Incomplete Markets, Heterogeneity and Macroeconomic Dynamics*

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Abstract

This paper solves a real business cycle model with heterogeneous agents and uninsurable income risk using perturbation methods. In contrast to the value function iteration-based solutions, this permits an analytic characterization of an agent's optimal decision rules that is accurate to the second order and render the implications of aggregation for macroeconomic dynamics transparent. Importantly, the role of cross-sectional holdings of capital in determining equilibruim dynamics can be directly assessed. Analysis discloses that an individual's optimal saving decisions are, to the second order, almost linear in their own capital stock giving rise to permanent income consumption behavior. This provides an explanation for the approximate aggregation properties of this model documented by Krusell and Smith (1998): the distribution of capital does not afffect aggregate dynamics. While the variance-covariance properties of endogenous variables are almost entirely determined by first order dynamics, the second order dynamics, which capture the evolution of the second order moments of the distribution of capital holdings, are nonetheless important for the determination of an individual's mean consumption and saving decisions and therefore the mean equilibrium capital stock. Policy evaluation exercises therefore need to take account of these higher order terms. Computation of the Euler equation errors induced by each approach reveals perturbation methods to give a more accurate model solution than value function iteration. Importantly, perturbation methods provide a flexible analytical tool for solving incomplete markets models even when approximate aggregation does not obtain.

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1 Introduction

This paper proposes a new approach to solving macroeconomic models with heterogeneous agents and incomplete markets. The study of such models is important for several reasons. First, the representative agent construct holds under stringent conditions unlikely to hold in reality. Second, heterogeneous agent models are of interest in their own right; potentially explain a range of economic phenomena; and address issues that are impossible to analyze in a representative agent framework. For example, such models may imply large costs of business cycles in contradistinction to the classic analysis of Lucas (1987). More generally, we observe important heterogeneity, but little is understood about its consequences. Of particular import is the extent to which heterogeneous agents are diversely affected by macroeconomic fluctuations. How should macroeconomic stabilization policies be designed when agents are differentially affected by policy?

A key obstacle to the analysis of such questions has been the availability of tractable solution methods. For instance, a stochastic growth model with heterogeneous agents, aggregate technology shocks and partially insurable labor income risk engenders a time varying distribution of capital holdings across agents. For agents to solve their optimization problem, knowledge of the stochastic properties of this wealth distribution is required to forecast future prices, as these prices depend on the aggregate capital stock. Solving such models is difficult.

This paper makes two contributions: one methodological and one substantive. The methodological contribution is to delineate a new approach to solving a stochastic growth model with heterogenous agents and incomplete markets. Building on the representative agent based analyses of Judd (1998, 2002), Kim, Kim, Schaumburg, and Sims (2003) and Schmitt-Grohe and Uribe (2004), a second order accurate solution to the model is developed. The approach can be readily extended to higher order approximations. The analysis makes clear that the approach applies to a broad class of alternative models which permit the analysis of a number of questions of interest such as optimal policy design in the presence of agent heterogeneity.

The use of perturbation methods requires confronting a number of conceptual issues. First, incomplete markets models often feature borrowing constraints which may be occasionally binding. These inequality constraints are not readily handled by perturbation methods, premised as they are on an appropriate degree of differentiability. Second, the set of relevant state variables that appear in a second order approximation must be determined. In heterogeneous agent models, aggregation constraints, relating individual decisions to aggregate conditions, induce new aggregate state variables that increase the dimensionality of the model. Third, aggregation also imposes constraints on the relationships between elasticities in individuals' optimal decisions and those characterizing aggregate dynamics. We show how each of these complications can be handled.

The analysis demonstrates that perturbation methods have several appealing features in application to heterogeneous agent models. The approach permits an analytic characterization of the evolution of the wealth distribution that is accurate up to the order of the approximation. This in turn permits a characterization of optimal decisions to the same order. Hence the elasticities of individual saving and consumption decisions in response to any state variable are determined. Because aggregation proceeds directly from these individual decision rules, the role of heterogeneity and the distribution of capital holdings in determining aggregate dynamics can be clearly and directly assessed.

As our methodology is analytic and based on standard methods for solving linear and quadratic systems of equations, solutions are generated in fractions of a second in contrast to existing numerical methods based on value function iteration. The analysis is not constrained in the manner in which uncertainty can be specified. While numerical procedures typically require uncertainty to be specified as a low-dimension discrete-state Markov process, perturbation methods readily handle continuously distributed random variables. Similarly, the analysis is not constrained by the number of state variables present in the model. Hence, the approach opens the way for econometric estimation of heterogenous agent models. As such, the framework provides a tractable laboratory for the study of optimal policy design in the presence of heterogeneity, as well as the quantification of the welfare costs associated with various sources of risk with imperfect insurance markets.

The substantive contribution of the paper is to give greater understanding of the role of heterogeneity in determining aggregate dynamics in a simple real business cycle model. In the benchmark calibration, optimal saving decisions are shown to be virtually linear in an individual's own holdings of the capital stock. There is very little curvature in optimal decisions due to second order characteristics of the cross-sectional distribution of capital held by agents. Agents are shown to be effectively permanent income consumers: they consume the returns on their capital holdings and keep the principle intact. In consequence the marginal propensities to save across individuals are almost equal and the model, therefore, displays the approximate aggregation property noted by Krusell and Smith (1998): the evolution of aggregate variables is largely determined by aggregate capital — the distribution of capital across individuals and therefore heterogeneity matters little for macroeconomic dynamics.

Notwithstanding this finding, we show that incomplete markets and heterogeneity do matter for understanding macroeconomic outcomes in this simple economy. The existence of borrowing constraints affect first order dynamics of the model economy, and these dynamics almost entirely determine the variance-covariance properties of all endogenous model variables. While it is not surprising that second order terms — terms that capture the evolving second order moments of the wealth distribution — are less important for aggregate variation than first order terms, evidence is adduced showing that these terms do matter for the determination of individual mean consumption and savings in equilibrium. These mean effects operate through two channels: first, the presence of risk leads to a constant adjustment in optimal decision rules, analogous to standard precautionary savings effects; second, the interaction of uncertainty and the non-linear mapping of states into decisions in a second order approximation, leads to Jensen inequality type effects on average consumption and savings. Indeed, the latter can be significantly larger than the former, depressing aggregate consumption by a fraction as large as 2 percent of steady state consumption on average in the simulations considered. This combined with significant observed variation in ex post individual consumption profiles, and therefore welfare, underscores the importance of incomplete markets and heterogeneity for macroeconomics.

The present analysis is most closely related to Krusell and Smith (1998). They present a novel solution algorithm for this class of problem using value function iteration-based methods. Because the wealth distribution is a high dimensional object, value function iteration methods must resort to solving an approximation to the true problem. To reduce the dimension of the state space, Krusell and Smith restrict the information set agents utilize in forecasting future prices. Analysis proceeds by conjecturing a boundedly rational law of motion for aggregate capital. Specifically, tomorrow's aggregate capital is assumed to be only a function of today's aggregate capital stock, and therefore depends only on the mean of the wealth distribution. Conditional on this conjectured aggregate capital accumulation equation, agents behave optimally. A central conclusion of their paper is that the model satisfies what they call approximate aggregation: aggregate dynamics do not depend on characteristics of the wealth distribution other than its mean (as would be the case under complete markets and the representative agent construct). The analysis presented here also finds approximate aggregation but our results are not implied by their findings. Perturbation methods represent a distinct solution method, approximating the model along a different dimension to solution procedures based on value function iteration.

Most importantly, the approach developed here remains valid even if the conditions for approximate aggregation do not obtain. As noted by Krusell and Smith (2006) on page 2, when discussing their solution algorithm based on value function iteration: "The key insight to solving the model with consumer heterogeneity using numerical methods is "approximate aggregation" in wealth". Furthermore, by including all state variables relevant to a second order approximation of the equilibrium dynamics, the perturbation approach permits a greater role for heterogeneity ex ante than does their algorithm which only permits the effects of heterogeneity on aggregate dynamics to be felt through the coefficients on the restricted law of motion for aggregate capital. The present analysis gives an analytical characterization of the problem, providing additional insight to the conditions required for approximate aggregation. Hence, the results presented here adduce new evidence on the importance or not of heterogeneity that originates from imperfect labor markets in explaining macroeconomic dynamics. Of course, while the Krusell and Smith algorithm may not provide an accurate characterization of aggregate dynamics when approximate aggregate fails to obtain, it does have the advantage of providing a global solution to the model, in contrast to perturbation methods which are necessarily a local characterization in the neighborhood of the model's steady state.[1] Nonetheless, the decisions determined by the perturbation approach are shown to induce smaller Euler equation errors than do value function iteration based solution methods.

[1] Though as shown by Swanson, Anderson, and Levin (2005), taking successively higher order approximations can, under suitable smoothness conditions satisfied by most macroeconomic models, deliver globally accurate characterizations of optimal decisions in the limit.

Following Krusell and Smith (1998), our analysis continues to build on earlier work on heterogeneous agent models by Bewley (1977, 1980), Huggett (1993) and Ayagari (1994). More recently Gourinchas (2000) analyzes an overlapping generations model which discloses the property of approximate aggregation as does Khan and Thomas (2005) in a model of firm investment dynamics with nonconvex costs of adjustment. Young (2005) further explores the robustness of the approximate aggregation result. Midrigan (2006) exploits this methodology in a study of firm pricing behavior. Further applications on the welfare costs of business cycles in heterogenous agent models include Storesletten, Telmer, and Yaron (2001) and Krusell and Smith (2002). Of particular relevance to the present study, though independently developed, is Kim, Kim, and Kollmann (2005). They analyze a Huggett-type economy with perturbation methods though make an approximation analogous to Krusell and Smith by only characterizing the equilibrium price to the first order — it is therefore not a second order accurate solution to the model. Finally, using projection methods, Gaspar and Judd (1997) solve for the consumption function that satisfies the household's Euler equation in two classes of heterogeneous agent models with finite number of households. Both models include only aggregate shocks, and agents, in contrast to the present analysis, are ex ante heterogenous: one model having agents with different preferences and the second with different initial wealth holdings. Our analysis advances this research by a adopting a new solution method – perturbation methods – and by providing a complete second order characterization of a stochastic growth model with a continuum of households that are ex ante homogeneous but ex post heterogeneous, aggregate and idiosyncratic shocks, and incomplete markets.

The paper proceeds as follows. The next section lays out the benchmark heterogeneous agent model. Section 3 deals with several conceptual issues relating to obtaining a second order accurate characterization of the model solution. Section 4 discusses some calibration exercises and highlights implications of heterogeneity and incomplete markets for macroeconomic dynamics. Section 5 gives further discussion of the conditions required for approximate aggregation and properties of solutions based on perturbation methods and value function iteration. The implied Euler equation errors in the true model induced by the approximate solution are computed to gauge the accuracy of the perturbation and value function iteration based solution methods. The final section offers some concluding remarks and a summary of on-going research grounded in the framework of this paper.

