Research Cycles

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PRELIMINARY AND INCOMPLETE

1 Introduction

This paper studies the dynamics of fundamental research. We observe that periods of intense innovations are followed by periods of exploitation of existing fields. We want to understand these dynamics and be able to study whether they are efficient from the point of view of social welfare.

A key aspect we are interested in is the credential one. Scientists derive utility from recognition from other scientists, which often takes the form of citations. In our model, the value derived by a scientist from a paper he has written is the sum of an "intrinsic" value of the paper, which depends on the field in which it is written and its order of appearance in that field, and a "citation premium" which depends on the number of subsequent papers written in that field.

We show that the economy can be either in a regime where new fields are constantly invented, and then converges to a steady state, or in a cyclical regime where periods of innovation alternate with periods where one only exploits existing fields. Furthermore, these cycles are very irregular and the duration of a cycle is "unpredictable" from the duration of the previous cycle – i.e. related to it by a very nonlinear function.

We are able to perform comparative statics in the convergence regime and show that a (i) higher citation premium raises the equilibrium rate of innovation, (ii) a mean-preserving spread in the distribution of the value of new fields reduces the equilibrium rate of innovation and (iii) a larger citation premium makes researchers less risk-averse, in that it alleviates (and potentially reverses) that effect.

2 The model

We consider an infinite horizon model with discrete time. At each date t there is a continuum of existing fields of research, which we index by i. Each field is characterized by a stock of contributions (or 'papers') $n_t(i)$ at the end of period t.

Papers are produced by researchers. Researchers live for two periods, hence we have an overlapping generation structure. In the first period of their life, researchers produce contributions. In the second period of their life, they enjoy the returns from their scientific "reputation", which defines their utility function. A researcher's scientific reputation is the sum of the contribution of each individual paper he or she has written. An individual paper written at date t in existing field i yields the following contribution to its author's reputation:

$$v_t(i) = \omega(i) - \beta(\ln\ n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)).$$

This reputation is the sum of two terms. The first term, $\omega(i)-\beta(\ln n_t(i)-\ln \bar{n})$, defines the *intrinsic* value of the paper. $\omega(i)$ is a field-specific constant which represents the field's value (or initial research potential) as a whole. The term $\beta(\ln\ n_t(i) - \ln \bar{n})$, where$\beta$ and $\bar{n}$ are positive parameters, captures the fact that there are decreasing returns to research: the larger the stock of knowledge in field i, the smaller the intrinsic value of additional contributions. The second term, $\theta(\ln n_{t+1}(i) - \ln n_t(i))$, is the *citation premium.* It tells us that the reputation obtained from papers written at t is greater, the greater the flow of further advances in the relevant field at $t+1$. Underlying this formulation is the idea that papers come in a given order, and that new papers cite previous papers, thus enhancing their author's reputation. Note that contemporaneous papers do not cite each other, so that what matters for citations is the log difference between the stock of papers written at the end of $t+1$ and that at the end of t.

The total mass of researchers per generation is normalized to 1. Each researcher is endowed with ν units of time. He allocates his time optimally between writing papers in different fields. In addition to that, one may create new fields.

When one writes the first paper in a new field, its potential $\omega(i)$ is drawn from some distribution, with cumulative density $f()$, such that all moments exist. The realization of $\omega(i)$ is unknown when one decides to write the paper. At the end of the period when the new field is created, its advancement level is set at the minimal value $\bar{n}$. Therefore, one must wait one period before making further contributions to a new field.

We assume that one unit of time produces either 1 paper in an existing field or γ papers in a new field.

We make two technical assumptions that we need to be able to solve the model:

Assumption A1 – If at date t*, there is a strictly positive measure of new fields invented, then all fields invented before date* t *no longer exist from date* $t+1$ *on.*

This assumption is a useful simplification that avoids having to keep track of all the fields ever invented at any date t. Only the fields invented in the last wave of innovation can be exploited at a given date.[1]

Assumption A2 – $\gamma < 1$.

This assumption states that inventing a new field requires more labor than writing a paper in an existingf field. It is a plausible, but merely technical assumption, required to prove the existence of an equilibrium for $\theta > 0$.[2]

3 Equilibrium

In this section, we show the existence of an equilibrium, and the conditions under which it is cyclical as opposed to converging to a steady state. We provide a result for uniqueness in the case where $\theta = 0$. We provide an economic interpretation of the results, as well as numerical examples in the $\theta = 0$ case. The section is organized as follows: we first discuss the equilibrium conditions of the model in the two regimes of interest. We state the paper's main result, whose proof is relegated to the Appendix. We discuss its economic interpretation using a graphical illustration, confining ourselves to the $\theta = 0$ case. We then work out our numerical example.

3.1 Equilibrium conditions

At any point in time, the economy may be in one of two regimes:

In Regime I, all the research input is allocated to improving existing fields. There exists a shadow value of time λ_t; a field is exploited if and only if the

[1] It is not necessary to make this assumption in the special case where $\theta = 0$. In such a case the value of inventing a new field is $V_N = \gamma\bar{\omega} = \gamma E(\omega)$, which is also the lower bound of the value of working on an existing field, since one could always produce new fields instead. Consequently, when new fields are invented, all previous fields reach their maximum advancement level, such that the value of the marginal paper is equal to V_N; they will not be exploited thereafter.

[2] It is again not needed for $\theta = 0$.

first paper written in the current period has a value greater than λ_t, that is:

$$\omega(i) - \beta(\ln n_{t-1}(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_{t-1}(i)) > \lambda_t.$$

The number of papers written in such a field, at t, must satisfy

$$\omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t.$$

The equilibrium value of λ_t must adjust so that the total mass of papers being written is equal to υ. Call s the last period where invention took place, and μ_s the mass of new fields invented at s. The evolution of one of these fields will clearly be a sole function of its intrinsic value ω. Thus n_t can be written as a sole function of ω rather than the field's specific index i. The preceding conditions are equivalent to

$$\omega > (\beta + \theta)(\ln n_{t-1}(\omega) - \beta \ln \bar{n}) - \theta \ln n_{t+1}(\omega) + \lambda_t, \tag{1}$$

and

$$n_t(\omega) = \bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+1}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega-\lambda_t}{\beta}}.$$

The full employment condition can therefore be written as

$$\mu_s \int_{\omega > (\beta+\theta)\ln n_{t-1}(\omega) - \beta \ln \bar{n}) - \theta \ln n_{t+1}(\omega) + \lambda_t} (\bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+1}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega-\lambda_t}{\beta}} - n_{t-1}(\omega)) f(\omega) d\omega = \upsilon. \tag{2}$$

Finally, the value of writing a new paper, denoted by V_{Nt}, must be lower than that of working on an existing field:

$$V_{Nt} < \lambda_t.$$

In Regime II, people exploit existing fields, and work on new fields as well. They must be indifferent between the two activities, so that one must have $\lambda_t = V_{Nt}$. An existing field is exploited if and only if

$$\omega > (\beta+\theta)\ln n_{t-1}(\omega) - \beta \ln \bar{n}) - \theta \ln n_{t+1}(\omega) + V_{Nt}.$$

Its advancement level then proceeds to

$$n_t(\omega) = \bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+1}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega - V_{Nt}}{\beta}}.$$

Total time devoted to existing fields cannot exceed υ :

$$\mu_s \int_{\omega > (\beta+\theta)\ln n_{t-1}(\omega) - \beta \ln \bar{n}) - \theta \ln n_{t+1}(\omega) + V_{Nt}} (\bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+1}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega - V_{Nt}}{\beta}} - n_{t-1}(\omega)) f(\omega) d\omega \leq \upsilon.$$

The remaining time endowment must be devoted to new fields; this determines the mass of new fields invented at t :

$$\mu_t = \gamma \left[\upsilon - \mu_s \int_{\omega > (\beta+\theta)\ln n_{t-1}(\omega) - \beta \ln \bar{n}) - \theta \ln n_{t+1}(\omega) + V_{Nt}} (\bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+1}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega - V_{Nt}}{\beta}} - n_{t-1}(\omega)) f(\omega) d\omega \right].$$

Because of Assumption (10), the existing fields will disappear at $t+1$ and be replaced by the mass μ_t of new fields, which will start with advancement level $\bar{n}$ at $t+1$.

Finally, in both regimes, the value of working in a new field V_{Nt} is determined as follows: If (1) holds at $t+1$, which is equivalent to

$$\omega > \theta \ln \bar{n} - \theta \ln n_{t+2}(\omega) + \lambda_t,$$

then the field will be active, and the inventor will benefit from citations. The value to the inventor is then given by

$$\begin{aligned} V_t(\omega) &= \omega + \theta(\ln n_{t+1}(\omega) - \bar{n}) \\ &= \omega \end{aligned}$$

$\bar{n}^{\frac{\beta}{\beta+\theta}} n_{t+2}(\omega)^{\frac{\theta}{\beta+\theta}} e^{\frac{\omega-\lambda_t}{\beta}}$

Otherwise, the field will not be active at $t+1$, and the inventor just gets the intrinsic value of the first paper:

$$V_t(\omega) = \omega.$$

Thus, the value of working on a new field at t is given by:

$$V_{Nt} = \gamma \left[\bar{\omega} + \frac{\theta}{\beta} \int_{\omega > \theta \ln \bar{n} - \theta \ln n_{t+2}(\omega) + \lambda_t} \left(\omega - \lambda_t + \frac{\beta\theta}{\theta+\beta} \ln \frac{n_{t+2}(\omega)}{\bar{n}} \right) f(\omega) d\omega \right].$$

3.2 Existence, uniqueness, and cycles

We now state the central results of the paper. To do so, we need to introduce the following two functions:

$$\Phi(y) = \gamma \left[\bar{\omega} + \frac{\theta}{\beta} \int_y^{+\infty} (\omega - y) f(\omega) d\omega \right], \quad (3)$$

and

$$I^*(y) = \int_y^{+\infty} (e^{\frac{\omega-y}{\beta}} - 1) f(\omega) d\omega.$$

Most of the equilibrium conditions involving the value of a new field V_N can be stated in terms of the $\Phi()$ function; for example, it can be easily shown that if the economy is in regime II in two consecutive periods t and $t+1$, then $V_{Nt} = \Phi(V_{Nt+1})$. As for $I^*()$, it shows up whenever one is adding progresses across exploited fields to get the associated total labor input. For example, if $\theta = 0$ and fields are exploited for the first time, then the LHS of (2) is equal to $\mu_s \bar{n} I^*(\lambda_t)$.

Let $\bar{V}_N$ be the fixed point of Φ :

$$\Phi(\bar{V}_N) = \bar{V}_N.$$

The paper's main result can be stated as follows:

PROPOSITION 1 – Assume that the economy starts at $t = 0$ *with an initial mass of fields* μ_{-1}*, whose intrinsic value is distributed with* $f()$*, and whose initial advancement level is given by* $\bar{n}$*. Then:*

(i) There exists an equilibrium path.

(ii) *If*

$$I^*(\bar{V}_N) > \frac{1}{\gamma\bar{n}}, \tag{4}$$

then any equilibrium is cyclical, i.e. periods in regime I alternate with periods in regime II. During periods in regime II, the mass of invented fields follows explosive oscillations, until the economy reverts to regime I. During periods in regime I, the set of exploited fields grows. The duration of a period in regime I cannot exceed $\gamma\bar{n}I^*(\gamma\bar{\omega})$*.*

(iii) If

$$I^*(\bar{V}_N) < \frac{1}{\gamma\bar{n}},$$

then there exists an equilibrium such that

-the economy is in regime II from $t = 0$ *on.*

-the value of working in a new field is equal to $\bar{V}_N$ *at all dates.*

-the mass of invented fields converges to its steady state value, given by

$$\bar{\mu} = \frac{\gamma\nu}{1 + \gamma\bar{n}I^*(\bar{V}_N)},$$

by dampened oscillations.

(iv) If $\theta = 0$*, equilibrium is unique*[3]*.*

PROOF – See Appendix.

[3] We conjecture that the equilibrium is unique for θ small enough, but cannot prove it.

4 Interpretation

To analyze the reason behind cycles, let us focus on the simpler case where $\theta = 0$. In the absence of a citation premium, inventors of new fields just get the intrinsic value of the field, ω, as a reward. Consequently, the value of a new field is pinned down and equal to $V_N = \gamma\bar{\omega}$ in any period.

Figure 1 plots the value of working in an existing field at any date t, λ_t, as a function of the total input in existing fields; that defines the LL schedule. This curve is downward-sloping, because of decreasing returns, captured by the $-\beta(\ln n_t(i) - \ln \bar{n})$ term in the utility function. For the same reason, its position is lower, the higher the initial advancement level of those fields, $n_{t-1}(i)$. Finally, given that level, its position is higher, the greater the mass of available fields μ_s, since the same total research input is now associated with a lower advancement level $n_t(i)$ in each field.

If, as is the case in Figure 1, that schedule intersects the horizontal line VV at $\lambda = V_N$, then the economy is in regime II. The horizontal distance AB determines the labor input into new fields, and hence the mass of fields being invented.

