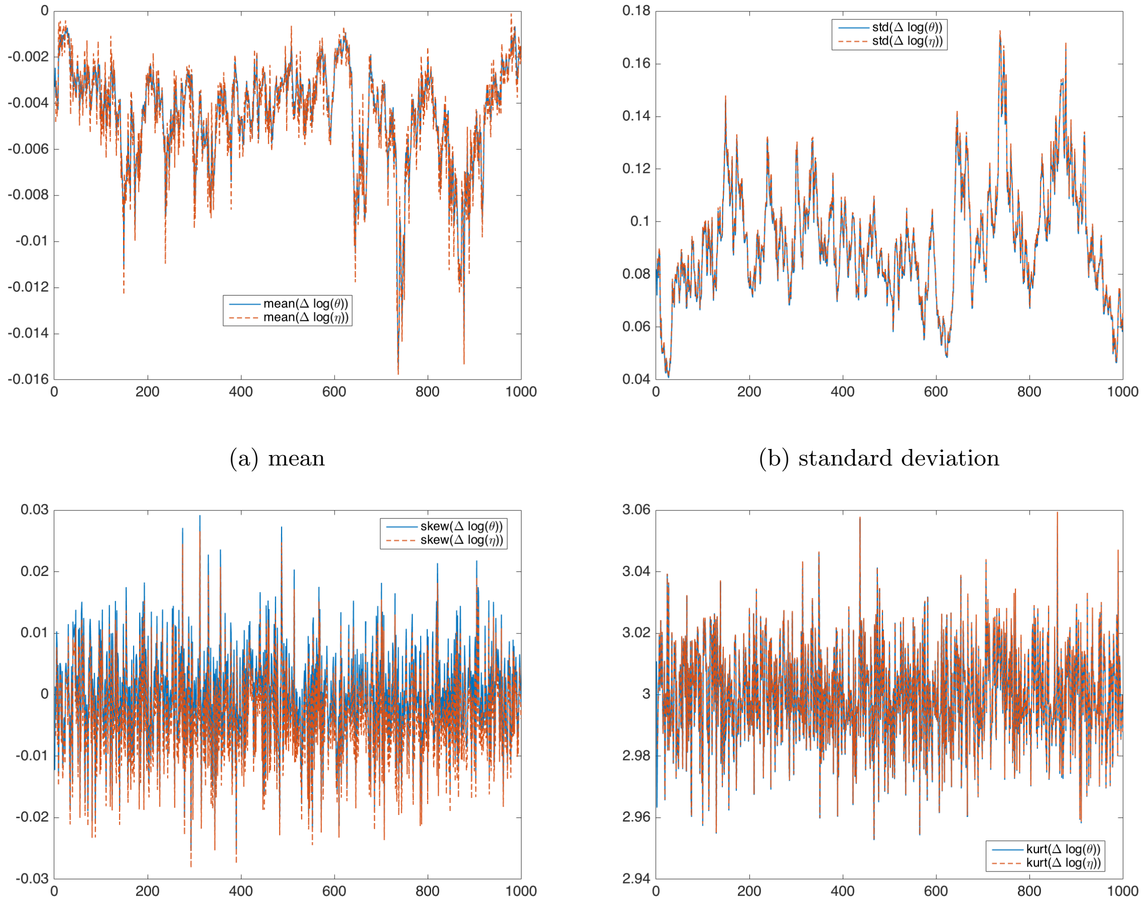


Figure 8: Histograms of $\log(g^{\theta}_{i,t})$, and the corresponding $\log(g^{\eta}_{i,t})$, conditional on $\eta_{i,t-1} = 1$ in the steady state of the baseline calibration of the flexible price heterogeneous household model; based on a sample of 10^6 draws.

The cross-sectional distributions of $g^{\eta}_{i,t}$ over time remain very close to that of $g^{\theta}_{i,t}$ when we allow the aggregate economy to evolve subject to aggregate TFP shocks. Figure 9 plots the time series of the first four moments in the cross-section of $\log(g^{\theta}_{i,t})$ and $\log(g^{\eta}_{i,t})$ for a simulation of the aggregate economy for $T = 1000$ quarters, given the baseline calibration with $\Upsilon_A = -14.3$, $\rho_\theta = \rho_A = 0.95$, and $\text{std}(\varepsilon^A_t) = 0.0072$, and given $\eta_{i,t-1} = 1$. The similarity between the series is stark, explained by the fact that over the sample, the terms $\Phi_{0i,t}$ and $\Phi_{1i,t}$ vary little, with the standard deviations across time being approximately $\text{std}(\Phi_{0i,t}) \approx \text{std}(\Phi_{1i,t}) \approx 8.2 \times 10^{-4}$.