2 The Model

This section describes a stochastic growth model, incorporating heterogenous agents that face partially uninsurable income risk. There are a continuum of agents with unit measure indexed by $i \in [0,1]$. Each household i seeks to maximize

$$E_t \sum_{T=t}^{\infty} \beta^{T-t} u\left(c_{i,T}\right)$$

for

$$u\left(c_{i,t}\right) = \frac{c_{i,t}^{1-\gamma} - 1}{1-\gamma}$$

where $\gamma > 0$ is the inverse intertemporal elasticity of substitution, $0 < \beta < 1$ the discount rate and $c_{i,t}$ household i's consumption of the economy's only available good. Maximization is subject to the flow budget constraint for capital

$$a_{i,t+1} = (1-\delta)\, a_{i,t} + y_{i,t} - c_{i,t}$$

where $0 < \delta < 1$ is the depreciation rate, $a_{i,t}$ denotes individual i's holdings of the capital stock and $y_{i,t}$ the income of individual i to be defined below.

Agents face partially insurable labor market income risk. Each agent is endowed with one unit of time. This endowment is transformed into labor input according to $l_{i,t} = e_{i,t}\bar{l}$ where $e_{i,t}$ is an idiosyncratic employment shock satisfying

$$e_{i,t+1} = (1-\rho_e)\, \mu_e + \rho_e e_{i,t} + \varepsilon_{i,t+1}^{e} \tag{1}$$

where $0 < \rho_e < 1$, $\mu_e > 0$ and $\varepsilon_{i,t+1}^{e}$ a bounded i.i.d. disturbance with mean and variance $\left(0, \sigma_e^2\right)$.[2] Subsequent sections will make clear a particular advantage of perturbation methods: stochastic components of the model need not be restricted to low dimensional discrete state Markov processes. $\bar{l} > 0$ is a normalizing constant.

Asset markets are incomplete with capital representing the only asset by which resources can be transferred over time. It is for this reason that employment risks are partially insurable.

[2] The analysis will later allow employment status to depend on the aggregate state. However, for simplicity of notation this correlation is presently ignored.

Following Ayagari (1994), capital holdings are restricted by a borrowing constraint

$$a_{i,t+1} + b \geqslant 0, \quad \forall\, i \in [0,1] \tag{2}$$

for borrowing limit $b \geq 0$, ensuring the repayment of loans and the absence of Ponzi schemes.

Given single economy-wide markets for labor and capital, wage and rental rates are determined by the aggregate production function and the aggregate quantities of the two inputs. The latter are defined by

$$k_t \equiv \int_0^1 a_{i,t} di \tag{3}$$

$$l_t \equiv \int_0^1 l_{i,t} di = \mu_e \bar{l} \tag{4}$$

where the final equality follows from the law of large numbers and implies that aggregate employment is equal to the mean of individual employment outcomes. Aggregate output is produced according to a Cobb-Douglas production technology $z_t k_t^{\alpha} l_t^{1-\alpha}$ taking as inputs the aggregate capital stock and labor supply. z_t is an aggregate technology shock, common to all households, and assumed to satisfy

$$z_{t+1} = (1 - \rho_z)\, \mu_z + \rho_z z_t + \varepsilon_{t+1}^z \tag{5}$$

where $0 < \rho_z < 1$, $\mu_z > 0$ and ε_{t+1}^z a bounded i.i.d. disturbance with mean and variance $\left(0, \sigma_z^2\right)$. These aggregate inputs imply market interest and wage rates equal to

$$\begin{aligned} r\,(k_t, l_t, z_t) &= \alpha z_t \,(k_t/l_t)^{\alpha-1} \\ w\,(k_t, l_t, z_t) &= (1-\alpha)\, z_t \,(k_t/l_t)^{\alpha}\,. \end{aligned}$$

Household i's income is then determined as

$$y_{i,t} = r\,(k_t, l_t, z_t)\, a_{i,t} + w\,(k_t, l_t, z_t)\, e_{i,t}\bar{l}.$$

To solve the optimization problem agents must forecast future prices. Under the maintained assumptions $\{l_t,\ z_t\}$ are governed by exogenously given stochastic processes. (In fact, relation (4) reveals l_t to be constant.) Therefore, to forecast future wage and rental rates, agents require knowledge of the stochastic process describing the evolution of the aggregate capital stock. However, the stochastic properties of the aggregate capital stock depend on the distribution of capital holdings in the population. Denote this distribution by Γ_t and associated law of motion

$$\Gamma_{t+1} = H(\Gamma_t, z_t). \tag{6}$$

This completes the description of the model.

To summarize, the model can be written as the following dynamic programming problem:

$$v\left(a_{i,t}, e_{i,t}; \Gamma_t, z_t\right) = \max_{c_{i,t}, a_{i,t+1}} \left[u\left(c_{i,t}\right) + \beta E_t v\left(a_{i,t+1}, e_{i,t+1}; \Gamma_{t+1}, z_{t+1}\right)\right] \tag{7}$$

subject to

$$a_{i,t+1} = (1-\delta)\, a_{i,t} + r\left(k_t, l_t, z_t\right) a_{i,t} + w\left(k_t, l_t, z_t\right) e_{i,t}\bar{l} - c_{i,t} \tag{8}$$

and relations (1), (2), (3), (4), (5) and (6). For later use, note that the first order conditions for optimality are given by the Kuhn-Tucker conditions

$$u_c\left(c_{i,t}\right) \geq \beta E_t\left[u_c\left(c_{i,t+1}\right)\left(r\left(k_{t+1}, l_{t+1}, z_{t+1}\right) + 1 - \delta\right)\right] \tag{9}$$

with equality if $a_{i,t+1} > 0$ combined with relations (1), (3), (4), (5), (6) and (8). This class of problem is difficult to solve because the law of motion for the wealth distribution is unknown and in principle an infinite dimensional object.

3 Perturbation Methods

This section describes the perturbation approach. Such methods seek to approximate the model solution to an arbitrary degree of accuracy in the neighborhood of some point of interest in the model space. This point is typically taken to be the model's steady state which is discussed further below

Several conceptual issues must be confronted. First, perturbation methods are not well equipped to handle inequality constraints of the kind implied by the borrowing constraint (2). Second, the aggregation conditions (3) and (4) impose significant structure on our model solution — structure that is not present in analogous representative agent models. The following sections deal with each of these issues in turn. We show how to modify the optimization problem so as to remove the inequality constraint implied by the restriction on borrowing. Perturbation methods are then described for a representative agent version of the model both to introduce notation and the basic solution method. The intricacies introduced by the presence of heterogenous agents and associated aggregation conditions for the solution are then delineated.

3.1 Borrowing Constraints

Because the optimality conditions for the model involve the complementary slackness conditions (9), the Euler equation does not hold with equality when the borrowing constraint binds. This presents a difficulty for perturbation methods which require model equations to be differentiable, at least to a degree commensurate with the degree of accuracy of the approximation.

To accommodate this requirement we make use of a long literature in the linear programming and non-linear optimization fields of applied mathematics on interior methods for optimization problems subject to inequality constraints — see Forsgren, Gill, and Wright (2002) for a review and detailed references therein. The idea is to replace the problem of maximizing the objective function (7) subject to the inequality constraint (2) with an unconstrained maximization problem. This is achieved by defining a composite function that reflects the properties of the unconstrained objective function and the constraint.

To this end, define the interior function

$$I\left(a_{i,t+1}\right) = \frac{1}{\left(a_{i,t+1} + b\right)^2}$$

which has the property that as individual asset holdings approach the borrowing constraint b the interior function approaches infinity. Now consider modifying the Bellman equation to give the composite function:

$$\tilde{v}\left(a_{i,t}, e_{i,t}; \Gamma_t, z_t\right) = \max_{c_{i,t}, a_{i,t+1}} \left[u\left(c_{i,t}\right) + \beta E_t \tilde{v}\left(a_{i,t+1}, e_{i,t+1}; \Gamma_{t+1}, z_{t+1}\right) - \phi I\left(a_{i,t+1}\right)\right].$$

It has the property that for small $\phi > 0$ the maximization problem behaves like the unconstrained maximization of (7). When $a_{i,t+1}$ approaches b the interior function tends to dominate the value function $v\left(a_{i,t}, e_{i,t}; \Gamma_t, z_t\right)$ leading to large negative values. The composite function therefore penalizes consumption-savings decisions that lead to an asset position near the borrowing limit. Importantly, the unconstrained problem retains all relevant differentiability properties of the original problem.

Forsgren, Gill, and Wright (2002) provide theorems under which the maximand of the composite function converge to the maximand of the original problem as $\phi \rightarrow 0$. Moreover, bounds can be determined on the magnitude of the error in the maximand obtained from the modified problem with small ϕ. We shall not develop the theory of interior methods further since we intend to take a second order approximation to this modified problem. However, several points should be underscored. Interior methods are very close in spirit to penalty functions used to solve the original Bellman equation when using numerical methods and value function iteration. Penalty functions heavily penalize the value function for decisions that violate the borrowing constraint. Hence, at this stage we have not departed in an important way from the recent approaches to solving this class of model — for example Krusell and Smith (1998) and Khan and Thomas (2005). The important departure is the use of perturbation methods which requires an explicit statement of the adopted penalty or barrier function. It is this departure that leads to different approximations relative to value function based methods. We later evaluate the magnitude of the induced approximation errors and show them to be comparable, if not smaller, to solutions based on value function iteration. Furthermore, in our simulation studies, we make sure that the penalty ϕ is sufficiently large to ensure agents do not violate the borrowing constraint.

Such penalty functions appear in various literatures. In related work Kim, Kim, and Kollmann (2005) directly introduce a penalty term in the utility function to enforce the same kind of borrowing constraint. Rotemberg and Woodford (1999) in an analysis of monetary policy concerned with the implications of the lower bound on nominal interest rates impose a penalty function on the central bank's objective to ensure that nominal interest rates are always non-negative.

An alternative approach is to introduce quadratic costs to adjusting capital holdings of the form $\phi\left(a_{i,t}+b\right)^{-2}$. As asset holdings approach the borrowing limit b the cost tends to infinity. The interpretation is that individuals that are close to their borrowing limit, therefore representing "bad credit risks", must expend greater resources to secure loans. Hence in equilibrium agents will never choose a sequence of $\{c_{i,t},\ a_{i,t}\}$ pairs that lead to (2) holding with equality. This approach is similar to the recent literature on small open economy macroeconomic models which has adopted debt sensitive interest rate premia to ensure stationarity of foreign debt holdings in equilibrium — see Benigno (2001), Kollmann (2002) and Schmitt-Grohe and Uribe (2003). This approach gives similar results to those we report for the interior method.

3.2 The Representative Agent Model

For ease of exposition, consider a representative agent version of the model described in Section 2. This will facilitate introduction of notation and the basics of the perturbation approach in obtaining a second order accurate characterization of the model. Our notation follows Schmitt-Grohe and Uribe (2004) though the analysis is otherwise identical to Judd (1998), Jin and Judd (2002) and Kim, Kim, Schaumburg, and Sims (2003).