If that is not the case, then the economy must be in regime I, and equilibrium determination is illustrated in Figure 2. At date t, all researchers work in existing fields. Advancement in these fields generate a downward shift in LL, and the intercept of the LL schedule for the next period must be equal to λ_t – which simply means that the value of the first marginal paper at $t+1$ in a given field is equal to the value of the last paper written in that field at t. The process continues until the LL schedule cuts the VV schedule, in which case one is back to regime II (at $t+2$ in the case of Figure 2). This must happen in finite time, otherwise decreasing returns would eventually drive VV below the $\lambda = 0$ line. Note that the λ_ts fall during the regime I period. That is the reason why the set of fields being exploited grows during that phase.[4]

What happens, next, in regime II? At each date, a given mass of fields is invented. The greater that mass, the greater the value of exploiting these fields next period (i.e. LL shifts up). On average, one field invented at date t, with a quality distribution $f(\omega)$, triggers an amount $\bar{n}I^*(V_N)$ of research input devoted to exploiting that field at date $t+1$. That reduces the amount of time devoted to innovation: the greater the mass of fields invented today, the lower the mass of fields invented tomorrow. The evolution of μ_t, the mass of fields invented at t, evolves according to

$$\mu_t = \gamma(\upsilon - \mu_{t-1}\bar{n}I^*(V_N)). \quad (5)$$

If these dynamics are stable ($\bar{n}I^*(V_N) < 1$, Figure 3), then the economy converges to a steady state. Otherwise, ($\bar{n}I^*(V_N) > 1$, Figure 4), the economy cannot remain in regime II forever: it will revert to regime I. As regime I itself cannot last forever, the two regimes must prevail alternatively.

The greater the quantity $I^*(V_N)$, the more invented fields are attractive, and the more likely it is that cycles arise.

4.1 Constructing an equilibrium

Constructing an equilibrium turns out to be easy in the $\theta = 0$ case. Consider Regime I. At any date, there are two kind of fields: those who are exploited, and those who have never been exploited since their invention. The latter are such that $\omega \leq \lambda_t$ have an advancement level equal to $\bar{n}$; the former, such that $\omega > \lambda_t$, reach an advancement level equal to $n_t(\omega) = \bar{n}e^{\frac{\omega - \lambda_t}{\beta}}$. At $t+1$, all these fields are exploited, plus a mass of extra fields such that $\lambda_t > \omega > \lambda_{t+1}$.

[4] Because of assumption A1, a field exploited during that phase must have been invented in the last period in regime II before the regime I phase. It enters regime I with an initial advancement level equal to $\bar{n}$. Using (1) with $\theta = 0$, it will therefore be exploited as soon as $\omega > \lambda_t$. Because the λs are falling, it will continue to be exploited until the economy reverts to regime II, when it becomes obsolete.

The research input into existing fields at date $t+1$ is therefore equal to

$$\begin{aligned} \upsilon_{At+1} &= \mu_s \int_{\omega>\lambda_{t+1}} (n_{t+1}(\omega) - n_t(\omega)) f(\omega) d\omega \\ &= \mu_s \int_{\lambda_t>\omega>\lambda_{t+1}} (\bar{n} e^{\frac{\omega-\lambda_t}{\beta}} - \bar{n}) f(\omega) d\omega + \mu_s \int_{\omega>\lambda_t} (\bar{n} e^{\frac{\omega-\lambda_{t+1}}{\beta}} - \bar{n} e^{\frac{\omega-\lambda_t}{\beta}}) f(\omega) d\omega \\ &= \bar{n}\mu_s (I^*(\lambda_{t+1}) - I^*(\lambda_t)). \end{aligned}$$

In the first period of regime I, $t = s+1$, no field has been exploited and the equation boils down to

$$\upsilon_{As+1} = \bar{n}\mu_s I^*(\lambda_{s+1}).$$

Regime I proceeds as long as $\upsilon_{At} = \upsilon$. In the first regime II period, T, after regime I, one must have $\lambda_T = V_N = \gamma\bar{\omega}$, and $0 \leq \upsilon_{AT} < \upsilon$. Therefore, it must be that

(i) $\lambda_t = I^{*-1}((t-s)\frac{\nu}{\bar{n}\mu_s})$.

(ii) The duration of the regime I phase is entirely pinned down by the condition $I^{*-1}((T-s)\frac{\nu}{\bar{n}\mu_s}) \leq V_N < I^{*-1}((T-1-s)\frac{\nu}{\bar{n}\mu_s})$, i.e.

$$T - s = INT(I^*(V_N)\frac{\bar{n}\mu_s}{\nu}) + 1 \tag{6}$$

. Note that one may have $T = s+1$, in which case there is no regime I phase. This happens if and only if

$$I^{*-1}(\frac{\nu}{\bar{n}\mu_s}) \leq V_N. \tag{7}$$

(iii) The economy enters regime II with an initial measure of new fields given

$$\begin{aligned} \mu_T &= \gamma(\upsilon - \upsilon_{At}) \\ &= \gamma(\upsilon - \bar{n}\mu_s(I^*(V_N) - I^*(\lambda_{T-1}))) \\ &= \gamma\upsilon(1 - DEC(I^*(V_N)\frac{\bar{n}\mu_s}{\upsilon})). \end{aligned} \tag{8}$$

Therefore, given an initial measure of invented fields at s, I can construct a unique, possibly empty, phase in regime I, and compute its length and the initial measure of invented fields in the first period T of the subsequent regime II phase.

The regime II phase is then constructed by mechanically applying (5). The economy asymptotically converges to the steady state if $\bar{n}I^*(V_N) < 1$; otherwise, it remains in regime II until date s' such that

$$\mu_{s'}\bar{n}I^*(V_N) > \upsilon \tag{9}$$

This gives the measure of invented fields available for the next regime I phase. One can then iterate the procedure. Note that s' is determined uniquely: on the one hand, the economy cannot remain in regime II after the first date which satisfies (9), as $\mu_{s'+1}$ would then be negative. On the other hand, if one picks up a date prior to s' for the transition to regime I, it would satisfy (7), thus leading to an empty regime I phase, which would not change the equilibrium.

It should be noted that equation (8), which related the mass of invented fields after a phase in regime I to the mass of invented fields before that phase, is discontinuous and highly non linear. Thus we expect cycles to be irregular: the characteristics of a cycle, such as the time spent in each regime and the mass of invented fields, are almost "unpredictable" from the characteristics of the preceding cycle.[5]

[5] A similar nonlinearity applies to the mass of invented fields at the end of phase II, and the duration of phase II, as a function of the initial mass of invented fields μ_T. Solving for (5) yields

$$\mu_{t+1} = \frac{\gamma\nu}{1+\gamma\bar{n}I^*} + \left(\mu_T - \frac{\gamma\nu}{1+\gamma\bar{n}I^*}\right)(-\gamma\bar{n}I^*)^{(t-T)}.$$

One has to distinguish between two cases.
If $\mu_T > \frac{\gamma\nu}{1+\gamma\bar{n}I^*}$, then $s' - T$ must be even, and one must have

4.2 Numerical illustration

In this section we provide some simulations in order to get a better idea of the irregular nature of the innovation cycles. We assume that the quality of a field ω is drawn from a uniform distribution over $[0, \omega_u]$, implying $\bar{\omega} = \omega_u/2$.

Figures 5 to 10 report the simulation results for the following set of parameters: $\bar{n} = 2$; $\omega = 1; \beta = 0.3$; $\gamma = 0.7; \nu = 1$. The initial measure of existing fields was taken as $\mu_c = 1$.

It is easy to show that (4) holds in this case, so that the equilibrium must be cyclical. The simulation shows that the economy follows cycles that are irregular, both in the duration spent in regime I and the duration spent in regime II. The time spent in regime I oscillates between 1 and 2 periods (Fig. 5), while time spent in regime II oscillates between 1 and up to 6 periods (Fig. 8)[6]. there are also chaotic oscillations in the stock of new fields available for exploration at the beginning of each regime I phase (Figure 6). Furthermore, as (6) predicts, there is a tight connection between that initial stock and the length of the time spent in period 1 (Fig. 7); the regime I cycle lasts for 2 periods if the initial stock of knowledge is $>\approx 0.6$, and for 1 period otherwise.

Figure 9 reports the average rate of innovation during the time spent in

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$$s' = T + 2INT\left(\frac{\ln \frac{\gamma\nu}{\mu_{T(\mu_c)+1}(1+\gamma n I^*)-\gamma\nu}}{2\ln(\gamma\bar{n}I^*)} + \frac{1}{2}\right).$$

If $\mu_T < \frac{\gamma\nu}{1+\gamma\bar{n}I^*}$, then

$$s' = 2INT\left(\frac{\ln \frac{\gamma\nu}{\mu_{T(\mu_c)+1}(1+\gamma n I^*)-\gamma\nu}}{2\ln(\gamma\bar{n}I^*)}\right) + 1.$$

As the initial measure μ_T changes, the duration of phase II will jump, and so will $\mu_{s'}$. Therefore, we also have discontinuities in the function which related $\mu_{s'}$ to μ_T.

[6] These figures report the 70 first cycles after the initial one.

regime II. We see that it exhibits irregular fluctuations. We also see (Figure 10), that cycles where a longer time is spent in regime II, have a lower rate of innovation. Intuitively, if a large number of researchers produce new fields, it is more likely that the economy reverts to regime I in the following period in order to exploit the potential of these new fields[7].

Relative to that benchmark simulation, we can perform some exercices. Figures 11 and 12 report the structure of cycles when we reduce the decreasing returns parameter from $\beta = 0.3$ to $\beta = 0.2$.[8] We see that overall, the economy spends more time in regime I and less time in regime II. In a cycle, regime I last between 1 and 5 periods, although that is quite often just 1 period, and regime II typically does not exceed 2 periods, although there are very rare occurences of cycles where the economy spends 3 periods in regime II.

In fact, while there is a maximum duration for the regime I phase, if the dynamics are truly chaotic one will have (very rare) regime II phases of arbitrary length. The reason is that the initial values of μ will span all the $[0, \gamma v]$ interval, becoming sometimes arbitrarily close to the unstable steady state value $\bar{\mu}$.

We are now in a position to analyze how the parameters of interest affect the equilibrium. We do it in two steps: first, we look at the case where (4) does not hold, and perform local comparative statics around the steady state. Second, we consider how the structure of cycles is affected by the model parameters when (4) holds.

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[7] Another interesting property of that simulation, is that cycles where regime I lasts for two periods, are such that the economy only spends 1 period in regime II. The explanation could be as follows: at the end of such cycles, fields are quite exhausted, and the value of working in new fields in regime II is quite high. Thus a large mass of innovation will take place during a short period of time, after which people revert to exploiting the new fields. However, this explanation is incomplete, since longer cycles are also those with a higher total initial potential. And that regularity is not robust to parameter changes.

[8] Given the richer results, simulation are reported over 140 cycles rather than 70.

5 Comparative statics

In this section, we perform local comparative statics around a regime II steady state. In particular, we are interested in how the equilibrium level of invention, $\bar{\mu}$, is affected by

- The citation premium θ,
- The strength of decreasing returns β, and
- The distribution of field quality $f(\omega)$; in particular: how does the riskiness of invention, measured by the variance of $f()$, affect the equilibrium allocation of effort between innovation and exploitation?

5.1 The effect of the citation premium

Equation (3) clearly implies that $\bar{V}_N$ is an increasing function of θ. Furthermore, one can straightforwardly check that $dI^*/d\bar{V}_N < 0$. Hence, $d\bar{\mu}/d\bar{V}_N > 0$. Consequently,

PROPOSITION 2 – More research input is devoted to new fields, the higher the citation premium θ.

This result is not totally obvious. In principle, the citation premium increases my incentives to work both in new fields and in existing fields. However, in this equilibrium, existing fields are only exploited during one period; thus one earns no citation premium on them. An increase in θ thus clearly increases the value of working on new fields.

5.2 The role of research uncertainty

Next, we look at the role of uncertainty in research; we want to know how the variance of ω – or any mean-preserving spread parameter denoted by σ – affects the arbitrage between working in new fields vs. existing fields. As we shall seen, option values intervene in two conflicting ways.

We first note that $I^*(V_N)$ can be written as $E(z(\omega))$, where $z(\omega) = \max(e^{\frac{\omega(i)-V_N}{\beta}} - 1, 0)$ is a convex function of $\omega(i)$. By Jensen's inequality, a mean-preserving spread in the distribution of ω raises $I^*(V_N)$ for any given V_N. If V_N were to remain unchanged, or move by only a little, μ would actually fall: more research uncertainty reduces the incentives to work in new fields.

If $\theta = 0$, it is actually true that V_N does not change in response to a mean-preserving change in the distribution of ω, for it is equal to $\gamma\bar{\omega}$. Similarly, eq. (**??**) implies that for θ arbitrarily small the change in V_N can be made arbitrarily small. Therefore:

PROPOSITION 3 – For θ small enough a mean-preserving spread in the distribution of ω reduces μ_∞.

$$\frac{\partial \mu_\infty}{\partial \sigma} < 0.$$

Uncertainty increases the value of exiting fields because one can select those of them with the highest potential. A greater variance of ω means that it is more valuable to work in the top field, while the value of working in the bottom fields is unchanged, because these fields are abandoned anyway. In contrast, the value of writing the first paper in an unknown field is increased is the field turns out to be good, but reduced if it turns out to be bad – if θ is small, then that value will roughly equal $\omega(i)$, regardless of the fate of the field after its invention. Hence greater uncertainty increases the value to work in known fields relative to unknown, new fields.

Against that logic, runs the fact that uncertainty increases the value of new fields, because of the citation premium. That is apparent from (**??**): a mean-preserving spread in ω increases its RHS. The option value of working in an existing field only if it is good enough also affects the value of working in new fields through the citation premium. When uncertainty goes up, I gain from my good ideas being cited more, but I do not lose from my bad, uncited ideas, being cited less. In other words, the higher the citation premium, the less risk-averse the researchers.

PROPOSITION 4 – A mean preserving spread in ω *increases* $\bar{V}_N$:

$$\frac{\partial \bar{V}_N}{\partial \sigma} > 0.$$

The effect is larger, the larger the citation premium:

$$\frac{\partial^2 \bar{V}_N}{\partial \sigma \partial \theta} > 0.$$

Consequently, the larger the citation premium, the lower the negative effect of uncertainty on the research input into new fields:

$$\frac{\partial^2 \mu_\infty}{\partial \theta \partial \sigma} > 0.$$

An interesting question is: can the reduction in risk aversion induced by the citation premium be so strong as to overturn the direct effect of uncertainty, so that one would have $\frac{\partial \mu_\infty}{\partial \sigma} > 0$?

5.3 The role of decreasing returns

6 Comparative dynamics

PROPOSITION 5 – (i) The equilibrium is less likely to be cyclical, the greater θ

(ii) Conditional on the initial mass of invented fields, the economy spends less time in the regime I phase for $\theta > 0$ *than for* $\theta = 0$*. Furthermore, if the amount of time spent in regime I is the same, then more invention takes place at the beginning of the subsequent regime II phase, if* $\theta > 0$.

7 Indeterminacy and "sunspots"

The greater θ, the more expectations about future citations have a strong effect on the decision to work on a given field. By analogy with the literature on indeterminacy, we can speculate that there are multiple equilibria for θ large enough. That is actually the case. The following result shows that there is local indeterminacy around the regime II steady state for large enough values of θ.

PROPOSITION 6 – Assume

$$\frac{\gamma\theta}{\beta}(1 - F(\bar{V}_N)) > 1$$

and

$$I^*(\bar{V}_N) < \frac{1}{\gamma\bar{n}}.$$

Then there exists a continuum of equilibria indexed by any initial value $V_{N0} = \bar{V}_N + v_t$*, for* v_t *sufficiently small.*

PROOF – See Appendix.

8 Conclusion

9 Appendix:

9.1 Proof of proposition 1

λ_t is the shadow cost of a paper at t. Clearly one must have $\lambda_t > \gamma\bar{\omega}$, which is a lower bound for V_{Nt}, the value of inventing a new field.

At date t, a field i is exploited if and only if

$$\omega(i) - \beta(\ln n_{t-1}(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_{t-1}(i)) < \lambda_t,$$

in which case n_t is determined by

$$\omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t.$$

A. One cannot forever remain in regime I

As if the $\theta = 0$ case, we first prove that one cannot stay forever in regime I. We do so by contradiction.

Assume the economy is always in regime I from t on. Call μ the total mass of existing fields, indexed by i, and $dF(i)$ their density. Assume a field is exploited at t. Then

$\omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t > \gamma\bar{\omega}$; therefore,

$$\ln n_{t+1}(i) \geq \frac{\beta + \theta}{\theta} \ln n_t(i) + \frac{\gamma\bar{\omega}}{\theta} - \frac{\omega(i)}{\theta} - \frac{\beta \ln \bar{n}}{\theta}.$$

A sufficient condition for $n_{t+1}(i) > n_t(i)$ is therefore

$$n_t(i) > \bar{n}e^{\frac{\omega(i) - \gamma\bar{\omega}}{\beta}} = k(i) \tag{10}$$

. Hence, if a field is exploited at t and satisfies (10), then it will be exploited at $t + 1$. Since $n_{t+1}(i)$ in turn also satisfies (10), one has $n_{t+2}(i) > n_{t+1}(i)$, and so on. Therefore:

If a field satisfies (10) and is exploited at t, i.e. $n_t(i) > n_{t-1}(i)$, then it is exploited forever.

Assume that at date t there is a set Γ with strictly positive measure of active fields such that (10) holds. Then, for each of these fields, $n_{t+s+1}(i) = n_{t+s}(i)\left(\frac{n_{t+s}(i)}{k(i)}\right)^{\frac{\beta}{\theta}}$. The quantity $n_{t+s+1}(i) - n_{t+s}(i)$ is growing without bounds, which contradicts the requirement that $\mu \int_\Gamma (n_{t+s+1}(i) - n_{t+s}(i))dF(i) \leq \nu$. Consequently, such a set cannot exist. Let then Γ_t be the set of all active fields at t. It must be that (10) is violated almost everywhere over Γ_t. Let $\Psi_t(i) = \max(k(i) - n_t(i), 0) \geq 0$. We have that

$$\int_{\Gamma_t} n_t(i)dF(i) = \int_{\Gamma_t} n_{t-1}(i)dF(i) + \frac{\nu}{\mu}$$

Furthermore, as $n_{t-1}(i) < n_t(i) < k(i)$ almost everywhere:

$$\begin{aligned}
\int_{\Gamma_t} \Psi_t(i)dF(i) &= \int_{\Gamma_t} (k(i) - n_t(i))dF(i) \\
&= \int_{\Gamma_t} (k(i) - n_{t-1}(i))dF(i) - \frac{\nu}{\mu} \\
&= \int_{\Gamma_t} \Psi_{t-1}(i)dF(i) - \frac{\nu}{\mu}.
\end{aligned}$$

On the other hand, $n_t(i) = n_{t-1}(i)$ for $i \notin \Gamma_t$, implying $\Psi_t(i) = \Psi_{t-1}(i)$. Therefore:

$$\int_{\Omega} \Psi_t(i)dF(i) = \int_{\Omega} \Psi_{t-1}(i)dF(i) - \frac{\nu}{\mu}.$$

We have constructed a sequence of positive functions whose integral eventually becomes negative, which is a contradiction.

B. Characterizing dynamics in regime I

Let then T be the date when regime I ends: at date T one is in regime II. Let V_{NT} be the value of working in a new field at T. By assumption, all fields invented prior to T are obsolete after $T+1$. Hence T must be the last period when fields active during the regime I phase are exploited. An existing field is active at T iff

$$\omega(i) - \beta(\ln n_{T-1} - \ln \bar{n}) > V_{NT},$$

in which case n_T is determined by

$$\omega(i) - \beta(\ln n_T - \ln \bar{n}) = V_{NT}.$$

Consider now a date $t < T$ in the regime I phase. Denoting by λ_t the shadow cost of a paper, a field is active iff

$$\omega(i) - \beta(\ln n_{t-1}(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_{t-1}) > \lambda_t$$

In which case

$$\omega(i) - \beta(\ln n_t(i) - \ln \bar{n}) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t.$$

We now construct a sequence $\hat{\lambda}_t$ such that the following property holds:

PROPERTY P1 – A field is active iff $\omega(i) - \beta(\ln n_{t-1} - \ln \bar{n}) > \hat{\lambda}_t$, in which case $\omega(i) - \beta(\ln n_t - \ln \bar{n})) = \hat{\lambda}_t$.

The sequence is constructed by backward induction, starting from $t = T$. We clearly can pick $\hat{\lambda}_T = V_{NT}$. Now, assume P1 holds for $t' > t$.

Assume $\lambda_t > \hat{\lambda}_{t+1}$. Then all active fields at t must also be active at $t+1$. To prove so, suppose there is a field i active at t and inactive at $t+1$. Then it must be that $n_{t+1}(i) = n_t(i)$. So that $\omega(i) - \beta(\ln n_t - \ln \bar{n})) \leq \hat{\lambda}_{t+1}$ and $\omega(i) - \beta(\ln n_t - \ln \bar{n})) + \theta(\ln n_{t+1}(i) - \ln n_t(i)) = \lambda_t = \omega(i) - \beta(\ln n_t - \ln \bar{n}))$, which violates the assumption that $\lambda_t > \hat{\lambda}_{t+1}$.

Then, a field is active at t if and only if

$$\omega(i) - \beta(\ln n_{t-1} - \ln \bar{n}) + \theta(\ln \bar{n} + \frac{\omega(i) - \hat{\lambda}_{t+1}}{\beta} - \ln n_{t-1}) > \lambda_t$$

If we define $\hat{\lambda}_t = \frac{\theta \hat{\lambda}_{t+1} + \beta \lambda_t}{\theta + \lambda}$, we get that this equation is equivalent to $\omega(i) - \beta(\ln n_{t-1} - \ln \bar{n}) > \hat{\lambda}_t$, and we can check that we then have $\omega(i) - \beta(\ln n_t - \ln \bar{n})) = \hat{\lambda}_t$.

Assume $\lambda_t \leq \hat{\lambda}_{t+1}$. Consider a field active at both t and $t+1$. Then we must have $\omega(i) - \beta(\ln n_t(i) - \ln \bar{n})) < \omega(i) - \beta(\ln n_t - \ln \bar{n})) + \theta(\ln n_{t+1}(i) - \ln n_t(i))$ $= \lambda_t \leq \hat{\lambda}_{t+1} = \omega(i) - \beta(\ln n_{t+1}(i) - \ln \bar{n}))$, implying that $n_{t+1}(i) < n_t(i)$, which cannot be. Therefore, all fields active at t must be inactive at $t+1$, in which case we just pick up $\hat{\lambda}_t = \lambda_t$.

To summarize, the $\hat{\lambda}_t$ sequence can be constructed as

$$\begin{aligned} \hat{\lambda}_T &= V_{NT}; \\ \hat{\lambda}_t &= \min(\frac{\theta \hat{\lambda}_{t+1} + \beta \lambda_t}{\theta + \beta}, \lambda_t). \end{aligned} \tag{11}$$

Let T_0 be the initial period of that phase in regime I. Denoting by $\mu_{T_0 - 1}$ the measure of exploitable fields, we can now get the evolution of the $\hat{\lambda}_t$s.

We have that

$$\mu_{T_0-1} \int_{\omega(i) > \hat{\lambda}_{T_0}} (\bar{n} e^{\frac{\omega(i) - \hat{\lambda}_{T_0}}{\beta}} - \bar{n}) = \nu,$$

or equivalently

$$\mu_{T_0-1} \bar{n} I^*(\hat{\lambda}_{T_0}) = \nu. \tag{12}$$

This equation allows to compute $\hat{\lambda}_{T_0}$ as a function of μ_{T_0-1}. At date T_0+1, there are two kinds of fields: those which were exploited at T_0, whose value of $n_{T_0}(i)$ satisfies (P1), and those which were not, such that $n_{T_0}(i) = \bar{n}$. If $\hat{\lambda}_{T_0+1} \geq \hat{\lambda}_{T_0}$, no field can be exploited at $t+1$, which is not possible. Therefore it must be that $\hat{\lambda}_{T_0+1} < \hat{\lambda}_{T_0}$. One can then compute $\hat{\lambda}_{T_0+1}$ as

$$\mu_{T_0-1} \int_{\omega(i)>\hat{\lambda}_{T_0}} (\bar{n}e^{\frac{\omega(i)-\hat{\lambda}_{T_0+1}}{\beta}} - \bar{n}e^{\frac{\omega(i)-\hat{\lambda}_{T_0}}{\beta}})dF(i) + \mu_{T_0-1} \int_{\hat{\lambda}_{T_0}>\omega(i)>\hat{\lambda}_{T_0+1}} (\bar{n}e^{\frac{\omega(i)-\hat{\lambda}_{T_0+1}}{\beta}} - \bar{n})dF(i) = \nu$$

or equivalently

$$\mu_{T_0-1}\bar{n}(I^*(\hat{\lambda}_{T_0+1}) - I^*(\hat{\lambda}_{T_0})) = \nu. \tag{13}$$

Similarly, assuming $\hat{\lambda}$ is falling between T_0 and t, at the beginning of $t+1$, fields can be split between those which were never exploited, so that $n_t(i) = \bar{n}$ and those which were exploited at t, such that $n_t(i) = \bar{n}e^{\frac{\omega(i)-\hat{\lambda}_t}{\beta}}$. Again, it must be that $\hat{\lambda}_{t+1} < \hat{\lambda}_t$, so that the $\hat{\lambda}$s must fall by induction. And they must again satisfy

$$\mu_{T_0-1}\bar{n}(I^*(\hat{\lambda}_{t+1}) - I^*(\hat{\lambda}_t)) = \nu. \tag{14}$$

Given that $\hat{\lambda}_{t+1} < \hat{\lambda}_t$, it must be that $\lambda_t > \hat{\lambda}_{t+1}$, so that active fields at t remain so until the end of regime I.

In regime I, the value of working on a new field must not exceed the value of working in existing fields. Consider a field invented at t. As this field would be infinitesimal, it would not make other fields obsolete at $t+1$. Its value at t is

$$\begin{aligned} W(i) &= \omega(i) + \theta(\ln n_{t+1}(i) - \ln \bar{n}) \\ &= \omega(i) + \frac{\theta}{\beta}(\omega(i) - \hat{\lambda}_{t+1}), \text{ if } \omega(i) > \hat{\lambda}_{t+1}, \end{aligned}$$

and

$$W(i) = \omega(i)$$

if not. Therefore, the value of working on a new field at t is equal to

$$\begin{aligned} V_{Nt} &= \gamma\left[\bar{\omega} + \frac{\theta}{\beta}\int_{\omega(i)>\hat{\lambda}_{t+1}} (\omega(i) - \hat{\lambda}_{t+1})\right] \\ &= \Phi(\hat{\lambda}_{t+1}). \end{aligned}$$

Hence for the economy to be in regime I, the following must hold:

$$\Phi(\hat{\lambda}_{t+1}) < \lambda_t. \tag{15}$$

C. Characterizing dynamics in regime II.

We now characterize the dynamics in regime II. Because a positive measure of new fields is invented at every period, fields invented at t are at most only exploited at $t+1$. Such a field i will be exploited at $t+1$ if and only if

$$\omega(i) > V_{Nt+1},$$

in which case

$$\ln n_{t+1} = \ln \bar{n} + \frac{\omega(i) - V_{Nt+1}}{\beta}.$$

Consequently, the value of a new field at t is

$$W(i) = \omega(i) + \frac{\theta}{\beta}(\omega(i) - V_{Nt+1}), \text{ if } \omega(i) > \hat{\lambda}_{t+1},$$

and

$$W(i) = \omega(i)$$

if not. Integrating, we get that

$$\begin{aligned} V_{Nt} &= \gamma\left[\bar{\omega} + \frac{\theta}{\beta}\int_{\omega(i)>\hat{\lambda}_{t+1}} (\omega(i) - V_{Nt+1})\right] \\ &= \Phi(V_{Nt+1}). \end{aligned} \tag{16}$$

This defines the dynamics of V_{Nt} in regime II. Denoting now by μ_t the measure of fields invented at t, the input into working in existing fields at $t+1$ is

$$\begin{aligned} \nu_{At+1} &= \mu_t \int_{\omega(i)>V_{Nt}} (\bar{n}e^{\frac{\omega(i)-V_{Nt+1}}{\beta}} - \bar{n}) \\ &= \bar{n}I^*(V_{Nt+1})\mu_t. \end{aligned}$$

Therefore the dynamics of μ are given by

$$\mu_{t+1} = \gamma(\nu - \bar{n}I^*(V_{Nt+1})\mu_t). \tag{17}$$

To remain in regime II this formula must yield a positive value of μ_{t+1} throughout.

D. The transition from regime I to regime II

Consider now the first period in regime II, T. We focus on the case where $V_{NT} < \hat{\lambda}_{T-1}$. The other possibility will be ruled out further below. That inequality is equivalent to

$$I^*(\hat{\lambda}_{T-1}) < I^*(V_{NT}) \tag{18}$$

Then, fields active during regime I are still exploited at T. The total input into active fields at T is given by

$$\mu_{T_0-1}\bar{n}(I^*(V_{NT}) - I^*(\hat{\lambda}_{T-1})) = \nu_{AT}.$$

For this to be consistent with equilibrium, we need that $\nu_{At} < \nu$, that is:

$$I^*(\hat{\lambda}_{T-1}) > I^*(V_{NT}) - \frac{\nu}{\mu_{T_0-1}\bar{n}}. \tag{19}$$

Because of (13), for a given V_{NT} and a given μ_{T_0-1}, there exists at most a unique value of T such that (19) and (18) simultaneously hold. Therefore, the duration (and characteristics) of the regime I phase are entirely pinned down by the initial measure of exploitable fields μ_{T_0-1} and the terminal value V_{NT}. The phase lasts at least one period if and only if $I^*(\hat{\lambda}_{T_0}) < I^*(V_{NT})$, or equivalently

$$I^*(V_{NT}) > \frac{\nu}{\mu_{T_0-1}\bar{n}}. \tag{20}$$

Otherwise, there cannot be a phase in regime I (we are in the special case where $T = T_0$.)

If (20) holds, then the initial measure of invented fields at the beginning of regime II is

$$\mu_T = \gamma(\nu - \mu_{T_0-1}\bar{n}(I^*(V_{NT}) - I^*(\hat{\lambda}_{T-1}))). \tag{21}$$

The economy then evolves as described above.

E. The transition from regime II to regime I

Next, consider the value of inventing a new field at date $T_0 - 1$. It is given by

$$V_{NT_0-1} = \Phi(\hat{\lambda}_{T_0}) \tag{22}$$

This defines a negative relationship between $\hat{\lambda}_{T_0}$ and V_{NT_0-1}. At the same time, (12) defines $\hat{\lambda}_{T_0}$ uniquely as an increasing function of μ_{T_0-1}. Thus, there must be a one-to-one, decreasing relationship between μ_{T_0-1} and V_{NT_0-1} :

$$V_{NT_0-1} = \Phi(I^{*-1}\left(\frac{\nu}{\mu_{T_0-1}\bar{n}}\right)). \tag{23}$$

F. Ruling out the case $V_{NT} > \hat{\lambda}_{T-1}$.

The above argument about the decreasing sequence $\hat{\lambda}_t$ does not apply to its terminal value V_{NT}, since it rests on the argument that all labor goes into existing fields. Let us now examine the case $V_{NT} > \hat{\lambda}_{T-1}$. If this holds, no existing field is exploited at T. Thus, it must be that $\mu_T = \gamma\nu$. Because of (11), it must also be that $\lambda_{T-1} = \hat{\lambda}_{T-1} > V_{NT-1} = \Phi(V_{NT})$. Therefore, $V_{NT} > \Phi(V_{NT})$, implying that $V_{NT} > \bar{V}_N$, where $\bar{V}_N$ is the steady state value V_N in regime II; $\bar{V}_N = \Phi(\bar{V}_N)$.

Assume the economy is still in regime II at $T+1$. Then, (17) implies that

$$\mu_{T+1} = \gamma(\nu - \bar{n}I^*(V_{NT+1})\gamma\nu).$$

Since $V_{NT+1} = \Phi^{-1}(V_{NT}) < \bar{V}_N$, $I^*(V_{NT+1}) > I^*(\bar{V}_N)$. A sufficient condition for the RHS of this equation to be negative is thus

$$\bar{n}I^*(\bar{V}_N)\gamma > 1 \tag{24}$$

When constructing a cyclical equilibrium, we will assume that (24) holds. In this case, the economy cannot be be in regime II at $T+1$.

Assume the economy is in regime I at $T+1$, and that (24). Then T is the last period in regime II before regime I. V_{NT} must therefore satisfy (23) for $\mu_T = \gamma\nu$

$$V_{NT} = \Phi(I^{*-1}\left(\frac{1}{\gamma\bar{n}}\right)).$$

Using (24) again, we see that $\bar{V}_N < I^{*-1}(\frac{1}{\gamma\bar{n}})$, implying $\bar{V}_N = \Phi(\bar{V}_N) > \Phi(I^{*-1}(\frac{1}{\gamma\bar{n}})) = V_{NT}$, which contradicts the observation that $V_{NT} > \bar{V}_N$. Under assumption (24), it can therefore never be that $V_{NT} > \hat{\lambda}_{T-1}$.

G. Constructing an equilibrium

Let t be a period and s the last period in regime II before t. Given V_{Nt} and the number of fields invented in the previous regime II episode, μ_s, we can compute exactly whether or not there will be a period in regime I between s and t.

We know that if (20) holds, i.e. if $V_{Nt} < I^{*-1}(\frac{\nu}{\bar{n}\mu_s})$, there is a regime-I episode. Its duration must satisfy (19) and (18), and the new mass of fields invented is given by (21). Finally, the $\hat{\lambda}$ sequence must satisfy (14). Putting these things together, we see that the duration of the cycle must be equal to $INT(\frac{\mu_s \bar{n}}{\nu} I^*(V_{Nt}))$ and that the new value of μ_t must be equal to

$$\mu_t = \gamma\nu(1 - DEC\left(\frac{\mu_s \bar{n}}{\nu} I^*(V_{Nt})\right)) = m(\mu_s, V_{Nt}). \tag{25}$$

which is the equivalent of (**??**), and the I^* function is the equivalent of the $h(.)$ function in that example.

If (20) is violated, i.e. if $V_{Nt} > I^{*-1}(\frac{\nu}{\bar{n}\mu_s})$, there is no regime I period between s and t. One must then have $t = s + 1$. μ_t is computed using regime II dynamics, i.e. (17), which, given that $\frac{\mu_s \bar{n}}{\nu} I^*(V_{Nt}) < 1$, is equivalent to (25).

Therefore, it must be that

$$\mu_t = m(\mu_s, V_{Nt}).$$

The discontinuity points of $m(\mu_s, .)$ are given by $I^*(V_N) = \frac{kv}{\mu_s \bar{n}}$, i.e. they are precisely equal to the successive values of $\hat{\lambda}_t$ during the regime I phase. The value of k corresponding to the discontinuity point immediately above V_{Nt} must be equal to the duration of the regime I phase. As $V_{Nt} \geq \gamma\bar{\omega}$, this duration cannot exceed $\mu_s \bar{n} I^*(\gamma\bar{\omega})/v \leq \gamma\bar{n} I^*(\gamma\bar{\omega})$.

To continue the construction of the equilibrium, we must pick the value of V_{Nt} such that in the last period in regime II, the equilibrium condition for the transition from regime II to regime I, (23), holds. We prove that such a V_{Nt} exists as follows.

For $u = t, ..., T$, an *admissible* sequence of pairs $\{(x_u, y_u), u = t, ..., T\}$ is a sequence of real numbers such that

$$\begin{aligned} x_u &= \gamma(\nu - \bar{n}x_{u-1}I^*(y_u)), u > t \\ y_u &= \Phi(y_{u+1}), t \leq u < T; \\ y_T &\in (\gamma\bar{\omega}, (\gamma + \frac{\theta}{\beta})\bar{\omega}) \end{aligned} \tag{26}$$

Note that, a priori, we do not rule out negative values for x_u.

An admissible sequence is *feasible* iff

$$x_u \geq 0, \forall u; \tag{27}$$

$$x_t = m(\mu_s, y_t). \tag{28}$$

If we find a feasible sequence such that

$$y_T = \Phi(I^{*-1}\left(\frac{\nu}{x_T\bar{n}}\right)), \tag{29}$$

then we can construct a phase in regime II during $T-t$ periods starting from T, such that the transitional condition (23) holds, by choosing $\mu_t = x_t$ and $V_{Nt} = y_t$.

A feasible sequence is *maximal* iff

$$\bar{n}x_T I^*(\Phi^{-1}(y_T)) > \nu. \tag{30}$$

That condition implies that there cannot be another feasible sequence $\{(x'_u, y'_u), u = t, ..., T'\}$ such that $T' > T$ and $(x'_u, y'_u) = (x_u, y_u)$ for all $u \leq T$, because the implied value of x_{T+1} would be negative.

PROPERTY P2 – Any feasible sequence is such that

$$\bar{n}x_u I^*(\Phi^{-1}(y_u)) \leq \nu, \text{ for all } u < T.$$

PROOF – For $u < T$ one must have $0 \leq x_{u+1} = \gamma(\nu - \bar{n}x_u I^*(y_{u+1})) = \gamma(\nu - \bar{n}x_u I^*(\Phi^{-1}(y_u)))$. QED.

Let us now denote by $K = \{I^{*-1}(\frac{k\upsilon}{\mu_s \bar{n}}),\ k = 1, ...\}$ the set of discontinuity points of $m()$. Then for any admissible sequence such that $y_t \notin K$, x_u and y_u are locally C^1 functions of y_t. Furthermore, as y_t varies, the following properties hold:

PROPERTY P3 – Assume $y_t \notin K$. Then:
(i) $\frac{dy_u}{dy_t} > 0$ if $u - t$ is even, < 0 if $u - t$ is odd.
(ii) $\frac{dy_u}{dy_t}\frac{dx_u}{dy_t} > 0$.

PROOF – Property (i) derives trivially from the fact that $y_u = \Phi(y_{u+1})$. Property (ii) can be proved by induction. It is clearly true for $u = t$, as $\partial m / \partial V_N > 0$. Assume it holds for $u - 1$. Differentiating (26), we get

$$dx_u = -dx_{u-1}\frac{\bar{n}\gamma}{\nu}I^*(y_u) - \frac{\bar{n}\gamma}{\nu}x_{u-1}I^{*'}(y_u)dy_u.$$

By assumption, $sign(dx_{u-1}) = sign(dy_{u-1}) = -sign(dy_u)$. Consequently, the first term has the same sign as dy_u, and so does the second, as $I^{*'}() < 0$. QED.

Thus, as the initial value y_t varies, subsequent contemporaneous values of x and y move in the same direction.

PROPERTY P4 – Assume that there exist two feasible sequences $\{(x_{0u}, y_{0u}), u = t, ..., T\}$, and $\{(x_{1u}, y_{1u}), u = t, ..., T\}$ such that for some integer k :

$$I^{*-1}(\frac{k\upsilon}{\mu_s \bar{n}}) < y_{0t} < y_{1t} < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s \bar{n}}).$$

Then there exists a family of mappings $X_u(y)$ (resp. $Y_u(y)$) from $[y_{0t}, y_{1t}]$ to $[x_{0u}, x_{1u}]$ (resp. $[y_{0u}, y_{1u}]$) such that

(i) $X_u(y)$ and $Y_u(y)$ are continuously differentiable in y

(ii) $\{(X_u(y), Y_u(y), u = t, ..., T\}$ is feasible;

(iii) $X_t(y) = m(\mu_{s-1}, y); Y_t(y) = y$

(iv) $X_u(y_{0t}) = x_{0u}$; $Y_u(y_{0t}) = y_{0u}; X_u(y_{1t}) = x_{1u}$; $Y_u(y_{1t}) = y_{1u}$

(v) $X'_u > 0, Y'_u > 0$ for $u - t$ even, and $X'_u < 0, Y'_u < 0$ for $u - t$ odd.

PROOF – The condition implies that y_{0t} and y_{1t} are between two consecutive discontinuity points. $m(\mu_s, .)$ is therefore C^1 over $[y_{0t}, y_{1t}]$. We can then construct $Y_u()$ recursively as $Y_t(y) = y$ and $Y_u(y) = \Phi^{-1}(Y_{u-1}(y))$; similarly, $X_u()$ is constructed recursively as $X_t(y) = m(\mu_{s-1}, y)$, $X_u(y) = \gamma(\nu - \bar{n}X_{u-1}(y)I^*(Y_u(y)))$. Thus, (i),(iii) and (iv) trivially hold. The monotonicity properties (v) in turn are a consequence of property P3. Finally, the feasibility property (ii) is a consequence of monotonicity: while (28) holds by construction, (27) derives from the fact that $X_u(y)$ is between x_{0u} and x_{1u}, which are both nonnegative. QED.