Although the above has established that given $\eta_{i,t-1} = 1$, the behaviors of $g^{\eta}_{i,t}$ and $g^{\theta}_{i,t}$ are virtually indistinguishable, things might not be as clear when we allow for the cross-sectional distribution of $\theta_{i,t}$ and $\eta_{i,t}$ to evolve. Large discrepancies from such similarities can arise especially because the cross-sectional dispersion in $\eta_{i,t}$ can lead to large differences in the implied labor productivity variance that households face. Moreover, because $\log(g^{\theta}_{i,t})$ follows a random walk, $\theta_{i,t}$ and $\eta_{i,t}$ do not have stationary cross-sectional distributions. To facilitate stationarity of the distribution of $\theta_{i,t}$ and $\eta_{i,t}$, one can directly follow Constantinides and Duffie (1996) and assume that each household has a conditional probability of survival $\varpi \in (0, 1)$ between periods t and $t + 1$. If a household dies between t and $t + 1$, it receives no utility from period $t + 1$ onwards, and its financial wealth is transferred to a "newborn", with "identity



(a) mean

(b) standard deviation

(c) skewness

(d) kurtosis

Figure 9: Time series of the first four moments of the cross-section of $\log(g^{\theta}_{i,t})$ and $\log(g^{\eta}_{i,t})$ at different points in time for a simulation of the aggregate economy for $T = 1000$ quarters and $N = 10^5$ individuals, given baseline calibration with $\Upsilon_A = -14.3$, $\rho_\theta = \rho_A = 0.95$, $\text{std}(\varepsilon^A_t) = 0.0072$, and given $\eta_{i,t-1} = 1$

label" i, for whom the initial $\eta_{i,t+1}$ is the productivity consistent with $\theta_{i,t+1} = 1$. If the households in this specification have a time discount factor $\bar{\beta} = \tilde{\beta}/\varpi$, then the behavior of the aggregate economy is identical to that of the model introduced in Section 4. At the same time, the distribution of $\theta_{i,t}$ and $\eta_{i,t}$ is stationary, with each agent's expected lifetime being $\frac{1}{1-\varpi}$ periods.

To examine the behavior of $\eta_{i,t}$ given a stationary distribution, I choose $\varpi = 0.9815$ to imply that 95% of the households are expected to have an active work life of at most 160 quarters, or 40 years. I then generate a stationary distribution of $\theta_{i,t}$ in the model's nonstochastic steady state, and compare both the distributions of $\theta_{i,t}$ and $\eta_{i,t}$, and the implied distributions of $g^{\theta}_{i,t}$ and $g^{\eta}_{i,t}$ that these agents face. Figure 10 plots the stationary distributions of $\log(\theta_{i,t})$ and $\log(\eta_{i,t})$, truncated at the 0.01% and 99.9% levels.[25] The implied stationary distributions of the idiosyncratic components of consumption and individual labor productivities are again barely distinguishable.

Finally, Figure 11 depicts a sample of the cross-section of growth rates $\log(g^{\theta}_{i,t})$ and $\log(g^{\eta}_{i,t})$ conditional on a stationary distribution of households. That is, for each of the 10^5 households in the stationary distribution of $(\theta_{i,t-1}, \eta_{i,t-1})$, I draw an idiosyncratic shock $\varepsilon_{i,t}$, and compute the implied cross-section

[25] Because $s_{ss} < 0$, it can potentially happen that if household i's $\theta_{i,t} \to 0$, the implied labor productivity $\eta_{i,t} < 0$. However, in the simulated sample, such cases are highly unlikely, occurring for about 0.02% of the individuals in the stationary distribution.

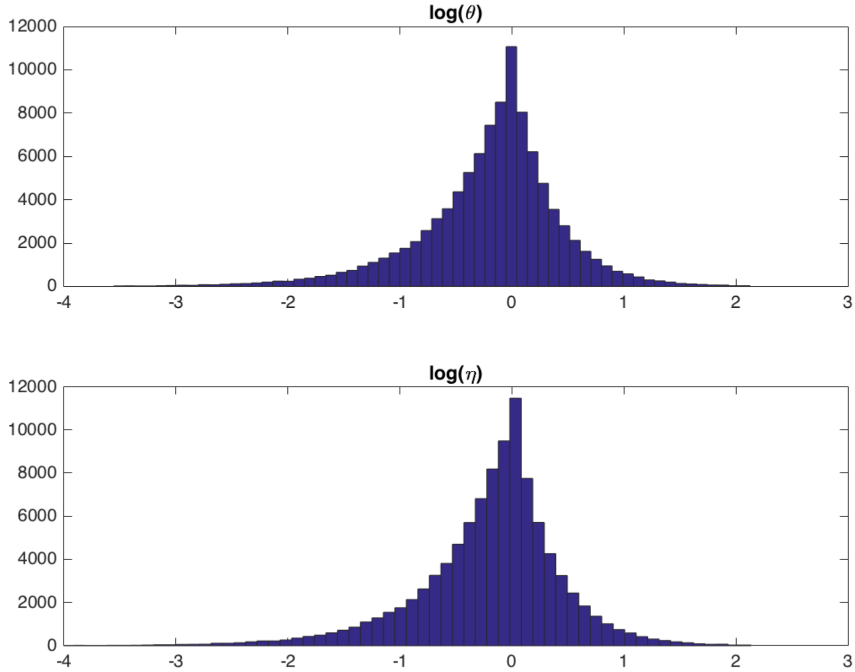


Figure 10: Stationary distributions of $\log(\theta_{i,t})$, and the corresponding $\log(\eta_{i,t})$, in the steady state of the baseline calibration of the flexible price heterogeneous household model; based on a sample of 10^5 individuals, given survival probability $\varpi = 0.9815$.

of growth rates $g^{\theta}_{i,t}$ and $g^{\eta}_{i,t}$. Clearly, because $g^{\theta}_{i,t}$ is i.i.d. across agents and time by assumption, the top panel in this Figure must be identical to the top panel in Figure 8, up to sampling variation. The only difference between Figures 8 and 11 is the distribution of $\eta_{i,t-1}$ which we are conditioning upon. What is striking, however, is that the implied cross-sectional distribution of $g^{\eta}_{i,t}$ in Figure 11 is still indistinguishable from that of $g^{\theta}_{i,t}$. It is noteworthy because, as seen above, $\eta_{i,t-1}$ affects the conditional distribution of $g^{\eta}_{i,t}$ faced by household i. For example, the distributions of $g^{\theta}_{i,t}$ and $g^{\eta}_{i,t}$ would visibly differ if we conditioned the computations in Figure 11 on, say $\eta_{i,t-1} = 0.01, \forall i$. But even though the stationary distribution of $\eta_{i,t}$ exhibits nontrivial variation, as seen in the bottom panel of Figure 10, this variation is not large enough to have any significant effects on the "average" distribution of growth rates of $\eta_{i,t}$ faced by the households. Thus, we can conclude that on the whole, there is very little difference in the stochastic properties of the idiosyncratic consumption component $\theta_{i,t}$ and the idiosyncratic labor productivity process $\eta_{i,t}$ faced by the households in the economy, and the treatment of the variance of growth in $\theta_{i,t}$ as a measure of households' idiosyncratic labor risk in Sections 4 and 5 is validated.

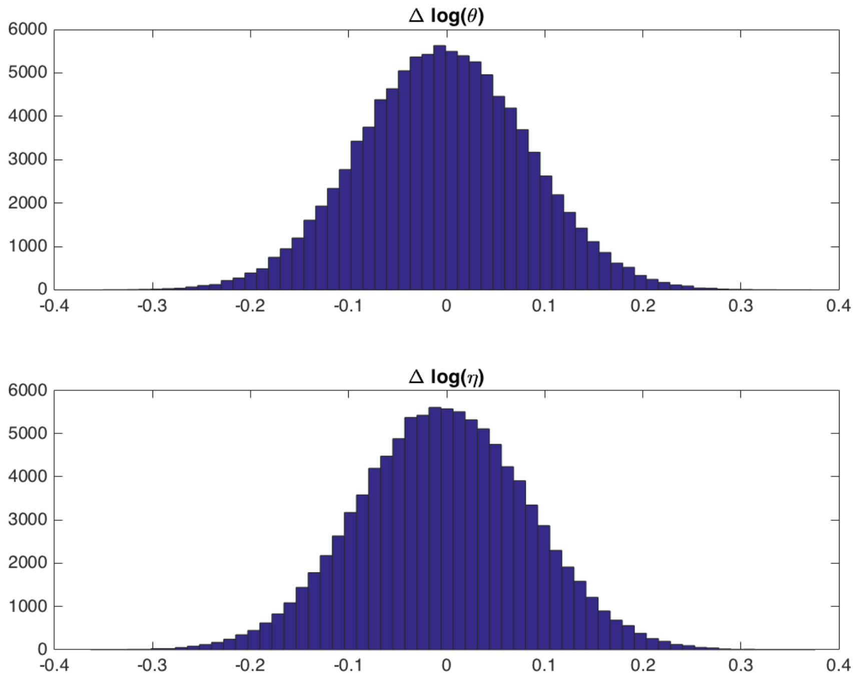


Figure 11: Cross-sectional distributions of growth rates $\log(g^{\theta}_{i,t})$ and $\log(g^{\eta}_{i,t})$ faced by households given the stationary distributions of $\theta_{i,t}$ and $\eta_{i,t}$, in the steady state of the baseline calibration of the flexible price heterogeneous household model; based on a sample of 10^5 individuals, given survival probability $\varpi = 0.9815$.

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