To generate a representative agent model assume that there are no idiosyncratic labor employment shocks and that each household inelastically supplies a unit of labor. Hence all agents will be identical ex ante and ex post so that $a_{i,t} = k_t$ for all i in equilibrium. The equilibrium for this model is determined by the optimality conditions

$$\begin{aligned} u_c(c_t) &= \beta E_t \left[u_c(c_{i,t+1}) \left(r(k_{t+1}, l_{t+1}, z_{t+1}) + 1 - \delta \right) \right] + \frac{2\phi}{(k_{t+1} + b)^3} \\ k_{t+1} &= (1-\delta) k_t + r(k_t, l_t, z_t) k_t + w(k_t, l_t, z_t) \bar{l} - c_t \end{aligned}$$

and relation (5). These conditions can then be summarized by

$$\begin{aligned} E_t F(c_{t+1}, c_t, x_{t+1}, x_t) &= E_t \begin{bmatrix} c_t^{-\gamma} - \beta c_{i,t+1}^{-\gamma} \left(r(k_{t+1}, l_{t+1}, z_{t+1}) + 1 - \delta \right) - \frac{2\phi}{(k_{t+1}+b)^3} \\ k_{t+1} - (1-\delta) k_t - r(k_t, l_t, z_t) k_t - w(k_t, l_t, z_t) \bar{l} + c_t \\ z_{t+1} - (1-\rho_z)\mu_z - \rho_z z_t - \varepsilon_{t+1}^z \end{bmatrix} \\ &= 0 \end{aligned} \tag{10}$$

where

$$x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}.$$

The solution to this model is of the form

$$\begin{aligned} c_t &= g(x_t, \sigma) \\ x_{t+1} &= h(x_t, \sigma) + \eta \sigma \varepsilon_{t+1} \end{aligned} \tag{11}$$

for unknown functions g and h with dimension (1×1) and (2×1) respectively. $\sigma > 0$ scales the degree of uncertainty in ε_{t+1}, itself a (2×1) vector, and η is a (2×2) selection matrix, designating how primitive shocks enter the state equations. This solution represents a generalization of the standard state-space representation of a linear rational expectations model. The first relation gives the policy function for the endogenous decision variables while the second describes the evolution of the model's state variables. In contrast to the linear case, the solution is here given by an arbitrary non-linear mapping from current state variables to the optimal allocations for consumption and future states. Perturbation methods seek to approximate the functions g and h in the neighborhood of the model's steady state $(\bar{c}, \bar{x})$, defined by the relations $\bar{c} = g(\bar{x}, 0)$ and $\bar{x} = h(\bar{x}, 0)$.

The second order approximation of the functions g and h around the steady state $(x_t, \sigma) = (\bar{x}, 0)$ yields

$$\begin{aligned} g(x,\sigma) &= g(\bar{x},0) + \sum_m g_{x_m}(\bar{x},0)(x_m - \bar{x}_m) + g_\sigma(\bar{x},0)\sigma \\ &+ \frac{1}{2}\sum_{m,n} g_{x_m x_n}(\bar{x},0)(x_m - \bar{x}_m)(x_n - \bar{x}_n) \\ &+ \frac{1}{2}\sum_m g_{x_m\sigma}(\bar{x},0)(x_m - \bar{x}_m)\sigma + \frac{1}{2}\sum_m g_{\sigma x_m}(\bar{x},0)(x_m - \bar{x}_m)\sigma \\ &+ \frac{1}{2} g_{\sigma\sigma}(\bar{x},0)\sigma^2 \end{aligned} \tag{12}$$

and

$$\begin{aligned} h(x,\sigma)^j &= h(\bar{x},0)^j + \sum_m h_{x_m}(\bar{x},0)^j (x_m - \bar{x}_m) + h_\sigma(\bar{x},0)^j \sigma \\ &+ \frac{1}{2}\sum_{m,n} h_{x_n x_m}(\bar{x},0)^j (x_m - \bar{x}_m)(x_n - \bar{x}_n) \\ &+ \frac{1}{2}\sum_m h_{x_m\sigma}(\bar{x},0)^j (x_m - \bar{x}_m)\sigma + \frac{1}{2}\sum_m h_{\sigma x_m}(\bar{x},0)^j (x_m - \bar{x}_m)\sigma \\ &+ \frac{1}{2} h_{\sigma\sigma}(\bar{x},0)^j \sigma^2 \end{aligned} \tag{13}$$

where $j,\ m,\ n = 1, 2$. Here j indexes the law of motion of the predetermined variable under consideration — either the capital stock or the technology shock — and therefore selects a particular element of the vector of non-linear functions given by $h(x, \sigma)$ in (11). m and n index the same two state variables in the construction of the approximation. For instance, $h^1_{x_1 x_2}$ gives the cross partial derivative with respect to k and z for the non-linear law of motion for k.

The unknowns in these Taylor expansions are given by the set of first order derivatives

$$g_{x_m},\ g_\sigma,\ h^j_{x_m},\ h^j_\sigma, \tag{14}$$

and the second order derivatives

$$g_{x_n x_m},\ g_{x_m\sigma},\ g_{\sigma x_m},\ g_{\sigma\sigma},\ h^j_{x_n x_m},\ h^j_{x_m\sigma},\ h^j_{\sigma x_m},\ h^j_{\sigma\sigma} \tag{15}$$

for $j,\ m,\ n = 1, 2$. Hence there are 9 unknown first order terms and 17 second order terms. These unknown coefficients can be solved for by taking the corresponding first and second derivatives of (10) with respect to x and σ and evaluating the resulting expression at the steady state $(x_t, \sigma) = (\bar{x}, 0)$.

Consider the first order terms. Taking derivatives of (10) with respect to x and σ yields

$$F_{x_m} = 0 \text{ and } F_\sigma = 0 \text{ for } m = 1, 2.$$

The first set of conditions provide 6 quadratic equations in the 6 unknowns g_{x_m} and $h^j_{x_m}$. The solution of these coefficients follows immediately from standard methods, such as a Schur decomposition or eigenvalue decomposition problem, and are analogous to solving for a unique determinate equilibrium in linear rational expectations models. The second restriction provides three equations in the three unknowns g_σ and h^j_σ for $j = 1, 2$.

The second order coefficients can similarly be determined by computing the second order derivatives of F to give

$$F^j_{x_m x_k} = 0\ ;\quad F^j_{\sigma\sigma} = 0;\qquad F^j_{x_m\sigma} = 0$$

for $j = 1, 2, 3$ and $m, n = 1, 2$. This gives 17 linear equations in the 17 unknowns. Worth noting is that $g_{\sigma\sigma}$ and $h^j_{\sigma\sigma}$ provide corrections to the mean of each variable due to the presence of uncertainty. In a first order approximation certainty equivalence holds and uncertainty does not affect the elasticities of optimal decision rules. The elasticities on second order terms are similarly unaffected. The effects of uncertainty on the model solution are fully captured by constant adjustments to the mean of each variable. We will prove a similar result for the heterogenous agent model.

Having determined the unknowns (14) and (15), relations (12) and (13) completely characterize a second-order accurate solution to the model. In particular, (13), once substituted into (11), provides a law of motion for the aggregate capital stock. This stochastic process determines the time series distribution of aggregate capital used to forecast future prices.

3.3 Heterogeneous Agent Model

Our task is to determine the probability distribution characterizing the stochastic evolution of the aggregate capital stock that is accurate to the second order. In contrast to the representative agent model, this computation is complicated by the presence of heterogeneity in capital holdings across households. Indeed, individual consumption and saving decisions, and therefore the aggregate capital stock, can now depend on an additional set of state variables relevant to describing the evolving distribution of wealth in the economy.

3.3.1 The Steady State

Before proceeding to the analysis of the relevant set of state variables, a few words on the steady state of the model are appropriate. Following Kydland and Prescott (1982), and as often done in the modern macroeconomics literature, an approximation to the model is sought in the neighborhood of the model's deterministic steady state. The steady state is characterized by a situation in which there are no aggregate shocks and no idiosyncractic shocks. In the absence of idiosyncractic shocks all agents are both ex ante and ex post identical. It follows that in the deterministic steady state the wealth distribution is degenerate: all agents hold the same quantity of the aggregate capital stock. Hence the cross-sectional distribution has unit probability mass on this aggregate quantity of capital. For example, the cross sectional variance of capital holdings is equal to zero in this steady state as is the cross sectional covariance between capital holdings and employment status. These two moments of the wealth distribution are revealed to be central to our second order approximation developed in the sequel. What the second order approximation does, is approximate the wealth distribution in the neighborhood of this degenerate wealth distribution. It does not seek to approximate the dynamics around some ergodic wealth distribution that might obtain with idiosyncractic shocks but no aggregate shocks as in the analysis of Huggett (1993).

3.3.2 State Variables Defined

To understand the set of possible state variables relevant to the evolution of the aggregate capital stock, consider the set of state variables relevant to individual i's decision problem at the first order. They are: $\{a_{i,t},\ e_{i,t},\ z_t\}$. Noting that optimal decisions will be linear in these state variables in our approximation, the aggregation constraint (3) then delivers a fourth state variable in the aggregate capital stock, k_t. Since individual decisions will depend on these variables, so too will the aggregate capital stock as it follows directly from summing individual saving decisions. Because individual decisions are linear in these four state variables, it is immediate that the aggregate capital stock at the first order can only depend on $\{k_t, z_t\}$ once the properties of (3) are applied. The sequel will discuss further the relation between the coefficients in the individual decision rules and the aggregate capital accumulation equation.

Which second order terms are relevant to the household's saving decision? In principle, decisions could depend on all pair-wise combinations of $\{a_{i,t},\ e_{i,t},\ z_t,\ k_t\}$ appearing in a second order polynomial of these first-order state variables. Hence the set of second order terms in deviations from steady state values are

$$
\begin{aligned}
&(a_{i,t}-\bar{a})(e_{i,t}-\bar{e}),\ (a_{i,t}-\bar{a})\left(k_t-\bar{k}\right),\ (a_{i,t}-\bar{a})(z_t-\bar{z}),\ (a_{i,t}-\bar{a})^2,\ (e_{i,t}-\bar{e})^2,\\
&(e_{i,t}-\bar{e})\left(k_t-\bar{k}\right),\ (e_{i,t}-\bar{e})(z_t-\bar{z}),\ \left(k_t-\bar{k}\right)^2,\ \left(k_t-\bar{k}\right)(z_t-\bar{z}),\ (z_t-\bar{z})^2. \qquad (16)
\end{aligned}
$$

Again, because the dynamic equation describing individual saving decisions is linear in these state variables, and because individual decisions must satisfy the aggregation constraint, the aggregate capital stock can only depend on the following six objects at the second order:

$$
\begin{aligned}
&\int_0^1 (a_{i,t}-\bar{a})(e_{i,t}-\bar{e})\,di,\quad \int_0^1 (a_{i,t}-\bar{a})^2\,di,\quad \left(k_t-\bar{k}\right)^2,\\
&\left(k_t-\bar{k}\right)(z_t-\bar{z}),\quad (z_t-\bar{z})^2,\quad \int_0^1 (e_{i,t}-\bar{e})^2\,di
\end{aligned}
$$

on noting that

$$
\begin{aligned}
\int_0^1 (a_{i,t}-\bar{a})\left(k_t-\bar{k}\right)di &= \left(k_t-\bar{k}\right)^2;\ \int_0^1 (a_{i,t}-\bar{a})(z_t-\bar{z})\,di = \left(k_t-\bar{k}\right)(z_t-\bar{z});\\
\int_0^1 (e_{i,t}-\bar{e})\left(k_t-\bar{k}\right)di &= \int_0^1 (e_{i,t}-\bar{e})(z_t-\bar{z})\,di = 0.
\end{aligned}
$$

The first five terms represent the aggregate second order state variables while the final term is a constant, representing a correction to the mean aggregate capital stock due to the presence of idiosyncratic risk.

Hence in addition to the second order polynomial terms $\left\{\hat{k}_t^2,\ \hat{k}_t\hat{z}_t,\ \hat{z}_t^2\right\}$ (introducing the notation that for any variable x, $\hat{x}_t = (x_t - \bar{x})$ gives the deviation from steady state) two new state variables are induced from aggregation:

$$\Phi_t \equiv \int_0^1 (a_{i,t} - \bar{a})^2\, di \text{ and } \Psi_t \equiv \int_0^1 (a_{i,t} - \bar{a})(e_{i,t} - \bar{e})\, di. \tag{17}$$

The former represents the cross-sectional variance of capital holdings while the latter gives the cross-sectional covariance between asset holdings and employment status. The dynamics of these two state variables will be central to characterizing the evolution of the wealth distribution. Note also that by construction these variables take a value of zero in the deterministic steady state. Our second order approximation characterizes the dynamics of these state variables in the neighborhood of a degenerate wealth distribution.

Because the aggregate capital stock can depend on these variables, it follows that in equilibrium individual household decisions must similarly depend on these second order objects. Hence, the set of primitive objects relevant to individual household decisions are given by

$$\{a_{i,t},\ e_{i,t},\ z_t,\ k_t,\ \Phi_t,\ \Psi_t\}. \tag{18}$$

Optimal decision rules to a second order will then depend on a second order polynomial in these state variables. Note that there will be no cross-product terms involving Φ_t and Ψ_t since these are inherently second order objects implying all such terms are third order or higher and therefore irrelevant to our second order approximation. Since decisions are linear in these state variables, aggregation then ensures that the aggregate capital stock depends only on the aggregate quantities

$$\left\{\ \hat{z}_t,\ \hat{k}_t,\ \hat{k}_t^2,\ \hat{k}_t\hat{z}_t,\ \hat{z}_t^2,\ \hat{\Phi}_t,\ \hat{\Psi}_t\right\}.$$

The model characterized in Section 2 can now be reformulated. Because (18) completely characterizes the primitive state variables relevant to household decisions to the second order, rewrite the model as

$$u_c(c_{i,t}) = \beta E_t\left[u_c(c_{i,t+1})\left(r(k_{t+1},l_{t+1},z_{t+1})+1-\delta\right)\right]+2\phi(a_{i,t}+b)^{-3} \quad (19)$$

$$a_{i,t+1} = (1-\delta)a_t + r(k_t,l_t,z_t)a_{i,t}+w(k_t,l_t,z_t)\bar{l}e_{i,t}-c_{i,t} \quad (20)$$

the exogenous processes

$$e_{i,t+1} = (1-\rho_e)\mu_e+\rho_e e_{i,t}+\varepsilon^e_{i,t+1}$$

$$z_{t+1} = (1-\rho_z)\mu_z+\rho_z z_t+\varepsilon^z_{t+1}$$

and the laws of motion of the endogenously determined aggregate state variables

$$k_{t+1} = h^k(k_t,z_t,\Phi_t,\ \Psi_t,\ \sigma) \equiv \int_0^1 a_{i,t+1}di \quad (21)$$

$$\Phi_{t+1} = h^\Phi(k_t,z_t,\Phi_t,\ \Psi_t,\ \sigma) \equiv \int_0^1 (a_{i,t+1}-\bar{a})^2 di \quad (22)$$

$$\Psi_{t+1} = h^\Psi(k_t,z_t,\Phi_t,\ \Psi_t,\ \sigma) \equiv \int_0^1 (a_{i,t+1}-\bar{a})(e_{i,t+1}-\bar{e})\,di. \quad (23)$$

Stacking these relations then permits the model to be written as

$$E_tF(c_{t+1},c_t,x_{t+1},x_t)=0$$

redefining the state vector as $x_t=\{a_{i,t},\ e_{i,t},\ z_t,\ k_t,\ \Phi_t,\ \Psi_t\}$. The solution to this model again takes the form

$$c_t = g(x_t,\sigma)$$

$$x_{t+1} = h(x_t,\sigma)+\eta\sigma\varepsilon_{t+1} \quad (24)$$

where $h(x_t,\sigma)$ is now a (6×1) vector corresponding to the states x_t. Hence, relative to the original problem, the reformulated problem has exchanged the true law of motion for the wealth distribution (6) with an approximate law of motion embodied in the dynamics of the final three relations described in (21) - (23). Worth underscoring is that the modified problem nonetheless provides a second-order accurate characterization of (6) given the arguments presented above.

3.4 Further Restrictions From Aggregation

So far we have discussed how heterogeneity and aggregation interact to determine the set of state variables relevant to the dynamics and stochastic properties of the aggregate capital stock. However, aggregation also imposes restrictions on the equilibrium coefficients that can obtain in the individual and aggregate capital accumulation equations.

Under the reformulated model, the equilibrium laws of motion for individual capital holdings in a second order approximation have the general form

$$\begin{aligned} h(x,\sigma)^a &= h(\bar{x},0)^a + \sum_m h^a_{x_m}(\bar{x},0)(x_m - \bar{x}_m) + h^a_\sigma(\bar{x},0)\sigma \\ &+ \frac{1}{2}\sum_{n,m} h^a_{x_n x_m}(\bar{x},0)(x_m - \bar{x}_m)(x_n - \bar{x}_n) \\ &+ \frac{1}{2}\sum_m h^a_{x_m \sigma}(\bar{x},0)(x_m - \bar{x}_m)\sigma + \frac{1}{2}\sum_m h^a_{\sigma x_m}(\bar{x},0)(x_m - \bar{x}_m)\sigma \\ &+ \frac{1}{2}h^a_{\sigma\sigma}(\bar{x},0)\sigma^2 + h^a_\Phi(\bar{x},0)\left(\Phi - \bar{\Phi}\right) + h^a_\Psi(\bar{x},0)\left(\Psi - \bar{\Psi}\right) \end{aligned}$$

where $x = \{a_{i,t},\ e_{i,t},\ z_t,\ k_t\}$. The expansion is taken around the model's steady state $(\bar{x},0)$ with no aggregate or idiosyncratic uncertainty (i.e. $\sigma = 0$). Note that because $\{\Phi_t,\ \Psi_t\}$ capture second order variation, these variables only appear in the final two terms. Similarly, aggregate capital satisfies

$$\begin{aligned} h(x,\sigma)^k &= h(\bar{x},0)^k + \sum_m h^k_{x_m}(\bar{x},0)(x_m - \bar{x}) + h^k_\sigma(\bar{x},0)\sigma \\ &+ \frac{1}{2}\sum_{n,m} h^k_{x_n x_m}(\bar{x},0)(x_m - \bar{x}_m)(x_n - \bar{x}_n) \\ &+ \frac{1}{2}\sum_m h^k_{x_m \sigma}(\bar{x},0)(x_m - \bar{x}_m)\sigma + \frac{1}{2}\sum_m h^k_{\sigma x_m}(\bar{x},0)(x_m - \bar{x}_m)\sigma \\ &+ \frac{1}{2}h^k_{\sigma\sigma}(\bar{x},0)\sigma^2 + h^k_\Phi(\bar{x},0)\left(\Phi - \bar{\Phi}\right) + h^k_\Psi(\bar{x},0)\left(\Psi - \bar{\Psi}\right). \end{aligned}$$

Aggregation then requires

$$\int_0^1 a_{i,t+1} di = \int_0^1 h(x_t,\sigma)^a \, di = h(x_t,\sigma)^k = k_{t+1}. \tag{25}$$

Using the facts that $\int_0^1 (a_{i,t} - \bar{a})\, di = \left(k_t - \bar{k}\right)$ and $\int_0^1 (e_{i,t} - \bar{e}) di = 0$ relation (25) imposes the following restrictions on the model solution. The first order coefficients must satisfy

$$h^k_a = h^k_e = 0,\ h^k_z = h^a_z \text{ and } h^k_k = h^a_a + h^a_k. \tag{26}$$

The second order coefficients are similarly shown to satisfy

$$h^k_{kk} = h^a_{ak} + h^a_{kk};\ h^k_{zz} = h^a_{zz};\ h^k_{kz} = h^a_{az} + h^a_{zk};\ h^k_\Phi = h^a_\Phi + h^a_{ae};\ h^k_\Psi = h^a_\Psi + h^a_{aa} \tag{27}$$

and

$$h^k_{\sigma\sigma} = h^a_{\sigma\sigma} + h^a_{ee} \int_0^1 (e_{i,t} - \bar{e})^2 \, di = h^a_{\sigma\sigma} + h^a_{ee}\sigma^2_e. \tag{28}$$

The latter restriction represents the correction to the mean capital stock. It comprises two components: one due to the aggregation of idiosyncratic risk and one due to aggregate risk given by $h^a_{\sigma\sigma}$. All remaining coefficients on second order terms are equal to zero.

The two equations (22) and (23) similarly impose structure on the coefficients of the second order approximation. The appendix shows that the following coefficient restrictions must be satisfied:

$$h^\Phi_\Phi = (h^a_a)^2\,;\; h^\Phi_\Psi = 2h^a_e h^a_a;\; h^\Phi_{kk} = h^a_k(h^a_a + h^a_k);\; h^\Phi_{zz} = (h^a_z)^2\,;\; h^\Phi_{zk} = 2h^a_z\,(h^a_a + h^a_k) \tag{29}$$

will all other coefficients equal to zero. The aggregation of individual specific risk also introduces a correction to the mean equal to

$$h^\Phi_{\sigma\sigma} = (h^a_a)^2 \int_0^1 (e_{i,t} - \bar{e})^2 \, di = (h^a_a)^2\,\sigma^2_e. \tag{30}$$

Finally, the law of motion for Ψ_t provides the restriction

$$h^\Psi_\Psi = \rho_e h^a_a \tag{31}$$

with all other coefficients equal to zero. Again, there is a correction to the mean from the aggregation of individual specific risk equal to

$$h^\Psi_{\sigma\sigma} = h^a_e \rho_e \int_0^1 (e_{i,t} - \bar{e})^2 \, di = h^a_e \rho_e \sigma^2_e. \tag{32}$$

Note that the dynamics for Φ_t and Ψ_t depend only on the first order coefficients appearing in the individual and aggregate capital equations and so introduce no new unknowns to be determined.

3.5 The Solution

Given the reformulated problem and the restrictions imposed by aggregation on the permissible elasticities in the individual and aggregate laws of motion for capital, the solution can proceed as for the representative agent case. Analogous to the analyses of Jin and Judd (2002) and Kim, Kim, Schaumburg, and Sims (2003) and theorem 1 of Schmitt-Grohe and Uribe (2004) for representative agent models, we prove the following result for heterogenous agent models.

Theorem 1 *All elasticities in the second order approximation to (24) — comprising the laws of motion (19) - (23) and the two exogenous disturbance processes — are independent of uncertainty. That is*

$$g_{\sigma}(\bar{x}, 0) = h_{\sigma}(\bar{x}, 0) = g_{x\sigma}(\bar{x}, 0) = h_{x\sigma}(\bar{x}, 0) = 0$$

for all $x \in \{a_{i,t}, e_{i,t},\ z_t, k_t,\ \Phi_t,\ \Psi_t\}$.

The proof is in the appendix which also outlines in detail the solution method and the full set of restrictions that are required to solve for the unknown coefficients characterizing the second order approximation. It shows that in a second order approximation, the laws of motion for the cross sectional variance of asset holdings (22) and the cross sectional covariance between asset holdings and employment status (23) depend only on the first order elasticities implied by the dynamics of (19), (20) and (21). To solve for the remaining elasticities, relations (19) and (20) provide 36 restrictions in 54 unknowns. The remaining 18 restrictions are determined by the aggregation constraints implied by (21) and given in (26) - (28). An immediate implication of theorem 1 is that the direct impact of uncertainty on optimal decisions is reflected in the model solution via the constants $g_{\sigma\sigma}$ and $h^j_{\sigma\sigma}$ — terms which represent the impact of risk on mean decisions. For instance, $g_{\sigma\sigma}$ represents the correction to an individual's mean consumption relative to steady state due solely to the presence of uncertainty. It therefore represents precautionary savings. Note that the finding $g_{\sigma}(\bar{x}, 0) = h_{\sigma}(\bar{x}, 0) = 0$ is the usual certainty equivalence result associated with first order approximations and linear-quadratic models.

Perturbation methods present several advantages in solving this class of problem. First, the solution technique analytically determines individual decision rules which are optimal to the second order. As such, the implications of aggregation can easily be analyzed, with the role of various state variables in determining the evolution of the aggregate capital stock quickly identified. This permits a careful examination of which moments of the wealth distribution are important for aggregate dynamics. Second, there are considerable practical advantages. Because the solution is based on standard analytical methods for solving quadratic and linear systems of equations, the model can be solved in fractions of a second in contrast to value function iteration-based methods. Similarly, because the approach can handle high dimension state spaces, arbitrary specification of the exogenous disturbance processes can be handled.

4 Results

The following section delineates some properties of the model solution. While fully analytic solutions could be presented, they are cumbersome and not readily interpretable. The following therefore exploits a calibration study of the model. Our benchmark calibration is discussed in detail, highlighting some of the qualitative properties of the model solution. The dependency of optimal decisions on the evolving wealth distribution is discussed and the implications for aggregation and aggregate dynamics made transparent. The quantitative predictions of the model are then explored for a number of alternative calibrations.

4.1 The Calibration and Steady State

The time period is one quarter. The intertemporal discount factor β is set equal to 0.98 and the depreciation rate δ to 0.025. The relative risk aversion parameter γ equals 2 and the share of capital α is 0.36. The normalizing constant $\bar{l}$ is set equal to 0.32 so that agents work a third of their available hours in steady state. The aggregate technology shock is specified by $\mu_z = 1$, $\rho_z = 0.75$ and $\sigma_z = 0.0132$ to correspond to the process adopted by Krusell and Smith (1998). The law of motion for individual's employment status is modified to

$$e_{i,t+1} = (1 - \rho_e)\,\mu_e + \rho_e e_{i,t} + \rho_{ze}(z_{t+1} - \bar{z}) + \varepsilon^e_{i,t+1}$$

to allow for the state of the labor market to depend on the aggregate state. This does not affect the solution method in any way, though does affect the determined elasticities on z_t in the optimal decision rules. The individual's employment status is specified as $\mu_e = 0.93$, $\rho_e = 0.70$, and $\sigma_e = 0.05$. Initially we take $\rho_{ze} = 0$ so that an individual's employment status is not correlated with the aggregate state. This facilitates comparison to a representative agent version of the model, isolating implications of incomplete markets and heterogeneity. We later choose ρ_{ze} to give an average unemployment rate of 7 percent, with 4 percent and

10 percent unemployment rate on average when there is positive and negative one standard deviation shock to technology leaving the remaining parametric assumptions unchanged.

The analysis assumes agents are constrained to hold positive quantities of the capital stock and therefore face a borrowing constraint of the form $a_{i,t+1} \geq 0$ so that the borrowing limit is $b = 0$. The parameter ϕ governing the sensitivity to the borrowing constraint in the modified utility function is set equal to 0.05. This ensures that no agent violates the borrowing constraint. Our results are similar for a range of values for ϕ though we note here that it has implications for the properties of the cross-sectional wealth distribution. The steady state is chosen to be the non-stochastic solution of the model in which all agents own the same amount of capital, so that $\bar{a}_i = \bar{k}$ for all i.

4.2 Optimal Decision Rules

Optimal saving decisions imply the following second order accurate law of motion for individual capital holdings:

$$\begin{aligned}\hat{a}_{i,t+1} \quad = \quad & 0.0003 + 0.9993\hat{a}_{i,t} + 0.6288\hat{e}_{i,t} + 0.8574\hat{z}_t - 0.0278\hat{k}_t \\ & +0.0002\hat{a}_{i,t}^2 + 0.0006\hat{a}_{i,t}\hat{e}_{i,t} + 0.0458\hat{a}_{i,t}\hat{z}_t - 0.0031\hat{a}_{i,t}\hat{k}_t \\ & +0.0006\hat{e}_{i,t}^2 - 0.6465\hat{e}_{i,t}\hat{z}_t + 0.0300\hat{e}_{i,t}\hat{k}_t \\ & +0.0036\hat{z}_t^2 - 0.0010\hat{z}_t\hat{k}_t + 0.0025\hat{k}_t^2 - 0.0009\hat{\Phi}_t - 0.00005\hat{\Psi}_t \end{aligned} \tag{33}$$

with all variables interpreted as deviations from steady state. The consumption allocation rule is not presented to conserve space.

Several points are worthy of note. First, the optimal decision rule depends on all state variables at the first and second order. Hence, optimal consumption and saving decisions depend on all variables relevant to the evolution of the wealth distribution — no elasticities are analytically found to be zero. While the coefficients on some second order terms are quite small, with the coefficient on $\hat{\Phi}_t$ being zero to the third decimal point, this does not necessarily imply they are irrelevant as will be made clear in the next subsection. Second, the constant in the decision rule arises due to the effects of precautionary saving. In the presence of partially insurable risk, both aggregate and idiosyncratic, agents tend to hold more capital.

Third, this decision rule implicitly determines an individual's marginal propensity to save. In general this marginal propensity to save will vary across individuals according to their specific history of employment shocks and asset accumulation decisions. For the allocation of capital to matter in this economy is must be the case that marginal propensities to save differ across individuals so that different allocations of wealth engender differing consumption and savings decisions in the aggregate, a point to which we shall return.

Applying the aggregation constraint (3) determines the aggregate capital accumulation equation as

$$\begin{aligned}\hat{k}_{t+1} &= 0.0003 + 0.8573\hat{z}_t + 0.9714\hat{k}_t + 0.0036\hat{z}_t^2 + 0.0449\hat{z}_t\hat{k}_t - 0.0006\hat{k}_t^2 \\ &\quad -0.0007\hat{\Phi}_t + 0.0006\hat{\Psi}_t. \end{aligned} \tag{34}$$

The law of motion for aggregate capital inherits many of the properties of the individual laws of motion but only depends on aggregate states. Importantly, there are five second order terms relevant to the evolution of the aggregate capital stock

$$\left\{\hat{k}_t^2,\ \hat{k}_t\hat{z}_t,\ \hat{z}_t^2,\ \hat{\Phi}_t,\ \hat{\Psi}_t\right\}.$$

The coefficients on the state variables $\{\Phi_t, \Psi_t\}$ have negative and positive coefficients respectively. To understand why there is a negative coefficient on the variance of cross-sectional capital holdings, consider an increase in this variance holding total capital fixed. As the variance rises more capital is being held by individuals with a lower marginal propensity to save. Because this reallocation of capital results in higher aggregate consumption and lower saving aggregate capital must fall in the next period. The positive coefficient on the covariance of cross-sectional capital holdings with employment status reflects the fact that a higher positive correlation implies individuals with lower capital holdings also have worse employment outcomes. This risk leads individuals to save more resulting in higher aggregate capital.

To further understand the implications of heterogeneity for aggregate dynamics, consider the associated quasi-representative agent model derived under the assumption of no idiosyncratic employment shocks, though maintaining the assumption that agents face a borrowing constraint.[3] The aggregate capital dynamics are given by

$$\hat{k}_{t+1} = 0.0001 + 0.8573\hat{z}_t + 0.9714\hat{k}_t + 0.0033\hat{z}_t^2 + 0.0446\hat{z}_t\hat{k}_t - 0.0005\hat{k}_t^2 \quad (35)$$

and depends on the same set of aggregate state variables, with the exception of the cross sectional variance of capital holdings and covariance between capital holdings and employment status. Comparison with the heterogeneous agent case yields several important insights. First, precautionary savings effects, which lead to higher capital accumulation and are captured in the constant of the equilibrium laws of motion, while small, are some three times larger in the heterogeneous agent model than in the representative agent case. While the magnitudes are in large part a product of features of this specific model and the calibration (to be discussed in the sequel), in general the presence of partially insurable idiosyncractic risks leads to greater accumulation of capital. Note also that the theory of section 2 showed that this correction is determined by the relation

$$h^k_{\sigma\sigma} = h^a_{\sigma\sigma} + h^a_{ee}\sigma_e^2.$$

It follows that the second term of the correction, which arises due to the aggregation of second order variation in the idiosyncratic shocks, is very small under this calibration.

Second, heterogeneity does not affect the model solution at the first order: all elasticities on first order dynamics are identical across the representative and heterogeneous agent models — compare (34) and (35). To understand this, note that i) the first order elasticities are determined independently of the second order properties of the model (though these second order properties depend on the first order elasticities) and ii) a first order approximation to the heterogeneous agent model is equivalent to only keeping track of the mean of the wealth distribution and therefore aggregate capital. For these two reasons, at the first order, the distribution of wealth across agents is irrelevant to dynamics.

Third, and related, to see the effects of heterogeneity on dynamics, we must look to the second order terms. The coefficients on $\left\{\hat{z}^2,\ \hat{z}\hat{k},\ \hat{k}^2\right\}$ are broadly of the same magnitude, though the first and third coefficients are respectively 25 and 50 percent larger in the heterogeneous agent model. And of course, the cross sectional properties of the wealth distribution

[3] The true underlying representative agent models does not require a borrowing constraint determined by the natural debt limit as in Ayagari (1994) – the usual No-Ponzi condition suffices.

are also relevant to dynamics in this case. In this sense heterogeneity matters qualitatively though further work must be done to establish the quantitative implications of these terms.

4.3 Further Model Properties

To interpret the magnitudes of the reported coefficients on these second order terms and their implications for macroeconomic dynamics, Table 1 reports model implied statistics for aggregate consumption, the cross-sectional holdings of capital and welfare. For both the representative and heterogeneous agent models a first and second order approximate solution are given. In each case, the statistics are generated using the same simulated path for technology shocks. For the heterogeneous agent model we simulate 2000 sequences of idiosyncratic shocks of length 5000. The first 1000 observations are dropped to remove the effects of initial conditions.

Only by considering a second order approximation of the model can the effects of risk on optimal decisions be assessed. This precautionary savings effect is captured in the constant of the optimal decision rule. As mentioned, in the heterogeneous agent model the precautionary savings effect is 300 percent larger reflecting the presence of idiosyncractic risk in this model. Note, however, that under the present calibration there is little precautionary saving measured as a fraction of steady state consumption.

Analysis of the variance and first order serial correlation of aggregate consumption makes clear that these properties are largely determined by first order dynamics. Indeed, for the representative agent model, the standard deviations and serial correlation are identical at the third decimal place under both a first and second order approximation, while in the heterogeneous agent model the standard deviation shows a small discrepancy at the third decimal point for the standard deviation. Hence, aggregate dynamics appear to be almost entirely determined by first order model properties. Moving to a second order approximation, which introduces terms relevant to describing the evolving properties (aside from the mean) of the wealth distribution, adds little to our predictions concerning aggregate consumption dynamics. Hence under this calibration Krusell and Smith (1998) approximate aggregation emerges once more: it seems enough for predicting aggregate dynamics to keep track of the mean of the wealth distribution.

However, this does not imply incomplete markets are irrelevant, since borrowing constraints do affect first order dynamics. Several further caveats should also be underscored. Even though aggregate dynamics appear to be little affected by second order variation, heterogeneity is nonetheless relevant to the model solution. Average aggregate consumption is lower in the heterogeneous agent model in a second order approximation when compared to the representative agent model. That consumption is depressed on average in the presence of greater risk suggests welfare ought to be lower. Computing average ex post welfare measures – assigning each household an equal weight in the social objective function and computing the present discounted value of utility implied by the determined consumption allocations – supports this conjecture: conditional on given path of technology the representative agent model gives a present discounted utility of -14.56; in contrast, the heterogeneous agent model yields -14.92. The standard deviation of these ex post welfare measures are 4.279 and 4.379 respectively. Hence, not only is welfare lower on average across agents in the incomplete markets model, but perhaps more importantly, there is significant variation in the welfare of individual agents when faced with incomplete markets and idiosyncractic risk. The relevance of this observation for policy design is immediate.

To make this point more starkly, consider the final two columns which report identical statistics for a risky version of the economy. Here the idiosyncractic employment process is specified according to the parameterization $\rho_e = 0.75$ and $\sigma_e = 0.05$ and $\rho_{ze} = 0.45$. It is therefore more persistent than the benchmark case and also allows for employment status to be correlated with the aggregate state. Note that given the evidence provided by Storesletten, Telmer, and Yaron (2004) one could easily rationalize an income process with significantly greater persistence in the idiosyncratic shock.

The key insight emerging from this calibration is that more risky economies lead to important welfare consequences. Indeed, the variance in ex post welfare outcomes rises significantly. For the benchmark calibration the standard deviation was 4.379 while in the risky economy it is 5.228. Hence risk has non-trivial consequences when contemplating welfare costs of business cycles. Moreover, it seems clear that in more general modeling frameworks such heterogeneity will have important implications for policy design.

It is also clear that in order to conduct policy evaluation exercises using model consistent welfare measures requires a second order approximation to the complete model. A recent macroeconomics literature has highlighted that a linear-quadratic approximation to the true non-linear policy problem may not accurately rank alternative policies in terms of their implications for welfare — see Kim and Kim (2003), Kim, Kim, Schaumburg, and Sims (2003) and Woodford (2003) for discussions.

Finally, a property of the present model is that agents make a single decision: how much to save or consume. The only friction in the model is the incompleteness of financial markets. Consistently with Krusell and Smith (2006), it is our conjecture that in more general models in which agents face several economic decisions in the presence of a range of frictions, first order dynamics may be significantly affected. For example, Roca (2006) in an application of the methodology presented here, shows in a real business cycle model with labor market search that the presence of incomplete markets leads to important differences in first order dynamics relative to a complete markets setting. Because an household's wealth affects their wage bargain, heterogeneity affects dynamics to the first order. The absence of large first order effects in the analysis here likely reflects the sparse economic environment and the near linearity of the real business cycle model. As such it is probably not the most useful laboratory for the study of incomplete markets and heterogeneity.

4.4 Marginal Propensities to Save

Given the above discussion, it is clear that the capital accumulation equation of household i is almost linear in their own holdings of capital. There is very little curvature in the decision rule at the second order, with the exception of the elasticity on the $\hat{a}_{i,t}\hat{z}_t$ term, which takes a coefficient of 0.0722, though even this term represents relatively small variation. But this does not necessarily mean that the marginal propensity to save is close to unity. To make this clear, write the law of motion (33) as

$$\hat{a}_{i,t+1} = \tau_t + \tau_{i,t} + 0.9986\hat{a}_{i,t} + 0.0003\hat{a}_{i,t}^2 + 0.0013\hat{a}_{i,t}\hat{e}_{i,t} + 0.0463\hat{a}_{i,t}\hat{z}_t - 0.0033\hat{a}_{i,t}\hat{k}_t \quad (36)$$

where τ_t and $\tau_{i,t}$ collect aggregate and individual specific terms respectively that do not depend on individual wealth holdings. Note that τ_t necessarily depends on the terms $\hat{\Phi}_t$ and $\hat{\Psi}_t$.

Now if all agents are permanent income consumers then this function will be a ray out of the origin with slope approximately equal to one. Of course, given that perturbation methods account for variation in many dimensions we cannot plot $a_{i,t+1}$ against $a_{i,t}$ in two dimensions without making specific assumptions about the values of all other state variables, undermining the utility of the approach. Perhaps more useful is to note the slope of the function (36) with respect to current wealth holdings is

$$\frac{\partial a_{i,t+1}}{\partial a_{i,t}} = 0.9986 + 0.0006\hat{a}_{i,t} + 0.0013\hat{e}_{i,t} + 0.0463\hat{z}_t - 0.0033\hat{k}_t.$$

It depends on a constant that is close to unity, the individual's capital holdings and employment status, and also the aggregate state variables z_t and k_t. Hence changes in future wealth given variations in current wealth vary across individuals according to differences in asset holdings and employment status. However, the effects of these four first order state variables are small. For instance, the typical variation in $\hat{z}_t$ is of the order 0.02 making the term $0.0463\hat{z}_t$ relevant only at the third decimal point. As this is true of all terms, the slope under this baseline calibration is essentially given by the constant 0.9986. But this does not mean that the marginal propensity to save is equal to unity. The marginal propensity is determined by both the slope and location of the schedule (33). The location differs across individuals according to differences in the stochastic constant $\tau_{i,t}$. As individuals experience different employment histories $\tau_{i,t}$ will vary across agents giving rise to variations in the marginal propensity to save. Furthermore, the location of the schedule depends on the wealth distribution through τ_t.

Analysis of the benchmark calibration indicates that there is limited variation in $\tau_{i,t}$. As it is in deviations from steady state it takes values approximately equal to zero. In this case, agents behave very much like permanent income consumers, having a marginal propensity to save equal to unity. Agents consume the return on capital holdings each period but leave the principle intact. Despite facing idiosyncractic income uncertainty, having access to capital markets provides households with adequate self insurance. It is this feature of agents decision rules that gives rise to approximate aggregation.

5 Approximate Aggregation

Krusell and Smith (1998) provide a novel and clever solution to the model of section 2. Noting that the law of motion for the wealth distribution is in principle an infinite dimensional object, they propose solving a simplified version of the model. Agents, rather than forecasting future prices using the true distribution describing the evolution of aggregate capital, instead use a boundedly rational law of motion of the form

$$\begin{aligned} k_{t+1} &= \alpha_{g,0} + \alpha_{g,1} k_t \\ k_{t+1} &= \alpha_{b,0} + \alpha_{b,1} k_t \end{aligned} \tag{37}$$

which describes the evolution in good times and bad times respectively. This assumption serves to dramatically reduce the state space of the model so that value function iteration-based methods can be used to solve agents' dynamic programming problem. They find that the model displays an approximate aggregation property — future prices can be well forecasted using only the mean capital stock. Because most agents in the economy behave like permanent income consumers their saving decisions are almost linear in their own capital holdings. On aggregating, tomorrow's aggregate capital stock is then only a function of today's aggregate capital stock (the mean of the cross-sectional distribution of capital holdings). No other characteristics of this distribution are present.

Despite this cleverness, value function iteration methods still suffer the curse of dimensionality as the size of the state space increases and can therefore handle only low dimension state space models. As a result, their analysis only reports laws of motion for the aggregate capital stock that depend on the past aggregate capital and its square. As shown above, in a second order accurate approximation to the model, the aggregate capital stock depends on some 5 second order moments of the wealth distribution. The use of boundedly rational laws of motion for the aggregate capital stock omits significant information relevant to forecasting future prices, and may engender rather different equilibrium dynamics to the true model.

While the previous section provides evidence that equilibrium dynamics may not be much influenced by second order terms, two points are worth noting. First, it does not mean heterogeneity itself is unimportant. The model solution certainly depends on the higher order characteristics of the wealth distribution and welfare may critically depend on such terms. Second, the finding of approximate aggregation is only a quantitative result for a particular model under a particular calibration. What is appealing about the perturbation approach is that it ex ante permits a greater role for heterogeneity to matter for aggregate dynamics. Because the Krusell and Smith algorithm imposes a particular law of motion for aggregate capital that depends only on past capital many higher order terms relevant to the wealth distribution are excluded. This restriction on the model solution directly limits the manner in which heterogeneity can be relevant to aggregate dynamics. Indeed, the effects of heterogeneity on aggregate capital dynamics can only be felt through different coefficients on the law of motion for aggregate capital (see equation (37) above). In contrast, the solution method delineated here allows an additional set of state variables to exert their influences on the dynamics of the aggregate capital stock. Hence, heterogeneity, in addition to influencing aggregate capital through the mean and the previous period's aggregate capital as in Krusell and Smith, will also affect dynamics according to the dynamics of the variance of aggregate capital, the cross sectional variance of individual capital holdings and the cross sectional covariance of individual asset holdings with employment status.

Perhaps most importantly, the perturbation approach does not rely on approximate aggregation holding or not to give an accurate second order characterization of aggregate dynamics. As discussed by Krusell and Smith (2006), because the value function iteration based solution method relies on the conjecture of a restricted law of motion for aggregate capital that depends only on past aggregate capital, it will in general only be valid if approximate aggregation holds and the distribution of wealth is irrelevant to dynamics. But since the solution method is a quantitative result, without theoretical foundation, it will typically be difficult to verify in general settings whether this is true or not given that i) we do not know the true solution to the model and ii) value function iteration methods can only consider state spaces of limited dimension.[4] While approximate aggregation has been found to hold in a number of model settings, such as the stochastic growth model presented here, this need not be a property of other classes of models with incomplete markets and heterogeneous agents — see Gour-

[4]Krusell and Smith (1998) consider more general laws of motion that include an additional moment over and above the mean capital stock, such as the volatility of aggregate capital or some measure of dispersion of capital holdings in the population. However, to our knowledge they have not considered a law of motion based on all moments relevant to the second order approximation discussed here.

inchas (2000), and Krueger and Kubler (2004) for examples where approximate aggregation is weak. We therefore view the perturbation approach as an additional analytical framework to complement solution algorithms of the kind proposed by Krusell and Smith to help better understand conditions under which market incompleteness and heterogeneous agents might have important implications for aggregate dynamics.

5.1 Further Insights on Approximate Aggregation

To give additional insight as to what lies behind the finding of approximate aggregation, consider solving the model ignoring the borrowing constraint (2). The following result is easy to verify.

Theorem 2 *If the borrowing constraint is ignored then there exists a solution to the individual's Euler equation that has a unit root in each individual i's capital accumulation equation so that* $h_a^a = 1$.

This result only demonstrates that there exists a solution of this form to the household's Euler equation. It does not, however, ensure satisfaction of the intertemporal budget constraint. The intuition for this result is as follows. The steady state of the model is consistent with the permanent income hypothesis. At this point, and in the absence of uncertainty, individuals consume precisely their permanent income. Hence agents' optimal saving decisions are linear in their own capital holdings. On introducing uncertainty, local to this "permanent income point" agents continue to exhibit permanent income type behavior. Note also that one can verify that h_{aa}^a, h_{az}^a, h_{ae}^a, h_{Ψ}^a and h_{Φ}^a are equal to zero in this case. So what is perhaps surprising is that there is little second order curvature.

To reconcile the results with the optimal capital accumulation equation given by (33) consider the implications of reintroducing the borrowing constraint. This restricts the extent of private agents' indebtedness. Indeed, agents must have non-negative capital holdings. This gives rise to a stronger precautionary motive leading to higher average capital holdings. However, by continuity; the fact that most agents hold a quantity of capital significantly above zero; and the absence of significant second order curvature, the optimal accumulation equation for individual capital implies small coefficients for h_{aa}^a, h_{az}^a, h_{ae}^a, h_{Ψ}^a and h_{Φ}^a in the adopted calibration, consistent with the discussion of marginal propensities to save in section 4.4.

This result underscores that in this model incomplete markets seem to matter little. Given an infinite horizon, transitory shocks and limited risk aversion, agents can engage effective self insurance using the economy's only asset: capital. For this reason, the borrowing constraint will only affect a small number of agents who happened to be subject to an highly unlucky sequence of employment shocks. Only in this case does the borrowing constraint become relevant, consistent with departures from permanent income behavior and therefore approximate aggregation.

5.2 Solution Accuracy and Evaluation

Having discussed the characteristics of the optimal decision rules, the analysis now turns to a final exercise to evaluate the accuracy of the perturbation solution. Following Judd (1992) and Krueger and Kubler (2004), we compute the Euler equations errors implied by each of the solution methods so as to gauge the accuracy of the solution. That is, given the computed consumption allocations implied both by perturbation methods and the Krusell and Smith algorithm, we compute the errors that are implied by the true model's Euler equation given by (19) and (9) respectively.

Given optimal decisions we compute the Euler equation errors according to

$$e\left(s^{t}\right) = 1 - \frac{u'^{-1}\left(\beta\Pi\left(s_{t+1}|s_{t}\right)\tilde{r}\left(s^{t+1}\right)u'\left(\hat{c}^{s}_{t+1}\left(s^{t+1}\right)\right)\right)}{\hat{c}^{s}_{t}\left(s^{t}\right)}.$$

Here s denotes the set of possible states (our discrete state approximation to the laws of motion for technology and individual employment status is discussed below). The history to time t is denoted s^t. $\Pi\left(s_{t+1}|s_{t}\right)$ gives the transition density of the markov process describing the evolution of the states and

$$\tilde{r}\left(s^{t+1}\right) = r\left(k_{t+1}\left(s^{t+1}\right), z_{t+1}\left(s^{t+1}\right)\right) + 1 - \delta.$$

Finally $\hat{c}^{s}_{t}\left(s^{t}\right)$ denotes the optimal decisions computed under the assumed calibration.

Because the error measure is unit free it permits comparison across the models solved by value function iteration and the perturbation approach. The models differ in each case due to the introduction of the penalty function in the perturbation approach. Note also that the perturbation approach adopts continuously distributed disturbances which implies constructing the above measure would be computationally burdensome. We therefore parameterize the exogenous disturbances processes

$$
\begin{aligned}
e_{i,t+1} &= (1-\rho_e)\,\mu_e + \rho_e e_{i,t} + \rho_{ze}\,(z_t - \mu_z) + \varepsilon^e_{i,t+1} \\
z_{t+1} &= (1-\rho_z)\,\mu_z + \rho_z z_t + \varepsilon^z_{t+1}
\end{aligned}
$$

so as to imply the same discrete four state markov process used in Krusell and Smith (1998) to describe the evolution of the technology and employment status processes. This then requires integrating out only four states in computing the Euler equation errors. In particular, as a benchmark, we assume that the technology shock and employment status are uncorrelated and that $\rho_e = 0.5$ and $\sigma_e = 0.05$ and $\rho_{ze} = 0.45$. The technology process is as specified in section 4.

Table 2 reports the Euler equation errors for the perturbation and value function iteration based approaches when solved subject to the above specified exogenous disturbances. Both root mean square errors and the mean absolute deviations are reported. It is immediate that the perturbation approach leads to smaller errors under both criteria. Indeed they are typically an order of magnitude smaller than those for the value function iteration procedure. For instance, under the perturbation approach the mean Euler equation error under the RMSE criterion is 0.0005 as compared with 0.0016 using value function iteration based methods. Hence perturbation methods yield an error which is on average a third of the value function iteration based approach. Consistent with this, the maximum errors are a fifth as large and the minimal errors two fifths as large. This suggests the perturbation approach to be an effective solution method for models with incomplete markets and heterogeneous agents. Even though it is a local approximation method, globally the induced errors are not that large.[5]

6 Conclusion

This paper solves a real business cycle model with heterogenous agents. Private agents face partially insurable labor income risk and aggregate technology shocks. Solving such models

[5] In principle solution methods based on value function function could be more accurate by appropriate choice of grid space etc. However, for a given penalty parameter ϕ in our interior method approach, perturbation methods can give a globally accurate characterization — see Anderson, Levin and Swanson (2005).

is difficult as the equilibrium depends on the wealth distribution. As complement to the contribution of Krusell and Smith, this paper proposes solving such models using perturbation methods. We show how to contend with non-differentiability due to borrowing constraints and how to construct an equilibrium which characterizes optimal behavior to the second order.

Like Krusell and Smith (1998), we find the model displays an approximate aggregation property. For the benchmark model and the same calibration as that paper, the aggregate capital stock exhibits little dependency on properties of the cross-sectional distribution of capital holdings. This finding is a direct implication of there being little curvature in the optimal saving decisions of individual households. Indeed, saving is close to being linear in own holdings of the capital stock. However, this paper contributes to our understanding by providing analytical foundations for approximate aggregation.

Despite the similarity in results, it is worth noting the following. First, our approach provides independent evidence of heterogeneity not mattering for aggregate dynamics in the benchmark model considered by Krusell and Smith (1998). It presents a distinct solution method that approximates the model solution on a different dimension to the solution algorithm proposed by Krusell and Smith. Second, the similarity in findings may well not hold for alternative calibrations or alternative models.

Third, the framework has considerable tractability and one that can be applied to a broad class of problems in economics. Because the approach relies on analytical methods for solving systems of linear and quadratic equations, solving the model takes seconds rather than hours in the case of value function iteration. While it comes at the cost of not providing a global solution (albeit one that introduces approximations to the model along different dimensions) we perceive the trade-off to be favorable. Indeed, the induced Euler equations errors reveal the perturbation approach to be as accurate in a global sense as a solution based on value function iteration. Furthermore, -the approach remains valid even in models for which approximate aggregation does not obtain.

A Appendix

A.1 Proof of Theorem 1

The proof of theorem 1 will serve two purposes. One is to establish the desired result. The second is to exposit the solution method for a heterogenous agents model with incomplete markets. A second order approximation to the following system is required:

$$F = \begin{bmatrix} F^c \\ F^a \\ F^k \\ F^\Phi \\ F^\Psi \end{bmatrix} = \begin{bmatrix} \beta E_t \left[u_c \left(c_{i,t+1} \right) \left(r \left(k_{t+1}, l_{t+1}, z_{t+1} \right) + 1 - \delta \right) \right] + \frac{2\phi}{\left(a_{i,t+1} + b \right)^3} - u_c \left(c_{i,t} \right) \\ \left(1 - \delta \right) a_t + r \left(k_t, l_t, z_t \right) a_{i,t} + w \left(k_t, l_t, z_t \right) \bar{l} e_{i,t} - c_{i,t} - a_{i,t+1} \\ \int_0^1 a_{i,t+1} di - k_{t+1} \\ \int_0^1 \left(a_{i,t+1} - \bar{a} \right)^2 di - \Phi_{t+1} \\ \int_0^1 \left(a_{i,t+1} - \bar{a} \right) \left(e_{i,t+1} - \bar{e} \right) di - \Psi_{t+1} \end{bmatrix} = 0$$

where the desired solution has the general form

$$\begin{aligned} c_t &= g \left(a_{i,t},\ e_{i,t},\ k_t,\ z_t,\ \Phi_t,\ \Psi_t,\ \sigma \right); \quad a_{i,t+1} = h^a \left(a_{i,t},\ e_{i,t},\ k_t,\ z_t,\ \Phi_t,\ \Psi_t,\ \sigma \right) \\ k_{t+1} &= h^k \left(k_t, z_t, \Phi_t,\ \Psi_t,\ \sigma \right);\ \Phi_{t+1} = h^\Phi \left(k_t, z_t, \Phi_t,\ \Psi_t,\ \sigma \right);\ \Psi_{t+1} = h^\Psi \left(k_t, z_t, \Phi_t,\ \Psi_t,\ \sigma \right) \end{aligned}$$

and for the purposes of exposition the penalty terms are ignored to simplify the notation somewhat. The presented numerical results of course incorporate these terms. Following Schmitt-Grohe and Uribe (2004), differentiate the first two rows of F to obtain:

$$\begin{aligned} F_a^c &= \beta u_{cc} g_a h_a^a \left(r + 1 - \delta \right) - u_{cc} g_a - 6\phi \left(a + b \right)^{-4} h_a^a \\ F_k^c &= \beta u_{cc} \left[g_a h_k^a + g_k h_k^k \right] \left(r + 1 - \delta \right) + \beta u_c r_k h_k^k - u_{cc} g_k - 6\phi \left(a + b \right)^{-4} h_k^a \\ F_z^c &= \beta u_{cc} \left[g_a h_z^a + g_k h_z^k + g_z \rho_z \right] \left(r + 1 - \delta \right) + \beta u_c r_k \left[h_z^k + \rho_z \right] - u_{cc} g_z 6\phi \left(a + b \right)^{-4} h_z^a \\ F_e^c &= \beta u_{cc} g_e \rho_e \left(r + 1 - \delta \right) - u_{cc} g_e - 6\phi \left(a + b \right)^{-4} h_e^a \end{aligned} \tag{38}$$

and

$$\begin{aligned} F_a^a &= \left(r + 1 - \delta \right) - h_a^a - g_a \\ F_e^a &= w - g_e - h_e^a \\ F_k^a &= r_k a_{i,t} + w_k e_{i,t} - g_k - h_k^a \\ F_z^a &= r_z a_{i,t} + w_z e_{i,t} - g_z - h_z^a \end{aligned} \tag{39}$$

which must all equal zero. To solve for the 12 first order coefficients

$$g_a, g_e, g_k, g_z, h^a_a, h^a_e, h^a_k, h^a_z, h^k_a, h^k_e, h^k_k, h^k_z \tag{40}$$

12 restrictions are required. The above gives 8 in the 12 unknowns. The final four come from derivatives of the aggregation constraint F^k. Recall

$$k_{t+1} = h^k\left(k_t, z_t, \Phi_t,\ \Psi_t\right) = \int_0^1 a_{i,t+1}di. \tag{41}$$

A first order expansion of the final term provides:

$$\begin{aligned}\int_0^1 a_{i,t+1}di &\doteq \int_0^1 \left[h^a_a\left(a_{i,t}-\bar{a}\right)+h^a_e\left(e_{i,t}-\bar{e}\right)+h^a_k\left(k_t-\bar{k}\right)+h^a_z\left(z_t-\bar{z}\right)\right]di \\ &= \left(h^a_a+h^a_k\right)\left(k_t-\bar{k}\right)+h^a_z\left(z_t-\bar{z}\right)\end{aligned}$$

using (41), $\bar{a}=\bar{k}$ and $\int_0^1 e_{i,t}di=\bar{e}$. Similarly the second term must satisfy the approximation

$$h^k\left(k_t, z_t, \Phi_t,\ \Psi_t\right) \doteq h^k_k\left(k_t-\bar{k}\right)+h^k_z\left(z_t-\bar{z}\right)$$

implying the restrictions

$$h^k_a = h^k_e = 0;\ h^k_k = h^a_a + h^a_k;\ h^k_z = h^a_z.$$

Together with the eight restrictions given by (38) and (39) the 12 first order unknown coefficients (40) can be determined. Note that all other coefficients are known, determined by household preferences or the firm's production function.

There are three other first order coefficients to determine: g_σ, h^a_σ and h^k_σ. Two restrictions come from the constraints

$$\begin{aligned}F^a_\sigma &= -g_\sigma - h^a_\sigma \\ F^c_\sigma &= \beta u_{cc}g_\sigma\left(r+1-\delta\right)-u_{cc}g_\sigma+\beta u_c r_k h^k_\sigma - 6\phi\left(a+b\right)^{-4}h^a_\sigma.\end{aligned}$$

The third constraint comes from the aggregation restriction. Because

$$h^k\left(k_t, z_t, \Phi_t,\ \Psi_t,\ \sigma\right) = \int_0^1 h^a\left(a_{i,t},\ e_{i,t},\ k_t,\ z_t,\ \Phi_t,\ \Psi_t,\ \sigma\right)di$$

it is immediate that

$$h^a_\sigma = h^k_\sigma.$$

providing three equations in three unknowns. As the system is linear and homogeneous, if there is a unique solution it must be the case that

$$g_\sigma = h^a_\sigma = h^k_\sigma = 0.$$

This established the first part of theorem 1: at the first order, uncertainty does not affect any of the first order elasticities. This is the usual certainty equivalence result.

Solving for the second order terms proceeds in much the same way: exploit the second order cross partial derivatives of F^c and F^a with respect to pairs of $\{a,\ e,\ k,\ z\}$ and the derivatives with respect to $\{\Phi,\ \Psi\}$. This provides 36 restrictions in 54 unknowns. The remaining 18 restrictions again come from the aggregation constraint (41). To give a flavor of the calculations, note that the second order partials F^a_{ij} can be directly computed. For example

$$\begin{aligned} F^a_{aa} &= -h^a_{aa} - g_{aa} \\ F^a_{ae} &= -h^a_{ae} - g_{ae} \\ F^a_{ak} &= r_k - h^a_{ak} - g_{ak} \\ F^a_{az} &= r_z - h^a_{az} - g_{az} \end{aligned}$$

must all equal zero. The remaining 12 cross partials are easily computed and generate restrictions that depend on the 16 unknown $g'_{ij}s$ and 16 unknown $h^{a\prime}_{ij}s$. The derivatives with respect to $\{\Phi,\ \Psi\}$ provide two additional restrictions:

$$\begin{aligned} F^a_\Phi &= -g_\Phi - h^a_\Phi \\ F^a_\Psi &= -g_\Psi - h^a_\Psi. \end{aligned} \qquad (42)$$

Turning to the second order terms relating to F^c 16 restrictions are again obtained from the cross partials in the variables $\{a,\ e,\ k,\ z\}$. As the algebra is somewhat tedious, and because these computations are not central to the conceptual heart of the solution method, not all coefficients are presented. For example, and for simplicity ignoring terms in the penalty function,

$$\begin{aligned} F^c_{aa} &= \beta(u_{ccc}\left(g_a h^a_a\right)^2 + u_{cc}\left[g_{aa}h^a_a + g_a h^a_{aa}\right])\left(r+1-\delta\right) - u_{ccc}g_a^2 h^a_a - u_{cc}g_{aa} \\ F^c_{ae} &= \beta(u_{ccc}g_a h^a_a\left(g_a h^a_e + g_e\rho_e\right) + u_{cc}\left(g_{ae}\rho_e h^a_a + g_{aa}h_e h^a_a + g_a h^a_{ae}\right))\left(r+1-\delta\right) \\ &\quad -u_{ccc}g_a\left(g_a h^a_e + g_e\varepsilon_e\right) - ucc\left(g_{aa}h^a_e + g_{ae}\rho_e\right) \end{aligned}$$

give 2 of the 16 cross partials which are again a function only of the 16 unknown $g_{ij}'s$, 16 unknown $h_{ij}^{a\prime}s$ and 16 unknown $h_{ij}^{k\prime}s$. The derivatives with respect to $\{\Phi,\ \Psi\}$ provide two additional restrictions:

$$\begin{aligned} F_{\Phi}^{c} &= \beta u_{cc}\left(g_a h_{\Phi}^{a} + g_k h_{\Phi}^{k} + g_{\Phi} h_{\Phi}^{\Phi} + g_{\Psi} h_{\Phi}^{\Psi}\right)(r+1-\delta) + \beta u_c r_k h_{\Phi}^{k} \\ &\quad - \frac{6\phi}{(b+\hat{a})^4} h_{\Phi}^{a} - u_{cc} g_{\Phi} \\ F_{\Psi}^{c} &= \beta u_{cc}\left(g_a h_{\Psi}^{a} + g_k h_{\Psi}^{k} + g_{\Phi} h_{\Psi}^{\Phi} + g_{\Psi} h_{\Psi}^{\Psi}\right)(r+1-\delta) + \beta u_c r_k h_{\Psi}^{k} \\ &\quad - \frac{6\phi}{(b+\hat{a})^4} h_{\Psi}^{a} - u_{cc} g_{\Psi} \end{aligned} \tag{43}$$

Relations (42) and (43) provide 4 constraints but introduce 10 more unknowns written as

$$g_{\Phi}, g_{\Psi}, h_{\Phi}^{a}, h_{\Phi}^{k}, h_{\Psi}^{a}, h_{\Psi}^{k}, h_{\Phi}^{\Phi}, h_{\Phi}^{\Psi}, h_{\Psi}^{\Phi}, h_{\Psi}^{\Psi}.$$

To proceed, suppose that the final 4 terms $\left\{h_{\Phi}^{\Phi}, h_{\Phi}^{\Psi}, h_{\Psi}^{\Phi}, h_{\Psi}^{\Psi}\right\}$ are in fact known (we will show below that they are completely determined by the first order coefficients already determined. Then the above determine 36 restrictions in 54 unknowns. The final 18 restrictions follow from aggregation constraint (41). A second order expansion gives

$$\begin{aligned} \int_0^1 a_{i,t+1} di &\doteq \bar{a} + \int_0^1 [h_a^a \hat{a} + h_e^a \hat{a} + h_k^a \hat{k} + h_z^a \hat{z} + h_{\Phi}^a \hat{\Phi} + h_{\Psi}^a \hat{\Psi} \\ &\quad + \frac{1}{2}[h_{aa}^a \hat{a} + h_{ae}^a \hat{a}\hat{e} + h_{ak}^a \hat{a}\hat{k} + h_{az}^a \hat{a}\hat{z} + h_{ea}^a \hat{e}\hat{a} + h_{ee}^a \hat{e}^2 + h_{ek}^a \hat{e}\hat{k} + h_{ez}^a \hat{e}\hat{z} \\ &\quad + h_{ka}^a \hat{k}\hat{a} + h_{ke}^a \hat{k}\hat{e} + h_{kk}^a \hat{k}^2 + h_{kz}^a \hat{k}\hat{z} + h_{za}^a \hat{z}\hat{a} + h_{ze}^a \hat{e}\hat{z} + h_{zk}^a \hat{z}\hat{k} + h_{zz}^a \hat{z}^2]]di \\ &= \bar{a} + \left(h_a^a + h_k^a\right)\hat{k} + h_z^a \hat{z} + \left(h_{\Phi}^a + \frac{h_{aa}^a}{2}\right)\left(\Phi_t - \bar{\Phi}\right) + \left(h_{ae}^a + h_{\Psi}^a\right)\left(\Psi_t - \bar{\Psi}\right) \\ &\quad + \frac{1}{2}\left(h_{ak}^a + h_{ka}^a + h_{kk}^a\right)\hat{k}^2 + \left(h_{az}^a + h_{za}^a + h_{kz}^a + h_{zk}^a\right)\hat{k}\hat{z} + h_{zz}^a \hat{z}^2 + h_{ee}^a \int_0^1 \hat{e}^2 di. \end{aligned}$$

Similarly, a second order expansion of

$$h^k\left(k_t, z_t, \Phi_t,\ \Psi_t\right) = \bar{k} + h_k^k \hat{k} + h_z^k \hat{z} + h^k \hat{\Phi} + h^k \hat{\Psi} + \frac{1}{2}\left[h_{kk}^k \hat{k}^2 + h_{zk}^k \hat{z}\hat{k} + h_{kz}^k \hat{k}\hat{z} + h_{zz}^k \hat{z}^2\right].$$

Ignoring the first order terms already discuss, matching coefficients then gives the 18 required restrictions

$$
\begin{aligned}
h_{\Phi}^{k} &= \frac{h_{aa}^{a}}{2} + h_{\Phi}^{a} \\
h_{\Psi}^{k} &= h_{ae}^{a} + h_{\Psi}^{a} \\
h_{kk}^{k} &= 2h_{ak}^{a} + h_{kk}^{a} \\
h_{kz}^{k} &= h_{az}^{a} + h_{kz}^{a} \\
h_{kz}^{k} &= h_{zk}^{k} \\
h_{zz}^{k} &= h_{zz}^{a}.
\end{aligned}
$$

The remaining coefficients satisfy

$$
h_{aa}^{k} = h_{ae}^{k} = h_{ak}^{k} = h_{az}^{k} = h_{ea}^{k} = h_{ee}^{k} = h_{ek}^{k} = h_{ke}^{k} = h_{ez}^{k} = h_{ka}^{k} = h_{za}^{k} = h_{ze}^{k} = 0.
$$

Finally note that aggregation induces a further correction to the mean capital stock since

$$
\bar{k} = \bar{a} + h_{ee}^{a} \int_{0}^{1} \hat{e}^{2} di
$$

where the latter is a constant by the law of large numbers.

Two tasks remain. One is to determine $\left\{h_{\Phi}^{\Phi}, h_{\Phi}^{\Psi}, h_{\Psi}^{\Phi}, h_{\Psi}^{\Psi}\right\}$. The second concerns accounting for uncertainty and solving for the second order partials involving σ. Take these in turn. