Optimal Targeting Rules for Monetary and Fiscal Policy*

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PRELIMINARY

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While there are by now substantial literatures seeking to characterize optimal monetary and fiscal policy respectively, the two literatures have largely developed in isolation, and upon apparently contradictory foundations. The modern literature on dynamically optimal fiscal policy often abstracts from monetary aspects of the economy altogether, and so implicitly allows for no useful role for monetary policy. When monetary policy is considered within the theory of optimal fiscal policy, it is most often in the context of models with flexible prices; in these models, monetary policy matters only (i) because the level of nominal interest rates (and hence the opportunity cost of holding money) determines the size of certain distortions that result from the attempt to economize on money balances, and (ii) because the way in the price level varies in response to real disturbances determines the state-contingent real payoffs on (riskless) nominally-denominated government debt, which may facilitate tax-smoothing in the case that explicitly state-contingent debt is not available. The literature on optimal monetary policy has instead been mainly concerned with quite distinct objectives for monetary stabilization policy, namely the minimization of the distortions that result from prices or wages that do not adjust quickly enough to clear markets. At the same time, this literature typically ignores the fiscal consequences of alternative monetary policies; the characterizations of optimal monetary policy that are obtained are thus strictly correct only for a world in which lump-sum taxes are available.

Here we wish to consider the way in which the conclusions reached in each of these two familiar literatures must be modified if one takes simultaneous account of the basic elements of the policy problems addressed in each literature. On the one hand, we wish to consider how conventional conclusions with regard to the nature of an optimal monetary policy rule must be modified if one recognizes that the government's only sources of revenue are distorting taxes, so that the fiscal consequences of monetary policy matter for welfare. And on the other hand, we wish to consider how conventional conclusions with regard to optimal tax policy must be modified if one recognizes that prices do not instantaneously clear markets, so that output determination depends on aggregate demand, in addition to the supply-side factors stressed in the conventional theory of optimal taxation.

A number of recent papers have also sought to jointly consider optimal monetary and fiscal policy, in the context of models with sticky prices; important examples include Correia *et al.,* (2001), Schmitt-Grohé and Uribe (2001), and Siu (2001). Our approach differs from those taken in these papers, however, in several respects. First,

we model price stickiness in a different way than in any of these papers, namely, by assuming staggered pricing of the kind introduced by Calvo (1983). This particular form of price stickiness has been widely used both in analyses of optimal monetary policy in models with explicit microfoundations (*e.g.,* Goodfriend and King, 1997; Clarida *et al.,* 1999; Woodford, 2003) and in the empirical literature on optimizing models of the monetary transmission mechanism (*e.g.,* Rotemberg and Woodford, 1997; Gali and Gertler, 1999; Sbordone, 2002).

Perhaps more importantly, we obtain analytical results rather than purely numerical ones. To obtain these results, we propose a linear-quadratic approach to the characterization of optimal monetary and fiscal policy, that allows us to nest both conventional analyses of optimal monetary policy, such as that of Clarida *et al.* (1999), and analyses of optimal tax-smoothing in the spirit of Barro (1979), Lucas and Stokey (1983), and Aiyagari *et al.* (2002) as special cases of our more general framework. We show how a linear-quadratic policy problem can be derived which yields a correct linear approximation to the optimal policy rules from the point of view of the maximization of expected discounted utility in a dynamic stochastic general-equilibrium model, building on the work of Benigno and Woodford (2003) for the case of optimal monetary policy when lump-sum taxes are available.

Finally, we do not content ourselves with merely characterizing the optimal dynamic responses of our policy instruments (and other state variables) to shocks under an optimal policy, given one assumption or another about the nature and statistical properties of the exogenous disturbances to our model economy. Instead, we also wish to derive *policy rules* that the monetary and fiscal authorities may reasonably commit themselves to follow, as a way of implementing the optimal equilibrium. In particular, we seek to characterize optimal policy in terms of *optimal targeting rules* for monetary and fiscal policy, of the kind proposed in the case of monetary policy by Svensson (1999), Svensson and Woodford (2003), and Giannoni and Woodford (2002, 2003). The rules are specified in terms of a target criterion for each authority; each authority commits itself to use its policy instrument each period in whatever way is necessary in order to allow it to project an evolution of the economy consistent with its target criterion. As discussed in Giannoni and Woodford (2002), we can derive rules of this form that are not merely consistent with the desired equilibrium responses to disturbances, but that in addition (i) imply a *determinate* rational-expectations equilibrium, so that there are not other equally possible (but less desirable) equilibria

consistent with the same policy; and (ii) bring about optimal responses to shocks regardless of the character of and statistical properties of the exogenous disturbances in the model.

1 The Policy Problem

Here we describe our assumptions about the economic environment and pose the optimization problem that jointly optimal monetary and fiscal policies are intended to solve. The approximation method that we use to characterize the solution to this problem is then presented in the following section. Further details of the derivation of the structural equations of our model of nominal price rigidity can be found in Woodford (2003, chapter 3).

The goal of policy is assumed to be the maximization of the level of expected utility of a representative household. In our model, each household seeks to maximize

$$U_{t_0} \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[\tilde{u}(C_t;\xi_t) - \int_0^1 \tilde{v}(H_t(j);\xi_t)dj \right],$$

where C_t is a Dixit-Stiglitz aggregate of consumption of each of a continuum of differentiated goods,

$$C_t \equiv \left[\int_0^1 c_t(i)^{\frac{\theta}{\theta-1}} di \right]^{\frac{\theta-1}{\theta}},$$

with an elasticity of substitution equal to $\theta > 1$, and $H_t(j)$ is the quantity supplied of labor of type j. Each differentiated good is supplied by a single monopolistically competitive producer. There are assumed to be many goods in each of an infinite number of "industries"; the goods in each industry j are produced using a type of labor that is specific to that industry, and also change their prices at the same time. The representative household supplies all types of labor as well as consuming all types of goods.[1] For each value of the disturbances ξ_t, $\tilde{u}(\cdot;\xi_t)$ is an increasing concave function and $\tilde{v}(\cdot;\xi_t)$ is an increasing convex function. The vector of exogenous disturbances ξ_t may contain several elements, so that no assumption is made about correlation of the exogenous shifts in the functions $\tilde{u}$ and $\tilde{v}$.

[1] We might alternatively assume specialization across households in the type of labor supplied; in the presence of perfect sharing of labor income risk across households, household decisions regarding consumption and labor supply would all be as assumed here.

We assume a common technology for the production of all goods, in which (industry-specific) labor is the only variable input,

$$y_t(i) = A_t f(h_t(i)),$$

where A_t is an exogenously varying technology factor and $f(\cdot)$ is an increasing concave function. We can then write the expected utility of the representative household as a function of the economy's production plan in the form

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[u(Y_t; \xi_t) - \int_0^1 v(y_t(i); \xi_t) di \right], \tag{1.1}$$

where[2]

$$u(Y_t; \xi_t) \equiv \tilde{u}(Y_t - G_t; \xi_t),$$
$$v(y_t(i); \xi_t) \equiv \tilde{v}(f^{-1}(y_t(i)/A_t); \xi_t),$$

G_t is the (exogenously given) level of government purchases of the composite good[3], and the disturbance vector ξ_t now includes the technology factor and the level of government purchases as well as the preference shocks.

The producers in each industry fix the prices of their goods in monetary units for a random interval of time, as in the model of staggered pricing introduced by Calvo (1983). We let $0 \leq \alpha < 1$ be the fraction of prices that remain unchanged in any period. A supplier that changes its price in period t chooses its new price $p_t(i)$ to maximize

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi(p_t(i), p_T^j, P_T; Y_T, \tau_T, \xi_T) \right\}, \tag{1.2}$$

where $Q_{t,T}$ is the stochastic discount factor by which financial markets discount random nominal income in period T to determine the nominal value of a claim to such

[2] Note that in the definition of $v(\cdot; \xi)$, we make use of the fact that under our assumptions, every good i in a given industry j will be produced in the same quantity in each state of the world, so that we can write the demand for labor of any type j as a function of the common quantity $y_t(i)$ that is produced of any of the goods using that type of labor. The existence of "industries" consisting of more than one good is introduced only to motivate our assumption that firms are wage-takers, despite the existence of segmented labor markets.

[3] The government is assumed to need to obtain an exogenously given quantity of the Dixit-Stiglitz aggregate each period, and to obtain this in a cost-minimizing fashion. Hence the government allocates its purchases across the suppliers of differentiated goods in the same proportion as do households.

income in period t, and α^{T-t} is the probability that a price chosen in period t will not have been revised by period T. In equilibrium, this discount factor is given by

$$Q_{t,T} = \beta^{T-t} \frac{\tilde{u}_c(C_T;\xi_T)}{\tilde{u}_c(C_t;\xi_t)} \frac{P_t}{P_T},$$

where

$$P_t \equiv \left[\int_0^1 p_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}}$$

is the Dixit-Stiglitz aggregate price index. The function

$$\begin{aligned} \Pi(p, p^j, P; Y, \tau, \xi) \;\; &\equiv \;\; (1-\tau) p Y (p/P)^{-\theta} \\ &- \;\; \mu_t^w \frac{\tilde{v}_h(f^{-1}(Y(p^j/P)^{-\theta}/A);\xi)}{\tilde{u}_c(Y-G;\xi)} P f^{-1}(Y(p/P)^{-\theta}/A) \end{aligned} \quad (1.3)$$

indicates the after-tax nominal profits of a supplier with price p, in an industry with common price p^j,[4] when the aggregate price index is equal to P, aggregate demand is equal to Y, and sales revenues are taxed at rate τ.[5] Here the real wage demanded for labor of type j is assumed to be given by

$$w_t(j) = \mu_t^w \frac{\tilde{v}_h(H_t(j);\xi)}{\tilde{u}_c(C_t;\xi_t)},$$

where $\mu_t^w \geq 1$ is an exogenous markup factor in the labor market (allowed to vary over time, but assumed common to all labor markets),[6] and firms are assumed to be wage-takers. We allow for wage markup variations in order to include the possibility of a "pure cost-push shock" that affects equilibrium pricing behavior while implying no change in the efficient allocation of resources. Note that variation in the tax rate τ_t has a similar effect on this pricing problem (and hence on supply behavior); this is the sole distortion associated with tax policy in the most basic version of our model.

[4] In equilibrium, each of the goods in a given industry has the same price. However, in considering the optimal pricing decision of each supplier, we must consider the consequences for that firm's profits of choosing a price p that differs from the choice p^j of the other firms in the industry.

[5] Note that we assume here that the price that is sticky is the price inclusive of the tax, as in the case of a European-style VAT. Our methods can equally easily be applied to other sorts of taxes, but we illustrate our method using this one.

[6] In the case that we assume that $\mu_t^w = 1$ at all times, our model is one in which both households and firms are wage-takers, or there is efficient contracting between them.

Each of the suppliers that revise their prices in period t choose the same new price p_t^*, which is implicitly defined by the first-order condition

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi_1(p_t^*, p_t^*, P_T; Y_T, \tau_T, \xi_T) \right\} = 0. \tag{1.4}$$

The price index then evolves according to a law of motion

$$P_t = \left[(1-\alpha) p_t^{*1-\theta} + \alpha P_{t-1}^{1-\theta} \right]^{\frac{1}{1-\theta}}. \tag{1.5}$$

The requirement that the joint evolution of prices and output be consistent with (1.4) – (1.5) is the critical constraint on possible production plans under alternative monetary and fiscal policies. In the case of flexible prices ($\alpha = 0$), this reduces to the requirement that

$$\Pi_1(P_t, P_t, P_t; Y_t, \tau_t, \xi_t) = 0, \tag{1.6}$$

which determines the equilibrium level of output (and the common production of each good) as a function of τ_t and the exogenous disturbances, given that Π_1 is homogeneous of degree zero in its first three arguments. It follows that monetary policy is irrelevant to welfare, except insofar as its fiscal effects can help to make possible a desired path for the tax rate $\{\tau_t\}$. When $\alpha > 0$, instead, monetary policy can affect the real allocation of resources even when the path $\{\tau_t\}$ is fixed.

To simplify the algebraic form of our results, we restrict attention in this paper to the case of isoelastic functional forms,

$$\tilde{u}(C_t; \xi_t) \equiv \frac{C_t^{1-\tilde{\sigma}^{-1}} \bar{C}_t^{\tilde{\sigma}^{-1}}}{1-\tilde{\sigma}^{-1}},$$

$$\tilde{v}(H_t; \xi_t) \equiv \frac{\lambda}{1+\nu} H_t^{1+\nu} \bar{H}_t^{-\nu},$$

$$A_t f(h_t) = A_t h_t^{1/\phi},$$

where $\tilde{\sigma}, \lambda, \nu > 0, \phi > 1$ are constant coefficients, and $\{\bar{C}_t, \bar{H}_t, A_t\}$ are positive-valued exogenous random variables. In this case, the first-order condition (1.4) has a closed-form solution

$$\frac{p_t^*}{P_t} = \left(\frac{K_t}{F_t} \right)^{\frac{1}{1+\omega\theta}},$$

where $\omega \equiv \phi(1+\nu) - 1 > 0$ is the elasticity of real marginal cost in an industry with respect to industry output, and F_t and K_t are aggregate variables of the form

$$F_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} (1-\tau_T) f(Y_T;\xi_T) \left(\frac{P_T}{P_t}\right)^{\theta-1},$$

$$K_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} k(Y_T;\xi_T) \left(\frac{P_T}{P_t}\right)^{\theta(1+\omega)},$$

for certain functions f and k defined in the appendix.[7] Substitution of this solution into (1.5) implies that equilibrium inflation in any period is given by

$$\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha} = \left(\frac{F_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}}, \tag{1.7}$$

where $\Pi_t \equiv P_t/P_{t-1}$. This defines a short-run aggregate supply relation between inflation and output, given the current tax rate τ_t, current disturbances ξ_t, and expectations regarding future inflation, output, taxes and disturbances.

Under our assumptions of staggered price adjustment and segmented labor markets, the representative household's disutility of labor will depend on the degree of dispersion of prices across industries, in addition to aggregate real activity. In the case of the functional forms just assumed, we can write

$$\int_0^1 v(y_t(i);\xi_t)di = \frac{\lambda}{1+\nu}\frac{Y_t^{1+\omega}}{A_t^{1+\omega}\bar{H}_t^{1+\nu}}\Delta_t,$$

where

$$\Delta_t \equiv \int_0^1 \left(\frac{p_t(i)}{P_t}\right)^{-\theta(1+\omega)} di \geq 1$$

is a measure of price dispersion at date t. This allows us to write the utility flow to the representative household each period in the form $U(Y_t, \Delta_t; \xi_t)$, in terms of only aggregate variables, and hence to write our objective (1.1) as

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t; \xi_t). \tag{1.8}$$

[7] In the case of the function $k(\cdot;\xi_t)$, the disturbance vector ξ_t is now understood to include the current value of the wage markup μ_t^w.

Because the relative prices of the industries that do not change their prices in period t remain the same (except for rescaling by the factor Π_t), we obtain

$$\Delta_t = \alpha \Delta_{t-1} \Pi_t^{\theta(1+\omega)} + (1-\alpha) \left(\frac{p_t^*}{P_t} \right)^{-\theta(1+\omega)}.$$

Using (1.5) to substitute for the relative price p_t^*/P_t as a function of Π_t, we obtain a law of motion of the form

$$\Delta_t = h(\Delta_{t-1}, \Pi_t) \tag{1.9}$$

for the dynamics of price dispersion. This is the source in our model of welfare losses from inflation or deflation.

We abstract here from any monetary frictions that would account for a demand for central-bank liabilities that earn a substandard rate of return; we nonetheless assume that the central bank can control the riskless short-term nominal interest rate i_t,[8] which is in turn related to other financial asset prices through the arbitrage relation

$$1 + i_t = [E_t Q_{t,t+1}]^{-1}.$$

We shall assume that the zero lower bound on nominal interest rates never binds under the optimal policies considered below,[9] so that we need not introduce any additional constraint on the possible paths of output and prices associated with a need for the chosen evolution of prices to be consistent with a non-negative nominal interest rate.

Our abstraction from monetary frictions, and hence from the existence of seignorage revenues, does not mean that monetary policy has no fiscal consequences, for interest-rate policy and the equilibrium inflation that results from it have implications for the real burden of government debt. In our baseline case, we assume that all public debt consists of riskless nominal one-period bonds. The nominal value B_t of end-of-period public debt then evolves according to a law of motion

$$B_t = (1 + i_{t-1}) B_{t-1} + P_t s_t, \tag{1.10}$$

where the real primary budget surplus is given by

$$s_t \equiv \tau_t Y_t - G_t - \nu_t.$$

[8]For discussion of how this is possible even in a "cashless" economy of the kind assumed here, see Woodford (2003, chapter 2).

[9]This can be shown to be true in the case of small enough disturbances, given that the nominal interest rate is equal to $\bar{r} = \beta^{-1} - 1 > 0$ under the optimal policy in the absence of disturbances.

Here τ_t, the share of the national product that is collected by the government as tax revenues in period t, is the key fiscal policy decision each period; the real value of (lump-sum) government transfers ν_t is treated as exogenously given, as are government purchases G_t. (We introduce the additional type of exogenously given fiscal needs so as to be able to analyze the consequences of a "purely fiscal" disturbance, with no implications for the real allocation of resources beyond those that follow from its effect on the government budget.)

Under the assumption of only riskless one-period nominal government debt, $b_{t-1} \equiv (1+i_{t-1})B_{t-1}/P_{t-1}$ is a predetermined state variable in period t. Rational-expectations equilibrium requires that the expected path of government surpluses must satisfy an intertemporal solvency condition

$$b_{t-1}\frac{P_{t-1}}{P_t} = E_t \sum_{T=t}^{\infty} R_{t,T} s_T \tag{1.11}$$

in each state of the world that may be realized at date t, where $R_{t,T} \equiv Q_{t,T}P_T/P_t$ is the stochastic discount factor for a real income stream.[10] This condition restricts the possible paths that may be chosen for the tax rate $\{\tau_t\}$. Monetary policy can affect this constraint, however, both by affecting the period t inflation rate (which affects the left-hand side) and (in the case of sticky prices) by affecting the discount factors $\{R_{t,T}\}$.

In considering the optimal policy to follow from some date t_0 onward, then, one must restrict attention to state-contingent paths $\{\Pi_t, Y_t, \tau_t, \Delta_t\}$ that satisfy (1.7), (1.9) and (1.11), given initial government debt b_{t_0-1} and price dispersion Δ_{t_0-1}. One might expect that an optimal policy should maximize (1.8) subject only to these constraints. However, the t_0-optimal policy in this sense will not be *time consistent;* the same criterion would lead one to choose a policy at the next date other than the continuation of the paths chosen at date t_0. This results from the forward-looking character of the constraints (1.7) and (1.11). As an example, in the case of positive initial government debt, it will ordinarily be judged optimal to carry government

[10] See Woodford (2003, chapter 2) for derivation of this condition from household optimization together with market clearing. The condition should not be interpreted as an *a priori* constraint on possible government policy rules, as discussed in Woodford (2001). However, when we consider the problem of choosing an optimal plan from the among the possible rational-expectations equilibria, this condition must be imposed among the constraints on the set of equilibria that one may hope to bring about.

debt into future periods, and to commit not to inflate away the value of that debt when those periods are reached. Yet it will also be judged optimal to choose a large inflation in period t_0, in order to reduce the pre-existing debt burden, and similar reasoning would lead one to choose high inflation at the later dates as well.

We here seek instead to characterize policy rules that are "optimal from a timeless perspective," as advocated by Woodford (1999).[11] This means that we wish to find time-invariant policy rules with the property that a commitment to follow these rules from date t_0 onward determines a rational-expectations equilibrium which maximizes (1.8), subject to certain constraints on initial outcomes that are of a kind to which one should have wished to commit oneself at an earlier date. In the model set out above, the feasible short-run tradeoffs among the values of Π_t, Y_t, and τ_t implied by constraints (1.7) and (1.11) depend on expectations at t that can be summarized[12] by expectations regarding the state-contingent value of a vector X_{t+1}, the elements of which are F_{t+1}, K_{t+1}, and W_{t+1}, where

$$W_t \equiv \frac{b_{t-1}}{\Pi_t}\tilde{u}_c(C_t;\xi_t).$$

Hence state-contingent commitments regarding the values of these three variables at date $t+1$ completely specify all aspects of the future regarding which it would be desirable for commitment to be possible at date t. We accordingly require only that the equilibrium implied by our recommended policy rules maximize (1.1) looking forward from any date t_0, subject to the constraints that (1.7), (1.9) and (1.11) be satisfied for each $t \geq t_0$, given the predetermined values of b_{t_0-1} and Δ_{t_0-1}, *and* the additional constraint that X_{t_0} take a particular value.

If we suppose that policy is chosen each period subject to predetermined values of F_t, K_t and W_t with which policy must be consistent — and that the policy decisions in period t include the choice of state-contingent commitments $X_{t+1} = X(\xi_{t+1})$ for the following period[13] — then the time-consistency problem is eliminated. The policy problem just described has a recursive form, and its solution satisfies a Bellman

[11]For further discussion, see Giannoni and Woodford (2002) and Woodford (2003, chapter 7).

[12]See the appendix for demonstration of this.

[13]Here we use the notation ξ_{t+1} to refer to a complete description of the exogenous state of the world in period $t+1$, including not just that period's disturbances to tastes, technology, and so on, but all information at date $t+1$ about the subsequent evolution of these disturbances. If the disturbance vector ξ_t referred to above follows a Markov process, then this involves no extension of the notation.

equation

$$V(b_{t-1}, \Delta_{t-1}, X_t; \xi_t) = \max_{x_t, X(\cdot)} \left\{ U(Y_t, \Delta_t; \xi_t) + \beta E_t[V(b_t, \Delta_t, X_{t+1}; \xi_{t+1})] \right\}, \qquad (1.12)$$

where $V(b_{t-1}, \ldots)$ indicates the maximum attainable value of (1.1) in period t given the initial constraints, and the maximization is over the values of $x_t \equiv (\Pi_t, Y_t, \tau_t, b_t, \Delta_t)$ and the state-contingent commitments $X(\cdot)$ for period $t+1$, subject to the constraints listed above. (See the appendix for details.)

Here, however, we shall not assume that optimal policy is literally implemented in this way. Instead, we shall suppose that the monetary and fiscal authorities choose their policy instruments each period so as to ensure that the economy's projected evolution from that time onward satisfies certain target criteria. These *targeting rules,* however, are optimal in the sense that the equilibrium that they determine from any date t_0 onward solves problem (1.12) for some specification of the initial commitments F_{t_0}, K_{t_0} and W_{t_0}. Furthermore, the way in which these initial commitments are (implicitly) chosen by the policy rules is *self-consistent* in the sense that in the solution to the recursive policy problem (1.12) one would choose state-contingent commitments in all later periods that would also conform to the same principle.[14] In order to derive rules for monetary and fiscal policy with this property, we must first characterize the state-contingent evolution of the endogenous variables in the solution to an optimization problem of the form (1.12).

2 A Linear-Quadratic Approximate Problem

Our goal here is to characterize timelessly optimal policy (as defined in the previous section) in the case that the random disturbances are small enough to make a purely local analysis a reasonable approximation. We first consider the purely deterministic case, in which the exogenous disturbances $\bar{C}_t$, G_t, $\bar{H}_t$, A_t, μ_t^w, ν_t each take constant values $\bar{C}, \bar{G}, \bar{H}, \bar{A}, \bar{\mu}^w > 0$ and $\bar{\nu} \geq 0$ for all $t \geq t_0$, and assume initial conditions $b_{t_0-1} = \bar{b} > 0$ and $\Delta_{t_0-1} = 1$. We shall furthermore suppose that in this case, there

[14]Note that the rule for choosing initial commitments in a self-consistent way need not be unique, as discussed in Woodford (2003, chapter 7). Hence the targeting rules that are optimal from a timeless perspective need not be uniquely defined, as Giannoni and Woodford (2002) show. Here we derive particular approximate policy rules of this kind that take an especially simple form.

exist initial constraints $X_{t_0} = \bar{X}$ such that the solution to the constrained optimization problem (1.12) involves a constant policy $x_t = \bar{x}, X_{t+1} = \bar{X}$ each period, where $\bar{\Pi} = 1$, $\bar{\Delta} = 1$, and $\bar{b}$ is equal to the initial real debt. (We show in the appendix that the first-order conditions for this problem have a solution of this form, and we verify below that the second-order conditions for a local optimum are also satisfied.) Note that our conclusion that the optimal steady-state inflation rate is zero generalizes the result of Benigno and Woodford (2003) for the case in which taxes are lump-sum at the margin. We may furthermore assume without loss of generality that the constant values of $\bar{C}$ and $\bar{H}$ are chosen (given the initial government debt $\bar{b}$) so that in the optimal steady state, $C_t = \bar{C}$ and $H_t = \bar{H}$ each period.[15] The associated steady-state tax rate is given by

$$\bar{\tau} = s_G + \frac{\bar{\nu} + (1-\beta)\bar{b}}{\bar{Y}},$$

where $\bar{Y} = \bar{C} + \bar{G} > 0$ is the steady-state output level, and $s_G \equiv \bar{G}/\bar{Y} < 1$ is the steady-state share output purchased by the government. As shown in the appendix, this solution necessarily satisfies $0 < \bar{\tau} < 1$.

We next wish to characterize the optimal responses to small perturbations of the initial conditions and small fluctuations in the disturbance processes around the above values. To do this, we compute a linear-quadratic approximate problem, the solution to which represents a linear approximation to the solution to policy problem (1.12), using the method introduced in Benigno and Woodford (2003). An important advantage of this approach is that it allows direct comparison of our results with those obtained in other analyses of optimal monetary stabilization policy. Other advantages are that it makes it straightforward to verify whether the second-order conditions hold that are required in order for a solution to our first-order conditions to be at least a local optimum,[16] and that it provides us with a welfare measure with which to rank alternative sub-optimal policies, in addition to allowing computation of the optimal policy.

We begin by computing a Taylor-series approximation to our welfare measure (1.8), expanding around the steady-state allocation defined above, in which $y_t(i) = \bar{Y}$

[15]Note that we may assign arbitrary positive values to $\bar{C}, \bar{H}$ without changing the nature of the implied preferences, as long as the value of λ is appropriately adjusted.

[16]Benigno and Woodford (2003) show that these conditions can fail to hold, so that a small amount of arbitrary randomization of policy is welfare-improving, but argue that the conditions under which this occurs in their model are not empirically plausible.

for each good at all times and $\xi_t = 0$ at all times.[17] As a second-order (logarithmic) approximation to this measure, we obtain[18]

$$U_{t_0} = \overline{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Phi \widehat{Y}_t - \frac{1}{2} u_{yy} \widehat{Y}_t^2 + \widehat{Y}_t u_\xi \xi_t - \frac{1}{2}(1-\Phi)(\theta^{-1}+\omega)\mathrm{var}_i \log y_t(i)$$
$$+ \quad \text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (2.1)$$

where $\hat{Y}_t \equiv \log(Y_t/\bar{Y})$ measures deviations of aggregate output from the steady-state level, the term "t.i.p." collects terms that are independent of policy (constants and functions of exogenous disturbances) and hence irrelevant for ranking alternative policies, and $||\xi||$ is a bound on the amplitude of our perturbations of the steady state.[19] Here the coefficient

$$\Phi \equiv 1 - \frac{\theta - 1}{\theta} \frac{1 - \bar{\tau}}{\bar{\mu}^w} < 1$$

measures the steady-state wedge between the marginal rate of substitution between consumption and leisure and the marginal product of labor, and hence the inefficiency of the steady-state output level $\bar{Y}$. Under the assumption that $\bar{b} > 0$, we necessarily have $\Phi > 0$, meaning that steady-state output is inefficiently low. The coefficients u_{yy} and u_ξ are defined in the appendix.

The log-linear demand curve for each good implied by Dixit-Stiglitz preferences allows us to write the dispersion of output levels as a function of the dispersion of prices,

$$\mathrm{var}_i \log y_t(i) = \theta^2 \mathrm{var}_i \log p_t(i).$$

[17]Here the elements of ξ_t are assumed to be $\bar{c}_t \equiv \log(\bar{C}_t \bar{C})$, $\bar{h}_t \equiv \log(\bar{H}_t/\bar{H})$, $a_t \equiv \log(A_t/\bar{A})$, $\hat{\mu}_t^w \equiv \log(\mu_t^w/\bar{\mu}^w)$, $\hat{G}_t \equiv (G_t - \bar{G})/\bar{Y}$, and $\hat{\nu}_t \equiv (\nu_t - \bar{\nu})/\bar{Y}$, so that a value of zero for this vector corresponds to the steady-state values of all disturbances. The perturbations $\hat{G}_t$ and ν_t are not defined to be logarithmic so that we do not have to assume positive steady-state values for these variables.

[18]See the appendix for details. Our calculations here follow closely those of Woodford (2003, chapter 6) and Benigno and Woodford (2003).

[19]Specifically, we use the notation $\mathcal{O}(||\xi||^k)$ as shorthand for $\mathcal{O}(||\xi, \hat{b}_{t_0-1}, \hat{\Delta}_{t_0-1}^{1/2}, \hat{X}_{t_0}||^k)$, where in each case hats refer to log deviations from the steady-state values of the various parameters of the policy problem (1.12). We treat $\hat{\Delta}_{t_0}^{1/2}$ as an expansion parameter, rather than $\hat{\Delta}_{t_0}$ because (1.9) implies that deviations of the inflation rate from zero of order ϵ only result in deviations in the dispersion measure Δ_t from one of order ϵ^2. We are thus entitled to treat the fluctuations in Δ_t as being only of second order in our bound on the amplitude of disturbances, since if this is true at some initial date it will remain true thereafter. We also assume in deriving (2.1) that deviations of $y_t(i)$ from $\bar{Y}$ are only of order $\mathcal{O}(||\xi||)$ in each industry; this is shown to be true of the optimal state-contingent evolution characterized below.

In addition, the Calvo assumption about the distribution of intervals between price changes allows us to relate the dispersion of prices to the overall rate of inflation,

$$\sum_{t=t_0}^{\infty} \beta^t \operatorname{var}_i \log p_t(i) = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} \sum_{t=t_0}^{\infty} \beta^t \pi_t^2 + \text{t.i.p.} + \mathcal{O}(||\xi||^3).$$

With these substitutions, we can then approximate our welfare measure by

$$\begin{aligned} U_{t_0} &= \bar{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\Phi \widehat{Y}_t - \frac{1}{2} u_{yy} \widehat{Y}_t^2 + \widehat{Y}_t u_\xi \xi_t - u_\pi \pi_t^2] \\ &+ \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{2.2}$$

for a certain coefficient $u_\pi > 0$ defined in the appendix. Note that we can now write our stabilization objective purely in terms of the evolution of the aggregate variables $\{\hat{Y}_t, \pi_t\}$ and the exogenous disturbances.

We note that when $\Phi > 0$, there is a non-zero linear term in (2.2), which means that we cannot expect to evaluate this expression to second order using only an approximate solution for the path of aggregate output that is accurate only to first order. Thus we cannot determine optimal policy, even up to first order, using this approximate objective together with approximations to the structural equations that are accurate only to first order. Rotemberg and Woodford (1997) avoid this problem by assuming an output subsidy (*i.e.*, a value $\bar{\tau} < 0$) of the size needed to ensure that $\Phi = 0$. Here we do not wish to make this assumption, because we assume that lump-sum taxes are unavailable, in which case $\Phi = 0$ would be possible only in the case of a particular initial level of government assets $\bar{b} < 0$. Furthermore, we are more interested in the case in which government revenue needs are more acute than that would imply.

Benigno and Woodford (2003) propose an alternative way of dealing with this problem, which is to use a second-order approximation to the aggregate-supply relation to eliminate the linear terms in the quadratic welfare measure. We show in the appendix that to second order, equation (1.7) can be written

$$\begin{aligned} V_t &= \kappa(c_x' x_t + c_\xi' \xi_t + \frac{1}{2} x_t' C_x x_t + x_t' C_\xi \xi_t + \frac{1}{2} c_\pi \pi_t^2) + \beta E_t V_{t+1} \\ &+ \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned} \tag{2.3}$$

Here the notation "s.o.t.i.p." indicates terms independent of policy that are entirely

of second or higher order, and we have defined

$$x_t \equiv \begin{bmatrix} \hat{\tau}_t \\ \hat{Y}_t \end{bmatrix},$$

in which $\hat{\tau}_t \equiv \ln(\tau/\bar{\tau})$; and

$$V_t \equiv \pi_t + \frac{1}{2}v_\pi \pi_t^2 + v_z \pi_t Z_t,$$

where

$$Z_t \equiv E_t \sum_{T=t} (\alpha\beta)^{T-t}[z_x' x_T + z_\pi \pi_T + z_\xi' \xi_T];$$

for certain coefficients defined in the appendix. Note that to first order (2.3) reduces simply to

$$\pi_t = \kappa[\hat{Y}_t + \psi \hat{\tau}_t + c_\xi' \xi_t] + \beta E_t \pi_{t+1}, \tag{2.4}$$

for certain coefficients $\kappa, \psi > 0$. This is the familiar "New Keynesian Phillips curve" relation, extended here to take account of the effects of variations in the level of distorting taxes on supply costs.

Integrating forward equation (2.3), we obtain a relation of the form

$$V_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \kappa [c_x' x_t + \frac{1}{2} x_t' C_x x_t + x_t' C_\xi \xi_t + \frac{1}{2} c_\pi \pi_t^2] + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \tag{2.5}$$

In the case that taxes are lump-sum, $\psi = 0$, and the linear terms $c_x' x_t$ in this expression involve only $\hat{Y}_t$. We can then use (2.5), as Benigno and Woodford (2003) show, to write the discounted sum of output terms in (2.2) as a function of purely quadratic terms, up to a residual of third order. However, in the present case, (2.5) also contains linear terms involving the expected discounted value of variations in the tax rate, and we must eliminate these linear terms as well.

The same method can be applied, however, using a second-order approximation to the intertemporal government budget constraint to solve for the expected discounted value of the variations in the tax rate. We show in the appendix that a second-order Taylor-series approximation to (1.11) can be written in the form

$$\begin{aligned} \tilde{W}_t &= (1-\beta)[b_x' x_t + b_\xi' \xi_t + \frac{1}{2} x_t' B_x x_t + x_t' B_\xi \xi_t] + \beta E_t \tilde{W}_{t+1} \\ &+ \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned} \tag{2.6}$$

where $\tilde{W}_t \equiv (W_t - \bar{W})/\bar{W}$. Note that to first order, the deviation of W_t from its steady-state value is given by

$$\tilde{W}_t = \hat{b}_{t-1} - \pi_t + w_x' x_t + w_\xi' \xi_t + \mathcal{O}(||\xi||^2),$$

where the coefficients w_x, w_ξ are also defined in the appendix. Hence a first-order approximation (2.6) is simply

$$\begin{aligned} \hat{b}_{t-1} - \pi_t &+ w_x' x_t + w_\xi' \xi_t = (1-\beta)[b_x' x_t + b_\xi' \xi_t] \\ &+ \beta E_t[\hat{b}_t - \pi_{t+1} + w_x' x_{t+1} + w_\xi' \xi_{t+1}]. \end{aligned} \tag{2.7}$$

This is another important constraint in our linear-quadratic policy problem.

Integrating forward (2.6), we obtain

$$\tilde{W}_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [b_x' x_t + \frac{1}{2} x_t' B_x x_t + x_t' B_\xi \xi_t] + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \tag{2.8}$$

as a second-order approximation to the intertemporal government budget constraint. We can now combine (2.5) and (2.8) in a way to eliminate the linear term in (2.2). Specifically, we show in the appendix that it is possible to find ϑ_1, ϑ_2 such that

$$\vartheta_1 b_x' + \vartheta_2 c_x' = a_x' \equiv [0 \ \Phi].$$

We can then write the discounted sum of output terms appearing in our approximate welfare measure as

$$\begin{aligned} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Phi \hat{Y}_t &= E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\vartheta_1 b_x' + \vartheta_2 c_x'] x_t \\ &= -E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\frac{1}{2} x_t' D_x x_t + x_t' D_\xi \xi_t + \frac{1}{2} d_\pi \pi_t^2] \\ &\quad + \vartheta_1 \tilde{W}_{t_0} + \vartheta_2 V_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned} \tag{2.9}$$

where

$$D_x \equiv \vartheta_1 B_x + \vartheta_2 C_x, \quad \text{etc.}$$

We can then rewrite (2.2) as

$$
\begin{aligned}
U_{t_0} &= \Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ a_x' x_t - \frac{1}{2} x_t' A_x x_t - x_t' A_\xi \xi_t - \frac{1}{2} a_\pi \pi_t^2 \right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} x_t' Q_x x_t + x_t' Q_\xi \xi_t + \frac{1}{2} q_\pi \pi_t^2 \right\} + T_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y (\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{1}{2} q_\pi \pi_t^2 \right\} + T_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (2.10)
\end{aligned}
$$

where again all coefficients are defined in the appendix. Here the second line uses (2.9) to substitute for the linear terms in the first line, while the third line uses the fact that the matrices

$$
Q_x = \begin{bmatrix} 0 & 0 \\ 0 & q_y \end{bmatrix}, \qquad Q_\xi = \begin{bmatrix} 0 \\ q_\xi' \end{bmatrix},
$$

take a simple form, and introduces the notation

$$
\widehat{Y}_t^* = -q_y^{-1} q_\xi' \xi_t. \qquad (2.11)
$$

The factor Ω is necessarily positive, while the term

$$
T_{t_0} \equiv (1 - \Phi) \bar{Y} \bar{u}_c [\vartheta_1 \tilde{W}_{t_0} + \vartheta_2 V_{t_0}]
$$

is a transitory component.

Once again, we are interested in characterizing optimal policy from a timeless perspective. We observe from the form of the structural relations (2.3) and (2.6) and the definition of V_t that the aspects of the expected future evolution of the endogenous variables that affect the feasible set of values for inflation, output and tax rates in any period t can be summarized (in our second-order approximation to the structural relations) by the expected values of V_{t+1}, Z_{t+1}, and $\tilde{W}_{t+1}$. Hence the only commitments regarding future outcomes that can be of value in improving stabilization outcomes in period t can be summarized by commitments at t regarding the state-contingent values of those three variables in the following period. It follows that we are interested in characterizing optimal policy from any date t_0 onward subject to the constraint that given values for V_{t_0}, Z_{t_0} and $\tilde{W}_{t_0}$ be satisfied,[20] in addition to the constraints represented by the structural equations and the predetermined state variable $\hat{b}_{t_0-1}$.

[20]Note that a specification of initial values for these three variables corresponds, in our quadratic approximation to the structural equations, to a specification of initial values for the three variables F_{t_0}, K_{t_0} and W_{t_0} in section 1.

But given predetermined values for V_{t_0} and $\tilde{W}_{t_0}$, the value of the transitory component T_{t_0} is predetermined. Hence, over the set of admissible policies, higher values of (2.10) correspond to lower values of

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} q_y (\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{1}{2} q_\pi \pi_t^2 \right\}. \tag{2.12}$$

It follows that we may rank policies in terms of the implied value of the discounted quadratic loss function (2.12). Because this loss function is purely quadratic (*i.e.*, lacking linear terms), it is possible to evaluate it to second order using only a first-order approximation to the equilibrium evolution of inflation and output under a given policy. Hence the log-linear approximate structural relations (2.4) and (2.7) are sufficiently accurate for our purposes. Similarly, it suffices that we use log-linear approximations to the variables V_{t_0} and $\tilde{W}_{t_0}$ in describing the initial commitments, which are given by

$$\begin{aligned} \hat{V}_{t_0} &= \pi_{t_0}, \\ \hat{W}_{t_0} &= \hat{b}_{t_0-1} - \pi_{t_0} + w_x' x_{t_0} + w_\xi' \xi_{t_0} \\ &= \hat{b}_{t_0-1} - \pi_{t_0} - \sigma^{-1}(\hat{Y}_{t_0} - g_{t_0}), \end{aligned}$$

where $\sigma > 0$ is the intertemporal elasticity of substitution of aggregate expenditure, and g_t is a composite shift (due either to government purchases or to taste shocks) in the relation between the marginal utility of income and aggregate real income. Then an optimal policy from a timeless perspective is a policy from date t_0 onward that minimizes the quadratic loss function (2.12) subject to the constraints implied by the linear structural relations (2.4) and (2.7) holding in each period $t \geq t_0$, given the initial value $\hat{b}_{t_0-1}$, and subject also to the constraints that certain predetermined values for $\hat{V}_{t_0}$ and $\hat{W}_{t_0}$ (or alternatively, for π_{t_0} and for $\hat{Y}_{t_0}$) be achieved.[21]

In order for this linear-quadratic problem to have a bounded solution (which then approximates the solution to the exact problem), we must verify that our quadratic objective is convex. We show in the appendix that $q_y, q_\pi > 0$, so that the objective is convex, as long as the steady-state tax rate $\bar{\tau}$ and share of government purchases s_G in the national product are below certain positive bounds. We shall here assume

[21] The constraint associated with a predetermined value for Z_{t_0} can be neglected, in a first-order characterization of optimal policy, because the variable Z_t does not appear in the first-order approximation to the aggregate-supply relation.

that these conditions are satisfied, *i.e.*, that the government's fiscal needs are not too severe. Note that in this case, our quadratic objective turns out to be of a form commonly assumed in the literature on monetary policy evaluation; that is, policy should seek to minimize the discounted value of a weighted sum of squared deviations of inflation from an optimal level (here zero) and squared fluctuations in an "output gap" $y_t \equiv \hat{Y}_t - \hat{Y}_t^*$, where the target output level $\hat{Y}_t^*$ depends on the various exogenous disturbances in a way discussed in the appendix. It is also perhaps of interest to note that a "tax smoothing" objective of the kind postulated by Barro (1979) and Bohn (1990) does not appear in our welfare measure as a separate objective. Instead, tax distortions are relevant *only* insofar as they result in "output gaps" of the same sort that monetary stabilization policy aims to minimize.

It is useful to write the linear constraints represented by our structural equations in terms of the output gap as well. The aggregate-supply relation (2.4) can be written as

$$\pi_t = \kappa[y_t + \psi\hat{\tau}_t + u_t] + \beta E_t \pi_{t+1}, \tag{2.13}$$

where u_t is composite "cost-push" shock, indicating the degree to which the exogenous disturbances preclude simultaneous stabilization of inflation, the welfare-relevant output gap, and the tax rate. Alternatively we can write

$$\pi_t = \kappa[y_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*)] + \beta E_t \pi_{t+1}, \tag{2.14}$$

where $\hat{\tau}_t^* \equiv -\psi^{-1} u_t$ indicates the tax change needed at any time to offset the "cost-push" shock, in order to allow simultaneous stabilization of inflation and the output gap (the two stabilization objectives reflected in (2.12)).

The effects of the various exogenous disturbances on the "cost-push" term u_t are explained in the appendix. It is worth noting that under certain conditions, u_t is unaffected by some disturbances. In the case that $\Phi = 0$, then the cost-push term is given by

$$u_t = u_{\xi 5}\hat{\mu}_t^w, \tag{2.15}$$

where in this case, $u_{\xi 5} = q_y^{-1} > 0$. Thus the cost-push term is affected *only* by variations in the wage markup $\hat{\mu}_t$; it does not vary in response to taste shocks, technology shocks, government purchases, or variations in government transfers. The reason is that when $\Phi = 0$ and neither taxes nor the wage markup vary from their steady-state values, the flexible-price equilibrium is efficient; it follows that level of output

consistent with zero inflation is also the one that maximizes welfare, as discussed in Woodford (2003, chapter 6).

Even when $\Phi > 0$, if there are no government purchases (so that $s_G = 0$) and no fiscal shocks (meaning that $\hat{G}_t = 0$ and $\hat{\nu}_t = 0$), then the u_t term is again of the form (2.15), but with $u_{\xi 5} = (1 - \Phi)q_y^{-1}$, as discussed in Benigno and Woodford (2003). Hence in this case neither taste or technology shocks have "cost-push" effects. The reason is that in this "isoelastic" case, if neither taxes nor the wage markup ever vary, the flexible-price equilibrium value of output and the efficient level vary in exactly the same proportion in response to each of the other types of shocks; hence inflation stabilization also stabilizes the gap between actual output and the efficient level. Another special case is the limiting case of linear utility of consumption ($\sigma^{-1} = 0$); in this case, u_t is again of the form (2.15), for a different value of $u_{\xi 5}$. In general, however, when $\Phi > 0$ and $s_G > 0$, all of the disturbances shift the flexible-price equilibrium level of output (under a constant tax rate) and the efficient level of output to differing extents, resulting in "cost-push" contributions from all of these shocks.

Similarly, the intertemporal budget constraint of the government[22] can be written as

$$\hat{b}_{t-1} - \pi_t - \sigma^{-1} y_t = -f_t \ + (1 - \beta) E_t \sum_{T=t}^{\infty} \beta^{T-t} [b_y y_T + b_\tau (\hat{\tau}_T - \hat{\tau}_T^*)] \tag{2.16}$$

where the coefficients b_y, b_τ are defined in the appendix, as is f_t, is a composite measure of exogenous "fiscal stress." We note then, that the only reason why it should *not* be possible to completely stabilize both inflation and the output gap from some date t onward is if the sum $\hat{b}_{t-1} + f_t$ is non-zero. The composite disturbance f_t therefore completely summarizes the information at date t about the exogenous disturbances that determines the degree to which stabilization of inflation and output is not possible; and under an optimal policy, the state-contingent evolution of the inflation rate, the output gap, and the real public debt depend solely on the evolution of the single composite disturbance process $\{f_t\}$.

This result contrasts with the standard literature on optimal monetary stabilization policy, in which (in the absence of a motive for interest-rate stabilization, as here) it is instead the cost-push term u_t that summarizes the extent to which exogenous

[22]This is a forward-integrated version of (2.7). Note that if we restrict our attention to bounded solutions, then (2.7) holding each period is equivalent to (2.16) holding each period.

disturbances require that fluctuations in inflation and in the output gap should occur. Note that in the case that there are no government purchases and no fiscal shocks, u_t corresponds simply to (2.15). Thus, for example, it is concluded (in a model with lump-sum taxes) that there should be no variation in inflation in response to a technology shock (Khan *et al.*, 2002; Benigno and Woodford, 2003). But even in this simple case, the fiscal stress is given by an expression of the form

$$f_t \equiv h'_\xi \xi_t - (1-\beta) E_t \sum_{T=t}^{\infty} \beta^{T-t} f'_\xi \xi_T, \tag{2.17}$$

where the expressions $h'_\xi \xi_t$ and $f'_\xi \xi_t$ both generally include non-zero coefficients on preference and technology shocks, in addition to the markup shock, as shown in the appendix. Hence many disturbances that do not have cost-push effects nonetheless result in optimal variations in both inflation and the output gap.

3 Optimal Responses to Shocks: The Case of Flexible Prices

In considering the solution to the problem of stabilization policy just posed, it may be useful to first consider the simple case in which prices are fully flexible. This is the limiting case of our model in which $\alpha = 0$, with the consequence that $q_\pi = 0$ in (2.12), and that $\kappa^{-1} = 0$ in (2.14). Hence our optimization problem reduces to the minimization of

$$\frac{1}{2} q_y E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} y_t^2 \tag{3.1}$$

subject to the constraints

$$y_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*) = 0 \tag{3.2}$$

and (2.16). It is easily seen that in this case, the optimal policy is one that achieves $y_t = 0$ at all times. Because of (3.2), this requires that $\hat{\tau}_t = \hat{\tau}_t^*$ at all times. The inflation rate is then determined by the requirement of government intertemporal solvency,

$$\pi_t = \hat{b}_{t-1} + f_t.$$

This last equation implies that unexpected inflation must equal the innovation in the fiscal stress,

$$\pi_t - E_{t-1}\pi_t = f_t - E_{t-1} f_t.$$

Expected inflation, and hence the evolution of nominal government debt, are indeterminate. If we add to our assumed policy objective a small preference for inflation stabilization, when this has no cost in terms of other objectives,[23] then the optimal policy will be one that involves $E_t\pi_{t+1} = 0$ each period, so that the nominal public debt must evolve according to

$$\hat{b}_t = -E_t f_{t+1}.$$

If, instead, we were to assume the existence of small monetary frictions (and zero interest on money), the tie would be broken by the requirement that the nominal interest rate equal zero each period.[24] The required expected rate of inflation (and hence the required evolution of the nominal public debt) would then be determined by the variation in the equilibrium real rate of return implied by a real allocation in which $\hat{Y}_t = \hat{Y}_t^*$ each period. That is, one would have $E_t\pi_{t+1} = -r_t^*$, where r_t^* is the (exogenous) real rate of interest associated output at the target level each period, and so

$$\hat{b}_t = -r_t^* - E_t f_{t+1}.$$

We thus obtain simple conclusions about the determinants of fluctuations in inflation, output and the tax rate under optimal policy. Unexpected inflation variations occur as needed in order to prevent taxes from ever having to be varied in order to respond to variations in fiscal stress, as in the analyses of Bohn (1990) and Chari and Kehoe (1999). This allows a model with only riskless nominal government debt to achieve the same state-contingent allocation of resources as the government would choose to bring about if it were able to issue state-contingent debt, as in the model of Lucas and Stokey (1983).

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[23]Note that this preference can be justified in terms of our model, in the case that α is positive though extremely small. For there will then be a very small positive value for q_π, implying that reduction of the expected discounted value of inflation is preferred to the extent that this does not require any increase in the expected discounted value of squared output gaps.

[24]The result relies upon the fact that the distortions created by the monetary frictions are minimized in the case of a zero opportunity cost of holding money each period, as argued by Friedman (1969). Neither the existence of effects of nominal interest rates on supply costs (so that an interest-rate term should appear in the aggregate-supply relation (3.2)) nor the contribution of seignorage revenues to the government budget constraint make any difference to the result, since unexpected changes in revenue needs can always be costlessly obtained through unexpected inflation, while any desired shifts in the aggregate-supply relation to offset cost-push shocks can be achieved by varying the tax rate.

Because taxes do not have to adjust in response to variations in fiscal stress, as in the tax-smoothing model of Barro (1979), it is possible to "smooth" them across states as well as over time. However, the sense in which it is desirable to "smooth" tax rates is that of minimizing variation in the gap $\hat{\tau}_t - \hat{\tau}_t^*$, rather than variation in the tax rate itself.[25] In other words, it is really the "tax gap" $\hat{\tau}_t - \hat{\tau}_t^*$ that should be smoothed. Under certain special circumstances, it will not be optimal for tax rates to vary in response to shocks; these are the conditions, discussed above, under which shocks have no cost-push effects, so that there is no change in $\hat{\tau}_t^*$. For example, if there are no government purchases and there is no variation in the wage markup, this will be the case. But more generally, all disturbances will have some cost-push effect, and result in variations in $\hat{\tau}_t^*$. There will then be variations in the tax rate in response to these shocks under an optimal policy. However, there will be no unit root in the tax rate, as in the Barro (1979) model of optimal tax policy. Instead, as in the analysis of Lucas and Stokey (1983), the optimal fluctuations in the tax rate will be stationary, and will have the same persistence properties as the real disturbances (specifically, the persistence properties of the composite cost-push shock).

Variations in fiscal stress will instead require changes in the tax rate, as in the analysis of Barro (1979), if we suppose that the government issues only riskless indexed debt, rather than the riskless nominal debt assumed in our baseline model. (Again, for simplicity we assume that only one-period riskless debt is issued.) In this case the objective function (2.12) and the constraints (2.16) and (3.2) remain the same, but $\underline{b}_{t-1} \equiv \hat{b}_{t-1} - \pi_t$, the real value of private claims on the government at the beginning of period t, is now a predetermined variable. This means that unexpected inflation variations are no longer able to relax the intertemporal government solvency condition. In fact, rewriting the constraint (2.16) in terms of $\underline{b}_{t-1}$, we see that the path of inflation is now completely irrelevant to welfare.

The solution to this optimization problem is now less trivial, as complete stabi-

[25] A number of authors (*e.g.*, Chari *et al.*, 1991, 1994; Hall and Krieger, 2000; Aiyagari *et al.*, 2002) have found that in calibrated flexible-price models with state-contingent government debt, the optimal variation in labor tax rates is quite small. Our results indicate this as well, in the case that real disturbances have only small cost-push effects, and we have listed earlier various conditions under which this will be the case. But under some circumstances, optimal policy may involve substantial volatility of the tax rate, and indeed, more volatility of the tax rate than of inflation. This would be the case if shocks occur that have large cost-push effects while having relatively little effect on fiscal stress.

lization of the output gap is not generally possible. The optimal state-contingent evolution of output and taxes can be determined using a Lagrangian method, as in Woodford (2003, chapter 7). The Lagrangian for the present problem can be written as

$$\begin{aligned}\mathcal{L}_{t_0} \quad = \quad & E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\frac{1}{2}q_y y_t^2+\varphi_{1t}[y_t+\psi\hat{\tau}_t]+\varphi_{2t}[(\underline{\mathrm{b}}_{t-1}-\sigma^{-1}y_t) \\ & -(1-\beta)(b_y y_t+b_\tau\hat{\tau}_t)-\beta(\underline{\mathrm{b}}_t-\sigma^{-1}y_{t+1})]\}+\sigma\varphi_{2,t_0-1}y_{t_0}, \qquad (3.3)\end{aligned}$$

where $\varphi_{1t}, \varphi_{2t}$ are Lagrange multipliers associated with constraints (3.2) and (2.16) respectively, for each $t \geq t_0$, and $\sigma\varphi_{2,t_0-1}$ is the notation used for the multiplier associated with the additional constraint that $y_{t_0} = \bar{y}_{t_0}$. The latter constraint is added in order to characterize optimal policy from a timeless perspective, as discussed at the end of section 2; the particular notation used for the multiplier on this constraint results in a time-invariant form for the first-order conditions, as seen below.[26] We have dropped terms from the Lagrangian that are not functions of the endogenous variables y_t and $\hat{\tau}_t$, *i.e.*, products of multipliers and exogenous disturbances, as these do not affect our calculation of the implied first-order conditions.

The resulting first-order condition with respect to y_t is

$$q_y y_t = -\varphi_{1t} + [(1-\beta)b_y + \sigma^{-1}]\varphi_{2t} - \sigma^{-1}\varphi_{2,t-1}; \qquad (3.4)$$

that with respect to $\hat{\tau}_t$ is

$$\psi\varphi_{1t} = (1-\beta)b_\tau\varphi_{2t}; \qquad (3.5)$$

and that with respect to $\underline{\mathrm{b}}_t$ is

$$\varphi_{2t} = E_t\varphi_{2,t+1}. \qquad (3.6)$$

Each of these conditions must be satisfied for each $t \geq t_0$, along with the structural equations (3.2) and (2.16) for each $t \geq t_0$, for given initial values $\underline{\mathrm{b}}_{t_0-1}$ and y_{t_0}. We look for a bounded solution to these equations, so that (in the event of small enough disturbances) none of the state variables leave a neighborhood of the steady-state values, in which our local approximation to the equilibrium conditions and our

[26]It should be recalled that in order for policy to be optimal from a timeless perspective, the state-contingent initial commitment $\bar{y}_{t_0}$ must be chosen in a way that conforms to the state-contingent commitment regarding y_t that will be chosen in all later periods, so that the optimal policy can be implemented by a time-invariant rule. Hence it is convenient to present the first-order conditions in a time-invariant form.

welfare objective remain accurate.[27] Given the existence of such a bounded solution, the transversality condition is necessarily satisfied, so that the solution to these first-order conditions represents an optimal plan.

An analytical solution to these equations is easily given. Using equation (3.2) to substitute for $\hat{\tau}_t$ in the forward-integrated version of (2.16), then equations (3.4) and (3.5) to substitute for y_t as a function of the path of φ_{2t}, and finally using (3.6) to replace all terms of the form $E_t\varphi_{2,t+j}$ (for $j \geq 0$) by φ_{2t}, we obtain an equation that can be solved for φ_{2t}. The solution is of the form

$$\varphi_{2t} = \frac{m_b}{m_b + n_b}\varphi_{2,t-1} - \frac{1}{m_b + n_b}[f_t + \underline{b}_{t-1}],$$

coefficients m_b, n_b are defined in the appendix. The implied dynamics of the government debt are then given by

$$\underline{b}_t = -E_t f_{t+1} - n_b\varphi_{2t}.$$

This allows a complete solution for the evolution of government debt and the multiplier, given the composite exogenous disturbance process $\{f_t\}$, starting from initial conditions $\underline{b}_{t_0-1}$ and φ_{2,t_0-1}.[28] Given these solutions, the optimal evolution of the output gap and tax rate are given by

$$y_t = m_\varphi\varphi_{2t} + n_\varphi\varphi_{2,t-1},$$

$$\hat{\tau}_t = \hat{\tau}_t^* - \psi^{-1}y_t,$$

[27] In the only such solution, the variables $\hat{\tau}_t$, $\underline{b}_t$ and y_t are all permanently affected by shocks, even when the disturbances are all assumed to be stationary (and bounded) processes. Hence a bounded solution exists only under the assumption that random disturbances occur *only in a finite number of periods.* However, our characterization of optimal policy does not depend on a particular bound on the number of periods in which there are disturbances, or which periods these are; in order to allow disturbances in a larger number of periods, we must assume a tighter bound on the amplitude of disturbances, in order for the optimal paths of the endogenous variables to remain within a given neighborhood of the steady-state values. Aiyagari *et al.* (2002) discuss the asymptotic behavior of the optimal plan in the exact nonlinear version of a problem similar to this one, in the case that disturbances occur indefinitely.

[28] The initial condition for φ_{2,t_0-1} is in turn chosen so that the solution obtained is consistent with the initial constraint $y_{t_0} = \bar{y}_{t_0}$. Under policy that is optimal from a timeless perspective, this initial commitment is in turn chosen in a self-consistent fashion, as discussed further in section 5. Note that the specification of φ_{2,t_0-1} does not affect our conclusions about the optimal responses to shocks, emphasized in this section.

where m_φ, n_φ are again defined in the appendix. The evolution of inflation remains indeterminate. If we again assume a preference for inflation stabilization when it is costless, optimal policy involves $\pi_t = 0$ at all times.

In this case, unlike that of nominal debt, inflation is *not affected* by a pure fiscal shock (or indeed any other shock) under the optimal policy, but instead the output gap and the tax rate are. Note also that in the above solution, the multiplier φ_{2t}, the output gap, and the tax rate all follow *unit root* processes: a temporary disturbance to the fiscal stress permanently changes the level of each of these variables, as in the analysis of the optimal dynamics of the tax rate in Barro (1979) and Bohn (1990). However, the optimal evolution of the tax rate is not in general a pure random walk as in the analysis of Barro and Bohn. Instead, the tax gap is an IMA(1,1) process, as in the local analysis of Aiyagari *et al.* (2002); the optimal tax rate $\hat{\tau}_t$ may have more complex dynamics, in the case that $\hat{\tau}_t^*$ exhibits stationary fluctuations. In the special case of linear utility ($\sigma^{-1} = 0$), $n_\varphi = 0$, and both the output gap and the tax gap follow random walks (as both co-move with φ_{2t}). If the only disturbances are fiscal disturbances ($\hat{G}_t$ and $\hat{\nu}_t$), then there are also no fluctuations in $\hat{\tau}_t^*$ in this case, so that the optimal tax rate follows a random walk.

More generally, we observe that optimal policy "smooths" φ_{2t}, the value (in units of marginal utility) of additional government revenue in period t, so that it follows a random walk. This is the proper generalization of the Barro tax-smoothing result, though it only implies smoothing of tax rates in fairly special cases. We find a similar result in the case that prices are sticky, even when government debt is not indexed, as we now show.

4 Optimal Responses to Shocks: The Case of Sticky Prices

We turn now to the characterization of the optimal responses to shocks in the case that prices are sticky ($\alpha > 0$). The optimization problem that provides a first-order characterization of optimal responses in this case is that of choosing processes $\{\pi_t, y_t, \hat{\tau}_t, \hat{b}_t\}$ from date t_0 onward to minimize (2.12), subject to the constraints (2.14) and (2.16) for each $t \geq t_0$, together with initial constraints of the form

$$\pi_{t_0} = \bar{\pi}_{t_0}, \qquad y_{t_0} = \bar{y}_{t_0},$$

given the initial condition $\hat{b}_{t_0-1}$ and the exogenous evolution of the composite disturbances $\{\hat{\tau}_t, f_t\}$. The Lagrangian for this problem can be written as

$$\begin{aligned}\mathcal{L}_{t_0} &= E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\frac{1}{2}q_y y_t^2 + \frac{1}{2}q_\pi \pi_t^2 + \varphi_{1t}[-\kappa^{-1}\pi_t + y_t + \psi\hat{\tau}_t + \kappa^{-1}\beta\pi_{t+1}] + \\ &\quad +\varphi_{2t}[\hat{b}_{t-1} - \pi_t - \sigma^{-1}y_t - (1-\beta)(b_y y_t + b_\tau\hat{\tau}_t) - \beta(\hat{b}_t - \pi_{t+1} - \sigma^{-1}y_{t+1})]\},\end{aligned}$$

by analogy with (3.3).

The first-order condition with respect to π_t is given by

$$q_\pi \pi_t = \kappa^{-1}(\varphi_{1t} - \varphi_{1,t-1}) + (\varphi_{2t} - \varphi_{2,t-1}); \tag{4.1}$$

that with respect to y_t is given by

$$q_y y_t = -\varphi_{1t} + [(1-\beta)b_y + \sigma^{-1}]\varphi_{2t} - \sigma^{-1}\varphi_{2,t-1}; \tag{4.2}$$

that with respect to $\hat{\tau}_t$ is given by

$$\psi\varphi_{1t} = (1-\beta)b_\tau\varphi_{2t}; \tag{4.3}$$

and finally that with respect to $\hat{b}_t$ is given by

$$\varphi_{2t} = E_t\varphi_{2,t+1}. \tag{4.4}$$

These together with the two structural equations and the initial conditions are to be solved for the state-contingent paths of $\{\pi_t, \hat{Y}_t, \tau_t, \hat{b}_t, \varphi_{1t}, \varphi_{2t}\}$. Note that the last three first order conditions are the same as for the flexible-price model with indexed debt; the first condition (4.1) replaces the previous requirement that $\pi_t = 0$. Hence the solution obtained in the previous section corresponds to a limiting case of this problem, in which q_π is made unboundedly large; for this reason the discussion above of the more familiar case with flexible prices and riskless indexed government debt provides insight into the character of optimal policy in the present case as well.

In the unique bounded solution to these equations, the dynamics of government debt and of the shadow value of government revenue φ_{2t} are again of the form

$$\varphi_{2t} = \frac{\tilde{m}_b}{\tilde{m}_b + n_b}\varphi_{2,t-1} - \frac{1}{\tilde{m}_b + n_b}[f_t + \hat{b}_{t-1}],$$

$$\hat{b}_t = -E_t f_{t+1} - n_b\varphi_{2t},$$

though the coefficient $\tilde{m}_b$ now has a different definition from that of m_b, also given in the appendix. The implied dynamics of inflation and output gap are then given by

$$\pi_t = -\omega_\varphi(\varphi_{2t} - \varphi_{2,t-1}), \tag{4.5}$$

$$y_t = m_\varphi \varphi_{2t} + n_\varphi \varphi_{2,t-1}, \tag{4.6}$$

where m_φ, n_φ are defined as before, and ω_φ is defined in the appendix. The optimal dynamics of the tax rate are those required to make these inflation and output-gap dynamics consistent with the aggregate-supply relation (2.14). Once again, the optimal dynamics of inflation, the output gap, and the public debt depend only on the evolution of the fiscal stress variable $\{f_t\}$; the dynamics of the tax rate also depend on the evolution of $\{\hat{\tau}_t^*\}$.

We now discuss the optimal response of the variables to a disturbance to the level of fiscal stress. The laws of motion just derived for government debt and the Lagrange multiplier imply that temporary disturbances to the level of fiscal stress cause a permanent change in the level of both the Lagrange multiplier and the public debt. This then implies a permanent change in the level of output as well, which in turn requires (since inflation is stationary) a permanent change in the level of the tax rate. Since inflation is proportional to the change in the Lagrange multiplier, the price level moves in proportion to the multiplier, which means a temporary disturbance to the fiscal stress results in a permanent change in the price level, as in the flexible-price case analyzed in the previous section. Thus in this case, the price level, output gap, government debt, and tax rate *all* have unit roots, combining features of the two special cases considered in the previous section. Both price level and φ_{2t} are random walks. They jump immediately to new permanent level in response to change in fiscal stress. In the case of purely transitory (white noise) disturbances, government debt also jumps immediately to a new permanent level. Given the dynamics of the price level and government debt, the dynamics of output and tax rate then jointly determined by the aggregate-supply relation and the government budget constraint.

We further find that the degree to which fiscal stress is relieved by a price-level jump (as in the flexible-price, nominal-debt case) as opposed to an increase in government debt and hence a permanently higher tax rate (as in the flexible-price, indexed-debt case) depends on the degree of price stickiness. We illustrate this with a numerical example. We calibrate a quarterly model by assuming that $\beta = 0.99, \omega = 0.473, \sigma^{-1} = 0.157$, and $\kappa = 0.0236$, in accordance with the estimates

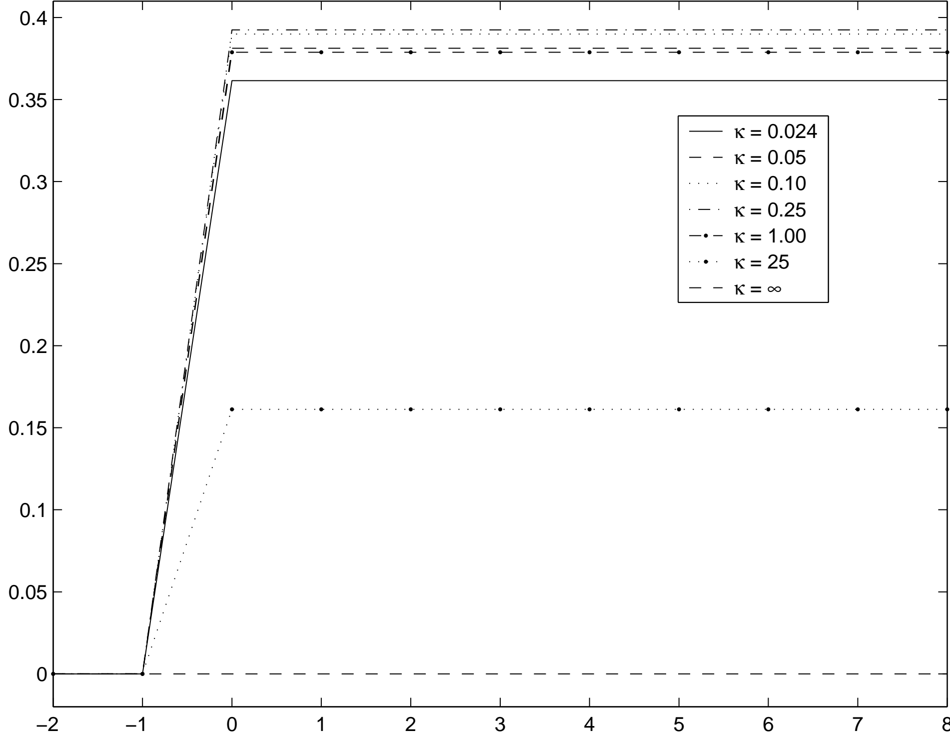

Figure 1: Impulse response of the public debt to a pure fiscal shock, for alternative degrees of price stickiness.

of Rotemberg and Woodford (1997). We furthermore assume an elasticity of substitution among alternative goods of $\theta = 10$, an overall level of steady-state distortions $\Phi = 1/3$, a steady-state tax rate of $\bar{\tau} = 0.2$, and a steady-state debt level $\bar{b}/\bar{Y} = 2.4$ (debt equal to 60 percent of a year's GDP). Given the assumed degree of market power of producers (a steady-state gross price markup of 1.11) and the assumed size of the tax wedge, the value $\Phi = 1/3$ corresponds to a steady-state wage markup of $\bar{\mu}^w = 1.08$. If we assume that there are no government transfers in the steady state, then the assumed level of tax revenues net of debt service would finance steady-state government purchases equal to a share $s_G = 0.176$ of output.

Let us suppose that the economy is disturbed by an exogenous increase in transfer programs $\hat{\nu}_t$, equal to one percent of aggregate output, and expected to last only for the current quarter. Figure 1 shows the optimal impulse response of the government debt $\hat{b}_t$ to this shock (where quarter zero is the quarter of the shock), for each of 7 different values for κ, the slope of the short-run aggregate-supply relation, maintaining

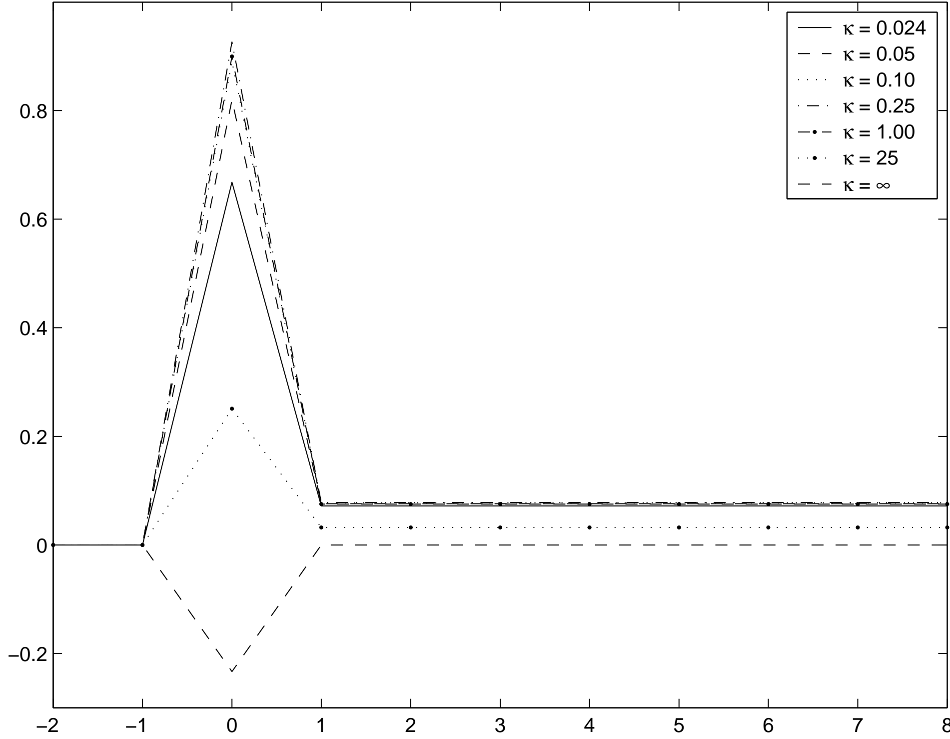

Figure 2: Impulse response of the tax rate to a pure fiscal shock.

the values just stated for the other parameters of the model. The solid line indicates the optimal response in the case of our baseline value for κ, based on the estimates of Rotemberg and Woodford; the other cases represent progressively greater degrees of price flexibility, up to the limiting case of fully flexible prices (the case $\kappa = \infty$). Figures 2 and 3 similarly show the optimal responses of the tax rate and the inflation rate to the same disturbance, for each of the same 7 cases.[29]

We see that the volatility of both inflation and tax rates under optimal policy depends greatly on the degree of stickiness of prices. Table 1 reports the initial quarter's response of both the tax rate and inflation for each of the 7 cases. The

[29]In figure 1, a response of 1 means a one percent increase in the value of b_t, from 60 percent to 60.6 percent of a year's GDP. In figure 2, a response of 1 means a one percent decrease in τ_t, from 20 percent to 20.2 percent. In figure 4, a response of 1 means a one percent per annum increase in the inflation rate, or an increase of the price level from 1 to 1.0025 over the course of a quarter (given that our model is quarterly). The responses reported in Table 1 are measured in the same way.

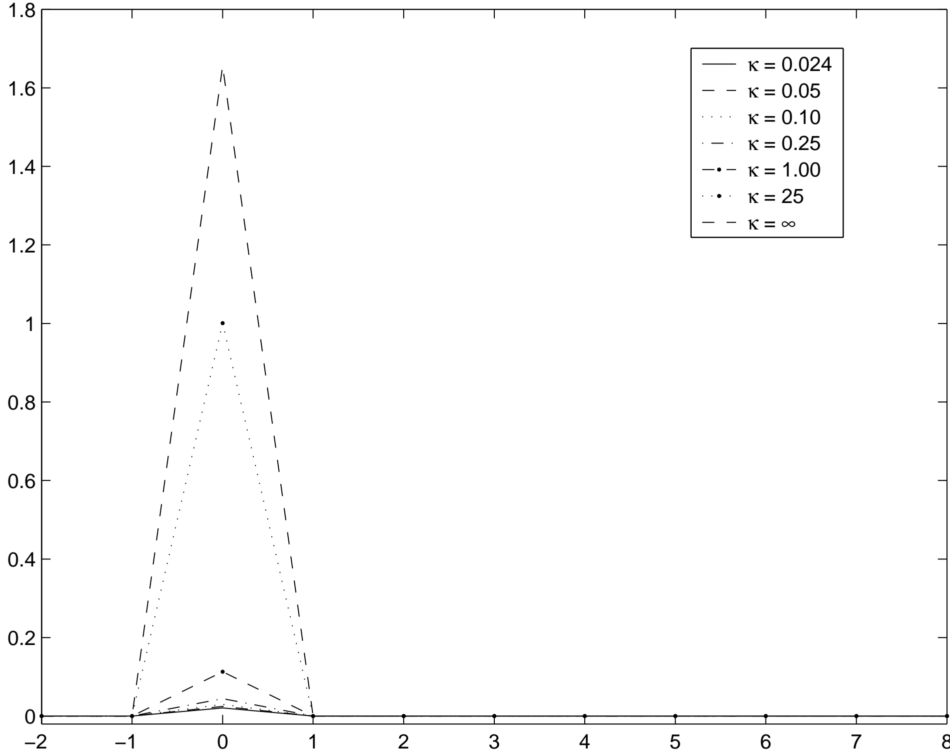

Figure 3: Impulse response of the inflation rate to a pure fiscal shock.

table also indicates for each case the implied average time (in weeks) between price changes, $T \equiv (-\log\alpha)^{-1}$, where $0 < \alpha < 1$ is the fraction of prices unchanged for an entire quarter implied by the assumed value of κ.[30] We first note that our baseline calibration implies that price changes occur only slightly less frequently than twice per year, which is consistent with survey evidence.[31] Next, we observe that even were we to assume an aggregate-supply relation several times as steep as the one estimated using U.S. data, our conclusions with regard to the size of the optimal responses of the tax rate and the inflation rate would be fairly similar. At the same time, the optimal responses with fully flexible prices are quite different: the response of inflation is 80

[30]We have used the relation between α and T for a continuous-time version of the Calvo model in order to express the degree of price stickiness in terms of an average time between price changes.

[31]The indicated average time between price changes for the baseline case is shorter than that reported in Rotemberg and Woodford (1997), both because we here assume a slightly larger value of θ, implying a smaller value of α, and because of the continuous-time method used here to convert α into an implied average time interval.

Table 1: Immediate responses for alternative degrees of price stickiness.

κ	T	$\hat{\tau}_0$	π_0
.024	29	.668	.021
.05	20	.820	.024
.10	14	.891	.030
.25	9	.927	.045
1.0	5.4	.900	.113
25	2.4	.251	1.001
∞	0	-.233	1.656

times as large as under the baseline sticky-price calibration (implying a variance of inflation 6400 times as large), while the tax rate responds with the opposite sign.[32] But even a small degree of stickiness of prices makes a dramatic difference in the optimal responses; for example, if prices are revised only every five weeks on average, the variance of inflation is reduced by a factor of more than 200, while the optimal response of the tax rate to the increased revenue need is a strong increase, rather than a decrease. Thus we find, as do Schmitt-Grohé and Uribe (2001) in the context of a calibrated model with convex costs of price adjustment, that the conclusions of the flexible-price analysis are quite misleading if prices are even slightly sticky. Under a realistic calibration of the degree of price stickiness, inflation should be quite stable, even in response to disturbances with substantial consequences for the government's budget constraint, while tax rates should instead respond substantially (and with a unit root) to variations in fiscal stress.

We can also compare our results with those that arise when taxes are lump-sum. In this case, $\psi = 0$, and the first-order condition (4.3) requires that $\varphi_{2t} = 0$. The remaining first-order conditions reduce to

$$q_\pi \pi_t = \kappa^{-1}(\varphi_{1t} - \varphi_{1,t-1}),$$

$$q_y y_t = -\varphi_{1t}$$

[32] Under flexible prices, as discussed above, the tax rate does not respond to variations in fiscal stress at all. Because the increase in government transfers raises the optimal level of output $\hat{Y}_0^*$, for reasons explained in the appendix, the optimal tax rate actually falls, in order to induce equilibrium output to increase.

for each $t \geq t_0$ as in Clarida *et al.* (1999) and Woodford (2003, chapter 7). In this case the fiscal stress is no longer relevant for inflation or output-gap determination. Instead, only the cost-push shock u_t is responsible for incomplete stabilization. The determinants of the cost-push effects of underlying disturbances, and of the target output level $\hat{Y}_t^*$ are also somewhat different, because in this case $\vartheta_1 = 0$. For example, a pure fiscal shock has no cost-push effect, nor any effect on $\hat{Y}_t^*$, and hence no effect on the optimal evolution of either inflation or output.[33] Furthermore, as shown in the references just mentioned, the price level no longer follows a random walk; instead, it is a *stationary* variable. Increases in the price level due to a cost-push shock are subsequently undone by period of deflation.

Note that the familiar case from the literature on monetary stabilization policy does not result simply from assuming that sources of revenue that do not shift the aggregate-supply relation are *available*; it is also important that the sort of tax that *does* shift the AS relation (like the sales tax here) is *not* available. We could nest both the standard model and our present baseline case within a single, more general framework by assuming that revenue can be raised using either the sales tax or a lump-sum tax, but that there is an additional convex cost (perhaps representing "collection costs", assumed to reduce the utility of the representative household but not using real resources) of increases in either tax rate. The standard case would then appear as the limiting case of this model in which the collection costs associated with the sales tax are infinite, while those associated with the lump-sum tax are zero; the baseline model here would correspond to an alternative limiting case in which the collection costs associated with the lump-sum tax are infinite, while those associated with the sales tax are zero. In intermediate cases, we would continue to find that fiscal stress affects the optimal evolution of both inflation and the output gap, as long as there is a positive collection cost for the lump-sum tax. At the same time, the result that the shadow value of additional government revenue follows a random walk under optimal policy (which would still be true) will not in general imply, as it does here, that the price level should also be a random walk; for the perfect co-movement of φ_{1t} and φ_{2t} that characterizes optimal policy in our baseline case will not be implied by the first-order conditions except in the case that there are no collection costs associated with the sales tax. Nonetheless, the price level will generally contain a unit root

[33]See Benigno and Woodford (2003) for detailed analysis of the determinants of u_t and $\hat{Y}_t^*$ in this case.

under optimal policy, even if it will not generally follow a random walk.

We also obtain results more similar to those in the standard literature on monetary stabilization policy if we assume (realistically) that it is not possible to adjust tax rates on such short notice in response to shocks as is possible with monetary policy. As a simple way of introducing delays in the adjustment of tax policy, suppose that the tax rate τ_t has to be fixed in period $t-d$. In this case, the first-order conditions characterizing optimal responses to shocks are the same as above, except that (4.3) is replaced by

$$\psi E_t \varphi_{1,t+d} = (1-\beta) b_\tau E_t \varphi_{2,t+d} \tag{4.7}$$

for each $t \geq t_0$. In addition, the first-order conditions and the structural relations are now to be solved for given initial tax rates $(\hat{\tau}_{t_0}, \ldots, \hat{\tau}_{t_0+d-1})$.

Figures 4 and 5 show the optimal state-contingent responses of the price level and the tax rate respectively to a pure fiscal shock (of the same size as in Figures 1 – 3), for each of several possible values of d, beginning with the case $d=0$ treated earlier. One observes that once delays in the adjustment of the tax rate are introduced, it ceases to be the case that optimal policy involves an immediate jump of the price level to its new permanent level. Instead, it is optimal to commit to a history-dependent policy under which unexpected increases in the price level will be partially reversed in subsequent quarters, as in the analysis of the optimal response to "cost-push shocks" in the case of lump-sum taxation.

However, even in the limiting case in which the delay d is made unboundedly long, the optimal responses when only distorting taxes exist are not the same as those obtained in an analysis that assumes lump-sum taxation. Even when tax policy is not a useful instrument for influencing the short-run dynamics of inflation or output, and can only be adjusted far in the future in the way necessary for intertemporal government solvency, it continues to be true that monetary policy should take account of the fiscal effects of alternative policies, given that the change in the tax rate that will eventually occur affects the size of the welfare losses resulting from tax distortions. In this limiting case, the conditions that determine the optimal state-contingent evolution are the first-order conditions (4.1), (4.2) and (4.4), together with an aggregate-supply relation of the form

$$\pi_t = \kappa[y_t + u_t] + \beta E_t \pi_{t+1} \tag{4.8}$$

and the intertemporal solvency condition (2.16). These conditions differ from those

of the model with lump-sum taxation in that $\{\varphi_{2t}\}$, the variation in the shadow value of additional government revenue, need not equal zero at all times, and in that the solvency condition remains relevant, determining the innovation each period in this multiplier.

One can show that first-order conditions (4.1) – (4.2) imply that in the optimal state-contingent evolution,

$$p_t + \phi y_t = \bar{p} + \gamma_1 \varphi_{2t} - \gamma_2 \varphi_{2,t-1} \tag{4.9}$$

where $p_t \equiv \log P_t$, $\bar{p}$ is a constant, and the coefficients $\phi > 0, 0 < \gamma_2 < \gamma_1$ are defined in the appendix.[34] In the case with lump-sum taxes, the terms involving the Lagrange multiplier are zero, and (4.9) implies that disturbances have only transitory effects on the price level, which is a stationary variable. But with distorting taxes, the multiplier φ_{2t} follows a random walk, the innovations in which are determined by the requirement that (2.16) be satisfied each period. Since the output gap continues to be a mean-zero stationary variable,[35] (4.9) implies the existence of a unit root in the price level. Thus while it continues to be true, as in the lump-sum case, that under optimal policy a positive innovation in inflation typically implies lower than average inflation subsequently, it is not true that optimal policy implies a stationary (or trend-stationary) price level.

The degree of accuracy of the recommendations of the traditional analysis (if in fact lump-sum taxes are unavailable, but tax rates cannot be changed except with long advance planning) then depends on the relative size of the fluctuations in the cost-push term u_t and the innovations $\Delta\varphi_{2t}$ in the Lagrange multiplier. If the latter series is much less variable than the former, then the optimal dynamic responses of inflation and output to real disturbances are essentially those indicated by the

[34]Strictly speaking, they imply that (4.9) must hold for all $t \geq t_0 + 1$. However, this suffices for our argument below about the unit root in the price level. Furthermore, in the case of policy that is optimal from a timeless perspective, the initial constraints imply initial Lagrange multipliers such that (4.9) is satisfied for $t = t_0$ as well.

[35]This is easily seen as follows. Equation (4.4) implies that φ_{2t} has a well-defined expected long-run value $\lim_{T\to\infty} E_t \varphi_{2T}$ at any time. It then follows from (4.9) that $p_t + \phi y_t$ has a well-defined expected long-run value. In any solution in which the output gap does not grow without bound, this requires an asymptotic expected inflation rate of zero. The AS relation (4.8) then implies that there must be a well-defined expected long-run value of the output gap, and that this must equal zero.

traditional analysis; if, instead, the variation in the latter series is greater, then the traditional analysis can be quite inaccurate. Thus the difference will be greatest in the case of disturbances that have substantial effects on fiscal stress while having only very small "cost-push" effects.

5 Optimal Targeting Rules for Monetary and Fiscal Policy

We now wish to characterize the *policy rules* that the monetary and fiscal authorities can follow in order to bring about the state-contingent responses to shocks described in the previous section. One might think that it suffices to solve for the optimal state-contingent paths for the policy instruments. But in general this is not a desirable approach to the specification of a policy rule, as discussed in Svensson (2003) and Woodford (2003, chapter 7). A description of optimal policy in these terms would require enumeration of all of the types of shocks that might be encountered later, indefinitely far in the future, which is not feasible in practice. A commitment to a state-contingent instrument path, even when possible, also may not determine the optimal equilibrium as the *locally unique* rational-expectations equilibrium consistent with this policy; many other (much less desirable) equilibria may also be consistent with the same state-contingent instrument path.

Instead, we here specify *targeting rules* in the sense of Svensson (1999, 2003) and Giannoni and Woodford (2003). These targeting rules are commitments on the part of the policy authorities to adjust their respective instruments so as to ensure that the projected paths of the endogenous variables satisfy certain *target criteria.* We show that under an appropriate choice of these target criteria, a commitment to ensure that they hold at all times will determine a unique non-explosive rational-expectations equilibrium, in which the state-contingent evolution of inflation, output and the tax rate solves the optimization problem discussed in the previous section. Moreover, we show that it is possible to obtain a specification of the policy rules that is *robust* to alternative specifications of the exogenous shock processes.

We apply the general approach of Giannoni and Woodford (2002), which allows the derivation of optimal target criteria with the properties just stated. In addition, Giannoni and Woodford show that such target criteria can be formulated that re-

fer only to the projected paths of the target variables (the ones in terms of which the stabilization objectives of policy are defined — here, inflation and the output gap). Briefly, the method involves constructing the target criteria by eliminating the Lagrange multipliers from the system of the system of first-order conditions that characterize the optimal state-contingent evolution, regardless of character of the (additive) disturbances. We are left with linear relations among the target variables, that do not involve the disturbances and with coefficients independent of the specification of the disturbances, that represent the desired target criteria.

Recall that the first-order conditions that characterize the optimal state-contingent paths in the problem considered in the previous section are given by (4.1) – (4.4). As explained in the previous section, the first three of these conditions imply that the evolution of inflation and of the output gap must satisfy (4.5) – (4.6) each period. We can solve (4.5) – (4.6) for the values of $\varphi_{2t}, \varphi_{2,t-1}$ *implied* by the values of π_t, y_t that are observed in an optimal equilibrium. We can then replace $\varphi_{2,t-1}$ in these two relations by the multiplier *implied* in this way by observed values of π_{t-1}, y_{t-1}. Finally, we can eliminate φ_{2t} from these two relations, to obtain a necessary relation between π_t and y_t, given π_{t-1} and y_{t-1}, given by

$$\pi_t + \frac{n_\varphi}{m_\varphi}\pi_{t-1} + \frac{\omega_\varphi}{m_\varphi}(y_t - y_{t-1}) = 0. \tag{5.1}$$

This target criterion has the form of a "flexible inflation target," similar to the optimal target criterion for monetary policy in model with lump-sum taxation (Woodford, 2003, chapter 7). It is interesting to note that, as in all of the examples of optimal target criteria for monetary policy derived under varying assumptions in Giannoni and Woodford (2003), it is only the projected *rate of change* of the output gap that matters for determining the appropriate adjustment of the near-term inflation target; the absolute *level* of the output gap is irrelevant.

The remaining first-order condition from the previous section, not used in the derivation of (5.1), is (4.4). By similarly using the solutions for $\varphi_{2,t+1}, \varphi_{2t}$ implied by observations of π_{t+1}, y_{t+1} to substitute for the multipliers in this condition, one obtains a further target criterion

$$E_t\pi_{t+1} = 0 \tag{5.2}$$

(Note that the fact that this always holds in the optimal equilibrium — *i.e.*, that the price level must follow a random walk — has already been noted in the previous

section.) We show in the appendix that policies that ensure that (5.1) – (5.2) hold for all $t \geq t_0$ determine a unique non-explosive rational-expectations equilibrium.

Moreover, this equilibrium solves the above first-order conditions for a particular specification of the initial lagged multipliers $\varphi_{1,t_0-1}, \varphi_{2,t_0-1}$, which are inferred from the initial values π_{t_0-1}, y_{t_0-1} in the way just explained. Hence this equilibrium minimizes expected discounted losses (2.12) given $\hat{b}_{t_0-1}$ and subject to constraints on initial outcomes of the form

$$\pi_{t_0} = \bar{\pi}(\pi_{t_0-1}, y_{t_0-1}), \tag{5.3}$$

$$y_{t_0} = \bar{y}(\pi_{t_0-1}, y_{t_0-1}). \tag{5.4}$$

Furthermore, these constraints are *self-consistent* in the sense that the equilibrium that solves this problem is one in which π_t, y_t are chosen to satisfy equations of this form in all periods $t > t_0$. Hence these time-invariant policy rules are *optimal from a timeless perspective*.[36] And they are optimal regardless of the specification of disturbance processes. Thus we have obtained *robustly* optimal target criteria, as desired.

We have established a pair of target criteria with the property that if they are expected to be jointly satisfied each period, the resulting equilibrium involves the optimal responses to shocks. This result in itself, however, does not establish which policy instrument should be used to ensure satisfaction of which criterion. Since the variables referred to in both criteria can be affected by both monetary and fiscal policy, there is not a uniquely appropriate answer to that question. However, the following represents a relatively simple example of a way in which such a regime could be institutionalized through separate targeting procedures on the part of monetary and fiscal authorities.

Let the central bank be assigned the task of maximizing social welfare through its adjustment of the level of short-term interest rates, *taking as given* the state-contingent evolution of the public debt $\{\hat{b}_t\}$, which depends on the decisions of the fiscal authority. Thus the central bank treats the evolution of the public debt as being outside its control, just like the exogenous disturbances $\{\xi_t\}$, and simply seeks to forecast its evolution in order to correctly model the constraints on its own policy.

[36]See Woodford (2003, chapters 7, 8) for further discussion of the self-consistency condition that the initial constraints are required to satisfy.

Here we do not propose a regime under which it is actually *true* that the evolution of the public debt would be unaffected by a change in monetary policy. But there is no inconsistency in the central bank's assumption (since a given bounded process $\{\hat{b}_t\}$ will continue to represent a *feasible* fiscal policy regardless of the policy adopted by the central bank), and we shall show that the conduct of policy under this assumption does not lead to a suboptimal outcome, as long as the state-contingent evolution of the public debt is correctly forecasted by the central bank.

The central bank then seeks to bring about paths for $\{\pi_t, y_t, \hat{\tau}_t\}$ from date t_0 onward that minimize (2.12), subject to the constraints (2.14) and (2.16) for each $t \geq t_0$, together with initial constraints of the form (5.3) – (5.4), given the evolution of the processes $\{\hat{\tau}_t^*, f_t, \hat{b}_t\}$. The first-order conditions for this optimization problem are given by (4.1), (4.2) and (4.4) each period, which in turn imply that (5.1) must hold each period, as shown above. One can further show that a commitment by the central bank to ensure that (5.1) holds each period determines the equilibrium evolution that solves this problem, in the case of an appropriate (self-consistent) choice of the initial constraints (5.3) – (5.4). Thus (5.1) is an optimal target criterion for a policy authority seeking to solve the kind of problem just posed; and since the problem takes as given the evolution of the public debt, it is obviously a more suitable assignment for the central bank than for the fiscal authority. The kind of interest-rate reaction function that can be used to implement a "flexible inflation target" of this kind is discussed in Svensson and Woodford (2003) and Woodford (2003, chapter 7).

Correspondingly, let the fiscal authority be assigned the task of choosing the level of government revenue each period that will maximize social welfare, taking as given the state-contingent evolution of output $\{y_t\}$, which it regards as being determined by monetary policy. (Again, it need not really be the case that the central bank ensures a particular state-contingent path of output, regardless of what the fiscal authority does. But again, this assumption is not inconsistent with our model of the economy, since it is possible for the central bank to bring about any bounded process $\{y_t\}$ that it wishes, regardless of fiscal policy, in the case that prices are sticky.) If the fiscal authority regards the evolution of output as outside its control, its objective reduces to the minimization of

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \pi_t^2. \tag{5.5}$$

But this is a possible objective for fiscal policy, given the effects of tax policy on

inflation dynamics (when taxes are not lump-sum) indicated by (2.14).

Forward integration of (2.14) implies that

$$\pi_t = \kappa E_t \sum_{T=t}^{\infty} \beta^{T-t} y_T + \kappa\psi E_t \sum_{T=t}^{\infty} \beta^{T-t} (\hat{\tau}_T - \hat{\tau}_T^*). \qquad (5.6)$$

Thus what matters about fiscal policy for current inflation determination is the present value of expected tax rates; but this in turn is constrained by the intertemporal solvency condition (2.16). Using (2.16) to substitute for the present value of taxes in (5.6), we obtain a relation of the form

$$\pi_t = \mu_1[\hat{b}_{t-1} - \sigma^{-1} y_t + f_t] + \mu_2 E_t \sum_{T=t}^{\infty} \beta^{T-t} y_T, \qquad (5.7)$$

for certain coefficients $\mu_1, \mu_2 > 0$ defined in the appendix. If the fiscal authority takes the evolution of output as given, then this relation implies that its policy in period t can have no effect on π_t. However, it can affect inflation in the following period through the effects the current government budget on $\hat{b}_t$. Furthermore, since the choice of $\hat{b}_t$ has no effect on inflation in later periods (given that it places no constraint on the level of public debt that may be chosen in later periods), $\hat{b}_t$ should be chosen so as to minimize $E_t \pi_{t+1}^2$.

The first-order condition for the optimal choice of $\hat{b}_t$ is then simply (5.2), which we find is indeed a suitable target criterion for the fiscal authority. The decision rule implied by this target criterion is seen to be

$$\hat{b}_t = -E_t f_{t+1} + \sigma^{-1} E_t y_{t+1} - (\mu_2/\mu_1) E_t \sum_{T=t+1}^{\infty} \beta^{T-t-1} y_T,$$

which expresses the optimal level of government borrowing as a function of the fiscal authority's projections of the exogenous determinants of fiscal stress and of future real activity. It is clearly possible for the fiscal authority to implement this target criterion, and doing so leads to a determinate equilibrium path for inflation, given the path of output. We thus obtain a pair of targeting rules, one for the central bank and one for the fiscal authority, that if both pursued will implement an equilibrium that is optimal from a timeless perspective. Furthermore, each individual rule can be rationalized as a solution to a constrained optimization problem that the particular policy authority is assigned to solve.

A Appendix

First-order conditions of the Calvo pricing model (equation (1.7))

In this section, we derive equation (1.7) in the main text and we define the variables F_t and K_t. In the Calvo model, a supplier that changes its price in period t chooses a new price $p_t(i)$ to maximize

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \Pi(p_t(i), p_T^j, P_T; Y_T, \tau_T, \xi_T) \right\},$$

where α^{T-t} is the probability that the price set at time t remains fixed in period T, $Q_{t,T}$ is the stochastic discount factor

$$Q_{t,T} = \beta^{T-t} \frac{\tilde{u}_c(C_T; \xi_T)}{\tilde{u}_c(C_t; \xi_t)} \frac{P_t}{P_T}$$

and $\Pi(\cdot)$ is defined as

$$\Pi(p, p^j, P; Y, \tau, \xi) \equiv (1-\tau) p Y (p/P)^{-\theta} - \mu_t^w \frac{\tilde{v}_h(f^{-1}(Y(p^j/P)^{-\theta}/A); \xi)}{\tilde{u}_c(Y - G; \xi)} P \cdot f^{-1}(Y(p/P)^{-\theta}/A).$$

The first-order condition for the optimal choice of the price $p_t(i)$ is then

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(\frac{p_t(i)}{P_T} \right)^{-\theta} Y_T \Psi_T(p_t(i), p_t^j) \right\} = 0,$$

with

$$\Psi_T(p, p^j) \equiv \left[(1 - \tau_T) - \frac{\theta}{\theta - 1} \mu_T^w \frac{\tilde{v}_h(f^{-1}(Y_T(p^j/P_T)^{-\theta}/A_T); \xi_T)}{\tilde{u}_c(Y_T - G_T; \xi_T) \cdot A_T f'(f^{-1}(Y_T(p/P_T)^{-\theta}/A_T))} \frac{P_T}{p} \right]$$

Using the definitions

$$u(Y_t; \xi_t) \equiv \tilde{u}(Y_t - G_t; \xi_t),$$

$$v(y_t(i); \xi_t) \equiv \tilde{v}(f^{-1}(y_t(i)/A_t); \xi_t) = \tilde{v}(H_t(i); \xi_t),$$

and noting that each firm in an industry will set the same price, so that $p_t(i) = p_t^j = p_t^*$, the common price of all goods with prices revised at date t, we can rewrite the above first-order condition as

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left(\frac{p_t^*}{P_T} \right)^{-\theta} Y_T \left[(1 - \tau_T) - \frac{\theta}{\theta - 1} \mu_T^w \frac{v_y(Y_T(p_t^*/P_T)^{-\theta}; \xi_T)}{u_c(Y_T; \xi_T)} \frac{P_T}{p_t^*} \right] \right\} = 0.$$

Substituting the equilibrium value for the discount factor, we finally obtain

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} u_c(Y_T;\xi_T) \left(\frac{p_t^*}{P_T}\right)^{-\theta} Y_T \left[\frac{p_t^*}{P_T}(1-\tau_T) - \frac{\theta}{\theta-1}\mu_T^w \frac{v_y(Y_T(p_t^*/P_T)^{-\theta};\xi_T)}{u_c(Y_T;\xi_T)} \right] \right\} = 0. \tag{A.8}$$

Using the isoelastic functional forms given in the text, we can simplify equation (A.8) to

$$\frac{p_t^*}{P_t} = \left(\frac{K_t}{F_t}\right)^{\frac{1}{1+\omega\theta}}, \tag{A.9}$$

where F_t and K_t are aggregate variables of the form

$$F_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t}(1-\tau_T) f(Y_T;\xi_T) \left(\frac{P_T}{P_t}\right)^{\theta-1},$$

$$K_t \equiv E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} k(Y_T;\xi_T) \left(\frac{P_T}{P_t}\right)^{\theta(1+\omega)},$$

in which expressions

$$f(Y;\xi) = u_c(Y;\xi)Y,$$

$$k(Y;\xi) = \frac{\theta}{\theta-1}\mu^w v_y(Y;\xi)Y,$$

and where in the function $k(\cdot)$, the vector of shocks has been extended to include the shock μ_t^w. Substitution of (A.9) into the law of motion for the Dixit-Stiglitz price index

$$P_t = \left[(1-\alpha)p_t^{*1-\theta} + \alpha P_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}} \tag{A.10}$$

yields a short-run aggregate-supply relation between inflation and output,

$$\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha} = \left(\frac{F_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}}, \tag{A.11}$$

where $\Pi_t \equiv P_t/P_{t-1}$, which is equation (1.7) in the text.

Recursive formulation of the policy problem

We have shown in the main text that we can represent the goal of policy as maximization of the objective

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t; \xi_t) \tag{A.12}$$

under the constraints

$$\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha} = \left(\frac{F_t}{K_t}\right)^{\frac{\theta-1}{1+\omega\theta}}, \tag{A.13}$$

$$\Delta_t = h(\Delta_{t-1}, \Pi_t), \tag{A.14}$$

$$W_t = u_c(Y_t; \xi_t)s_t + \beta E_t W_{t+1}, \tag{A.15}$$

where

$$s_t = \tau_t Y_t - G_t - \nu_t,$$

given the initial conditions $b_{t_0-1}, \Delta_{t_0-1}$.

We first note that

$$F_t = (1-\tau_t)f(Y_t; \xi_t) + \alpha\beta E_t\{\Pi_{t+1}^{\theta-1}F_{t+1}\}, \tag{A.16}$$

$$K_t = k(Y_t; \xi_t) + \alpha\beta E_t\{\Pi_{t+1}^{\theta(1+\omega)}K_{t+1}\}. \tag{A.17}$$

We can then conclude that the short-run trade-offs among the values of Π_t, Y_t, and τ_t implied by constraints (A.13) and (A.15) depend on expectations at t that can be summarized by expectations regarding the state-contingent value of a vector X_{t+1}, the elements of which are F_{t+1}, K_{t+1}, and W_{t+1}. Thus the only commitments regarding future policy that can improve outcomes in period t can be summarized by the values of these variables in the following period. To obtain a recursive formulation for the policy problem, we define an additional state variable for our problem, the vector X_t, and consider the problem in each period of choosing an optimal policy from that time onward given a pre-existing commitment regarding the value of this vector. We observe that at any time t, given the predetermined values of b_{t-1}, Δ_{t-1}, and a commitment regarding X_t, the policymaker should optimally choose the vector $x_t \equiv (\Pi_t, Y_t, \tau_t, b_t, \Delta_t)$ and the state-contingent values of the vector X_{t+1} in order to solve the problem

$$V(b_{t-1}, \Delta_{t-1}, X_t; \xi_t) = \max_{x_t, X(\cdot)} U(Y_t, \Delta_t; \xi_t) + \beta E_t[V(b_t, \Delta_t, X_{t+1}; \xi_{t+1})],$$

where the maximization is subject to (A.13), (A.14) and (A.15).

The deterministic steady state

We now consider a deterministic steady state in which the exogenous disturbances $\bar{C}_t$, G_t, $\bar{H}_t$, A_t, μ_t^w, ν_t each take constant values $\bar{C}$, $\bar{H}$, $\bar{A}$, $\bar{\mu}^w > 0$ and $\bar{G}, \bar{\nu} \geq 0$ for all

$t \geq t_0$, and we start from initial conditions $b_{t_0-1} = \bar{b} > 0$ and $\Delta_{t_0-1} = 1$. (The value of $\bar{b}$ is arbitrary, subject to an upper bound discussed below. We must, however, start from zero price dispersion in order to be in the steady state; otherwise, in the absence of disturbances, price dispersion will decline monotonically over time.) Given that we are interested in characterizing optimal policy from a timeless perspective, we also assume that there exists an initial commitment specifying the value of the vector X_{t_0}. The values of the initial commitments, $F_{t_0} = \bar{F}$, $K_{t_0} = \bar{K}$, and $W_{t_0} = \bar{W}$, consistent with a steady state remain to be determined. That is, we wish to show that for a certain choice of the initial commitment, the optimal policy is one in which all endogenous variables take constant values, including the constant values $\bar{F}, \bar{K}, \bar{W}$ for the elements of X_t.

We then consider the problem of maximizing

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} U(Y_t, \Delta_t) \tag{A.18}$$

subject to the constraints

$$K_t p(\Pi_t)^{\frac{1+\omega\theta}{\theta-1}} = F_t, \tag{A.19}$$

$$F_t = (1 - \tau_t) f(Y_t) + \alpha\beta\Pi_{t+1}^{\theta-1} F_{t+1}, \tag{A.20}$$

$$K_t = k(Y_t) + \alpha\beta\Pi_{t+1}^{\theta(1+\omega)} K_{t+1}, \tag{A.21}$$

$$W_t = u_c(Y_t)(\tau_t Y_t - \bar{G} - \bar{\nu}) + \beta W_{t+1}, \tag{A.22}$$

$$W_t = \frac{u_c(Y_t) b_{t-1}}{\Pi_t}, \tag{A.23}$$

$$\Delta_t = \alpha\Delta_{t-1}\Pi_t^{\theta(1+\omega)} + (1 - \alpha) p(\Pi_t)^{-\frac{\theta(1+\omega)}{1-\theta}}, \tag{A.24}$$

$$\Delta_t \geq 1, \tag{A.25}$$

and given the specified initial conditions $b_{t_0-1}, \Delta_{t_0-1}, X_{t_0}$, where we have defined

$$p(\Pi_t) \equiv \left(\frac{1 - \alpha\Pi_t^{\theta-1}}{1 - \alpha} \right).$$

We introduce Lagrange multipliers ϕ_{1t} through ϕ_{7t} corresponding to constraints (A.19) through (A.25) respectively. We also introduce multipliers dated t_0 corresponding to the constraints implied by the initial conditions; the latter multipliers are normalized in such a way that the first-order conditions take the same form at

date t_0 as at all later dates. The first-order conditions of the maximization problem are then the following. The one with respect to Y_t is

$$U_y(Y_t,\Delta_t)-(1-\tau_t)f_y(Y_t)\phi_{2,t}-k_y(Y_t)\phi_{3,t}-\tau_t f_y(Y_t)\phi_{4,t}-u_{cc}(Y_t)b_{t-1}\Pi_t^{-1}\phi_{5,t}=0; \tag{A.26}$$

that with respect to Δ_t is

$$U_\Delta(Y_t,\Delta_t)+\phi_{6,t}-\alpha\beta\Pi_t^{\theta(1+\omega)}\phi_{6,t+1}+\phi_{7,t}=0; \tag{A.27}$$

that with respect to Π_t is

$$\begin{aligned}&\frac{1+\omega\theta}{\theta-1}p(\Pi_t)^{\frac{\theta(1+\omega)}{\theta-1}}p_\pi(\Pi_t)\phi_{1,t}-\alpha(\theta-1)\Pi_t^{\theta-2}F_t\phi_{2,t}\\ &-\theta(1+\omega)\alpha\Pi_t^{\theta(1+\omega)-1}K_t\phi_{3,t}+u_c(Y_t)b_{t-1}\Pi_t^{-2}\phi_{5,t}+\\ -\theta(1+\omega)\alpha\Delta_{t-1}\Pi_t^{\theta(1+\omega)-1}\phi_{6,t}&+\frac{\theta(1+\omega)}{1-\theta}(1-\alpha)p(\Pi_t)^{-\frac{(1+\omega\theta)}{1-\theta}}p_\pi(\Pi_t)\phi_{6,t}=0;\end{aligned} \tag{A.28}$$

that with respect to τ_t is

$$\phi_{2,t}-\phi_{4,t}=0; \tag{A.29}$$

that with respect to F_t is

$$-\phi_{1,t}+\phi_{2,t}-\alpha\Pi_t^{\theta-1}\phi_{2,t-1}=0; \tag{A.30}$$

that with respect to K_t is

$$p(\Pi_t)^{\frac{1+\omega\theta}{\theta-1}}\phi_{1,t}+\phi_{3,t}-\alpha\Pi_t^{\theta(1+\omega)}\phi_{3,t-1}=0; \tag{A.31}$$

that with respect to W_t is

$$\phi_{4,t}-\phi_{4,t-1}+\phi_{5,t}=0; \tag{A.32}$$

that with respect to b_t is

$$\phi_{5,t}=0; \tag{A.33}$$

and finally, we have the complementary slackness condition

$$\phi_{7,t}(\Delta_t-1)=0. \tag{A.34}$$

We search for a solution to these first-order conditions in which $\Pi_t=1$, $\Delta_t=1$, $Y_t=\bar{Y}$, $\tau_t=\bar{\tau}$ and $b_t=\bar{b}$ at all times. Note that this solution implies that $\bar{F}=\bar{K}$, $\bar{W}=u_c(\bar{Y})\bar{b}$, $p(\bar{\Pi})=1$ and that $(1-\bar{\tau})f(\bar{Y})=k(\bar{Y})$ or $(1-\bar{\tau})\bar{u}_c=\frac{\theta}{\theta-1}\bar{\mu}^w\bar{v}_y$.

A steady-state solution of this kind also requires that the Lagrange multipliers take constant values. In order for this to be true, equations (A.26), (A.27), (A.28), (A.30), and (A.31) respectively require that

$$U_y(\bar{Y},1) - \bar{\tau} f_y(\bar{Y})\phi_2 = 0,$$

$$U_\Delta(\bar{Y},1) + (1-\beta\alpha)\phi_6 + \phi_7 = 0,$$

$$(1+\theta\omega)\bar{F}\phi_2 - \theta(1+\omega)\alpha\phi_6 = 0,$$

$$\phi_1 = (1-\alpha)\phi_2,$$

and

$$\phi_1 = -(1-\alpha)\phi_3,$$

from which we can obtain the steady-state values of the multipliers ϕ_1, ϕ_2, ϕ_3, ϕ_6, and ϕ_7. From (A.29) and (A.33) we similarly obtain the values of ϕ_4 and ϕ_5.

Moreover, in this steady-state solution the tax rate satisfies

$$\bar{\tau} = s_G + \frac{\nu + (1-\beta)\bar{b}}{\bar{Y}},$$

where s_G is the steady-state share of output purchased by the government, $s_G \equiv \bar{G}/\bar{Y}$. Given our assumption that $\bar{b} > 0$ and that $\bar{G}, \bar{\nu} \geq 0$, it is necessarily the case that $\bar{\tau} > 0$. Consider the steady-state relation

$$(1-\bar{\tau})u_c(\bar{Y}-\bar{G},0) = \frac{\theta}{\theta-1}\bar{\mu}^w v_y(\bar{Y},0)$$

which determines the steady-state level of output $\bar{Y} = Y(\bar{\tau},\bar{G})$ as a decreasing function of the tax rate and increasing function of the level of government purchases. When $\bar{\tau} \geq 1$, then $\bar{Y} = 0$ and revenues will be equal to zero. Thus any solution must involve $0 < \bar{\tau} < 1$. Over this range of tax rates, the revenues $\bar{\tau}\bar{Y}$ are bounded above, following a Laffer-curve type of argument. It follows that $\bar{b}$ must be within some finite positive upper bound in order for a steady state to exist consistent with that initial condition. But for any $0 < \bar{b} \leq \bar{b}_{\max}$, we may verify that a steady-state tax rate $0 < \bar{\tau} < 1$ consistent with it exists.

We have thus verified that a constant solution to the first-order conditions exists. With a method explained below, we will check that this solution is indeed at least a local optimum. Note that as asserted in the text, this deterministic steady state involves zero inflation.

A second-order approximation to utility (equations (2.1) and (2.2))

We now proceed to compute a second-order approximation to the expected discounted value of the utility of the representative household

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left[u(Y_t; \xi_t) - \int_0^1 v(y_t(i); \xi_t) di \right], \qquad \text{(A.35)}$$

following the treatment in Woodford (2003, chapter 6). Here we use $\mathcal{O}(||\xi||^k)$ as shorthand for $\mathcal{O}(||\xi, \hat{b}_{t_0-1}, \hat{\Delta}_{t_0-1}^{1/2}, \hat{X}_{t_0}||^k)$, where in each case hats refer to log deviations from the steady-state values of the various parameters of the policy problem. We treat $\hat{\Delta}_{t_0}^{1/2}$ as an expansion parameter, rather than $\hat{\Delta}_{t_0}$, because (A.14) implies that deviations of the inflation rate from zero of order ϵ only result in deviations in the dispersion measure Δ_t from one of order ϵ^2. We are thus entitled to treat the fluctuations in Δ_t as being only of second order in our bound on the amplitude of disturbances, since if this is true at some initial date it will remain true thereafter.

The first term in (A.35) can be approximated using a second-order Taylor expansion around the steady state defined in the previous section as

$$\begin{aligned} u(Y_t; \xi_t) &= \bar{u} + \bar{u}_c \tilde{Y}_t + \bar{u}_\xi \xi_t + \frac{1}{2} \bar{u}_{cc} \tilde{Y}_t^2 + \bar{u}_{c\xi} \xi_t \tilde{Y}_t + \frac{1}{2} \xi_t' \bar{u}_{\xi\xi} \xi_t + \mathcal{O}(||\xi||^3) \\ &= \bar{u} + \bar{Y} \bar{u}_c \cdot (\hat{Y}_t + \frac{1}{2} \hat{Y}_t^2) + \bar{u}_\xi \xi_t + \frac{1}{2} \bar{Y} \bar{u}_{cc} \hat{Y}_t^2 \\ &\qquad + \bar{Y} \bar{u}_{c\xi} \xi_t \hat{Y}_t + \frac{1}{2} \xi_t' \bar{u}_{\xi\xi} \xi_t + \mathcal{O}(||\xi||^3) \\ &= \bar{Y} u_c \hat{Y}_t + \frac{1}{2} [\bar{Y} \bar{u}_c + \bar{Y}^2 \bar{u}_{cc}] \hat{Y}_t^2 - \bar{Y}^2 \bar{u}_{cc} g_t \hat{Y}_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\ &= \bar{Y} \bar{u}_c \left\{ \hat{Y}_t + \frac{1}{2} (1 - \sigma^{-1}) \hat{Y}_t^2 + \sigma^{-1} g_t \hat{Y}_t \right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned} \qquad \text{(A.36)}$$

where a bar denotes the steady-state value for each variable, a tilde denotes the deviation of the variable from its steady-state value (*e.g.*, $\tilde{Y}_t \equiv Y_t - \bar{Y}$); and a hat refers to the log deviation of the variable from its steady-state value (*e.g.*, $\hat{Y}_t \equiv \ln Y_t / \bar{Y}$). We use ξ_t to refer to the entire vector of exogenous shocks,

$$\xi_t' \equiv \left[\begin{array}{ccccc} \hat{\nu}_t & \hat{G} & g_t & q_t & \hat{\mu}_t^w \end{array} \right],$$

in which $\hat{\nu}_t \equiv (\nu_t - \overline{\nu})/\overline{Y}$, $\hat{G}_t \equiv (G_t - \overline{G})/\overline{Y}$, $g_t \equiv \hat{G}_t + s_C \bar{c}_t$, $\omega q_t \equiv (1+\nu)\bar{h}_t + \phi(1+\nu) a_t$, $\hat{\mu}_t^w \equiv \ln \mu_t^w / \bar{\mu}^w$, $\bar{c}_t \equiv \ln \bar{C}_t / \bar{C}$, $a_t \equiv \ln A_t / \bar{A}$, $\bar{h}_t \equiv \ln \bar{H}_t / \bar{H}$. Moreover, we use the

definitions $\sigma^{-1} \equiv \tilde{\sigma}^{-1} s_C^{-1}$ with $s_C \equiv \bar{C}/\overline{Y}$ and $s_C + s_G = 1$. We have used the Taylor expansion

$$Y_t/\bar{Y} = 1 + \hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 + \mathcal{O}(||\xi||^3).$$

to get a relation for $\tilde{Y}_t$ in terms of $\hat{Y}_t$. Finally the term "t.i.p." denotes terms that are independent of policy, and may accordingly be suppressed as far as the welfare ranking of alternative policies is concerned.

We may similarly approximate $v(y_t(i); \xi_t)$ by

$$\begin{aligned} v(y_t(i); \xi_t) &= \bar{Y}\bar{v}_y \left\{ \hat{y}_t(i) + \frac{1}{2}(1+\omega)\hat{y}_t(i)^2 - \omega q_t \hat{y}_t(i) \right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\ &= (1-\Phi)\bar{Y}u_c \left\{ \hat{y}_t(i) + \frac{1}{2}(1+\omega)\hat{y}_t(i)^2 - \omega q_t \hat{y}_t(i) \right\} \\ &\quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{A.37}$$

where $\hat{y}_t(i) \equiv \log(y_t(i)/\bar{Y})$. The second line uses the steady state relation $\bar{v}_y = (1 - \Phi)\bar{u}_c$ to replace $\bar{v}_y$ by $(1 - \Phi_y)\bar{u}_c$, where

$$\Phi \equiv 1 - \left(\frac{\theta - 1}{\theta}\right)\left(\frac{1 - \bar{\tau}}{\bar{\mu}^w}\right) < 1$$

measures the inefficiency of steady-state output $\bar{Y}$. We also assume in deriving (A.37) that deviations of $y_t(i)$ from $\bar{Y}$ are only of order $\mathcal{O}(||\xi||)$ in each industry; this is shown to be true of the optimal state-contingent evolution characterized below.

Integrating this expression over the differentiated goods i, we obtain

$$\begin{aligned} \int_0^1 v(y_t(i); \xi_t) &= (1-\Phi)\bar{Y}\bar{u}_c \left\{ E_i\hat{y}_t(i) + \frac{1}{2}(1+\omega)[(E_i\hat{y}_t(i))^2 + \text{var}_i\hat{y}_t(i)] - \omega q_t E_i\hat{y}_t(i) \right\} \\ &\qquad + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\ &= (1-\Phi)\bar{Y}\bar{u}_c \left\{ \hat{Y}_t + \frac{1}{2}(1+\omega)\hat{Y}_t^2 - \omega q_t \hat{Y}_t + \frac{1}{2}(\theta^{-1} + \omega)\text{var}_i\hat{y}_t(i) \right\} \\ &\qquad + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{A.38}$$

using the notation $E_i\hat{y}_t(i)$ for the mean value of $\hat{y}_t(i)$ across all differentiated goods at date t, and $\text{var}_i\hat{y}_t(i)$ for the corresponding variance. In the second line, we use the Taylor series approximation to the aggregate output index obtaining

$$\hat{Y}_t = E_i\hat{y}_t(i) + \frac{1}{2}(1 - \theta^{-1})\text{var}_i\hat{y}_t(i) + \mathcal{O}(||\xi||^3),$$

to eliminate $E_i \hat{y}_t(i)$.

Combining (A.36) and (A.38), we finally obtain equation (2.1) in the text

$$\begin{aligned} U_{t_0} &= \overline{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \Phi \widehat{Y}_t - \frac{1}{2} u_{yy} \widehat{Y}_t^2 + \widehat{Y}_t u_\xi \xi_t - \frac{1}{2}(1-\Phi)(\theta^{-1}+\omega) \mathrm{var}_i \log y_t(i) \\ &+ \text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (A.39) \end{aligned}$$

where

$$\begin{aligned} u_{yy} &\equiv (\omega + \sigma^{-1}) - (1-\Phi)(1+\omega), \\ u_\xi \xi_t &\equiv [\sigma^{-1} g_t + (1-\Phi)\omega q_t]. \end{aligned}$$

We finally observe that demand curve implies

$$\mathrm{var}_i \log y_t(i) = \theta^2 \mathrm{var}_i \log p_t(i). \qquad (A.40)$$

and that the Calvo's pricing mechanism imply

$$\sum_{t=t_0}^{\infty} \beta^t \ \mathrm{var}_i \log p_t(i) = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} \sum_{t=t_0}^{\infty} \beta^t \ \pi_t^2 + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \qquad (A.41)$$

By substituting (A.40) and (A.41) into (A.39), we obtain

$$\begin{aligned} U_{t_0} &= \overline{Y}\bar{u}_c \cdot E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\Phi \widehat{Y}_t - \frac{1}{2} u_{yy} \widehat{Y}_t^2 + \widehat{Y}_t u_\xi \xi_t - u_\pi \pi_t^2] \\ &\quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned}$$

which coincides with equation (2.1) in text where we have further defined

$$\kappa \equiv \frac{(1-\alpha\beta)(1-\alpha)}{\alpha} \frac{(\omega+\sigma^{-1})}{(1+\theta\omega)}, \qquad u_\pi \equiv \frac{\theta(\omega+\sigma^{-1})}{\kappa}.$$

A second-order approximation to the AS equation (equation (2.3))

We now compute a second-order approximation to the aggregate supply equation (A.9), or equation (2.3) in the main text. We can write (A.9) as

$$\tilde{p}_t = \left(\frac{K_t}{F_t} \right)^{\frac{1}{1+\omega\theta}},$$

where $\widetilde{p}_t \equiv p_t^*/P_t$. As shown in Benigno and Woodford (2003), a second-order expansion of this can be expressed in the form

$$\begin{aligned}\frac{(1+\omega\theta)}{(1-\alpha\beta)}\hat{\tilde{p}}_t &= z_t + \alpha\beta\frac{(1+\omega\theta)}{(1-\alpha\beta)}E_t(\hat{\tilde{p}}_{t+1} - \hat{P}_{t,t+1}) + \frac{1}{2}z_t X_t + \\ &\quad -\frac{1}{2}(1+\omega\theta)\hat{\tilde{p}}_t Z_t + \frac{1}{2}\alpha\beta(1+\omega\theta)E_t\{(\hat{\tilde{p}}_{t+1} - \hat{P}_{t,t+1})Z_{t+1}\} \\ &\quad +\frac{\alpha\beta}{2(1-\alpha\beta)}(1-2\theta-\omega\theta)(1+\omega\theta)E_t\{(\hat{\tilde{p}}_{t+1} - \hat{P}_{t,t+1})\hat{P}_{t,t+1}\} \\ &\qquad + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \quad \text{(A.42)}$$

where we define

$$\hat{P}_{t,T} \equiv \log(P_t/P_T),$$

$$z_t \equiv \omega(\hat{Y}_t - q_t) + \tilde{\sigma}^{-1}(\hat{C}_t - \bar{c}_t) - \hat{T}_t + \widehat{\mu}_t^w,$$

$$Z_t \equiv E_t\left\{\sum_{T=t}^{+\infty}(\alpha\beta)^{T-t}[X_T + (1-2\theta-\omega\theta)\hat{P}_{t,T}]\right\},$$

and in this last expression

$$X_T \equiv (2+\omega)\hat{Y}_T - \omega q_T + \hat{\mu}_T^w + \hat{T}_T - \tilde{\sigma}^{-1}(\hat{C}_T - \bar{c}_T),$$

where $\hat{T}_t = \ln(1-\tau_t)/(1-\bar{\tau})$. Here "s.o.t.i.p." refers to second-order (or higher) terms independent of policy; the first-order terms have been kept as these will matter for the log-linear aggregate-supply relation that appears as a constraint in our policy problem.

We next take a second-order expansion of the law of motion (A.10) for the price index, obtaining

$$\hat{\tilde{p}}_t = \frac{\alpha}{1-\alpha}\pi_t - \frac{1-\theta}{2}\frac{\alpha}{(1-\alpha)^2}\pi_t^2 + \mathcal{O}(||\xi||^3), \quad \text{(A.43)}$$

where we have used the fact that

$$\hat{\tilde{p}}_t = \frac{\alpha}{1-\alpha}\pi_t + \mathcal{O}(||\xi||^2),$$

and $\hat{P}_{t-1,t} = -\pi_t$. We can then plug (A.43) into (A.42) obtaining

$$\begin{aligned}\pi_t &= \frac{1-\theta}{2}\frac{1}{(1-\alpha)}\pi_t^2 + \frac{\kappa}{(\omega+\sigma^{-1})}z_t + \beta E_t\pi_{t+1} - \frac{1-\theta}{2}\frac{\alpha\beta}{(1-\alpha)}E_t\pi_{t+1}^2 \\ &\quad +\frac{1}{2}\frac{\kappa}{(\omega+\sigma^{-1})}z_t X_t - \frac{1}{2}(1-\alpha\beta)\pi_t Z_t + \frac{\beta}{2}(1-\alpha\beta)E_t\{\pi_{t+1}Z_{t+1}\} \\ &\quad -\frac{\beta}{2}(1-2\theta-\omega\theta)E_t\{\pi_{t+1}^2\} + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned} \quad \text{(A.44)}$$

We note further that a second-order approximation to the identity $C_t = Y_t - G_t$ yields

$$\hat{C}_t = s_C^{-1}\hat{Y}_t - s_C^{-1}\hat{G}_t + \frac{s_C^{-1}(1-s_C^{-1})}{2}\hat{Y}_t^2 + s_C^{-2}\hat{Y}_t\hat{G}_t + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \qquad \text{(A.45)}$$

and that

$$\hat{T}_t = -\omega_\tau\hat{\tau}_t - \frac{\omega_\tau}{(1-\bar{\tau})}\hat{\tau}_t^2 + \mathcal{O}(||\xi||^3), \qquad \text{(A.46)}$$

where $\omega_\tau \equiv \bar{\tau}/(1-\bar{\tau})$. By substituting (A.45) and (A.46) into the definition of z_t in (A.44), we finally obtain a quadratic approximation to the AS relation.

This can be expressed compactly in the form

$$\begin{aligned} V_t &= \kappa(c_x' x_t + c_\xi'\xi_t + \frac{1}{2}x_t'C_x x_t + x_t'C_\xi\xi_t + \frac{1}{2}c_\pi\pi_t^2) + \beta E_t V_{t+1} \\ &\quad + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned} \qquad \text{(A.47)}$$

where we have defined

$$x_t \equiv \begin{bmatrix} \hat{\tau}_t \\ \hat{Y}_t \end{bmatrix},$$

$$c_x' = \begin{bmatrix} \psi & 1 \end{bmatrix},$$

$$c_\xi' = \begin{bmatrix} 0 & 0 & -\sigma^{-1}(\omega+\sigma^{-1})^{-1} & -\omega(\omega+\sigma^{-1})^{-1} & (\omega+\sigma^{-1})^{-1} \end{bmatrix},$$

$$C_x = \begin{bmatrix} \psi & (1-\sigma^{-1})\psi \\ (1-\sigma^{-1})\psi & (2+\omega-\sigma^{-1}) + \sigma^{-1}(1-s_C^{-1})(\omega+\sigma^{-1})^{-1} \end{bmatrix},$$

$$C_\xi = \begin{bmatrix} 0 & 0 & \psi\sigma^{-1} & 0 & 0 \\ 0 & \frac{\sigma^{-1}s_C^{-1}}{(\omega+\sigma^{-1})} & -\frac{\sigma^{-1}(1-\sigma^{-1})}{(\omega+\sigma^{-1})} & -\frac{\omega(1+\omega)}{(\omega+\sigma^{-1})} & \frac{(1+\omega)}{(\omega+\sigma^{-1})} \end{bmatrix},$$

$$c_\pi = \frac{\theta(1+\omega)(\omega+\sigma^{-1})}{\kappa}$$

$$V_t = \pi_t + \frac{1}{2}v_\pi\pi_t^2 + v_z\pi_t Z_t,$$

$$Z_t = z_x' x_t + z_\pi\pi_t + z_\xi'\xi_t + \alpha\beta E_t Z_{t+1},$$

in which the coefficients

$$\psi \equiv \omega_\tau/(\omega+\sigma^{-1}),$$

and

$$v_\pi \equiv \theta(1+\omega) - \frac{1-\theta}{2(1-\alpha)}, \qquad v_z \equiv \frac{(1-\alpha\beta)}{2},$$

$$v_k \equiv \frac{\alpha\beta}{1-\alpha\beta}(1-2\theta-\omega\theta),$$

$$z_x' \equiv \left[\begin{array}{cc} (2+\omega-\sigma^{-1}) + v_k(\omega+\sigma^{-1}) & -\omega_\tau(1-v_k) \end{array}\right],$$

$$z_\xi' \equiv \left[\begin{array}{ccccc} 0 & 0 & \sigma^{-1}(1-v_k) & -\omega(1+v_k) & (1-v_k) \end{array}\right],$$

$$z_\pi \equiv -v_k.$$

Note that in a first-order approximation, (A.47) can be written as simply

$$\pi_t = \kappa[\hat{Y}_t + \psi\hat{\tau}_t + c_\xi'\xi_t] + \beta E_t\pi_{t+1}, \tag{A.48}$$

where

$$c_\xi'\xi_t \equiv (\omega+\sigma^{-1})^{-1}[-\sigma^{-1}g_t - \omega q_t + \hat{\mu}_t^w].$$

We can also integrate (A.47) forward from time t_0 to obtain

$$\begin{aligned} V_{t_0} &= \sum_{t=t_0}^{\infty} \beta^{t-t_0}\kappa(c_x' x_t + \frac{1}{2}x_t' C_x x_t + x_t' C_\xi \xi_t + \frac{1}{2}c_\pi \pi_t^2) \\ &\quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{A.49}$$

where the term $c_\xi'\xi_t$ is now included in terms independent of policy. (Such terms matter when part of the log-linear constraints, as in the case of (A.48), but not when part of the quadratic objective.) This corresponds to equation (2.3) in the main text.

A second-order approximation to the intertemporal government budget constraint (equation (2.6))

We now take a second-order approximation of the intertemporal budget constraint of the government

$$W_t = E_t \sum_{T=t}^{\infty} \beta^{T-t}\tilde{u}_c(Y_T;\xi_T)s_T, \tag{A.50}$$

where

$$s_t = \tau_t Y_t - G_t - \nu_t, \tag{A.51}$$

given the definition

$$W_t \equiv \frac{b_{t-1}}{\Pi_t}\tilde{u}_c(C_t;\xi_t). \tag{A.52}$$

First, we take a second-order approximation of the term $\tilde{u}_c(C_t;\xi_t)s_t$ obtaining

$$\begin{aligned}
\tilde{u}_c(C_t;\xi_t)s_t &= \bar{s}\bar{\tilde{u}}_c + \bar{\tilde{u}}_{cc}\bar{s}\tilde{C}_t + \bar{\tilde{u}}_c\tilde{s}_t + \bar{s}\bar{\tilde{u}}_{c\xi}\xi_t + \frac{1}{2}\bar{s}\bar{\tilde{u}}_{ccc}\tilde{C}_t^2 + \bar{\tilde{u}}_{cc}\tilde{C}_t\tilde{s}_t + \bar{s}\tilde{C}_t\bar{\tilde{u}}_{cc\xi}\xi_t \\
&\quad + \tilde{s}_t\bar{\tilde{u}}_{c\xi}\xi_t + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3), \\
&= \bar{s}\bar{\tilde{u}}_c + \bar{\tilde{u}}_{cc}\bar{s}\bar{C}\hat{C}_t + \bar{\tilde{u}}_c\tilde{s}_t + \bar{s}\bar{\tilde{u}}_{c\xi}\xi_t + \frac{1}{2}\bar{s}(\bar{\tilde{u}}_{cc}\bar{C} + \bar{\tilde{u}}_{ccc}\bar{C}^2)\tilde{C}_t^2 + \bar{C}\bar{\tilde{u}}_{cc}\hat{C}_t\tilde{s}_t + \\
&\quad \bar{s}\bar{C}\bar{\tilde{u}}_{cc\xi}\xi_t\hat{C}_t + \bar{\tilde{u}}_{c\xi}\xi_t\tilde{s}_t + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= \bar{s}\bar{\tilde{u}}_c + \bar{\tilde{u}}_c[-\widetilde{\sigma}^{-1}\bar{s}\hat{C}_t + \tilde{s}_t + \bar{s}\bar{\tilde{u}}_c^{-1}\bar{\tilde{u}}_{c\xi}\xi_t + \frac{1}{2}\bar{s}\widetilde{\sigma}^{-2}\tilde{C}_t^2 - \widetilde{\sigma}^{-1}\tilde{s}_t\hat{C}_t + \\
&\quad \bar{s}\bar{C}\bar{\tilde{u}}_c^{-1}\bar{\tilde{u}}_{cc\xi}\xi_t\hat{C}_t + \bar{\tilde{u}}_c^{-1}\bar{\tilde{u}}_{c\xi}\xi_t\tilde{s}_t] + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= \bar{s}\bar{\tilde{u}}_c + \bar{\tilde{u}}_c[-\tilde{\sigma}^{-1}\bar{s}(\hat{C}_t - \bar{c}_t) + \tilde{s}_t + \frac{1}{2}\bar{s}\tilde{\sigma}^{-2}\tilde{C}_t^2 - \tilde{\sigma}^{-1}\tilde{s}_t(\hat{C}_t - \bar{c}_t) - \tilde{\sigma}^{-2}\bar{s}\bar{c}_t\hat{C}_t] \\
&\quad + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3)
\end{aligned} \tag{A.53}$$

where we have followed previous definitions and use the isoelastic functional forms assumed and note that we can write $\bar{\tilde{u}}_c^{-1}\bar{\tilde{u}}_{c\xi}\xi_t = \tilde{\sigma}^{-1}\bar{c}_t$ and $\bar{C}\bar{\tilde{u}}_c^{-1}\bar{\tilde{u}}_{cc\xi}\xi_t = -\tilde{\sigma}^{-2}\bar{c}_t$. Plugging (A.45) into (A.53) we obtain

$$\begin{aligned}
\tilde{u}_c(C_t;\tilde{\xi}_t)s_t &= \bar{s}\bar{\tilde{u}}_c[1 - \sigma^{-1}\hat{Y}_t + \sigma^{-1}g_t + \bar{s}^{-1}\tilde{s}_t + \frac{1}{2}[\sigma^{-1}(s_C^{-1} - 1) + \sigma^{-2}]\hat{Y}_t^2 + \\
&\quad -\sigma^{-1}\bar{s}^{-1}(\hat{Y}_t - g_t)\tilde{s}_t - \sigma^{-1}(s_C^{-1}\hat{G}_t + \sigma^{-1}g_t)\hat{Y}_t] + \text{s.o.t.i.p.} + \\
&\quad +\mathcal{O}(||\xi||^3)
\end{aligned} \tag{A.54}$$

where we have defined $g_t \equiv \hat{G}_t + s_C\bar{c}_t$ and $\sigma^{-1} \equiv \widetilde{\sigma}^{-1}s_C^{-1}$.

We recall now that the primary surplus is defined as

$$s_t = \tau_t Y_t - G_t - \nu_t$$

which can be expanded in a second-order expansion to get

$$\begin{aligned}
\bar{s}^{-1}\tilde{s}_t &= (1+\omega_g)(\hat{Y}_t + \hat{\tau}_t) - s_d^{-1}(\hat{G}_t + \hat{\nu}_t) + \frac{(1+\omega_g)}{2}(\hat{Y}_t + \hat{\tau}_t)^2 \\
&\quad + \text{s.o.t.i.p.} + \mathcal{O}(||\xi||^3),
\end{aligned} \tag{A.55}$$

where we have defined $s_d \equiv \bar{s}/\overline{Y}$, $\omega_g = (\overline{G} + \overline{\nu})/\bar{s}$. and $\hat{\nu}_t = (\nu_t - \overline{\nu})/\overline{Y}$. Substituting

(A.55) into (A.54), we obtain

$$\begin{aligned}
\tilde{u}_c(C_t;\tilde{\xi}_t)s_t &= \bar{s}\bar{\tilde{u}}_c[1-\sigma^{-1}\hat{Y}_t+(1+\omega_g)(\hat{Y}_t+\hat{\tau}_t)+\sigma^{-1}g_t-s_d^{-1}(\hat{G}_t+\hat{\nu}_t) \\
&+\frac{(1+\omega_g)}{2}\hat{\tau}_t^2+(1+\omega_g)(1-\sigma^{-1})\hat{\tau}_t\hat{Y}_t+ \\
&+\frac{1}{2}\left[1+\omega_g+\sigma^{-1}(s_C^{-1}-1)+\sigma^{-2}-2\sigma^{-1}(1+\omega_g)\right]\hat{Y}_t^2+ \\
&-\sigma^{-1}[s_C^{-1}\hat{G}_t+(\sigma^{-1}-1-\omega_g)g_t-s_d^{-1}(\hat{G}_t+\hat{\nu}_t)]\widehat{Y}_t \\
&+\sigma^{-1}(1+\omega_g)g_t\hat{\tau}_t]+\text{s.o.t.i.p.}+\mathcal{O}(||\xi||^3). \qquad \text{(A.56)}
\end{aligned}$$

Substituting (A.56) into (A.50), we obtain

$$\begin{aligned}
\tilde{W}_t &= (1-\beta)[b_x'x_t+b_\xi'\xi_t+\frac{1}{2}x_t'B_xx_t+x_t'B_\xi\xi_t]+\beta E_t\tilde{W}_{t+1} \\
&\text{s.o.t.i.p.}+\mathcal{O}(||\xi||^3) \qquad \text{(A.57)}
\end{aligned}$$

which corresponds to equation (2.6) in the text where $\tilde{W}_t \equiv (W_t-\bar{W})/\bar{W}$

$$b_x' = \begin{bmatrix} (1+\omega_g) & (1+\omega_g)-\sigma^{-1} \end{bmatrix},$$

$$b_\xi' = \begin{bmatrix} -s_d^{-1} & -s_d^{-1} & \sigma^{-1} & 0 & 0 \end{bmatrix},$$

$$B_x = \begin{bmatrix} (1+\omega_g) & (1-\sigma^{-1})(1+\omega_g) \\ (1-\sigma^{-1})(1+\omega_g) & (1+\omega_g)+(s_C^{-1}-1)\sigma^{-1}+\sigma^{-2}-2\sigma^{-1}(1+\omega_g) \end{bmatrix},$$

$$B_\xi = \begin{bmatrix} 0 & \sigma^{-1}(1+\omega_g) & \sigma^{-1}(1+\omega_g) & 0 & 0 \\ s_d^{-1}\sigma^{-1} & s_d^{-1}\sigma^{-1}-s_C^{-1}\sigma^{-1} & -\sigma^{-1}(\sigma^{-1}-1-\omega_g) & 0 & 0 \end{bmatrix},$$

We further note from (A.57) that

$$\tilde{W}_t \equiv (\hat{b}_{t-1}-\pi_t-\widetilde{\sigma}^{-1}\hat{C}_t+\bar{c}_t)+\frac{1}{2}(\hat{b}_{t-1}-\pi_t-\widetilde{\sigma}^{-1}\hat{C}_t+\bar{c}_t)^2+\mathcal{O}(||\xi||^3).$$

Substituting in (A.45), we obtain

$$\begin{aligned}
\tilde{W}_t &\equiv \hat{b}_{t-1}-\pi_t-\sigma^{-1}(\hat{Y}_t-g_t)-\frac{\sigma^{-1}(1-s_C^{-1})}{2}\hat{Y}_t^2-\sigma^{-1}s_C^{-1}\hat{Y}_t\hat{G}_t \\
&+\frac{1}{2}(\hat{b}_{t-1}-\pi_t-\sigma^{-1}(\hat{Y}_t-g_t))^2+\text{s.o.t.i.p.}+\mathcal{O}(||\xi||^3)
\end{aligned}$$

which can be written as

$$\begin{aligned}
\tilde{W}_t &= \hat{b}_{t-1}-\pi_t+w_x'x_t+w_\xi'\xi_t+\frac{1}{2}x_t'W_xx_t+x_t'W_\xi\xi_t+\frac{1}{2}[\hat{b}_{t-1}-\pi_t+w_x'x_t+w_\xi'\xi_t]^2 \\
&+\text{s.o.t.i.p.}+\mathcal{O}(||\xi||^3)
\end{aligned}$$

$$w_x' = \left[\begin{array}{cc} 0 & -\sigma^{-1} \end{array} \right],$$

$$w_\xi' = \left[\begin{array}{ccccc} 0 & 0 & \sigma^{-1} & 0 & 0 \end{array} \right],$$

$$W_x = \left[\begin{array}{cc} 0 & 0 \\ 0 & (s_C^{-1} - 1)\sigma^{-1} \end{array} \right],$$

$$W_\xi = \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & -s_C^{-1}\sigma^{-1} & 0 & 0 & 0 \end{array} \right].$$

Note that in the first-order approximation we can simply write (A.57) as

$$\begin{aligned} \hat{b}_{t-1} - \pi_t \quad &+ \quad w_x' x_t + w_\xi' \xi_t = (1-\beta)[b_x' x_t + b_\xi' \xi_t] \\ &+ \quad \beta E_t[\hat{b}_t - \pi_{t+1} + w_x' x_{t+1} + w_\xi' \xi_{t+1}]. \end{aligned} \tag{A.58}$$

More integrating forward (A.57), we obtain that

$$\tilde{W}_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [b_x' x_t + \frac{1}{2} x_t' B_x x_t + x_t' B_\xi \xi_t] + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{A.59}$$

where we have moved $b_\xi' \xi_t$ in t.i.p.

A quadratic policy objective (equation (2.10))

We now derive a quadratic approximation to the policy objective function. To this end, we combine equation (A.49) and (A.59) in a way to eliminate the linear term in (A.39). Indeed, we find ϑ_1, ϑ_2 such that

$$\vartheta_1 b_x' + \vartheta_2 c_x' = a_x' \equiv [0 \ \Phi].$$

The solution is given by

$$\begin{aligned} \vartheta_1 &= -\frac{\Phi \omega_\tau}{\Gamma}, \\ \vartheta_2 &= \frac{\Phi(1+\omega_g)}{\Gamma}, \end{aligned}$$

where

$$\Gamma = (\omega + \sigma^{-1})(1 + \omega_g) - \omega_\tau (1 + \omega_g) + \omega_\tau \sigma^{-1}.$$

We can write

$$
\begin{aligned}
E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\Phi\hat{Y}_t &= E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}[\vartheta_1 b_x' + \vartheta_2 c_x']x_t \\
&= -E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}[\frac{1}{2}x_t' D_x x_t + x_t' D_\xi \xi_t + \frac{1}{2}d_\pi \pi_t^2] \\
&\quad +\vartheta_1 \bar{W}^{-1}W_{t_0} + \vartheta_2 V_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3)
\end{aligned}
$$

where

$$
D_x \equiv \vartheta_1 B_x + \vartheta_2 C_x, \quad \text{etc.}
$$

Hence

$$
\begin{aligned}
U_{t_0} &= \Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{a_x' x_t - \frac{1}{2}x_t' A_x x_t - x_t' A_\xi \xi_t - \frac{1}{2}a_\pi \pi_t^2\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= -\Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{\frac{1}{2}x_t' Q_x x_t + x_t' Q_\xi \xi_t + \frac{1}{2}q_\pi \pi_t^2\right\} + T_{t_0} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= -\Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\left\{\frac{1}{2}q_y(\hat{Y}_t - \hat{Y}_t^*)^2 + \frac{1}{2}q_\pi \pi_t^2\right\} + T_{t_0} + \\
&\quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \qquad \text{(A.60)}
\end{aligned}
$$

In particular, we obtain that $\Omega = \bar{u}_c \bar{Y}$ and that

$$
Q_x = \begin{bmatrix} 0 & 0 \\ 0 & q_y \end{bmatrix},
$$

with

$$
\begin{aligned}
q_y \equiv\ & (1-\Phi)(\omega + \sigma^{-1}) + \Phi(\omega + \sigma^{-1})\frac{(1+\omega_g)(1+\omega)}{\Gamma} + \Phi\sigma^{-1}\frac{(1+\omega_\tau)(1+\omega_g)}{\Gamma} \\
& -\Phi\sigma^{-1}s_C^{-1}\frac{1+\omega_g+\omega_\tau}{\Gamma};
\end{aligned}
$$

moreover we have defined

$$
Q_\xi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ q_{\xi 1} & q_{\xi 2} & q_{\xi 3} & q_{\xi 4} & q_{\xi 5} \end{bmatrix},
$$

with

$$
q_{\xi 1} = -\frac{\Phi\omega_\tau}{\Gamma}s_d^{-1}\sigma^{-1},
$$

$$q_{\xi 2} = -\frac{\Phi\sigma^{-1}s_d^{-1}\omega_\tau}{\Gamma} + \frac{\sigma^{-1}s_C^{-1}\Phi(1+\omega_g+\omega_\tau)}{\Gamma},$$

$$q_{\xi 3} = -(1-\Phi)\sigma^{-1} - \frac{\sigma^{-1}\Phi(1+\omega)(1+\omega_g)}{\Gamma},$$

$$q_{\xi 4} = -(1-\Phi)\omega - \frac{\omega\Phi(1+\omega)(1+\omega_g)}{\Gamma},$$

$$q_{\xi 5} = \Phi\frac{1+\omega_g}{\Gamma}(1+\omega),$$

and

$$q_\pi = \frac{\Phi(1+\omega_g)\theta(1+\omega)(\omega+\sigma^{-1})}{\Gamma\kappa} + \frac{(1-\Phi)\theta(\omega+\sigma^{-1})}{\kappa}.$$

We note that under the assumption that $\omega + \sigma^{-1} > \omega_\tau = \bar{\tau}/(1-\bar{\tau})$, $\Gamma > 0$, which implies that $q_\pi > 0$. Moreover, if

$$s_C > \frac{\Phi\sigma^{-1}(1+\omega_g+\omega_\tau)}{(1-\Phi)(\omega+\sigma^{-1})\Gamma + \Phi(\omega+\sigma^{-1})(1+\omega_g)(1+\omega) + \Phi\sigma^{-1}(1+\omega_g)(1+\omega_\tau)},$$

then $q_y > 0$ and the objective function is concave. Since the expression on the right-hand side of this inequality is necessarily less than one (given that $\Gamma > 0$), the inequality is satisfied for all values of s_G less than a positive upper bound. We have further defined $\hat{Y}_t^*$, the desired level of output, as

$$\hat{Y}_t^* = -q_y^{-1}q_\xi'\xi_t.$$

Finally,

$$T_{t_0} \equiv (1-\Phi)\bar{Y}\bar{u}_c[\vartheta_1 W_{t_0} + \vartheta_2 V_{t_0}]$$

is a transitory component. Equation (A.60) corresponds to equation (2.10) in the main text.

A log-linear aggregate-supply relation and the cost-push shock

The AS equation (A.48) can be written as

$$\pi_t = \kappa[y_t + \psi\hat{\tau}_t + u_t] + \beta E_t\pi_{t+1}, \tag{A.61}$$

where u_t is composite "cost-push" shock defined as $u_t \equiv c_\xi'\xi_t + \hat{Y}_t^*$. We can write (A.61) as

$$\pi_t = \kappa[y_t + \psi(\hat{\tau}_t - \hat{\tau}_t^*)] + \beta E_t\pi_{t+1}, \tag{A.62}$$

where we have further defined

$$u_t = u'_\xi \xi_t \equiv \hat{Y}_t^* + c'_\xi \xi_t,$$

where

$$u_{\xi 1} \equiv \frac{\Phi\omega_\tau}{q_y\Gamma} s_d^{-1}\sigma^{-1},$$

$$u_{\xi 2} \equiv \frac{\Phi\sigma^{-1}s_d^{-1}\omega_\tau}{q_y\Gamma} - \frac{\sigma^{-1}s_C^{-1}\Phi(1+\omega_g+\omega_\tau)}{q_y\Gamma},$$

$$u_{\xi 3} \equiv -\Phi\sigma^{-2}\frac{(1+\omega_\tau)(1+\omega_g)}{q_y\Gamma(\omega+\sigma^{-1})} + \Phi\sigma^{-2}s_C^{-1}\frac{1+\omega_g+\omega_\tau}{q_y\Gamma(\omega+\sigma^{-1})},$$

$$u_{\xi 4} \equiv \omega\sigma u_{\xi 3},$$

$$u_{\xi 5} = -\sigma^2 u_{\xi 3} + \frac{(1-\Phi)}{q_y},$$

We finally define

$$\hat{\tau}_t^* \equiv -\psi^{-1}u_t$$

in a way that we can write (A.48)

$$\pi_t = \kappa[(\hat{Y}_t - \hat{Y}_t^*) + \psi(\hat{\tau}_t - \hat{\tau}_t^*)] + \beta E_t \pi_{t+1}, \tag{A.63}$$

which is equation (2.14) in the text.

A log-linear intertemporal solvency condition and the 'fiscal stress' shock

The flow budget constraint (A.58) can be solved forward to yield the intertemporal budget constraint of the government

$$\hat{b}_{t-1} - \pi_t - \sigma^{-1}y_t = -f_t + (1-\beta)E_t\sum_{T=t}^{\infty}\beta^{T-t}[b_y y_T + b_\tau(\hat{\tau}_T - \hat{\tau}_T^*)] \tag{A.64}$$

where f_t, the fiscal stress shock, is defined in the following way

$$\begin{aligned} f_t &\equiv \sigma^{-1}(g_t - \hat{Y}_t^*) - (1-\beta)E_t\sum_{T=t}^{\infty}\beta^{T-t}[b_y\hat{Y}_T^* + b_\tau\hat{\tau}_T^* + b'_\xi\xi_T] \\ &= \sigma^{-1}(g_t - \hat{Y}_t^*) + (1-\beta)E_t\sum_{T=t}^{\infty}\beta^{T-t}[\omega_\tau^{-1}\Gamma\hat{Y}_T^* - (b'_\xi - \psi^{-1}c'_\xi)\xi_T], \end{aligned}$$

which can be rewritten in a more compact way as

$$f_t \equiv h'_\xi \xi_t - (1-\beta) E_t \sum_{T=t}^{\infty} \beta^{T-t} f'_\xi \xi_T,$$

where

$$h_{\xi 1} \equiv -\frac{\Phi \omega_\tau}{q_y \Gamma} \frac{\sigma^{-2}}{s_d}$$

$$f_{\xi 1} \equiv \frac{\Phi}{q_y} \frac{\sigma^{-1}}{s_d} + \frac{1}{s_d}$$

$$h_{\xi 2} \equiv -\frac{\Phi \sigma^{-2} s_d^{-1} \omega_\tau}{\Gamma q_y} + \frac{\sigma^{-2} s_C^{-1} \Phi (1 + \omega_g + \omega_\tau)}{\Gamma q_y}$$

$$f_{\xi 2} \equiv \frac{\Phi \sigma^{-2} s_d^{-1}}{q_y} - \frac{\sigma^{-2} s_C^{-1} \Phi (1 + \omega_g + \omega_\tau)}{\omega_\tau q_y} + \frac{1}{s_d}$$

$$\begin{aligned} h_{\xi 3} &\equiv \Phi \sigma^{-2} \frac{(1+\omega_\tau)(1+\omega_g)}{q_y \Gamma} - \Phi \sigma^{-2} s_C^{-1} \frac{1 + \omega_g + \omega_\tau}{q_y \Gamma} \\ &\quad + \frac{(1-\Phi)\omega \sigma^{-1}}{q_y} + \frac{\omega \sigma^{-1} \Phi (1+\omega)(1+\omega_g)}{\Gamma q_y}, \end{aligned}$$

$$f_{\xi 3} \equiv \frac{\omega_\tau^{-1} \Gamma (1-\Phi) \sigma^{-1}}{q_y} + \frac{\omega_\tau^{-1} \sigma^{-1} \Phi (1+\omega)(1+\omega_g)}{q_y} - \sigma^{-1}(1 + \omega_\tau^{-1})$$

$$h_{\xi 4} \equiv -\frac{\sigma^{-1}(1-\Phi)\omega}{q_y} - \frac{\sigma^{-1} \omega \Phi (1+\omega)(1+\omega_g)}{\Gamma q_y},$$

$$f_{\xi 4} \equiv \frac{\omega_\tau^{-1} \sigma^{-1} \Gamma (1-\Phi) \omega}{q_y} + \frac{\omega_\tau^{-1} \sigma^{-1} \omega \Phi (1+\omega)(1+\omega_g)}{q_y} - \omega \omega_\tau^{-1}$$

$$h_{\xi 5} \equiv \sigma^{-1} \Phi \frac{(1+\omega_g)(1+\omega)}{q_y \Gamma}$$

$$f_{\xi 5} \equiv -\omega_\tau^{-1} \sigma^{-1} \Phi \frac{(1+\omega_g)(1+\omega)}{q_y} + \omega_\tau^{-1}$$

Definition of the coefficients in sections 3 and 4

The coefficients m_φ, n_φ, n_b, m_b, $\tilde{m}_b$, ω_φ are defined in the following ways

$$\begin{aligned} m_\varphi &\equiv -q_y^{-1} \psi^{-1} (1-\beta) b_\tau + q_y^{-1} [(1-\beta) b_y + \sigma^{-1}], \\ n_\varphi &\equiv -q_y^{-1} \sigma^{-1}, \end{aligned}$$

$$\begin{aligned} n_b &\equiv b_\tau(\psi^{-1}-1)(m_\varphi+n_\varphi), \\ m_b &\equiv -n_\varphi[(1-\beta)b_\tau\psi^{-1}-(1-\beta)b_y-\sigma^{-1}], \end{aligned}$$

$$\tilde{m}_b \equiv \sigma^{-1}n_\varphi+\omega_\varphi-(1-\beta)[b_\tau\psi^{-1}-b_y]n_\varphi+(1-\beta)\psi^{-1}\kappa^{-1}b_\tau\omega_\varphi,$$

$$\omega_\varphi \equiv -q_\pi^{-1}(\kappa^{-1}(1-\beta)b_\tau\psi^{-1}+1),$$

$$\phi \equiv \kappa^{-1}q_\pi^{-1}q_y,$$

$$\gamma_1 \equiv \kappa^{-1}q_\pi^{-1}[(1-\beta)b_y+\sigma^{-1}],$$

$$\gamma_2 \equiv \kappa^{-1}q_\pi^{-1}\sigma^{-1}.$$

Proof of determinacy of equilibrium under the optimal targeting rules

We now show that there is a determinate equilibrium if policy is conducted so as to ensure that the two target criteria

$$E_t\pi_{t+1}=0 \tag{A.65}$$

and

$$\Delta y_t+\omega_\varphi^{-1}(m_\varphi+n_\varphi)\pi_t-\omega_\varphi^{-1}n_\varphi\Delta\pi_t=0 \tag{A.66}$$

are satisfied in each period $t \geq t_0$. Note that (A.66) can be written as

$$\Delta y_t=\gamma_3\pi_t+\gamma_4\pi_{t-1} \tag{A.67}$$

where

$$\begin{aligned} \gamma_3 &\equiv -\omega_\varphi^{-1}m_\varphi, \\ \gamma_4 &\equiv -\omega_\varphi^{-1}n_\varphi. \end{aligned}$$

Use (A.65), combined with

$$\tau_t-\hat{\tau}_t^*=\kappa^{-1}\pi_t-\psi^{-1}y_t-\kappa^{-1}\beta E_t\pi_{t+1}. \tag{A.68}$$

and

$$E_t\Delta y_{t+1}=-\omega_\varphi^{-1}n_\varphi\pi_t$$

to eliminate $E_t\pi_{t+1}, E_ty_{t+1}$ and $\tau_t-\hat{\tau}_t^*$ from

$$\begin{aligned} \hat{b}_{t-1}-\pi_t &- \sigma^{-1}y_t+f_t=(1-\beta)[b_yy_t+b_\tau(\hat{\tau}_t-\hat{\tau}_t^*)] \\ &+ \beta E_t[\hat{b}_t-\pi_{t+1}-\sigma^{-1}\hat{y}_{t+1}+f_{t+1}]. \end{aligned}$$

Then further use (A.66) to eliminate y_t from the resulting expression. One obtains an equation of the form

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} + m_{41}\pi_t + m_{42}\pi_{t-1} + m_{43}y_{t-1} + \varepsilon_t,$$

where ε_t is an exogenous disturbance. The system consisting of this equation plus (A.66) and (A.65) can then be written as

$$E_t z_{t+1} = M z_t + N\varepsilon_t,$$

where

$$z_t \equiv \begin{bmatrix} \pi_t \\ \pi_{t-1} \\ y_{t-1} \\ \hat{b}_{t-1} \end{bmatrix}, \qquad M \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & \beta^{-1} \end{bmatrix}, \qquad N \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ n_{41} \end{bmatrix}.$$

Because M is lower triangular, its eigenvalues are the four diagonal elements: 0, 0, 1, and β^{-1}. Hence there is exactly one eigenvalue outside the unit circle, and equilibrium is determinate (but possesses a unit root). Because of the triangular form of the matrix, one can also easily solve explicitly for the elements of the left eigenvector

$$v' = [v_1 \ v_2 \ v_3 \ 1]$$

associated with the eigenvalue β^{-1}, where

$$\begin{aligned} v_1 \quad = \quad & (1+\omega_g)[\psi^{-1}-1]\beta\gamma_4 - (1-\beta\sigma^{-1}\gamma_4) - (1-\beta)(1+\omega_g)(\kappa\psi)^{-1} \\ & +(1+\omega_g)[\psi^{-1}-1]\gamma_3, \end{aligned}$$

$$v_2 = (1+\omega_g)[\psi^{-1}-1]\gamma_4,$$

$$v_3 = (1+\omega_g)[\psi^{-1}-1].$$

Pre-multiplying the vector equation by v', one obtains a scalar equation with a unique non-explosive solution of the form

$$v' z_t = -\sum_{j=0}^{\infty} \beta^{j+1} E_t \varepsilon_{t+j}.$$

In the case that $v_1 \neq 0$, this can be solved for π_t as a linear function of $\pi_{t-1}, y_{t-1}, \hat{b}_{t-1}$ and the exogenous state vector as it follows

$$\pi_t = -\frac{1}{v_1}\widehat{b}_{t-1} - \frac{v_2}{v_1}\pi_{t-1} - \frac{v_3}{v_1}y_{t-1} + \frac{1}{v_1}f_t. \tag{A.69}$$

The solution for π_t can then be substituted into the above equations to obtain the equilibrium dynamics of y_t and $\hat{b}_t$ as well, and hence of τ_t also.

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