Competitive bargaining equilibrium

J. Dávila[a, b], J. Eeckhout[c, *]

[a]*Centre d'Economie de la Sorbonne, CNRS, France*
[b]*ECARES, Université Libre de Bruxelles, Belgium*
[c]*Department of Economics, University of Pennsylvania, USA*

Abstract

In a simple exchange economy we propose a bargaining procedure that leads to a Walrasian outcome as the agents become increasingly patient. The competitive outcome therefore obtains even if agents have market power and are not price-takers. Moreover, where in other bargaining protocols the final outcome depends on bargaining power or relative impatience, the outcome here is determinate and depends only on preferences and endowments. Our bargaining procedure involves bargaining over prices and maximum quantity constraints, and it guarantees convergence to a Walrasian outcome for *any* standard exchange economy. In contrast, without quantity constraints we show that equilibrium is generically inefficient.
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1. Introduction

Price-taking behavior is typically invoked as a necessary requirement to obtain the competitive outcome. In this paper, we propose a bargaining foundation for the Walrasian equilibrium in a small exchange economy where agents are not price-takers. The bargaining procedure we analyze relates to those studied in Binmore [2] and Yildiz [17]. More specifically, in our set-up each agent alternatingly offers a price and a maximum amount to be exchanged, and the respondent either accepts and chooses the quantities to be traded at the terms of the offer, or rejects and makes an offer in the next period in which utilities are discounted. We show in this set-up that the competitive outcome obtains when bargaining frictions vanish, even without price-taking

* Corresponding author.
  E-mail addresses: julio.davila@univ-paris1.fr (J. Dávila), eeckhout@ssc.upenn.edu (J. Eeckhout).

behavior. This convergence result holds for any standard exchange economy. Moreover, the outcome does not depend on details such as relative impatience and bargaining power or outside options. Paradoxically, by explicitly introducing price-setting as a strategic variable in an otherwise standard bargaining environment, the competitive outcome is restored. Price-taking is therefore not a necessary requirement for attaining a perfectly competitive outcome.

The main implication of the convergence result is that, as discounting frictions vanish, the bargaining outcome does not depend on the exact specification of time preferences. Instead, the bargaining outcome converges to a Walrasian allocation which is determined by the preferences and endowments of the agents. It seems natural after all that the bargaining outcome is not exclusively determined by relative patience (or by exogenous bargaining power in axiomatic Nash bargaining) as is the case in the alternating-offers bargaining of Rubinstein [11] or Ståhl [16]. There, the relatively patient agent obtains a proportionally larger share of the surplus (see for example Binmore, Rubinstein and Wolinsky [1]). Rather, in many economic environments the bargaining outcome may depend, at least in part, on preferences and endowments, for example on the degree of substitutability between the goods consumed.

The bargaining procedure with price offers that guarantees our convergence results *necessarily* involves maximum trade constraints. This indicates that the details of the bargaining procedure are important (see also Binmore [2]). We show that the conditions for convergence obtained in Yildiz [17] for a bargaining procedure over prices *without maximum trade constraints* are too strong in the sense that almost no economy satisfies the assumptions made in Yildiz [17]. In general, for any economy in an open and dense subset of the set of standard exchange economies there will exist at least one stationary equilibrium of the bargaining game without quantity constraints that converges to an inefficient outcome. This inefficient outcome leaves each agent indifferent between two distinct allocations. The utility obtained when accepting a price offer on her own offer curve is the same as the utility obtained from her own price offer. In the latter case, her accepted offer induces the other agent to choose an allocation on his own offer curve.

Intuitively, the maximum constraint provides a credible tool for the offering agent to induce the outcome to be efficient. Without the constraint, the two agents may get locked into an inefficient outcome in which each agent is indifferent between accepting an unfavorable price now while choosing the quantity, and having a favorable price accepted tomorrow letting the quantity to be chosen by the other agent. The proposer of a price cannot prevent the responder from asking for his demand at this price, so that no agent can deviate by offering a Pareto-improving outcome within the lens formed by the indifference curves corresponding to the inefficient levels of utility. In contrast, the maximum trade constraint allows the offering agents to undo this inefficient outcome with a deviating offer of a price and maximum trade that induces the responder to accept and ask for an allocation within this lens of Pareto-improving allocations. This highlights the role played by the maximum quantity constraints in order to obtain efficiency. Such maximum constraints are common in commodity markets, limit orders in stock transactions, and wage bargaining.

The importance of maximum trade constraints is first established by Binmore [2] in the context of axiomatic bargaining. He is the first in the modern bargaining literature to connect the competitive equilibrium to bargaining outcomes in two person economies. He presents a modified Nash demand game with minimum prices and maximum quantity constraints and shows that the (large) set of Nash equilibria of this game includes the Walrasian allocation. [1]

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[1] See also Serrano and Volij [13] for the relation between axiomatic bargaining and Walrasian allocations.

In the next section, we lay out the model. In Section 3, we present in Theorem 1 the main result of convergence to a Walrasian allocation. We obtain first the result for equilibria in which there is immediate acceptance by showing that, without discounting, every stationary subgame perfect (SSP) equilibrium allocation is Walrasian and conversely. Then the convergence result in the second part of the theorem follows from the upper hemicontinuity of the equilibrium correspondence. We also establish that SSP equilibria with delay do not exist, from which the convergence result in Theorem 1 for *every* SSP equilibrium follows. We then study non-stationary subgame perfect equilibria. When there is a unique Walrasian allocation, Theorem 2 establishes that any non-stationary SP equilibrium converges to the Walrasian allocation. In Section 4 we consider the same bargaining procedure but without maximum trade constraints, as in Yildiz [17]. In this case, Theorem 3 shows the generic existence of asymptotically inefficient SSP equilibria, and the theorem also shows the impossibility of a unique SSP equilibrium converging to a Walrasian outcome. Section 5 provides a discussion. Section 6 finishes with some concluding remarks. Proofs and some propositions and lemmas are relegated to the appendix.

2. The model

Consider an exchange economy with two agents A and B, each with endowments e^A and e^B of n goods over which they have preferences represented by utility functions u^A and u^B satisfying Assumption A1.

Assumption A1.

(1) For all $i = A, B$, u^i is $\mathbb{R}_+$-valued; continuous in $\mathbb{R}_+^n$; differentiable in $\mathbb{R}_{++}^n$; monotonous in the sense that $Du^i(x) \in \mathbb{R}_{++}^n$ always; strictly differentiably quasi-concave in the sense that $D^2 u^i(x)$ is always negative definite in the normal space of $Du^i(x)$; and well-behaved at the boundary in the sense that $(u^i)^{-1}(a) \subset \mathbb{R}_{++}^n$ for any $a \in u^i(\mathbb{R}_{++}^n)$.
(2) $e^A, e^B \in \mathbb{R}_{++}^n$.

In general, for given endowments, the allocation is not Pareto-efficient and therefore there exist gains from trade. This paper addresses whether all gains from trade are realized and, if so, which efficient allocation is obtained.

Since the price-taking assumption is not easily justified in a two-person economy, we propose a bargaining procedure in which agents set prices that allow them to realize completely the gains from trade. More specifically, we consider an alternating-offers bargaining game in which, in any given period, one of the agents offers to the other a vector of relative prices at which he is willing to trade up to some maximum amount (the *maximum trade constraint* henceforth). Thus an offer by say A consists of a vector of prices p^A (in terms of say good 1) and a quantity constraint q^A. Without loss of generality, we assume the quantity constraint is on the amount traded of any of the goods. Upon the reception of an offer, the recipient, i.e., B in this case, can either accept the offer or reject it. If she accepts, she then chooses her most preferred consumption $\tilde{x}^B(p^A, q^A)$ at the offered price, without violating the maximum trade constraint expressed in the offer, i.e., $\tilde{x}^B(p^A, q^A)$ solves

$$
\begin{aligned}
\max_{x^B} \quad & u^B(x^B) \\
\text{s.t.} \quad & p^A(x^B - e^B) \leqslant 0, \\
& |x^B - e^B| \leqslant q^A,
\end{aligned}
\tag{1}
$$

where the norm used is the sup norm. If B rejects the offer, then B counter-offers another pair (p^B, q^B) with new prices and a new maximum trade constraint. The utility of both agents A and B is discounted for every iteration by positive discount factors δ^A and δ^B not bigger than 1. Not reaching an agreement entails a zero utility to both agents.

An SSP equilibrium with immediate acceptance of this game is characterized by a pair of offers (p^A, q^A) and (p^B, q^B) such that, in every subgame where A is called to make an offer, A offers (p^A, q^A) and this offer solves

$$
\max_{p,q} \quad u^A(e - \tilde{x}^B(p, q))
$$

$$
\text{s.t.} \quad u^B(\tilde{x}^B(p, q)) \geqslant \delta^B u^B(e - \tilde{x}^A(p^B, q^B)), \tag{2}
$$

given (p^B, q^B) (where $e = e^A + e^B$ denotes the total endowments) and similarly for (p^B, q^B) given (p^A, q^A). From subgame perfection, once B decides to accept any offer (p, q) from A, she will choose the consumption bundle $\tilde{x}^B(p, q)$ that maximizes her utility subject to the terms of the offer. Therefore, knowing that upon acceptance B chooses $\tilde{x}^B(p, q)$, A decides to make an offer (p^A, q^A) that maximizes his utility of consuming $e - \tilde{x}^B(p, q)$, provided that the offer induces B to accept it. This requires that B obtains at least as much utility from accepting the offer (p^A, q^A), i.e., $u^B(\tilde{x}^B(p^A, q^A))$, as she would get from rejecting the offer and waiting for her equilibrium offer (p^B, q^B) to be accepted in the next period, which gives her a utility $\delta^B u^B(e - \tilde{x}^A(p^B, q^B))$.

It turns out that there is no loss of generality in focusing on the SSP equilibria with immediate acceptance. We focus first on equilibria with immediate acceptance, and then we show that there are no SSP equilibria with delay whenever the agents are impatient.

3. The main result

In this section, we establish that the limit of every convergent sequence of allocations of SSP equilibria with immediate acceptance of the bargaining over prices and maximum trades is a Walrasian allocation. The key insight of the argument is that in exchanging price and quantity offers, the agents are actually bargaining over *some* allocations. Given subgame perfection, any agent making an offer anticipates the optimal acceptance behavior by the recipient, and therefore an offer (p^A, q^A) amounts to offering the allocation $(e - \tilde{x}^B(p^A, q^A), \tilde{x}^B(p^A, q^A))$, where $\tilde{x}^B(p^A, q^A)$ is the consumption chosen by B given the prices p^A and the maximum trade constraint q^A.

This allows us to characterize the allocations that might be accepted at an SSP equilibrium with immediate acceptance. Note first that in the absence of a maximum trade constraint q^A (or equivalently, when the constraint is slack), B's response to A's offer is to choose her demand $x^B(p^A)$ at the prices p^A. Note also that by means of the maximum trade constraint q^A, agent A can prevent agent B from attaining $x^B(p^A)$, forcing her to a lesser trade. Nevertheless, in no instance can A force B to exchange more than necessary to attain her desired demand $x^B(p^A)$ at those prices. As a consequence, an offer by A that is responded with an optimal acceptance decision by B results in an allocation $(e - x^B, x^B)$ such that

$$
Du^B(x^B)(x^B - e^B) \geqslant 0. \tag{3}
$$

As shown by Lemma A1 in the appendix, condition (3) characterizes the set of solutions to maximization problems of the class (1) above. It holds with equality if the maximum trade constraint does not effectively constrain B's choice and with strict inequality otherwise. A similar

condition holds for offers made by B that might be accepted by A at an SSP equilibrium with immediate acceptance.

As a consequence an SSP equilibrium with immediate acceptance of the alternating-offers bargaining over prices and maximum trades can be characterized by two feasible allocations (x_A^A, x_A^B) and (x_B^A, x_B^B)—where (x_A^A, x_A^B) denotes the allocation resulting from the acceptance of A's offer, and similarly for (x_B^A, x_B^B)—such that (x_A^A, x_A^B) solves

$$\max_{x^A, x^B} \quad u^A(x^A),$$
$$Du^B(x^B)(x^B - e^B) \geqslant 0,$$
$$u^B(x^B) \geqslant \delta^B u^B(x_B^B),$$
$$x^A + x^B = e^A + e^B, \tag{4}$$

given (x_B^A, x_B^B), and likewise for (x_B^A, x_B^B) given (x_A^A, x_A^B). Condition (4) is thus equivalent to condition (2).

This characterization allows us to establish in part (1) of Theorem 1 below that, for infinitely patient agents (i.e., $\delta^A = \delta^B = 1$), every Walrasian allocation is the allocation of an SSP equilibrium with immediate acceptance and conversely. As a consequence of this and of the upper hemicontinuity of the correspondence associating the SSP equilibrium allocations to the discount factors (whenever the latter is not empty-valued in some neighborhood of $(\delta^A, \delta^B) = (1, 1)$) it follows in part (2) of Theorem 1 that every convergent sequence of allocations of SSP equilibria with immediate acceptance, as the agents become arbitrarily patient, converges to a Walrasian allocation. The proof of Theorem 1 is provided in the appendix. Lemma A3 in the appendix establishes that no equilibrium with delay exists, so that the result in part (2) holds for *all* SSP equilibria. Proposition 1 in the appendix provides sufficient conditions for equilibrium existence.

Theorem 1. *For any economy satisfying A1,*

(1) *for infinitely patient agents, the set of Walrasian allocations and the set of SSP equilibrium allocations with immediate acceptance of the alternating-offers bargaining over prices and maximum trades coincide,*
(2) *the limit, as the agents become arbitrarily patient, of every convergent sequence of SSP equilibrium allocations of this bargaining is a Walrasian allocation.*

Theorem 1 considers only stationary strategies, therefore ruling out the use of non-stationary threats. Merlo and Wilson [10] show that such non-stationary strategies can lead to a continuum of SP equilibria.[2] In the light of such indeterminacy, stationarity is often invoked as a natural selection criterion because it acts as focal point within the set of SP equilibria, or because it requires a minimal number of states for automata to implement an SP equilibrium.

Nonetheless, for economies in which the Walrasian allocation is unique, it follows from Theorem 1 that every SSP equilibrium allocation converges to the unique Walrasian allocation. Theorem 2 below then follows from this convergence result and Theorem 8 in Merlo and Wilson [10]. The proof is provided in the appendix.[3] This result requires the following additional assumption:

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[2] Yildiz [17] also analyzes non-stationary strategies in the case of bargaining over prices without quantity constraints. We discuss his results in detail in the next section.

[3] We are grateful to Antonio Merlo for pointing us to their result in Merlo and Wilson [10].

Assumption A2. For all $(i, j) \in \{A, B\}^2$, the set

$$f^i \equiv \left\{ (v^i, v^j) \in \mathbb{R}_+^2 \,|\, (v^i, v^j) = \left(u^i(x^i(p)), u^j(e - x^i(p)) \right), \text{ for some } p \in \mathbb{R}_{++}^n \right\} \quad (5)$$

is single-peaked.

Theorem 2. *If an economy* $\{u^i, e^i\}_{i=A,B}$ *satisfying A1 and A2 has a unique Walrasian equilibrium, then all the SP equilibria of the bargaining over prices and maximum trades converge to the Walrasian equilibrium.*

4. Bargaining over prices only

As the agents exchange price offers in the bargaining protocol considered above, the maximum trades they express in their offers are essential for the convergence result to obtain. If the agents bargain over prices only (as in Yildiz [17]), then generically there exist SSP equilibria whose payoffs to the agents remain bounded away from the Pareto frontier, even in the limit as their discount factors δ^A and δ^B converge to 1. Acceptance is as before, and an offer now consists only of a price p.

The existence of an inefficient SSP equilibrium can be discerned, as in Yildiz [17], by looking at how the SSP equilibrium payoffs behave in the space of utilities as δ^A and δ^B converge to 1 in this case. [4] Note first that at an SSP equilibrium necessarily the only constraints the agents face making their offers (namely, the acceptability of their offers) must be binding, i.e.,

$$u^A(x^A(p^B)) - \delta^A u^A(e - x^B(p^A)) = 0,$$
$$u^B(x^B(p^A)) - \delta^B u^B(e - x^A(p^B)) = 0. \quad (6)$$

Equivalently, the payoffs of an SSP equilibrium of the bargaining over prices only must be intersections in the space of utilities of two curves $f^A_{\delta^B}$, $f^B_{\delta^A}$ parameterized by the relative price p and defined as

$$f^A_{\delta^B}(p) = (u^A(x^A(p)), \delta^B u^B(e - x^A(p))),$$
$$f^B_{\delta^A}(p) = (\delta^A u^A(e - x^B(p)), u^B(x^B(p))). \quad (7)$$

These curves are, for discount factors δ^A and δ^B close to 1, slightly continuous deformations of their counterparts f^A, f^B for $\delta^A, \delta^B = 1$ represented in Fig. 1 below.

The typical pattern of f^A, for instance, is that as A's utility increases when we move away from the endowment point along A's offer curve, B's utility initially increases too, but eventually decreases. [5] And similarly for f^B with the roles of the axes reversed. The two curves f^A and f^B intersect at any profile of Walrasian utilities like (u^{A*}, u^{B*}) on the Pareto frontier (in dashes) in

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[4] The analysis in Yildiz [17] is developed entirely in the space of utilities. As a matter of fact, even the assumptions in Yildiz [17] are assumptions made directly on derived concepts in this space, such as the "offer curves" in the space of utilities, instead of being made on the fundamentals of the economy. Unfortunately, it turns out that the assumptions in Yildiz [17] made this way are actually satisfied jointly only by a degenerate set of economies. Our analysis in the previous section has been made instead in the space of allocations and taking into account explicitly the fundamentals of the economy in order to avoid this problem.

[5] Note that the curve needs not be single-peaked in general. The single-peakedness stated in assumption A2 is only needed in Theorem 2.
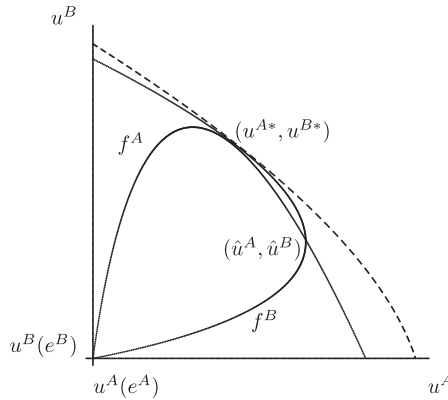
Fig. 1.

Fig. 1. Also it follows from the differentiability of both f^A and f^B, and of the Pareto frontier itself that all the three curves are tangent at any Walrasian profile of utilities such as (u^{A*}, u^{B*}). [6]

The intersection of f^A and f^B at a Walrasian intersection (u^{A*}, u^{B*}) has another important property, illustrated in Fig. 1 and stated in Lemma 1 below. That property plays a crucial role in showing the existence of SSP equilibria of the alternating-offers bargaining over prices (without quantity constraints) that remain inefficient even as $\delta^A, \delta^B \to 1$. The proof is provided in the appendix. In Lemma 1, and for the remainder of the paper, a property will be said to hold generically, or to be generic, in the space of economies satisfying A1 whenever it is satisfied within an open and dense set of this space with respect to the topology of C^n uniform convergence on compacts in the space of utility functions, for $n \geqslant 2$, and the usual topology in the space of endowments.

Lemma 1. *In the space of economies $\{u^i, e^i\}_{i=A,B}$ satisfying A1, the intersection without crossing of the curves f^A and f^B at any Walrasian profile (u^{A*}, u^{B*}) is a generic property.*

A short discussion is at this point in order. If one assumes on the contrary that f^A and f^B do actually cross at a unique Walrasian intersection (u^{A*}, u^{B*}), then it can be proved that the alternating-offers bargaining over prices has only one SSP equilibrium that moreover necessarily converges to the Walrasian equilibrium as δ^A and δ^B converge to 1. This has been established in Yildiz [17] under his assumptions A3 (both monopolistic outcomes are dominated by some allocation attainable along an offer curve) and A4 (there is a unique crossing of f^A and f^B within the interval defined by the profiles of utilities attained at the monopolistic outcomes). Nevertheless, while each of the two assumptions A3 and A4 in Yildiz [17] are not degenerate on their own, the requirement of both of them to hold simultaneously amounts to having a crossing of f^A and f^B at a Walrasian profile of utilities which, according to Lemma 1 above, makes them a degenerate set of assumptions.

Other results in Yildiz [17] do not rely on A3 and A4 holding simultaneously. Theorem 6 in Yildiz [17] establishes under general assumptions that as $\delta^A, \delta^B \to 1$ the SP equilibrium payoffs of the bargaining only over prices are within an arbitrarily small distance of the rectangle defined

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[6] See Proposition 9 in Dávila and Eeckhout [4] for a proof of this property.

by the smallest and biggest coordinates of the crossings of f^A and f^B. Still, as a consequence of our Theorem 3 below, this rectangle is generically "big'', i.e., it has a non-empty interior, even if the Walrasian equilibrium is unique. Convergence to the Walrasian outcome without quantity constraints cannot unfortunately be guaranteed then on the basis of Theorem 6 in Yildiz [17].

Lemma 1 has strong implications for the existence of asymptotically inefficient SSP equilibria of the alternating-offers bargaining over prices. Note for instance that Lemma 1, along with the behavior of f^A and f^B at the boundary, implies the existence of an intersection of f^A and f^B like $(\hat{u}^A, \hat{u}^B)$ in Fig. 1. This intersection does not correspond to a Walrasian equilibrium since it is inefficient. Note also that, as the discount factors δ^A and δ^B depart slightly from 1, by continuity a nearby intersection $(\hat{u}^A, \hat{u}^B)_{\delta^A, \delta^B}$ of $f^A_{\delta^B}$ and $f^B_{\delta^A}$ still exists. This intersection $(\hat{u}^A, \hat{u}^B)_{\delta^A, \delta^B}$ not only satisfies the necessary conditions for an SSP equilibrium of the bargaining over prices, but actually corresponds to such an equilibrium whenever both $f^A_{\delta^B}$ and $f^B_{\delta^A}$ have a negative slope there, which is the case for δ^A and δ^B close enough to 1, by continuity, for a non-empty open set of economies. [7] Finally, note that the intersection $(\hat{u}^A, \hat{u}^B)_{\delta^A, \delta^B}$ converges necessarily to $(\hat{u}^A, \hat{u}^B)$ as δ^A and δ^B converge to 1. As a consequence, the corresponding SSP equilibrium of the alternating-offers bargaining over prices is not only inefficient for every δ^A and δ^B close to 1, but it also remains bounded away from the Pareto frontier as δ^A and δ^B converge to 1.

The existence of asymptotically inefficient SSP equilibria of the alternating-offers bargaining over prices is a generic property of these economies. Theorem 3 below establishes this and its proof can be found in the appendix. It also addresses the issue of whether there still exist SSP equilibria of the bargaining over prices that do converge to a Walrasian equilibrium as the discount factors δ^A and δ^B converge to 1. This actually depends on how δ^A and δ^B approach 1. But whenever that is the case, there will be a multiplicity of such equilibria.

Theorem 3. *In the space of economies* $\{u^i, e^i\}_{i=A,B}$ *satisfying A1,*

(1) *generically there exists an SSP equilibrium of the bargaining over prices for* $\delta^A, \delta^B \rightarrow 1$ *that remains bounded away from efficiency, and*

(2) *for any Walrasian allocation* (x^{A*}, x^{B*})*, either there are multiple SSP equilibrium allocations converging to* (x^{A*}, x^{B*})*, or no SSP equilibrium allocation converges to* (x^{A*}, x^{B*})*.*

The necessity of maximum quantity constraints. We illustrate the existence of asymptotically inefficient SSP equilibria of the bargaining over prices without maximum quantity constraints in a simple Cobb–Douglas setup. Let $u^i(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, for $i = A, B$. The total resources are $e = (1, 1)$ and the distribution of initial endowments between A and B is $e^A = (0.9, 0.3)$ and $e^B = (0.1, 0.7)$. In this example, the contract curve is the diagonal of the Edgeworth box. [8]

An SSP equilibrium with immediate acceptance must satisfy the necessary conditions in (6) above. When there is no discounting, p^A and p^B being equal to the Walrasian relative price $p^* = 1$ leading to the allocation $x^{A*} = (0.6, 0.6)$, $x^{B*} = (0.4, 0.4)$, is a solution to Eqs. (6). Fig. 2 below

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[7] A crossing where either f^A or f^B is positively sloped leaves room for a mutually beneficial deviation that undoes the candidate equilibrium.

[8] The fact that the result in Yildiz [17] is non-generic translates here into the fact that only the economies with initial endowments on the anti-diagonal between the upper-left and lower-right corners of the Edgeworth box satisfy the assumptions made in Yildiz [17]. Any small deviation away from the endowments on the anti-diagonal, or a perturbation in the preferences, gives rise to asymptotically inefficient SSP equilibria.
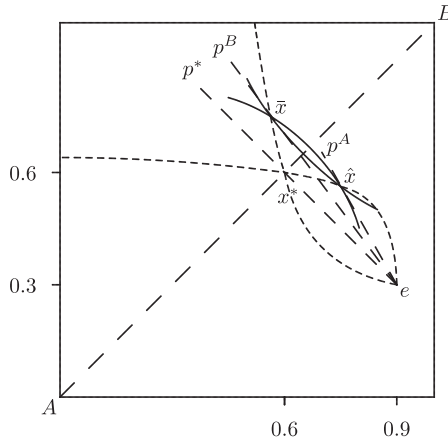
Fig. 2.

shows another solution to the system (6) with $\delta^A = \delta^B = 1$, namely $(p^A, p^B) = (1.750, 1.333)$ leading to the allocations $\bar{x}$ and $\hat{x}$ on A's and B's offer curves, respectively, with

$$\bar{x}^A = (\bar{x}_1^A(p^B), \bar{x}_2^A(p^B)) = (0.5625, 0.75),$$

$$\hat{x}^A = (e - \hat{x}_1^B(p^A), e - \hat{x}_2^B(p^A)) = (0.75, 0.5625), \tag{8}$$

and the complementary bundles for agent B. Note that, unlike the Walrasian solution, this other solution is not Pareto-efficient. Moreover, $(p^A, p^B) = (1.750, 1.333)$ are indeed SSP equilibrium prices since no agent can profitably deviate at any stage of the game. [9]

By continuity, for δ^A and δ^B close to 1, there exists as well a solution $(p^A_{\delta^A\delta^B}, p^B_{\delta^A\delta^B})$ to the system of equations (6) close to $(p^A, p^B) = (1.750, 1.333)$. That solution is still an SSP equilibrium. Note that as $\delta^A, \delta^B \to 1$, $(p^A_{\delta^A\delta^B}, p^B_{\delta^A\delta^B})$ converges to $(p^A, p^B) = (1.750, 1.333)$ and hence remains bounded away from efficiency.

5. Discussion

In this section, we provide first some intuition for the conjecture that, not only every SSP equilibrium outcome converges to a Walrasian outcome as stated in Theorem 1, but also every Walrasian outcome is reachable as an SSP equilibrium outcome as $\delta^A, \delta^B \to 1$. Then we provide an example illustrating the need for some of the assumptions in order to obtain the results. We consider an economy with non-differentiable utility functions where bargaining with quantity constraints does not necessary imply convergence to the Walrasian equilibrium, which stresses the importance of the differentiability conditions.

Is every Walrasian allocation reachable as an SSP equilibrium allocation as $\delta^A, \delta^B \to 1$? We have established in Theorem 2 that the limit of every convergent sequence of SSP equilibrium allocations of the alternating-offers bargaining over prices and maximum trades must be a Walrasian allocation as $\delta^A, \delta^B \to 1$. However another question one might be interested in is whether every Walrasian allocation is reachable as an SSP equilibrium allocation this way. This question

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[9] This is a consequence of the fact that no allocation on the offer curves Pareto-improves upon $\hat{x}, \bar{x}$.
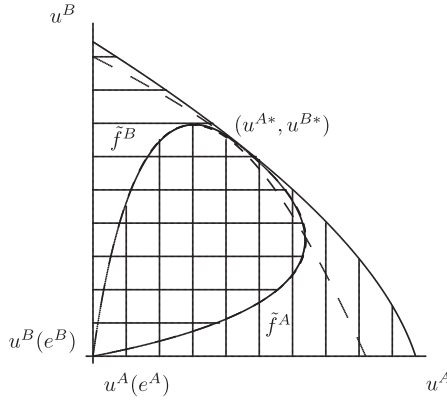
Fig. 3.

is, however, not addressed by the previous results. Still the intuition that this conjecture holds true is supported by the behavior of the SSP equilibrium payoffs as δ^A, $\delta^B \to 1$.

From agent A's problem in Eq. (4) and the corresponding problem for B, it follows that at an SSP equilibrium the feasible allocations (x_A^A, x_A^B) and (x_B^A, x_B^B) resulting from the offers by A and B, respectively, the second constraint of each agent's problem must be binding, i.e., $u^A(x_B^A) = \delta^A u^A(x_A^A)$ and $u^B(x_A^B) = \delta^B u^B(x_B^B)$. That is to say $(u^A(x_B^A), \delta^B u^B(x_B^B))$ and $(\delta^A u^A(x_A^A), u^B(x_A^B))$ must denote the same point of the intersection of the sets $\tilde{f}_{\delta^B}^A$ and $\tilde{f}_{\delta^A}^B$ of utility profiles that can be attained through an acceptance by A or B, respectively (these are slight continuous deformations of $\tilde{f}^A$ and $\tilde{f}^B$ in Fig. 3 for the case $\delta^A = \delta^B = 1$). [10]

In the case $\delta^A = \delta^B = 1$, the point $(u^A(x_B^A), \delta^B u^B(x_B^B)) = (\delta^A u^A(x_A^A), u^B(x_A^B))$ must be in the vertically and horizontally shaded area in Fig. 3 that is the intersection of $\tilde{f}^A$ and $\tilde{f}^B$. Nevertheless, note that not all the utilities in that area can be SSP equilibrium payoffs. Every profile Pareto-dominated by some point in either $\tilde{f}_{\delta^B}^A$ or $\tilde{f}_{\delta^A}^B$ corresponds to a situation in which there is room for a mutually beneficial deviation by some agent. The only profiles of utilities that are not Pareto-dominated this way, and correspond hence to SSP equilibrium payoffs, are the crossings of the *boundaries* of $\tilde{f}_{\delta^B}^A$ and $\tilde{f}_{\delta^A}^B$. In the case $\delta^A = \delta^B = 1$, these crossings correspond to the Walrasian payoffs. [11] By continuity, for any δ^A and δ^B converging to 1 there exists an undominated crossing of the boundaries of $\tilde{f}_{\delta^B}^A$ and $\tilde{f}_{\delta^A}^B$ converging to each Walrasian equilibrium crossing of the boundaries of $\tilde{f}^A$ and $\tilde{f}^B$ that corresponds to SSP equilibrium payoffs for discount factors close enough to 1.

---

[10] Note that whenever A's desired trade at some prices p^B by B is smaller than B's desired trade at those same prices, there is no way in which B can obtain a bigger trade than that resulting from A's demand at the prices p^B. Hence the utilities in $\tilde{f}^A$ attainable through A's acceptance are bounded above by the profiles along A's offer curve to the left of the Walrasian profile (u^{A*}, u^{B*}). On the contrary, whenever A's desired trade is bigger than B's, efficiency can be imposed by B by means of the maximum trade constraint of his offer (which explains that to the right of (u^{A*}, u^{B*}) the upper boundary of $\tilde{f}^A$ is the Pareto frontier). And similarly for $\tilde{f}^B$. See the proof of Theorem 1 for a formal definition of $\tilde{f}^A$ and $\tilde{f}^B$.

[11] Note that, while Lemma 1 establishes that f^A and f^B do not cross generically at the Walrasian payoffs, the boundaries of $\tilde{f}^A$ and $\tilde{f}^B$ do necessarily cross at (u^{A*}, u^{B*}).
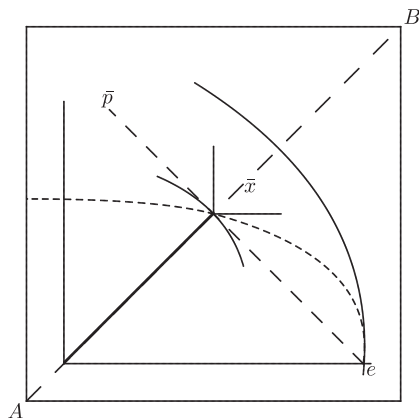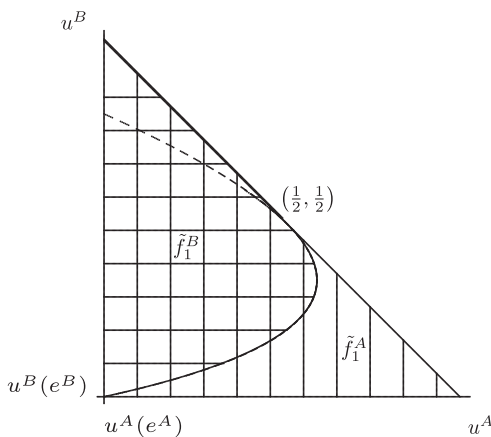
Fig. 4.

Fig. 5.

A counter-example in the case of non-differentiability. Consider an economy represented in the Edgeworth box in Fig. 4: $e^A = (0.9, 0.1)$, $e^B = (0.1, 0.9)$, $u^A(x_1, x_2) = \min\{x_1, x_2\}$, $u^B(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$. Its unique Walrasian equilibrium allocation is $\bar{x}^A = \bar{x}^B = \left(\frac{1}{2}, \frac{1}{2}\right)$, supported by the relative price $\bar{p} = 1$.

In this economy, the relevant part of A's offer curve coincides with the contract curve, which is the diagonal of the Edgeworth box (Fig. 4). When making an offer, A can impose a maximum constraint on B and therefore any offer that would induce B to consume more than the Walrasian equilibrium amount of good 2 will lead to an offer accepted on the contract curve. Therefore, offers accepted by B exactly coincide with offers accepted by A. The thick line segment on the diagonal in Fig. 4 represents those coinciding allocations. In the space of utilities in Fig. 5 below, this translates into a continuum of undominated intersections of the boundaries of $\tilde{f}_1^A$ and $\tilde{f}_1^B$ corresponding to utility levels $u^A \in [0, 1/2]$, $u^B \in [1/2, 1]$ such that $u^A + u^B = 1$. Each element in this continuum of intersections corresponds then to an SSP equilibrium with δ's equal to 1.

With discounting, when $\delta^A = \delta^B < 1$ and therefore the discount factors converge to 1 at the same rate, there is still a continuum of SSP equilibria. Clearly these equilibria need not converge to the Walrasian one as δ approaches 1. For δ's converging to 1 with δ^A/δ^B bounded away from 1, the SSP equilibrium converges either to the Walrasian allocation or to the corner solution where B extracts all the surplus from trade.

6. Concluding remarks

In this paper, we have proposed a simple bargaining procedure that achieves the competitive equilibrium allocation without assuming price-taking behavior. It relates to procedures studied in Binmore and Yildiz [2,17]. Extending the use of the quantity constraints of the Nash demand game in Binmore [2] to an alternating-offers bargaining game as in Yildiz [17], we obtain a convergence result for general economies as opposed to for the degenerate subset of economies characterized in Yildiz [17]. The procedure is commonly observed, in the sense that negotiating parties often bargain over a price with a quantity constraint, and then choose the quantity of trade separately. An interesting property of the main result of this paper is that, by always obtaining the Walrasian equilibrium, the outcome of the bargaining does not depend on specifics such as relative bargaining powers or impatience, but only on primitives, i.e., preferences and endowments.

In the context of the Nash demand game, Binmore [2] stresses the importance of quantity constraints. Bargaining procedures with maximum quantity constraints capture the main aspects of several existing price setting mechanisms. For example, in commodity future markets, the seller of future contracts will typically announce to a candidate buyer the price for the contract and how many contracts he has on offer. The candidate buyer can accept the price offer and choose the number of future contracts as long as it does not exceed the quantity constraint that was offered initially. The same is true for limit orders when selling stock. Your limit order guarantees a certain price for the stock, but you cannot be sure that the order will be filled. Only if there is sufficient demand at that price will your order be filled (either partially or completely). In addition, our bargaining procedure involves separation of the price-setting by the proposer from the quantity decision by the recipient. This price-quantity separation is obviously well known in negotiations and has common applications in several economic environments such as union-wage bargaining in the labor economics literature, [12] and standard buy-out provisions in two-person partnerships. [13]

Finally, Gale [7] (see also Kunimoto and Serrano [9]) establishes a bargaining foundation for the Walrasian equilibrium outcome in general exchange economies with a continuum of traders and random pairwise matching. The possibility of being matched later on to another agent offering better terms drives the convergence to the competitive outcome. That argument does not apply in a small economy like ours. Therefore, our results contribute to extending a bargaining foundation for Walrasian equilibrium to economies with a small number of agents. Despite the (only two) agents being price-setters, the perfectly competitive outcome still obtains. We conjecture that our results extend to the case of an arbitrarily finite number of agents. In the extreme case in which the number of agents increases to a continuum and pairs are formed through random matching, then the bargaining procedure proposed in Gale [7] leads to the Walrasian equilibrium outcome.

---

[12] See Solow and MacDonald [15] and Farber [5] amongst others who study and document such bargaining over wages where the union negotiates the wage and the employer chooses the level of employment.

[13] Cramton et al. [3], and Fiesseler et al. [6] model such buy-out provisions. When partners decide they want to separate, the provision prescribes that one partner chooses the price of the shares, and the other partner chooses the quantity traded, i.e., whether to buy or sell.

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Appendix A.

Lemma A1. *If u and e satisfy A1 and $x \in \mathbb{R}^n_{++}$ solves*

$$\begin{aligned}
&\max\ u(x),\\
&p(x - e) \leqslant 0,\\
&\|x - e\| \leqslant q,
\end{aligned} \tag{9}$$

where $\| \cdot \|$ stands for the Euclidean norm, [14] *then*

$$Du(x)(x - e) \geqslant 0, \tag{10}$$

and, conversely, if x satisfies (10), then there exist p, q for which x solves (9).

Proof. Assume $x \neq e$, otherwise (10) is trivially satisfied. Since x solves (9), then necessarily for some $\lambda, \mu \geqslant 0$

$$\begin{aligned}
&Du(x) = \lambda p + \mu(x - e),\\
&\lambda p(x - e) = 0,\\
&\mu[(x - e)(x - e) - q^2] = 0.
\end{aligned} \tag{11}$$

Therefore

$$\begin{aligned}
Du(x)(x - e) &= \lambda p(x - e) + \mu(x - e)(x - e)\\
&= \mu(x - e)(x - e) \geqslant 0.
\end{aligned} \tag{12}$$

Conversely, assume x satisfies (10). If x satisfies $Du(x)(x - e) = 0$, let

$$\begin{aligned}
&\lambda = 1,\\
&\mu = 0,\\
&p = Du(x),\\
&q^2 = (x - e)(x - e).
\end{aligned} \tag{13}$$

If x satisfies $Du(x)(x - e) > 0$ (which implies in particular that $x \neq e$), let

$$\begin{aligned}
&\lambda > 0,\\
&\mu = \frac{Du(x)(x - e)}{(x - e)(x - e)},\\
&p = \frac{1}{\lambda}\left[Du(x) - \frac{Du(x)(x - e)}{(x - e)(x - e)}(x - e)\right] \geqslant 0,\\
&q^2 = (x - e)(x - e),
\end{aligned} \tag{14}$$

---

[14] In Eq. (1) the maximum trade constraint was expressed using the sup norm. It is clear that any such constraint can be expressed equivalently as in (9).

where the weak inequality in (14) follows from Lemma A2 below. In both cases, then (10) holds. $\square$

Lemma A2. *If $a \in \mathbb{R}^n_{++}$ and $b \notin \mathbb{R}^n_+$ are such that $ab > 0$, then*

$$a - \frac{a \cdot b}{b \cdot b} b \geqslant 0. \tag{15}$$

Proof. It suffices to establish the claim in $\mathbb{R}^2$, since the general case can then be established in the 2-dimensional space spanned by the vectors a and b.

Since $a \in \mathbb{R}^2_{++}$ and $ab > 0$, then $b \notin -\mathbb{R}^2_+$. Since $b \notin -\mathbb{R}^2_+$ and $b \notin \mathbb{R}^2_+$, then $b_1 b_2 < 0$. Assume, without loss of generality that $b_1 < 0$ and $b_2 > 0$.

Note first that, for all $a \in \mathbb{R}^2_{++}$, the inequality $a - \frac{a \cdot b}{b \cdot b} b \geqslant 0$ holds if, and only if,

$$\frac{a}{\|a\|} - \frac{b}{\|b\|} \cos \widehat{ab} \geqslant 0. \tag{16}$$

Since $a \in \mathbb{R}^2_+$, $ab > 0$, and $b_1 < 0$, then it trivially holds

$$\frac{a_1}{\|a\|} - \frac{b_1}{\|b\|} \cos \widehat{ab} \geqslant 0. \tag{17}$$

Moreover, since $b_1 < 0$, then

$$b_1 \left(\frac{a_1}{\|a\|} - \frac{b_1}{\|b\|} \cos \widehat{ab} \right) \leqslant 0 \tag{18}$$

but since, for any $a, b \in \mathbb{R}^2$, it holds

$$\sum_{i=1}^{2} \frac{b_i}{\|b\|} \left(\frac{a_i}{\|a\|} - \frac{b_i}{\|b\|} \cos \widehat{ab} \right) = 0, \tag{19}$$

then

$$b_2 \left(\frac{a_2}{\|a\|} - \frac{b_2}{\|b\|} \cos \widehat{ab} \right) \geqslant 0 \tag{20}$$

and finally, since $b_2 > 0$, hence

$$\frac{a_2}{\|a\|} - \frac{b_2}{\|b\|} \cos \widehat{ab} \geqslant 0. \qquad \square \tag{21}$$

Proof of Theorem 1. Part (1): Let (x^A_A, x^B_A) (respectively, (x^A_B, x^B_B)) be the feasible allocation resulting from B's (resp., A's) acceptance of A's (resp., B's) offer of a price and maximum trade at an SSPE with immediate acceptance for infinitely patient agents, that is to say such that (x^A_A, x^B_A) solves

$$\max_{x^A, x^B} u^A(x^A),$$

$$Du^B(x^B)(x^B - e^B) \geqslant 0,$$

$$u^B(x^B) \geqslant u^B(x^B_B),$$

$$x^A + x^B = e^A + e^B, \tag{22}$$

given (x_B^A, x_B^B), and similarly for (x_A^A, x_A^B) given (x_A^A, x_A^B). Then necessarily there exist λ^A, μ^A, λ^B, $\mu^B \geqslant 0$ and v_i^A, v_i^B, for all $i = 1, \ldots, n$, such that the following $2n$ equations are satisfied

$$
\begin{pmatrix} Du^A(x_A^A) \\ 0 \end{pmatrix} + \lambda^A \begin{pmatrix} 0 \\ Du^B(x_A^B) \end{pmatrix}
$$

$$
+ \mu^A \begin{pmatrix} 0 \\ Du^B(x_A^B) + D^2u^B(x_A^B)(x_A^B - e^B) \end{pmatrix} + \sum_{i=1}^n v_i^A \begin{pmatrix} e_i \\ e_i \end{pmatrix} = 0, \tag{23}
$$

where e_i stands for the ith vector of the canonical basis of $\mathbb{R}^n$, or equivalently

$$
Du^A(x_A^A) = \lambda^A Du^B(x_A^B) + \mu^A \left[Du^B(x_A^B) + D^2u^B(x_A^B)(x_A^B - e^B) \right] \tag{24}
$$

and similarly for B's problem.

Assume that $(x_A^A, x_A^B) \neq (x_B^A, x_B^B)$. Since at an SSP equilibrium the second constraint in (22) is binding (and similarly for B's problem), [15] both allocations are on the same indifference surface for both agents. Moreover, given that u^A and u^B are strictly differentiably quasi-concave, these two indifference surfaces determine strictly convex upper contour sets. As a consequence, neither (x_A^A, x_A^B) nor (x_B^A, x_B^B) can be efficient whenever distinct. In particular, for some i, j it must hold

$$
\frac{D_i u^A(x_A^A)}{D_j u^A(x_A^A)} > \frac{D_i u^B(x_A^B)}{D_j u^B(x_A^B)} \tag{25}
$$

since otherwise (x_A^A, x_A^B) would be efficient. But then (24) above cannot hold for a non-negative μ^A. [16] In effect, λ^A, μ^A must solve (24) above, and in particular

$$
\begin{pmatrix} D_i u^B(x_A^B) & D_i u^B(x_A^B) + D_{ii} u^B(x_A^B)(x_{Ai}^B - e_i^B) + D_{ij} u^B(x_A^B)(x_{Aj}^B - e_j^B) \\ D_j u^B(x_A^B) & D_j u^B(x_A^B) + D_{ji} u^B(x_A^B)(x_{Ai}^B - e_i^B) + D_{jj} u^B(x_A^B)(x_{Aj}^B - e_j^B) \end{pmatrix}
$$

$$
\times \begin{pmatrix} \lambda^A \\ \mu^A \end{pmatrix} = \begin{pmatrix} D_i u^A(x_A^A) \\ D_j u^A(x_A^A) \end{pmatrix} \tag{26}
$$

from which

$$
\mu^A = \frac{\begin{vmatrix} D_i u^B(x_A^B) & D_i u^A(x_A^A) \\ D_j u^B(x_A^B) & D_j u^A(x_A^A) \end{vmatrix}}{\begin{vmatrix} D_i u^B(x_A^B) & D_i u^B(x_A^B) + D_{ii} u^B(x_A^B)(x_{Ai}^B - e_i^A) + D_{ij} u^B(x_A^B)(x_{Aj}^B - e_j^A) \\ D_j u^B(x_A^B) & D_j u^B(x_A^B) + D_{ji} u^B(x_A^B)(x_{Ai}^B - e_i^A) + D_{jj} u^B(x_A^B)(x_{Aj}^B - e_j^A) \end{vmatrix}}. \tag{27}
$$

---

[15] Since (x_A^A, x_A^B) satisfies the constraints of B's maximization problem (in particular $Du^A(x_A^A)(x_A^A - e_A) \geqslant 0$ since A will never choose at equilibrium to trade more than necessary to attain his demand at the implicit prices), necessarily $u^B(x_A^B) \leqslant u^B(x_B^B)$. Hence it cannot be that $u^B(x_A^B) > u^B(x_B^B)$.

[16] If the inequality (25) holds in the opposite direction, it is B's FOCs which cannot hold for a non-negative μ^B and the following argument applies with the obvious changes.

Since (25) implies that the numerator is positive, the denominator is strictly positive as well, which amounts to

$$\left(-D_j u^B(x_A^B) \quad D_i u^B(x_A^B) \right) \begin{pmatrix} D_{ii}u^B(x_A^B) & D_{ij}u^B(x_A^B) \\ D_{ji}u^B(x_A^B) & D_{jj}u^B(x_A^B) \end{pmatrix} \begin{pmatrix} x_{Ai}^B - e_i^B \\ x_{Aj}^B - e_j^B \end{pmatrix} > 0. \qquad (28)$$

But $D^2 u^B(x_A^B)$ is negative definite in the space orthogonal to $Du^B(x_A^B)$, and hence for any $(0, \ldots, 0, x_{Ai}^B - e_i^B, 0, \ldots, 0, x_{Aj}^B - e_j^B, 0, \ldots, 0)$ orthogonal to $Du^B(x_A^B)$, i.e., such that

$$D_i u^B(x_A^B)(x_{Ai}^B - e_i^B) + D_j u^B(x_A^B)(x_{Aj}^B - e_j^B) = 0 \qquad (29)$$

or equivalently collinear to $(-D_j u^B(x_A^B), D_i u^B(x_A^B))$, the left-hand side of (28) should be negative!

Therefore, at any SSP equilibrium both allocations coincide, i.e., $(x_A^A, x_A^B) = (x_B^A, x_B^B) = (x^A, x^B)$, whenever $\delta^A = \delta^B = 1$. Since x^B must solve

$$\begin{aligned} &\max_{\tilde{x}^B} u^A(e - \tilde{x}^B), \\ &Du^B(\tilde{x}^B)(\tilde{x}^B - e^B) \geqslant 0, \\ &u^B(\tilde{x}^B) \geqslant u^B(e - x^A), \end{aligned} \qquad (30)$$

given x^A, should the first constraint not be binding at x^B, then x^B would solve as well

$$\begin{aligned} &\max u^A(e - \tilde{x}^B), \\ &u^B(\tilde{x}^B) \geqslant u^B(e - x^A), \end{aligned} \qquad (31)$$

given x^A, because u^A is strictly monotone and u^B is strictly quasi-concave. Hence for some $\lambda > 0$, $Du^A(x^A) = \lambda Du^B(x^B)$. Since $Du^B(x^B)(x^B - e^B) > 0$, then $\frac{1}{\lambda} Du^A(x^A)(x^A - e^A) < 0$ which contradicts that x^A solves

$$\begin{aligned} &\max u^B(e - \tilde{x}^A), \\ &Du^A(\tilde{x}^A)(\tilde{x}^A - e^A) \geqslant 0, \\ &u^A(\tilde{x}^A) \geqslant u^A(e - x^B), \end{aligned} \qquad (32)$$

given x^B. Therefore, necessarily $Du^B(x^B)(x^B - e^B) = 0$ and similarly $Du^B(x^A)(x^A - e^A) = 0$, i.e., at the allocation (x^A, x^B) each agent gets his demand at the implicit prices, and hence (x^A, x^B) is a Walrasian allocation.

Conversely, let (x^A, x^B) be a Walrasian allocation supported by p^*. Let $p^A = p^* = p^B$ and q^A, q^B be slack. Then $(x^A, x^B) = (\tilde{x}^A(p^B, q^B), \tilde{x}^B(p^A, q^A))$ and, since (p^A, q^A) and (p^B, q^B) are such that they solve Eq. (2) for $\delta^A = 1 = \delta^B$, the Walrasian allocation (x^A, x^B) is the outcome of the following SSP equilibrium profile of strategies:

(1) A offers always p^A, q^A and accepts only offers p, q such that $u^A(\tilde{x}^A(p, q)) \geqslant u^A(\tilde{x}^A(p^B, q^B))$.
(2) B offers always p^B, q^B and accepts only offers p, q such that $u^B(\tilde{x}^B(p, q)) \geqslant u^B(\tilde{x}^B(p^A, q^A))$.

There are only four types of subgames in which either A makes an offer, or A replies to an offer and similarly for B. For instance, if it is A's turn to make an offer and A sticks to the strategy

above, then B will accept and A will get utility $u^A(e - \tilde{x}^B(p^A, q^A))$. While if A deviates, *in any possible way*, [17] then

(1) either they disagree forever,
(2) or A ends up accepting p^B, q^B *at the earliest* one period later,
(3) or B ends up accepting now or at any future period some p, q such that $u^B(\tilde{x}^B(p, q)) \geqslant u^B(\tilde{x}^B(p^A, q^A))$.

None of the possible outcomes after deviation in this exhaustive list makes A better off when $p^A = p^* = p^B$ and q^A, q^B are slack. In case (2) A gets the utility $u^A(\tilde{x}^A(p^B, q^B))$ which equals the no-deviation utility $u^A(e - \tilde{x}^B(p^A, q^A))$. In case (3) A gets the utility $u^A(e - \tilde{x}^B(p, q))$ for some p, q such that $u^B(\tilde{x}^B(p, q)) \geqslant u^B(\tilde{x}^B(p^A, q^A))$, which is again not bigger than the no-deviation utility because of the strict quasi-concavity of u^A and u^B.

If it is A's turn to respond, and A sticks to the strategy above, then A will accept (p^B, q^B) and will get the utility $u^A(\tilde{x}^A(p^B, q^B))$. While if A deviates, *in any possible way*, then

(1) either they disagree forever,
(2) or A accepts p^B, q^B now or at any future period,
(3) or B ends up accepting at the earliest next period some p, q such that $u^B(\tilde{x}^B(p, q)) \geqslant u^B(\tilde{x}^B(p^A, q^A))$.

Once more, none of the possible outcomes makes A better off when $p^A = p^* = p^B$ and q^A, q^B are slack. In case (2) A gets the no-deviation utility. In case (3) A gets the utility $u^A(e - \tilde{x}^B(p, q))$ for some p, q such that $u^B(\tilde{x}^B(p, q)) \geqslant u^B(\tilde{x}^B(p^A, q^A))$, which is again not bigger than the no-deviation utility because of the strict quasi-concavity of u^A and u^B.

Part (2): Consider Φ such that

$$
\begin{aligned}
\Phi(x_A^A, x_A^B, x_B^A, x_B^B; \delta^A, \delta^B) = \arg \max_{0 \leqslant x^A, x^B} u^A(x^A) \times \arg \max_{0 \leqslant x^A, x^B} u^B(x^B) \\
Du^B(x^B)(x^B - e^B) \geqslant 0, \qquad Du^A(x^A)(x^A - e^A) \geqslant 0, \\
u^B(x^B) \geqslant \delta^B u^B(x_B^B), \qquad u^A(x^A) \geqslant \delta^A u^A(x_A^A), \\
x^A + x^B = e^A + e^B, \qquad x^A + x^B = e^A + e^B, \\
\text{given } (x_B^A, x_B^B) \qquad \text{given } (x_A^A, x_A^B).
\end{aligned}
\tag{33}
$$

Note that, by the theorem of the maximum each of the arg max's on the right-hand side of Eq. (33) is a compact-valued, upper hemicontinuous correspondence that depends (in some cases trivially) on $x_A^A, x_A^B, x_B^A, x_B^B, \delta^A$ and δ^B. [18] And similarly for agent A's problem. Therefore, Φ is

---

[17] Observe that by allowing any deviation, this standard argument does not rely on the one-shot deviation principle, which does not apply when $\delta = 1$ since the game is not continuous in this case.

[18] Since, for instance, u^A depends continuously on x^A and also trivially on $x_A^A, x_A^B, x_B^A, x_B^B, \delta^A, \delta^B$, and the correspondence defined by the constraints

$$
\begin{aligned}
\Omega^A(x_A^A, x_A^B, x_B^A, x_B^B, \delta^A, \delta^B) = \{(x^A, x^B) \in \mathbb{R}^{2n} \,|\, Du^B(x^B)(x^B - e^B) \geqslant 0, \\
u^B(x^B) \geqslant \delta^B u^B(x_B^B), \\
x^A + x^B = e^A + e^B\}
\end{aligned}
$$

is continuous and compact-valued.

the cartesian product of compact-valued, upper hemicontinuous correspondences, and hence it is compact-valued and upper hemicontinuous itself.

Consider Γ such that

$$\Gamma(\delta^A, \delta^B) = \Big\{ (x_A^A, x_A^B, x_B^A, x_B^B,) \in \mathbb{R}^{2(2n)} |$$

$$(x_A^A, x_A^B, x_B^A, x_B^B,) \in \Phi(x_A^A, x_A^B, x_B^A, x_B^B; \delta^A, \delta^B) \Big\}. \tag{34}$$

Since Φ is compact-valued and upper hemicontinuous, then by Lemma A3 below, the correspondence mapping to each pair (δ^A, δ^B) the fixed points of $\Phi(\cdot, \cdot, \cdot, \cdot, \delta^A, \delta^B)$ is upper hemicontinuous.

Finally, note that Γ is the correspondence of SSP equilibrium allocations (without delay). Since this correspondence is upper hemicontinuous in particular at $(\delta^A, \delta^B) = (1, 1)$ and, according to Theorem 1, $\Gamma(1, 1)$ is the set of Walrasian allocations, then the claim follows. $\quad\square$

Lemma A3. *If X, Y are metric spaces and $\Phi \in \mathcal{P}(X)^{X \times Y}$ is compact-valued and upper hemicontinuous, then $\Gamma \in \mathcal{P}(X)^Y$ such that*

$$\Gamma(y) = \big\{ x \in X \,|\, x \in \Phi(x, y) \big\} \tag{35}$$

is upper hemicontinuous.

Proof. Assume that Γ is not upper hemicontinuous at some y. Then there exist $\{y_n\} \to y$, x and $\{x_n\} \to x$ such that $x_n \in \Gamma(y_n)$ for all $n \in \mathbb{N}$, while $x \notin \Gamma(y)$. That is to say, for all $n \in \mathbb{N}$, $x_n \in \Phi(x_n, y_n)$ while $x \notin \Phi(x, y)$. As a consequence, since Φ is compact-valued, then Φ is not upper hemicontinuous at (x, y)! $\quad\square$

Lemma A4. *If $\{u^i, e^i\}_{i=A,B}$ satisfies A1 and the agents are impatient (that is to say, $\delta^A, \delta^B < 1$), then there does not exist any SSP equilibrium with delay.*

Proof. Consider a candidate SSP equilibrium (p^A, q^A, p^B, q^B) in which, for instance, B rejects and A accepts. Let $(x_A^A, x_A^B) = (e - \tilde{x}^B(p^A, q^A), \tilde{x}^B(p^A, q^A))$ and $(x_B^A, x_B^B) = (\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B))$ be the allocations resulting from A's and B's offers if accepted. Then it must be the case that

(1) B's offer is rational, that is to say B is interested in A's acceptance since he obtains more utility this way than from A's offer one period later, i.e.,

$$u^B(x_B^B) > \delta^B u^B(x_A^B) \tag{36}$$

and also B's offer is his most preferred acceptable to A, i.e.,

$$(x_B^A, x_B^B) \in \arg \max_{x^A, x^B} u^B(x^B),$$

$$Du^A(x^A)(x^A - e^A) \geqslant 0,$$

$$u^A(x^A) \geqslant \delta^A u^A(x_A^A),$$

$$x^A + x^B = e^A + e^B, \tag{37}$$

given (x_A^A, x_A^B), and

(2) A's offer is rational, that is to say A is interested in B's rejection since A obtains more utility from B's offer one period later, i.e.,

$$\delta^A u^A(x_B^A) \geqslant u^A(x_A^A) \tag{38}$$

and accordingly makes an unacceptable offer to B, i.e.,

$$u^B(x_A^B) < \delta^B u^B(x_B^B), \tag{39}$$

which guarantees B's rejection.

Therefore, from (28) and the fact that (x_B^A, x_B^B) solves (37) above, it follows that whenever $\delta^A < 1$, necessarily

$$u^A(x_B^A) > \delta^A u^A(x_B^A) \geqslant u^A(x_A^A) > \delta^A u^A(x_A^A), \tag{40}$$

i.e., the second constraint in (37) is not binding. Since u^B is strictly monotone (and hence the solution cannot be interior), then necessarily the first constraint in (37) must be binding, i.e., $Du^A(x_B^A)(x_B^A - e^A) = 0$. As a consequence, $\tilde{x}^A(p^B, q^B) = x^A(p^B)$, that is to say q^B does not actually constrain A's demand (in the 2 goods case, (x_B^A, x_B^B) is on A's offer curve in the Edgeworth box).

Since u^A is strictly differentiably quasi-concave, then (x_B^A, x_B^B) is not efficient. Since the normal direction to A's offer curve (manifold, in general) is $Du^A(x_B^A) + D^2 u^A(x_B^A)(x_B^A - e_A)$, this is collinear to $Du^A(x_B^A)$ (which is necessary for $((x_B^A, x_B^B))$ to be efficient) only if

$$(1 - r)Du^A(x_B^A) + D^2 u^A(x_B^A)(x_B^A - e_A) = 0 \tag{41}$$

for some $r > 0$. But since $Du^A(x_B^A)(x_B^A - e_A) = 0$ and u^A is strictly differentiably quasi-concave in the space normal a $Du^A(x_B^A)$, that would imply that

$$\begin{aligned}
0 &> (x_B^A - e_A)^t D^2 u^A(x_B^A)(x_B^A - e_A) \\
&= (1 - r)(x_B^A - e_A)^t Du^A(x_B^A) + (x_B^A - e_A)^t D^2 u^A(x_B^A)(x_B^A - e_A) \\
&= (x_B^A - e_A)^t \left[(1 - r)Du^A(x_B^A) + D^2 u^A(x_B^A)(x_B^A - e_A) \right] = 0!
\end{aligned} \tag{42}$$

As a consequence, since (x_B^A, x_B^B) is not efficient, there is room for A deviating and making an offer that is Pareto improving with respect to x_B and that B would accept. [19] $\square$

Proposition 1. *If $\{u^i, e^i\}_{i=A,B}$ satisfies A1 and, for $i = A, B$, u^i is strongly concave,*[20] *then there exists an SSP equilibrium with immediate acceptance of the alternating-offers bargaining over prices and maximum trades, for any $\delta^A, \delta^B \in [0, 1]$.*

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[19] At any rate, for weaker assumptions on u^A (e.g. just differentiable quasi-concavity), if it was efficient, then A could deviate offering himself B's offer instead, since he will accept it anyway later, saving the cost of the delay in reaching an agreement.

[20] In the sense that

$$\det \left\{ 2D^2 u^i(x) + \left[\sum_{k=1}^n D_{hkj} u^i(x)(x_k - e_k^h) \right]_{hj} \right\}$$

does not change sign. This guarantees that the offer curve (or surface in general) does not ever change curvature (i.e., has no inflexion points) and hence the constrained domain delimited by the offer curve is convex. This condition is satisfied whenever the substitution effect dominates largely the wealth effect, and in particular by every CES utility function.

Proof. An SSP equilibrium is characterized by two allocations (x_A^A, x_A^B) and (x_B^A, x_B^B) solving Eq. (4) for both agents. Then letting $\Upsilon^A(x_A^A, x_A^B, x_B^A, x_B^B)$ be the set of maximizers solving (4) for A, and similarly $\Upsilon^B(x_A^A, x_A^B, x_B^A, x_B^B)$ for B, an SSP equilibrium of the bargaining game is a fixed point of the correspondence $\Upsilon^A \times \Upsilon^B$ that associates to every $(x_A^A, x_A^B, x_B^A, x_B^B)$ the set $\Upsilon^A(x_A^A, x_A^B, x_B^A, x_B^B) \times \Upsilon^B(x_A^A, x_A^B, x_B^A, x_B^B)$.

Since by the theorem of the maximum both Υ^A and Υ^B have closed graphs, so does $\Upsilon^A \times \Upsilon^B$. Also, since u^A and u^B are strongly concave, then $Du^i(x^i)(x^i - e^i) \geqslant 0$, for $i = A, B$, and the other constraints define a convex domain, so that Υ^A and Υ^B are both convex-valued and therefore so is $\Upsilon^A \times \Upsilon^B$.

The closed-graph, convex-valued correspondence $\Upsilon^A \times \Upsilon^B$ takes values in the nonempty, compact, convex set of ordered pairs of feasible allocations. Then by Kakutani's fixed point theorem, a fixed point of $\Upsilon^A \times \Upsilon^B$ exists that corresponds to an SSP equilibrium of the bargaining game. $\square$

Proof of Theorem 2. Letting $\bar{u}^A$, $\bar{u}^B$ be, respectively, the maximizers of the offer curves in the space of utilities f^A, f^B defined in assumption A2, note that $\bar{u}^A$ and $\bar{u}^B$ bound from below the SSP equilibrium payoffs of, respectively, A and B, since any positively sloped point of f^A or f^B allows for mutually beneficial deviations that undoes any would-be equilibrium there. Also, letting for all $(i, j) \in \{A, B\}^2$, $\tilde{f}^i$ be the set of utilities attainable through the acceptance of price-quantity offers, i.e.,

$$\tilde{f}^i \equiv \left\{ (v^i, v^j) \in \mathbb{R}_+^2 | (v^i, v^j) = \left(u^i(\tilde{x}^i(p, q)), u^j(e - \tilde{x}^i(p, q)) \right), \right.$$

$$\left. \text{for some } p \in \mathbb{R}_{++}^n, q \in \mathbb{R}_+ \right\} \tag{43}$$

the restrictions to $\mathbb{R}_+^2 + \{(\bar{u}^A, \bar{u}^B)\}$ of $\tilde{f}^A$ and $\tilde{f}^B$ (and hence by continuity those of $\tilde{f}_{\delta^B}^A$ and $\tilde{f}_{\delta^A}^B$, for δ^A, δ^B close to 1 as well) satisfy the assumption A1 in Merlo and Wilson [10]. Now, Theorem 8 in Merlo and Wilson [10] establishes under their assumption A1 [21] that the extreme payoffs of the set of SP equilibria are stationary, and that all SP equilibrium payoffs are contained within those extreme SSP payoffs.

Since there is a unique Walrasian equilibrium and every SSP equilibrium converges to it, these extreme payoffs (and hence every SP equilibrium payoff) converge to the Walrasian payoffs. The strict convexity of the preferences implies that the convergence takes place also in allocations and prices. $\square$

Proof of Lemma 1. Let $\bar{x}$ be an allocation on A's offer curve close to the Walrasian allocation x^*. Let $\hat{x}$ be the allocation distinct from $\bar{x}$ giving A and B the same utilities as $\bar{x}$. Then $\bar{x}$ and $\hat{x}$ are characterized by the equations (in terms of A's consumptions)

$$u^A(\hat{x}^A) - u^A(\bar{x}^A) = 0,$$
$$u^B(e^A + e^B - \hat{x}^A) - u^B(e^A + e^B - \bar{x}^A) = 0,$$
$$\phi(\bar{x}^A) = 0,$$
$$(p, 1)(\bar{x}^A - e^A) = 0, \tag{44}$$

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[21] There is a typo in the statement of Theorem 8 in Merlo and Wilson [10]: "given (A2) and $k = 2 \ldots$" should be "given (A1) and $k = 2 \ldots$". We thank the authors for the confirmation of this typo.
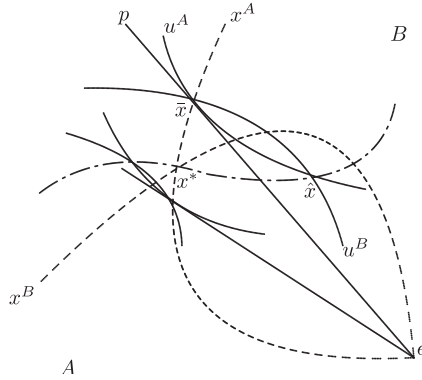
Fig. 6.

for some relative price p, where $\phi^A(\bar{x}^A) = Du^A(\bar{x}^A)(\bar{x}^A - e^A)$ and hence $\phi^A(x) = 0$ is the equation of A's offer curve. The system (44) defines implicitly $\hat{x}$ as a differentiable function of p in a neighborhood of the Walrasian price p^*, so that as p varies or, equivalently, as $\bar{x}$ runs along A's offer curve, $\hat{x}$ follows a smooth path that goes through x^* for $p = p^*$, for which $\bar{x} = \hat{x} = x^*$ (see Fig. 6).

The function that determines $\hat{x}^A$ for each p in the system (44) is the composition of the function $\bar{\xi}^A$ associating $\bar{x}^A$ to each p that is implicitly defined by the last two equations in (44) and the function $\hat{\xi}^A$ associating $\hat{x}^A$ to each $\bar{x}^A$ that is implicitly defined by the first two equations in (44). In order to see this, regarding $D\bar{\xi}^A(p)$ note first that in the Jacobian of the left-hand side of the last two equations in (44)

$$\begin{pmatrix} D_1\phi^A(\bar{x}^A) & D_2\phi^A(\bar{x}^A) & 0 \\ p & 1 & \bar{x}_1^A \end{pmatrix} \tag{45}$$

the first two columns are linearly independent, even at the Walrasian equilibrium allocation x^*,[22] and hence

$$D\bar{\xi}^A(p) = - \begin{vmatrix} D_1\phi^A(\bar{x}^A) & D_2\phi^A(\bar{x}^A) \\ p & 1 \end{vmatrix}^{-1} \begin{pmatrix} -D_2\phi^A(\bar{x}^A)\bar{x}_1^A \\ D_1\phi^A(\bar{x}^A)\bar{x}_1^A \end{pmatrix}. \tag{46}$$

As for $D\hat{\xi}^A(x^{A*})$, note that, although the Jacobian of the left-hand side of the first two equations in (44)

$$\begin{pmatrix} D_1u^A(\hat{x}^A) & D_2u^A(\hat{x}^A) & -D_1u^A(\bar{x}^A) & -D_2u^A(\bar{x}^A) \\ -D_1u^B(\hat{x}^B) & -D_2u^B(\hat{x}^B) & D_1u^B(\bar{x}^B) & D_2u^B(\bar{x}^B) \end{pmatrix} \tag{47}$$

drops rank at the Walrasian allocation x^*, the first two equations in (44) still define $\hat{x}^A$ as a function of $\bar{x}^A$ since, for strictly convex preferences and any given point $\bar{x}$, there exists a unique $\hat{x}$ where

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[22] This is a consequence of the strictly differentiably quasi-concavity of u^A.
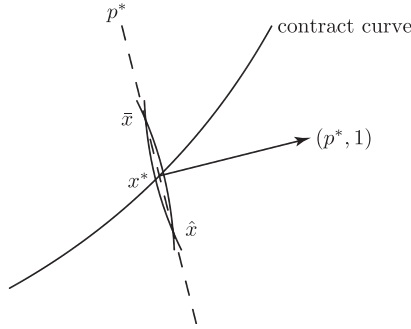
Fig. 7.

the two indifference curves through $\bar{x}$ cross each other again (if $\bar{x}$ happens to be efficient, then $\hat{x}$ actually coincides with $\bar{x}$). This function is not only differentiable off the contract curve (where the Jacobian is full rank and the implicit function theorem does apply), but also at x^* since, as $\bar{x}$ departs slightly from an efficient allocation x^* on the contract curve, the lens formed by the two indifference curves going through $\bar{x}$ cross again (almost) at a point $\hat{x}$ across the contract curve in the direction of the line supporting x^* as a Walrasian equilibrium (see Fig. 7).

The linear mapping approximating this function is

$$D\hat{\xi}^A(x^{A*}) = \begin{pmatrix} p^* & -1 \\ 1 & p^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -c^* \end{pmatrix} \begin{pmatrix} p^* & -1 \\ 1 & p^* \end{pmatrix}^{-1} \tag{48}$$

for some $c^* > 0$ that depends on the curvature of A's and B's indifference curves at x^*.[23] Therefore, since $\frac{d\hat{x}^A}{dp}(p^*) = D\hat{\xi}^A(x^{A*})D\bar{\xi}^A(p^*)$ it follows that

$$\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*}) = \frac{\frac{d\hat{x}_2^A}{dp}(p^*)}{\frac{d\hat{x}_1^A}{dp}(p^*)} = \frac{(1 - c^* p^{*2})D_1\phi^A(x^{A*}) - (1 + c^*)p^* D_2\phi^A(x^{A*})}{(c^* - p^{*2})D_2\phi^A(x^{A*}) + (1 + c^*)p^* D_1\phi^A(x^{A*})}. \tag{49}$$

If, as shown in Fig. 6, the slope of the path followed by $\hat{x}$ is at x^* smaller than the slope of B's offer curve, then around the Walrasian allocation, for any given level of utility u^A close to $u^{A*} = u^A(x^{A*})$, agent B attains on B's offer curve a higher utility than on A's, and hence around (u^{A*}, u^{B*}) the curve f^B is above the curve f^A, as shown in Fig. 1. And conversely if, on the contrary, the path followed by $\hat{x}$ had at x^* a slope bigger than B's offer curve. Only in the case in which the path followed by $\hat{x}$ had at x^* a slope equal to that of B's offer curve, i.e., only if the equation

$$-\frac{D_1\phi^B(x^{B*})}{D_2\phi^B(x^{B*})} = \frac{(1 - c^* p^{*2})D_1\phi^A(x^{A*}) - (1 + c^*)p^* D_2\phi^A(x^{A*})}{(c^* - p^{*2})D_2\phi^A(x^{A*}) + (1 + c^*)p^* D_1\phi^A(x^{A*})} \tag{50}$$

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[23] In words, $D\hat{\xi}^A(x^{A*})$ consists of the composition of (i) a change to an orthogonal basis containing the price vector $(p^*, 1)$, (ii) a jump across the first axis of that basis, and (iii) the undoing of the change of basis.
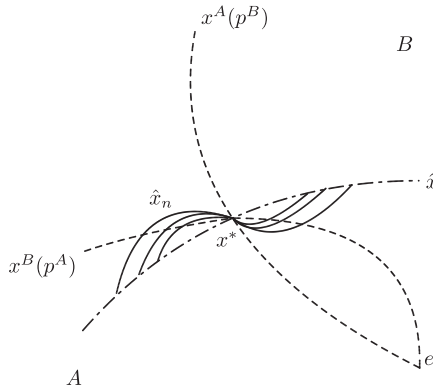
Fig. 8.

holds, could a crossing of f^A and f^B occur at (u^{A*}, u^{B*}). Note that Eq. (50) imposes a constraint on the second-order partial derivatives of the utility functions u^A and u^B at the Walrasian allocation x^*. Since the Walrasian allocation is completely characterized just by the first-order partial derivatives of the utility functions u^A and u^B, there exists arbitrarily close to u^A and u^B in any topology of C^n uniform convergence on compacts, for $n \geqslant 2$, utility functions v^A, v^B with the same first-order partial derivatives at x^* as u^A, u^B but different second-order partial derivatives so that the equilibrium equations are still satisfied but Eq. (50) does not hold. This establishes the density of the no-crossing property. The openness follows from the fact that for any pair of utility functions within a sufficiently small open neighborhood of (v^A, v^B) Eq. (50) does not hold either. [24]   □

Proof of Theorem 3. Given a Walrasian equilibrium allocation x^* of an economy $\{u^i, e^i\}_{i=A,B}$, consider a sequence $\{u_n^A\}_n$ such that (i) the corresponding sequence of paths followed by $\{\hat{x}_n\}_n$ as defined in Lemma 1 above converges pointwise to the path followed by $\hat{x}$ of $\{u^i, e^i\}_{i=A,B}$ around x^*, and (ii) all $\hat{x}_n$ have a common slope at x^* that reverses its order with respect to the slope of B's offer curve x^B at x^* so that each $\hat{x}_n$ intersects B's offer curve (see Fig. 8).

The pointwise convergence of $\{\hat{x}_n\}$ to $\hat{x}$ guarantees the pointwise convergence within a compact of the associated offer curves [25] $\{x_n^A(p^B)\}$ to $x^A(p^B)$. Also the (piecewise) monotone and pointwise convergence of $\{x_n^A(p^B)\}$ within a compact guarantees that their convergence to $x^A(p^B)$ is uniform indeed. As a consequence, the utility functions u_n^A generating these offer curves $x_n^A(p^B)$ converge in the topology of C^1 convergence on compacts toward the utility function u^A that generates the offer curve $x^A(p^B)$.

For such a sequence $\{u_n^A\}_n$ to exist, it suffices that the slope $\frac{d\hat{x}_2^A}{d\hat{x}_1^A}(x_1^{A*})$ of $\hat{x}$ at x^* (the right-hand side of (51) below) [26] may be made equal to the slope of B's offer curve at x^* (the left-hand side
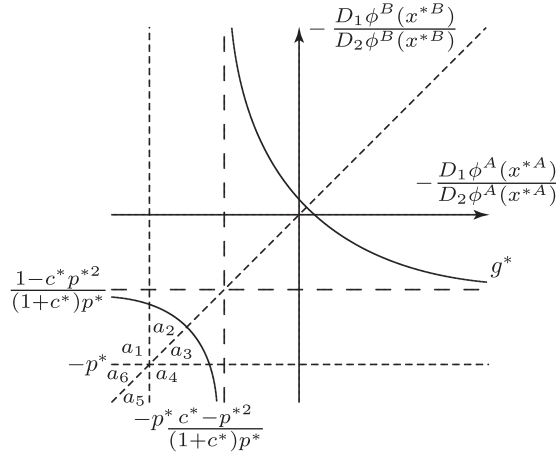

Fig. 9.

of (51) below) without inducing new Walrasian equilibria, i.e.,

$$-\frac{D_1\phi^B(x^{B*})}{D_2\phi^B(x^{B*})} = \frac{(1-c^*p^{*2})D_1\phi^A(x^{A*}) - (1+c^*)p^*D_2\phi^A(x^{A*})}{(1+c^*)p^*D_1\phi^A(x^{A*}) + (c^*-p^{*2})D_2\phi^A(x^{A*})}. \tag{51}$$

This is possible because the slope of A's offer curve can be perturbed as little as required in the C^1 topology for u^A in such a way that the pair of offer curves slopes at the Walrasian allocation $\left(-\frac{D_1\phi^A(x^{A*})}{D_2\phi^A(x^{A*})}, -\frac{D_1\phi^B(x^{B*})}{D_2\phi^B(x^{B*})}\right)$ is on the graph of

$$g(z) = \frac{(1-c^*p^{*2})z + (1+c^*)p^*}{(1+c^*)p^*z - (c^*-p^{*2})} \tag{52}$$

in Fig. 9 below (for the case $c^* = 1$ and $1 < p^{*2}$)[27] without ever crossing the boundaries (in short dashes) between the regions a_i, $i = 1, \ldots, 4$, which would imply new crossings of the offer curves that would correspond to new Walrasian equilibria (excluding a_5 and a_6 where A's and B's demand are simultaneously upward-sloped for both goods).[28]

Far enough in the sequence $\{\hat{x}_n\}$, A's marginal rate of substitution at the intersection of $\hat{x}_n$ with B's offer curve is close to p^*, and hence not bigger than the slope of B's offer curve at x^*. By continuity, the same is true for δ^A and δ^B close to 1. This guarantees that this intersection corresponds to an SSP equilibrium.

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[27] The relevant property is that, since for any c^*, $p^* > 0$, it holds true that $-p^* < \frac{c^*-p^{*2}}{(1+c^*)p^*}$ and $-p^* < \frac{1-c^*p^{*2}}{(1+c^*)p^*}$ always, the asymptotes of g^* (and hence g^* itself) intersect every region a_i in Fig. 9, except for a_5, a_6 see footnote 28 about these excluded cases.

[28] That is to say, in a_5, a_6, for $i = A, B$, it holds $\frac{dx_1^i}{dp} > 0$ and $\frac{dx_2^i}{dp^{-1}} > 0$ simultaneously for some range of prices. We think of this case in which demand increases for the good that is becoming more expensive and decreases for the good that is becoming cheaper as a non-observed pathological case. Note that this does not prevent backward-bending offer curves, and hence that any good may be inferior for some range of prices. It just excludes the possibility of both goods being inferior for the *same* range of prices.

We now establish the second part of the theorem concerning convergence to Walrasian alloca-
tions. Note first that since a Walrasian allocation is efficient, f^B is invertible around the profile
of Walrasian utilities (u^{A*}, u^{B*}). Hence so is $\delta^A f^B$ around u^{A*} for δ^A close enough to 1.

For given δ^A, δ^B close to 1, should $(\delta^A f^B)^{-1}(u^{A*})$ be smaller (respectively, bigger) than
$\delta^B f^A(u^{A*})$, and $f^A(u^A) \leqslant (f^B)^{-1}(u^{A*})$ for every u^A close enough to u^{A*}, then there would
exist two other (respectively, no other) intersections of $\delta^A f^A$ and $\delta^A f^B$. [29]

Now, clearly $\delta^B f^A(u^{A*}) = \delta^B u^{B*}$. As for $(\delta^A f^B)^{-1}(u^{A*})$, let $\tilde{f}^B(u^B, \delta^A) = \delta^A f^B(u^B)$.
Linearizing $\tilde{f}^B$ around $(u^{B*}, 1)$ it follows that $(\delta^A f^B)^{-1}(u^{A*})$ is the level of utility u^B for B
such that $0 \approx f^{B'}(u^{B*})(u^B - u^{B*}) + f^B(u^{B*})(\delta^A - 1)$, i.e.,

$$(\delta^A f^B)^{-1}(u^{A*}) \approx u^{B*} + \frac{u^{A*}}{f^{B'}(u^{B*})}(1 - \delta^A). \tag{53}$$

Therefore $(\delta^A f^B)^{-1}(u^{A*}) < \delta^B f^A(u^{A*})$ holds for δ^A, δ^B smaller but close to 1 if, and only if,

$$u^{B*} + \frac{u^{A*}}{f^{B'}(u^{B*})}(1 - \delta^A) < \delta^B u^{B*}, \tag{54}$$

i.e., if, and only if,

$$\frac{u^{B*}}{u^{A*}} < -\frac{1}{f^{B'}(u^{B*})} \frac{1 - \delta^A}{1 - \delta^B}. \tag{55}$$

Note that the range of values taken by $\frac{1-\delta^A}{1-\delta^B}$ in every neighborhood of $(\delta^A, \delta^B) = (1, 1)$ in
$(0, 1) \times (0, 1)$ is $\mathbb{R}_{++}$. Therefore there always exist discount factors δ^A, δ^B arbitrarily close
to 1 for which the condition (55) holds, as well as discount factors δ^A, δ^B arbitrarily close
to 1 for which the reversed inequality holds. Since, generically, either $f^A(u^A) \leqslant (f^B)^{-1}(u^{A*})$ or
$f^A(u^A) \geqslant (f^B)^{-1}(u^{A*})$ holds for all u^A close enough to u^{A*}, the conclusion follows. $\square$

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[29] Similarly $(\delta^B f^B)^{-1}(u^{A*}) > \delta^A f^A(u^{A*})$ (respectively, $<$) along with $f^A(u^A) \geqslant (f^B)^{-1}(u^{A*})$ guarantees the
existence of two other (respectively, no) intersections, for δ^A, δ^B close enough to 1.

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