With these properties in hand, we are now able to construct a feasible sequence such that (29) holds. Denoting again by $\bar{V}_N$ the fixed point of $\Phi()$, and assuming (24) holds, there necessarily exists a *maximal* sequence such that $y_u = \bar{V}_N$, for all u. Again, that is because the dynamics of x are then unstable if (24) holds. Call $\tilde{x}_u$ the values of x_u in that sequence. Because of (30), we must have

$$\bar{V}_N > \Phi(I^{*-1}(\frac{\nu}{\bar{n}\tilde{x}_T}))$$

Consider now the *admissible* sequence of equal length $T, \{(\hat{x}_u, \hat{y}_u), u = t, ..., T\}$, such that $\hat{y}_T = \Phi(I^{*-1}(\frac{\nu}{\bar{n}\tilde{x}_T}))$. It can be easily constructed by iterating $\Phi()$ backwards on $\hat{y}_T$, yielding some $\hat{y}_t$, and then computing the corresponding values of x_u by applying (26). Three possibilities arise:

G1. The admissible sequence is feasible and $\hat{y}_t$ is such that $I^{*-1}(\frac{k\upsilon}{\mu_s \bar{n}}) < \hat{y}_t < \bar{V}_N < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s \bar{n}})$ (for $T-t$ even), or $I^{*-1}(\frac{k\upsilon}{\mu_s \bar{n}}) < \bar{V}_N < \hat{y}_t < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s \bar{n}})$ (for $T-t$ odd).

In this case, we note that property P4 can be applied, using the feasible sequences $\{(\tilde{x}_u, \bar{V}_N)\}$ and $\{(\hat{x}_u, \hat{y}_u)\}$ as our boundaries. Because of the monotonicity property (v), it must be that $\hat{x}_T < \tilde{x}_T$, since $\hat{y}_T < \bar{V}_N$. Therefore $\hat{y}_T = \Phi(I^{*-1}(\frac{\nu}{\bar{n}\tilde{x}_T})) < \Phi(I^{*-1}(\frac{\nu}{\bar{n}\hat{x}_T}))$. Hence:

$$\hat{y}_T < \Phi(I^{*-1}(\frac{\nu}{\bar{n}\tilde{x}_T})).$$

Therefore, the function $Y_T(y) - \Phi(I^{*-1}(\frac{\nu}{\bar{n}X_T(y)}))$, which is continuous, becomes positive and negative as y varies between $\hat{y}_t$ and $\bar{V}_N$. There exists $y^* \in [\hat{y}_t, \bar{V}_N]$, such that it is equal to zero. The sequence $\{(X_u(y^*), Y_u(y^*), u = t, ..., T\}$ is feasible because of property (ii), and satisfies (29).

G2. The admissible sequence is not feasible, but $\hat{y}_t$ satisfies $I^{*-1}(\frac{k\upsilon}{\mu_s \bar{n}}) < \hat{y}_t < \bar{V}_N < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s \bar{n}})$ (for $T-t$ even), or $I^{*-1}(\frac{k\upsilon}{\mu_s \bar{n}}) < \bar{V}_N < \hat{y}_t < I^{*-1}(\frac{(k-1)\upsilon}{\mu_s \bar{n}})$ (for $T-t$ odd).

In this case, lack of feasibility must be due to the fact that (27) is violated. We can then construct a maximal sequence $\{(\hat{x}_u, \hat{y}_u), u = t, ..., T_2\}$ for some $T_2 < T$.[9]

Because of Property (P2), we have that

$$\bar{V}_N \leq \Phi(I^{*-1}(\frac{\upsilon}{\tilde{x}_{T_2}\bar{n}})).$$

Because of the maximality condition (30), we have that

$$\hat{y}_{T_2} > \Phi(I^{*-1}(\frac{\upsilon}{\hat{x}_{T_2}\bar{n}})).$$

[9] Furthermore $T - T_2$ has to be odd. That is because, by construction, $\hat{x}_u < \tilde{x}_u$ and $\hat{\omega}_t < \bar{V}_N$ if $T - T_2$ is even (due to Property P3), implying that $\bar{n}\hat{x}_{T_2} I^*(\Phi^{-1}(\hat{\omega}_u)) \leq \bar{n}\tilde{x}_{T_2} I^*(\Phi^{-1}(\bar{V}_N)) \leq \nu$, where the last inequality is due to (P2). That violates the maximality condition (30).

As $\{(\hat{x}_u, \hat{y}_u), u = t, ..., T_2\}$ is now feasible, we can apply the same reasoning as in case G1, but between t and T_2 instead of t and T.

G3. The admissible sequence does not satisfy $I^{*-1}(\frac{kv}{\mu_s \bar{n}}) < \hat{y}_t < \bar{V}_N < I^{*-1}(\frac{(k-1)v}{\mu_s \bar{n}})$ (for $T-t$ even), or $I^{*-1}(\frac{kv}{\mu_s \bar{n}}) < \bar{V}_N < \hat{y}_t < I^{*-1}(\frac{(k-1)v}{\mu_s \bar{n}})$ (for $T-t$ odd).

Assume $T-t$ is even. Then it must be that $\hat{y}_t < I^{*-1}(\frac{kv}{\mu_s \bar{n}}) < \bar{V}_N$. Consider now $y_m = I^{*-1}(\frac{kv}{\mu_s \bar{n}})+\eta$, for $\eta > 0$ arbitrarily small. Note that $\Phi^{-1}(y_m)$ exists, as $y_m \in [\hat{y}_t, \bar{V}_N]$. The two-period sequence $\{(x_{mt}, y_{mt}), (x_{mt+1}, y_{mt+1})\}$ defined by $x_{mt} = (m(\mu_s, y_m)$, $y_{mt} = y_m, x_{mt+1} = (\gamma(\nu - \bar{n}m(\mu_s, y_m) I^*(\Phi^{-1}(y_m))$, $y_{mt+1} = \Phi^{-1}(y_m))$, is clearly feasible, since $y_{mt+1} \in [\bar{V}_N, \hat{y}_{t+1}]$, $m(\mu_s, y_m)$ is positive and arbitrarily close to zero, and x_{mt+1} is thus arbitrarily close to $\gamma\nu$.

Next, note that the maximality condition holds for $t+1$, since

$$\bar{n}(\gamma\nu)I^*(\Phi^{-1}(y_{mt+1})) > \bar{n}(\gamma\nu)I^*(\Phi^{-1}(\bar{V}_N)) > \nu,$$

because of (24).

Thus, this 2 period sequence is maximal, and satisfies

$$y_{mt+1} > \Phi(I^{*-1}(\frac{v}{x_{mt+1}\bar{n}})),$$

while, again because of (P2):

$$\bar{V}_N < \Phi(I^{*-1}(\frac{v}{\tilde{x}_{t+1}\bar{n}})).$$

We are again in a position to apply the same continuity argument as in cases G1 and G2.

Assume $T - t$ is odd. Then it must be that $\bar{V}_N < I^{*-1}(\frac{(k-1)v}{\mu_s \bar{n}}) < \hat{y}_t$. Consider now $y_m = I^{*-1}(\frac{(k-1)v}{\mu_s \bar{n}}) - \eta$, for $\eta > 0$ arbitrarily small. The one-period sequence $\{(x_{mt}, y_{mt})\}$ defined by $x_{mt} = m(\mu_s, y_m)$, $y_{mt} = y_m$, is clearly feasible. Again, note that because of (24), the maximality condition holds for t, since

$$\bar{n}(\gamma\nu)I^*(\Phi^{-1}(y_{mt})) > \bar{n}(\gamma\nu)I^*(\Phi^{-1}(\bar{V}_N)) > \nu.$$

We have constructed a maximal sequence which satisfies $y_{mt} > \Phi(I^{*-1}(\frac{v}{x_{mt}\bar{n}}))$. Property (P2) again implies that $\bar{V}_N < \Phi(I^{*-1}(\frac{v}{\tilde{x}_t \bar{n}}))$. Since $I^{*-1}(\frac{kv}{\mu_s \bar{n}}) < \bar{V}_N < y_{mt} < I^{*-1}(\frac{(k-1)v}{\mu_s \bar{n}})$, we can again apply the continuity argument as in G1 to construct a one-period sequence matching the equilibrium condition (29).

Thus we are always able to construct a feasible sequence such that the equilibrium condition (29) holds.

The only equilibrium condition that remains to be checked is condition (15) in the regime I phase. Noting that, during that phase, $\lambda_u = \hat{\lambda}_u + \frac{\theta}{\beta}(\hat{\lambda}_u - \hat{\lambda}_{u+1})$, that condition is equivalent to

$$\Phi(\hat{\lambda}_{u+1}) < \hat{\lambda}_u + \frac{\theta}{\beta}(\hat{\lambda}_u - \hat{\lambda}_{u+1}). \tag{31}$$

Observe now that we have constructed the value of V_{Nt} at the end of phase I in such a way that V_{Nt} falls in the same $[I^{*-1}(\frac{kv}{\mu_s \bar{n}}), I^{*-1}(\frac{(k-1)v}{\mu_s \bar{n}})]$ as $\bar{V}_N$. Since $I^{*-1}(\frac{(k-1)v}{\mu_s \bar{n}}) = \hat{\lambda}_{t-1}$, it follows that $\hat{\lambda}_{t-1} > \bar{V}_N > \Phi(\hat{\lambda}_{t-1})$; since the $\hat{\lambda}$ are a decreasing sequence, we also have that $\hat{\lambda}_u > \bar{V}_N > \Phi(\hat{\lambda}_u)$ for all u. Consider now the function defined by $\Lambda(\lambda, y) = \lambda + \frac{\beta}{\lambda}(\lambda - y) - \Phi(y)$. Clearly, $\Lambda(\hat{\lambda}_u, \hat{\lambda}_u) = \hat{\lambda}_u - \Phi(\hat{\lambda}_u) > 0$. Furthermore, as $\gamma < 1$, $\frac{\partial \Lambda}{\partial y} = -\frac{\theta}{\beta} + \gamma\frac{\theta}{\beta}(1 - F(y)) < 0$. Thus, $\Lambda(\hat{\lambda}_u, y) > 0$ for all $y < \hat{\lambda}_u$. In particular, $\Lambda(\hat{\lambda}_u, \hat{\lambda}_{u+1}) > 0$, which is equivalent to (A22).

That concludes the proof that a cyclical equilibrium exists.

H. Constructing an equilibrium when (24) *does not hold.*

We have constructed a cyclical equilibrium when (24) holds. If it does not, we can easily construct an equilibrium in regime II which converges to the steady state. To do so, start from an inherited mass of invented fields μ_s at the beginning of the (potentially empty) phase in regime I, and choose $V_{Nt} = \bar{V}_N$, $\mu_t = m(\mu_s, \bar{V}_N)$. This defines an initial phase in regime I whose length equals the number of discontinuity points of $m(\mu_s, .)$ above $\bar{V}_N$. Thereafter, the measure of new fields evolves according to

$$\mu_{u+1} = \gamma(\nu - \bar{n}I^*(\bar{V}_N)\mu_u),$$

and, given that (24) is violated and that $m(\mu_s, \bar{V}_N) \in [0, \gamma\upsilon]$, it is straightforward to check that μ_u will converge to its steady-state value by damped oscillations, remaining in the feasible $[0, \gamma\upsilon]$ interval. Furthermore, the largest discontinuity point of $m()$ is at $I^{*-1}(\frac{\upsilon}{\mu_s\bar{n}}) < I^{*-1}(\frac{1}{\gamma\bar{n}}) < \bar{V}_N$. Thus the economy in fact does not spend any time in regime I.

I. Proof of uniqueness when $\theta = 0$.

If the feasible sequence constructed in G. is unique, then the equilibrium is unique. That is easy to prove in the $\theta = 0$ case. The initial value of V_{Nt}, at the beginning of regime II, must be equal to $\gamma\bar{\omega}$. Furthermore, the date at which the economy leaves regime II to revert to regime I is uniquely determined: If it were such that $\bar{n}x_uI^*(\gamma\bar{\omega}) \leq \upsilon$, then (20) would be violated for the next regime I phase, since one must have $V_N = \gamma\bar{\omega}$ at the end of that regime. Therefore, it must be equal to the first date such that $\bar{n}x_uI^*(\gamma\bar{\omega}) > \upsilon$, since the economy cannot feasibly remain in regime II after that date.

9.2 Proof of Proposition 5

9.3 Proof of Proposition 6

To prove Proposition 6, just differentiate the dynamics of V_{Nt} in regime II, $V_{Nt} = \Phi(V_{Nt+1})$, around the fixed point $\bar{V}_N$. Denoting by $v_t = V_{Nt} - \bar{V}_N$, we get

$$v_t = \frac{\gamma\theta}{\beta}(1 - F(\bar{V}_N))v_{t+1}.$$

If $\frac{\gamma\theta}{\beta}(1-F(\bar{V}_N)) > 1$, then we can construct an equilibrium for any initial value of v_t. QED.

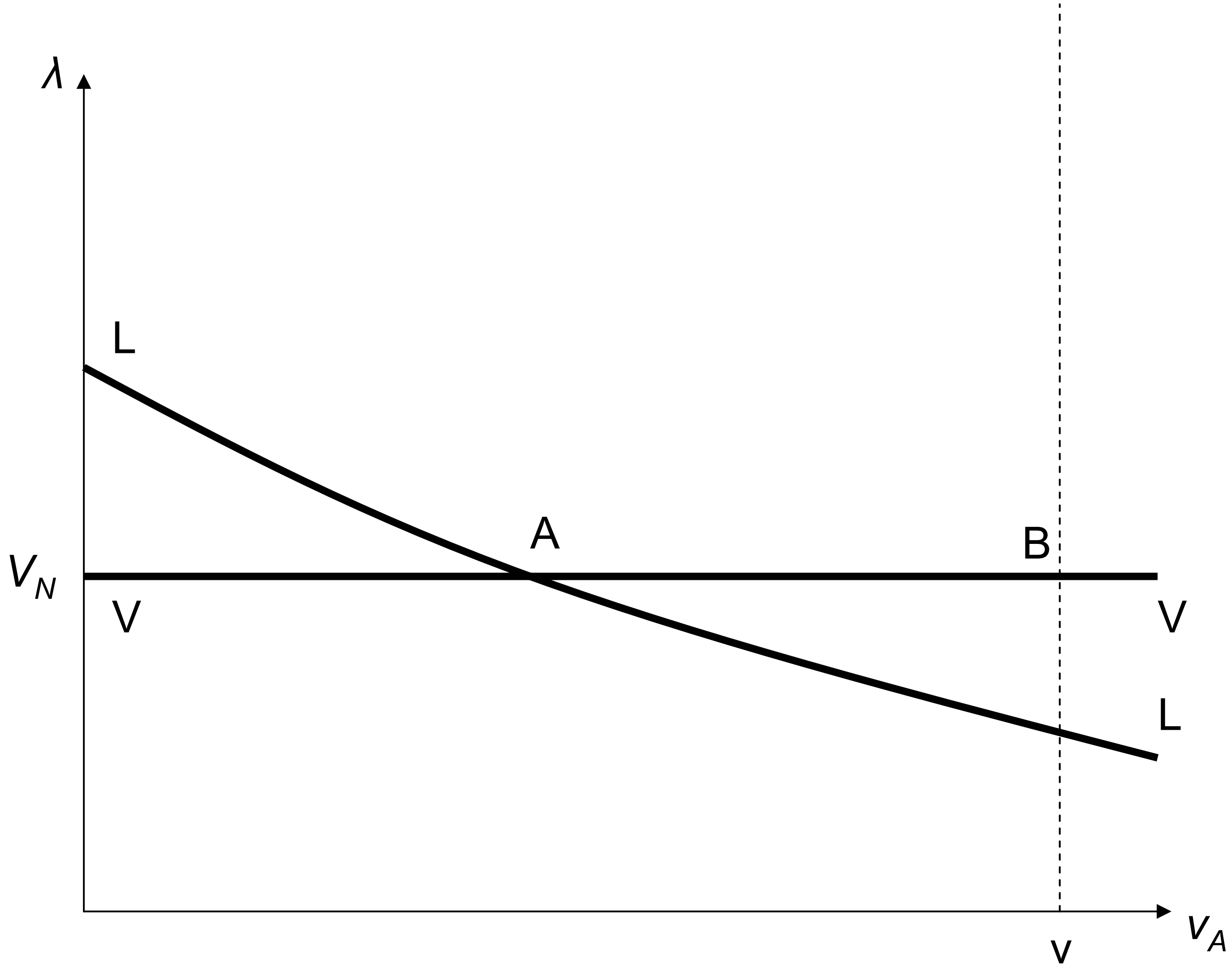


Figure 1 – Equilibrium determination in regime II

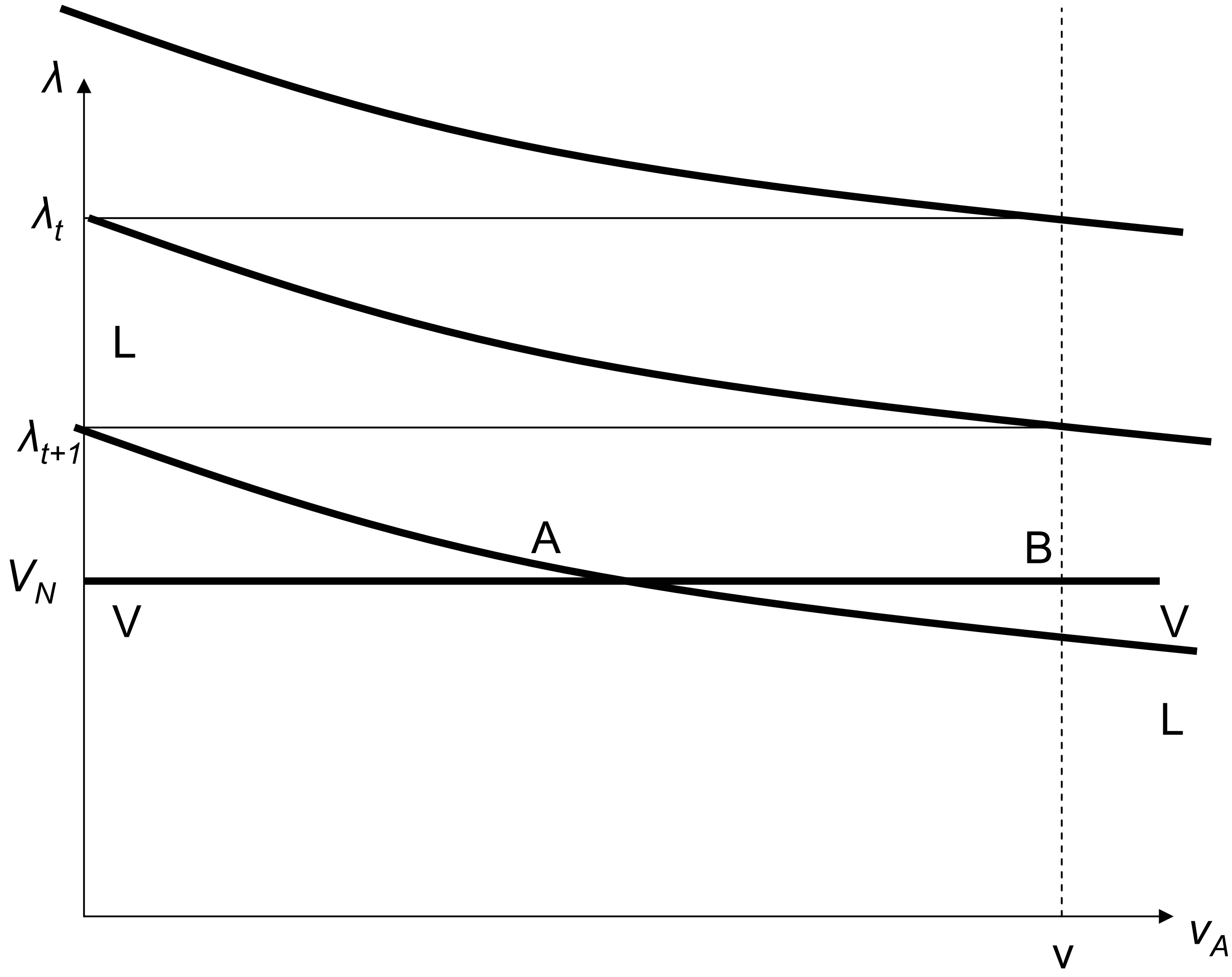


Figure 2 – Equilibrium determination in regime I

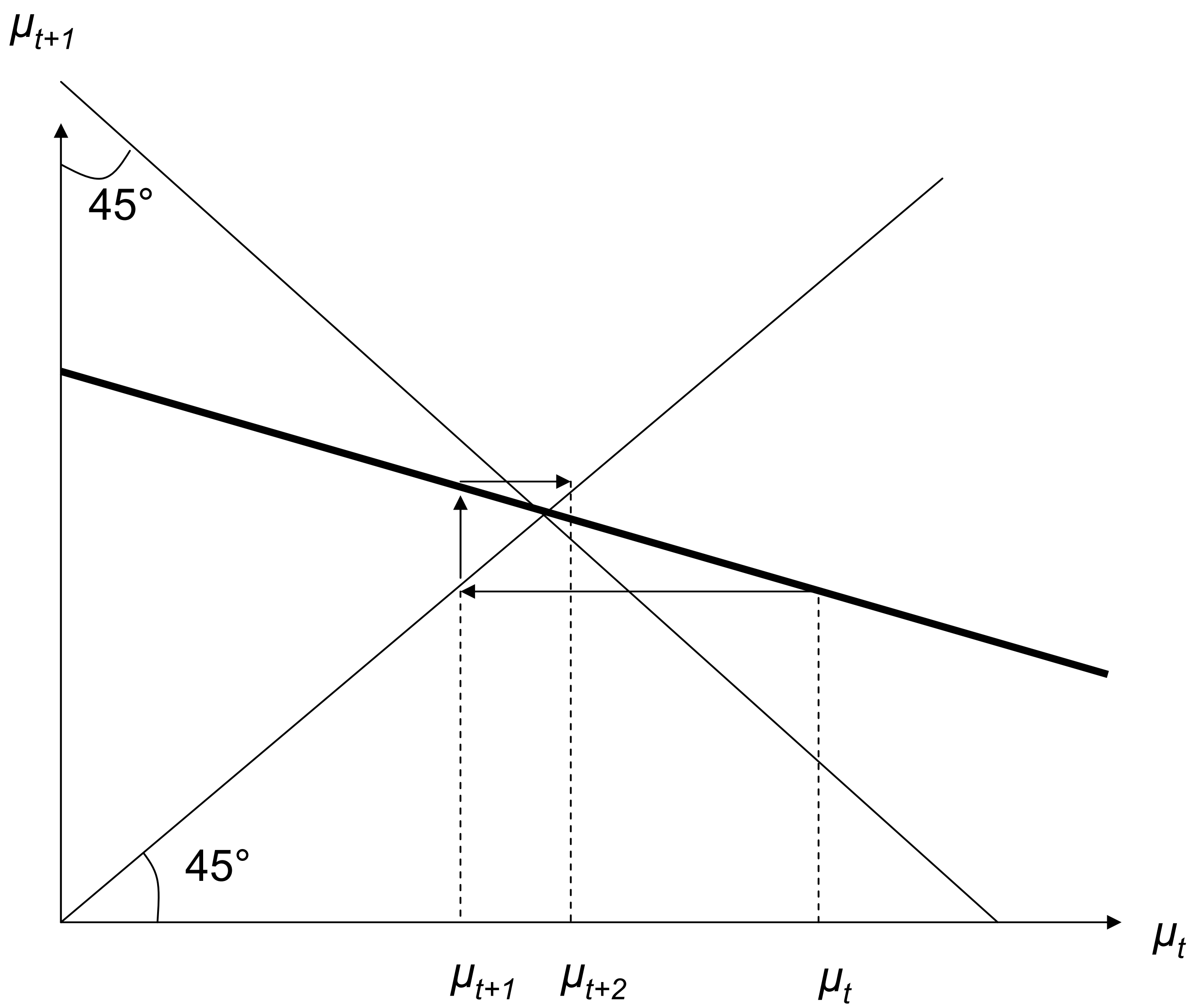


Figure 3 – Convergence to the regime II steady state

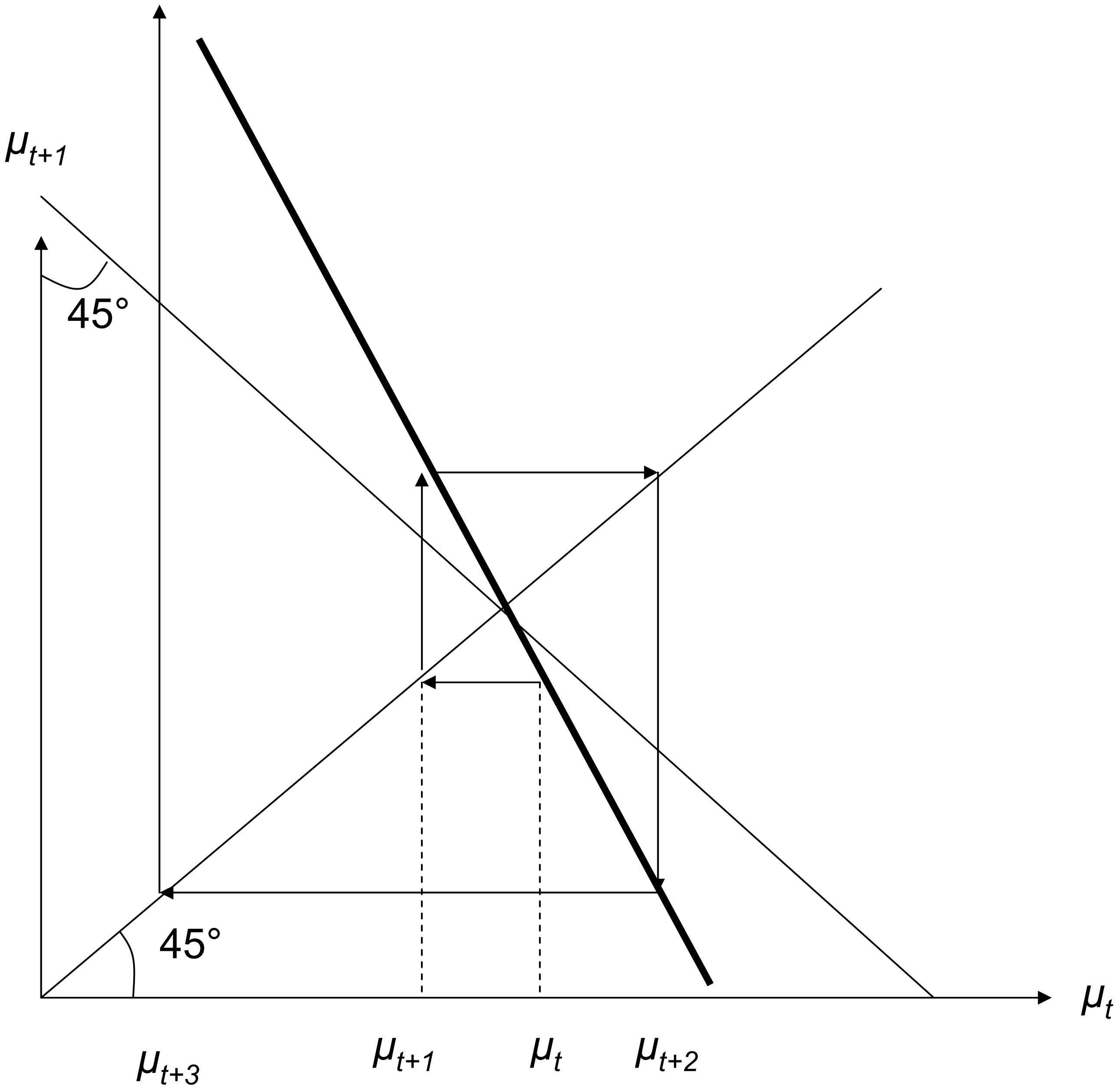


Figure 4 – The economy eventually leaves regime II

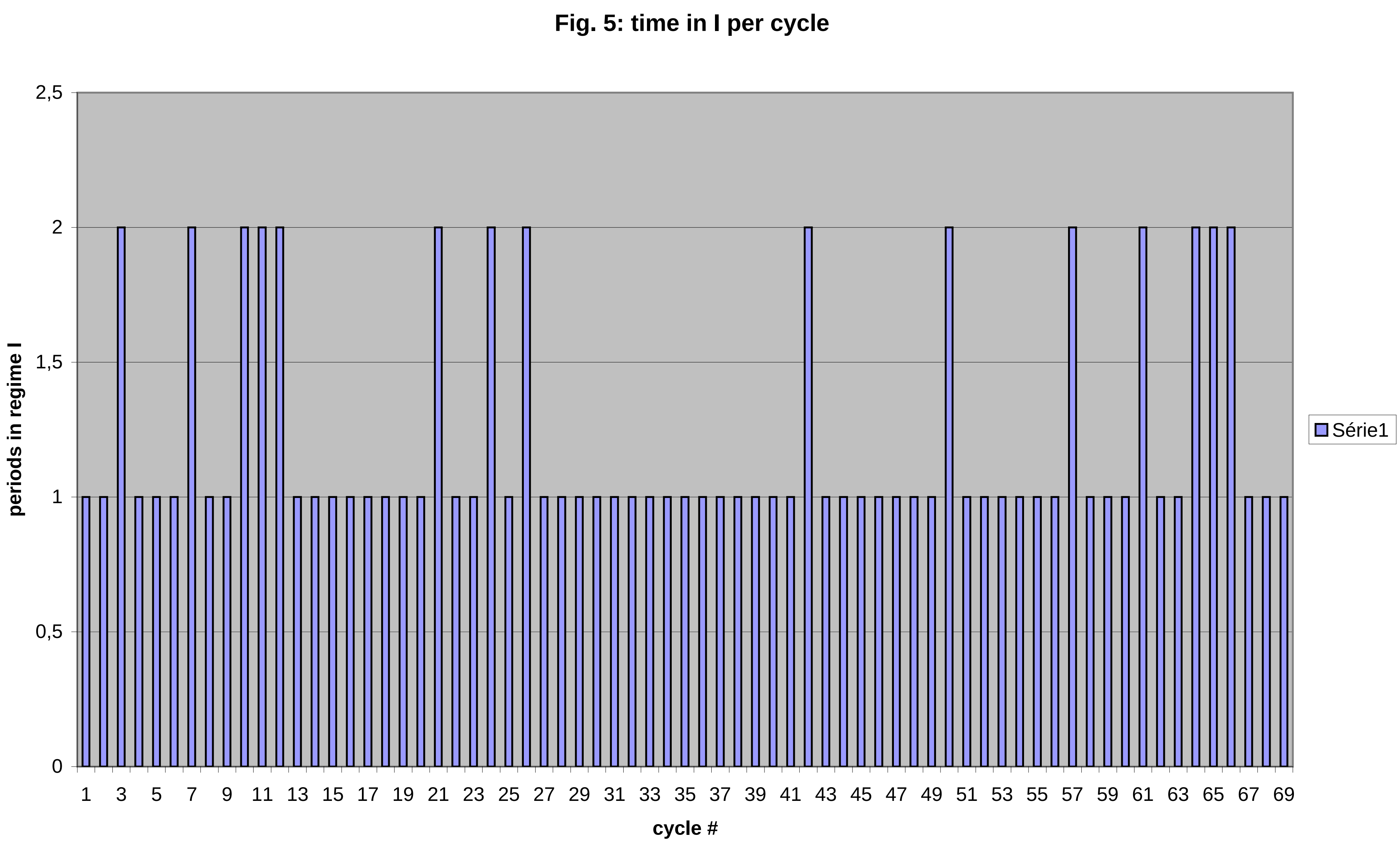


Fig. 5: time in I per cycle
periods in regime I
2,5
2
1,5
1
0,5
0
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49 51 53 55 57 59 61 63 65 67 69
cycle #
Série1

Fig 6: mass of new fields per cycle

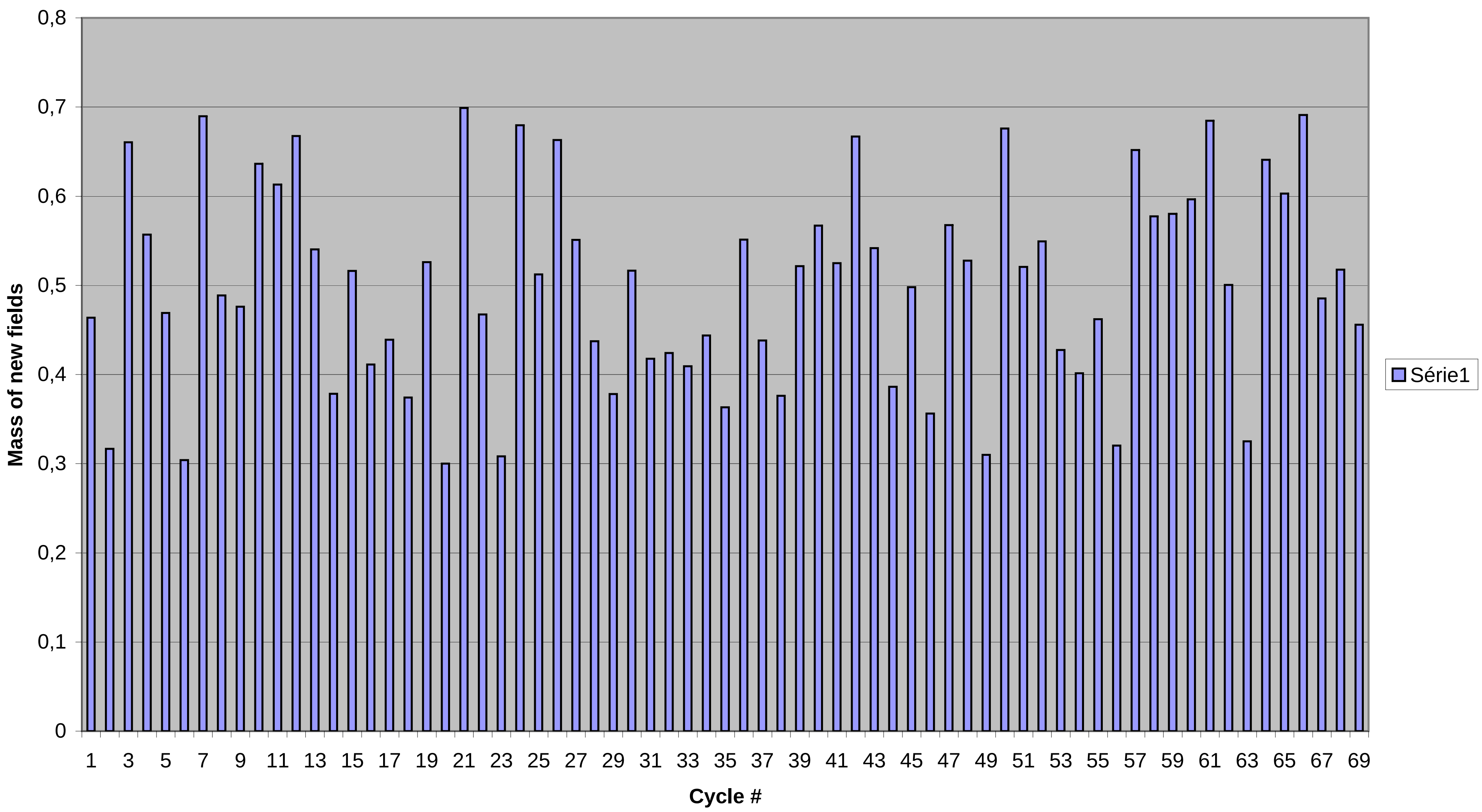
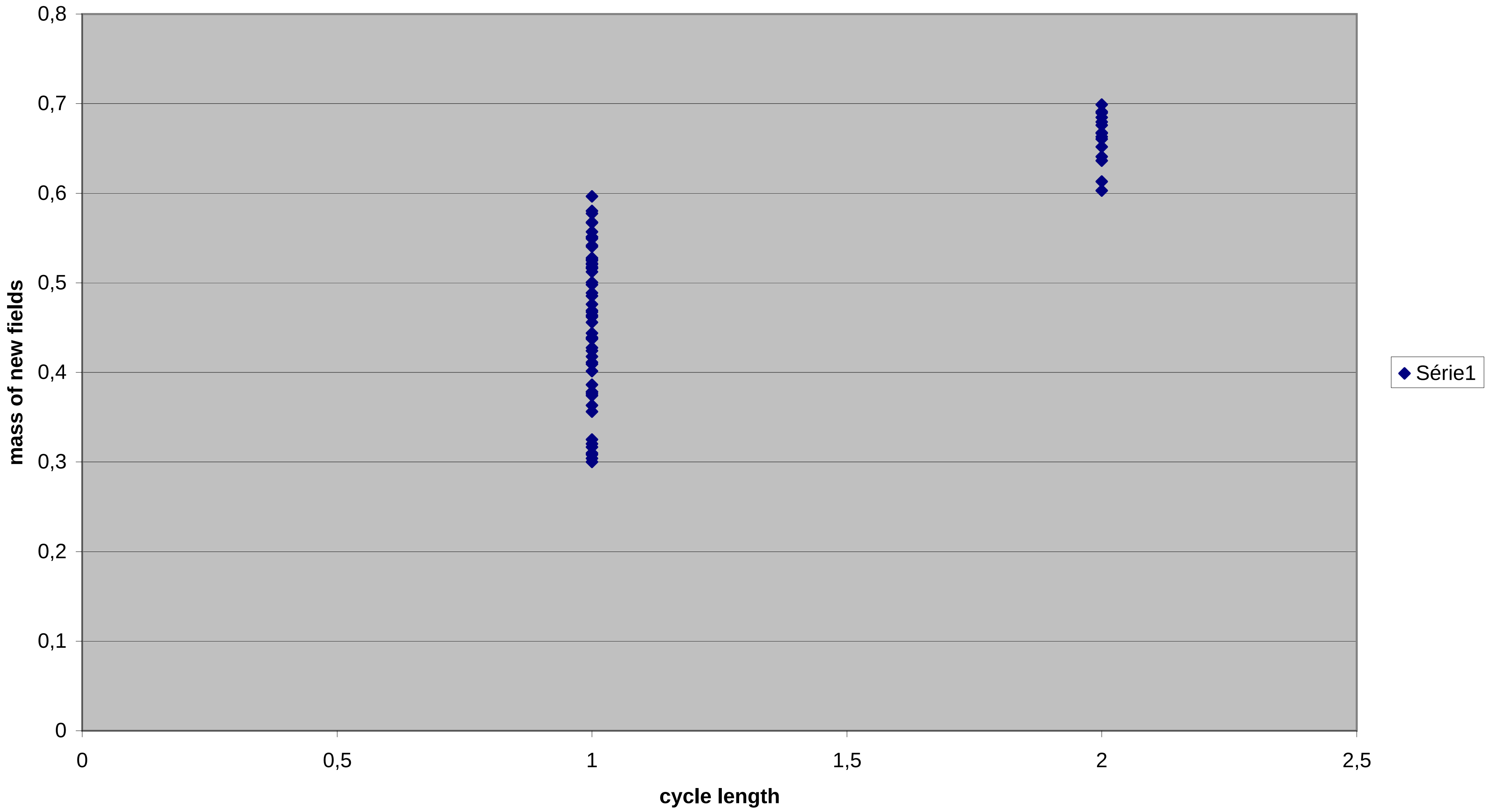


Fig. 7 cycle length and mass of new fields



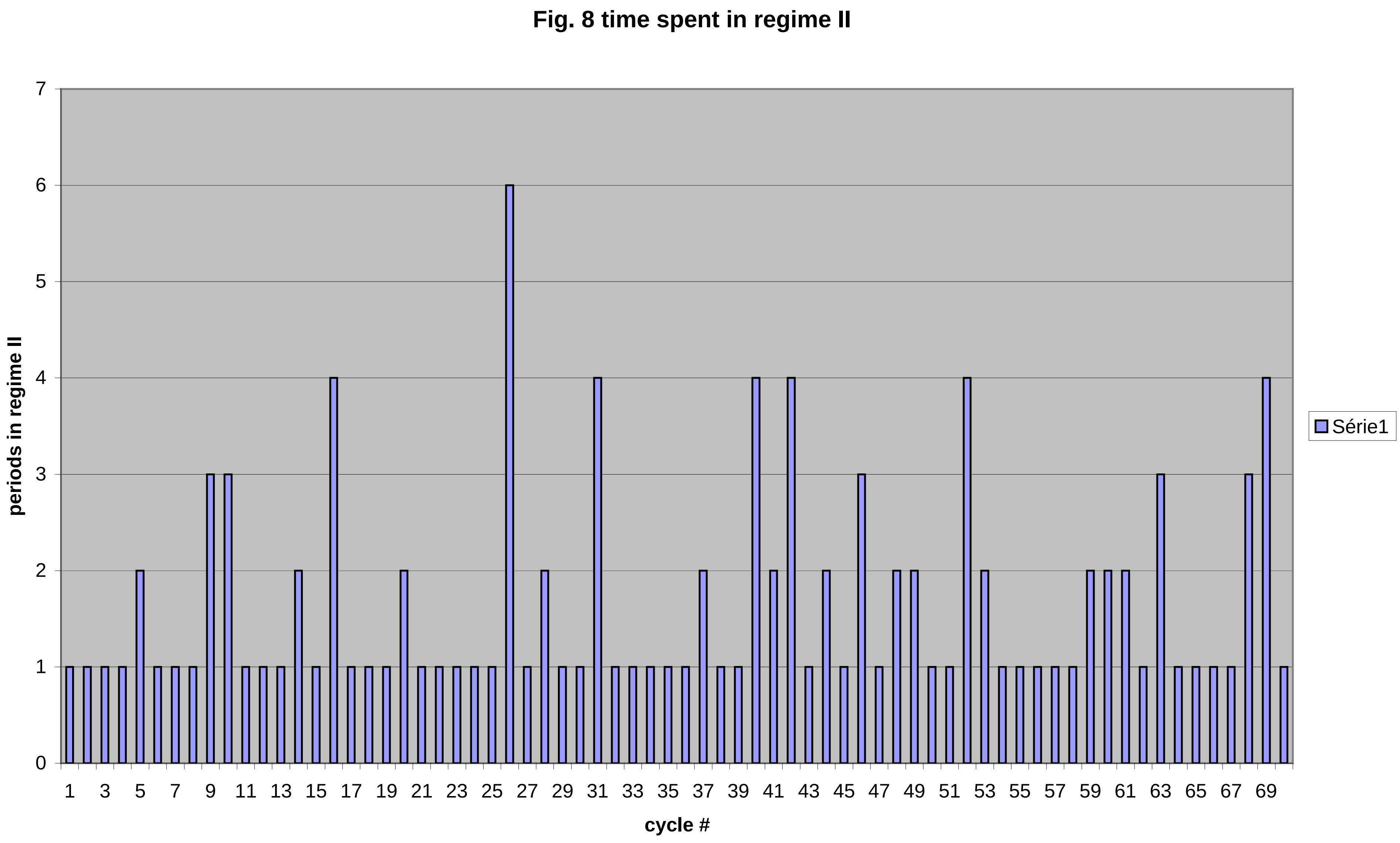


Fig. 8 time spent in regime II
periods in regime II
0
1
2
3
4
5
6
7
1
3
5
7
9
11
13
15
17
19
21
23
25
27
29
31
33
35
37
39
41
43
45
47
49
51
53
55
57
59
61
63
65
67
69
cycle #
Série1

Fig. 9: Average production of new fields in regime II per cycle

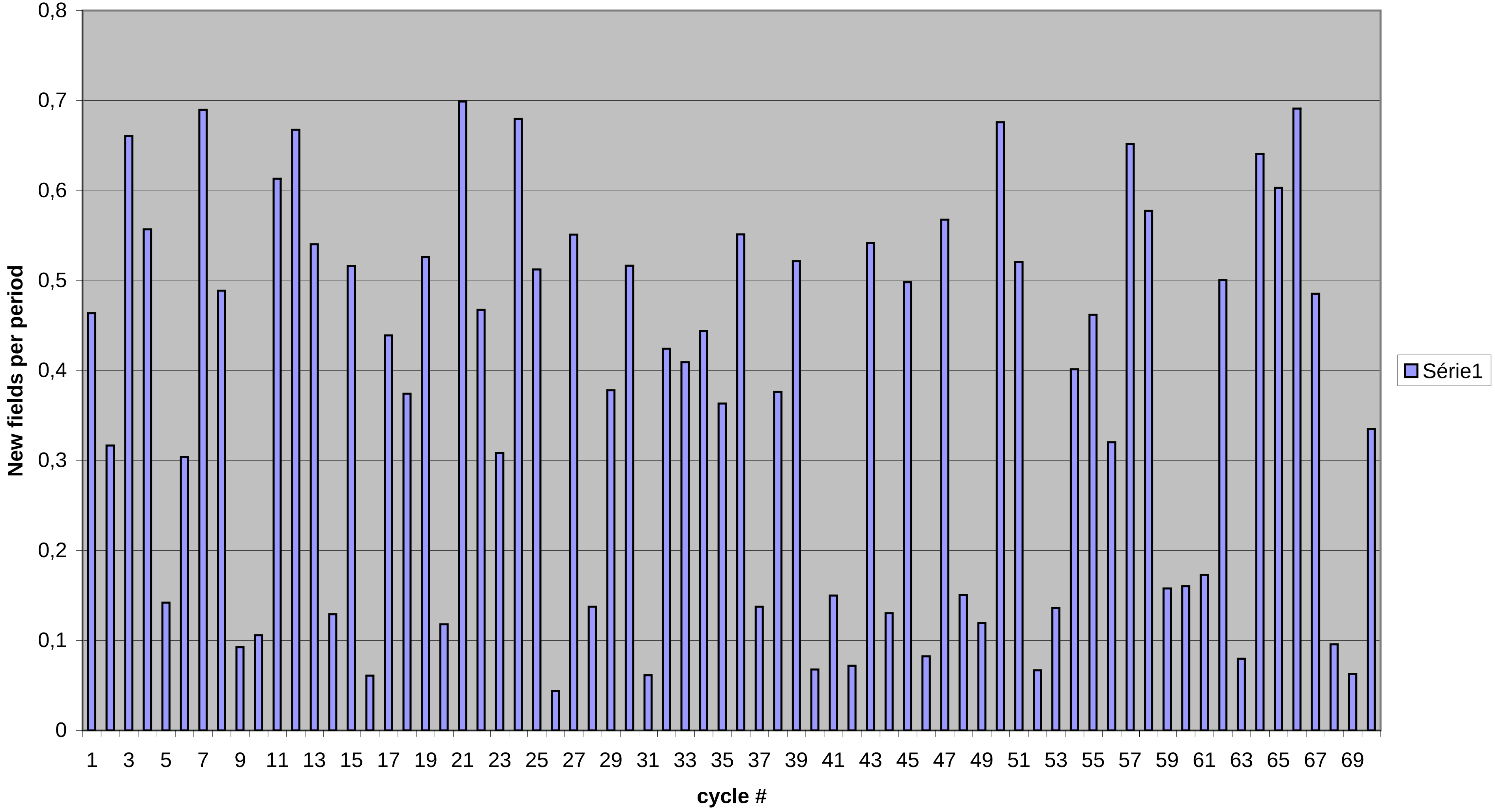


Fig. 10: Time in regime II and average innovation

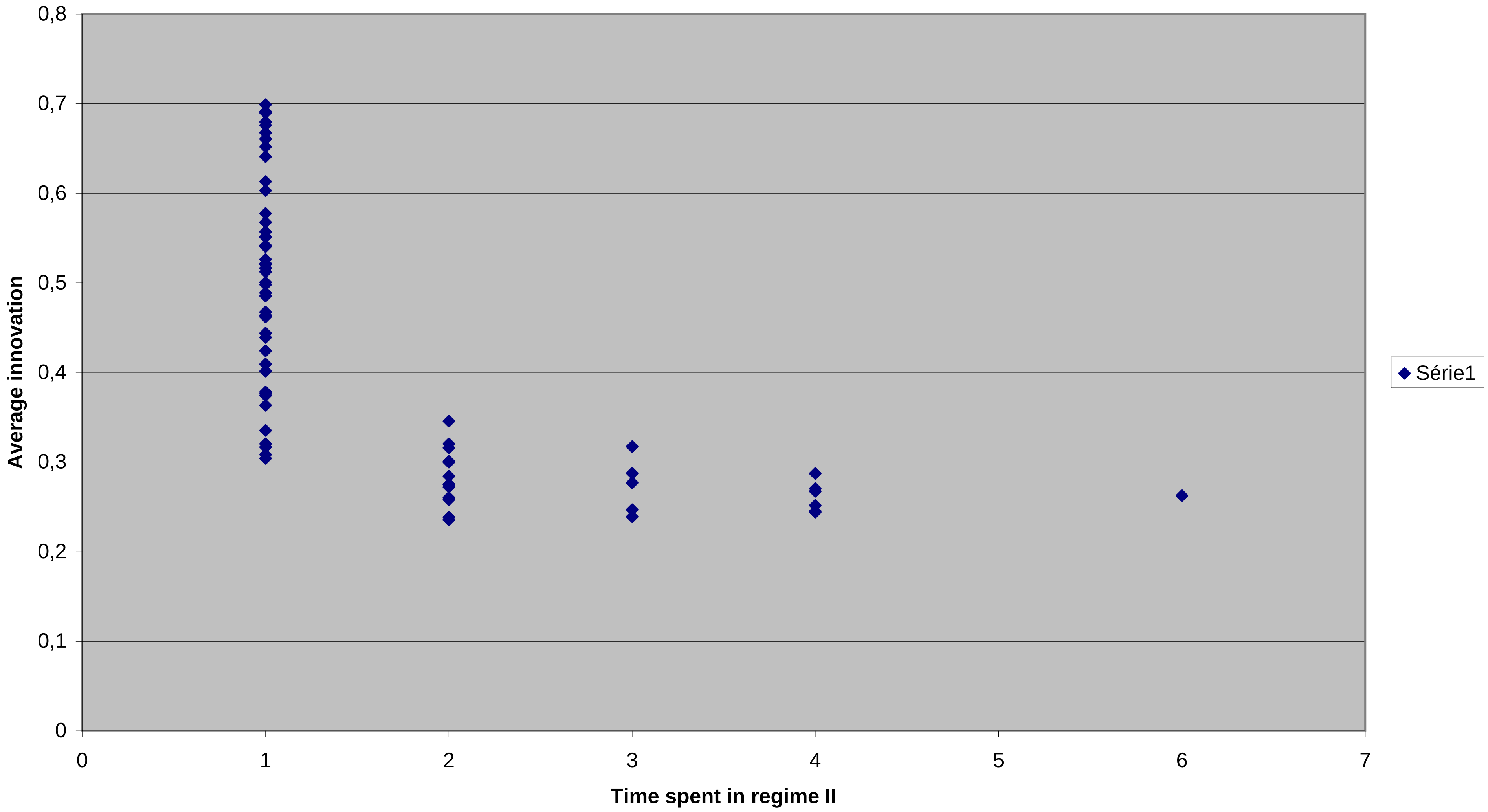


Fig. 11: cycle duration in reg. I, beta = 0.2

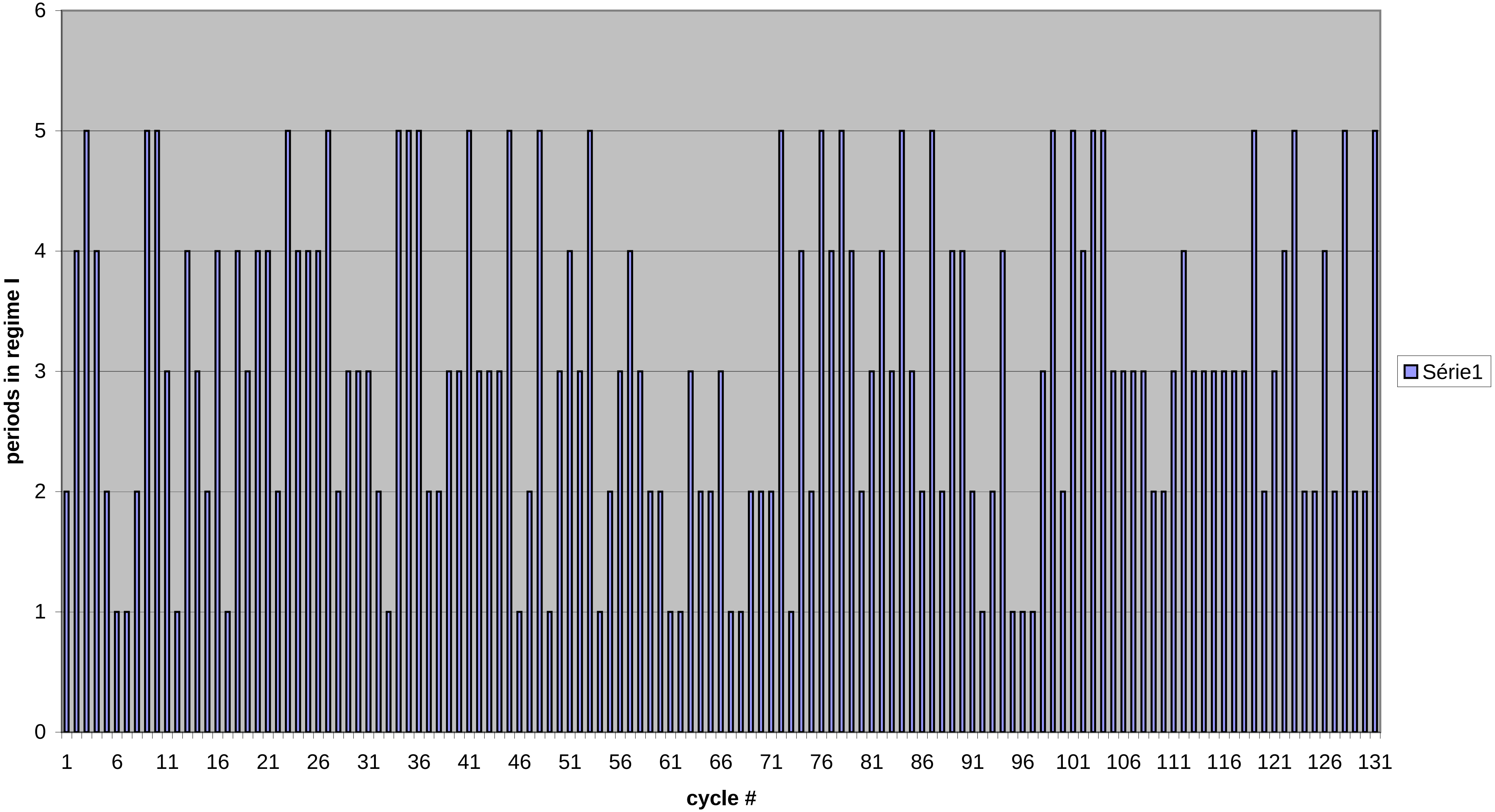


Fig. 12, time spent in regime II per cycle, beta = 0.2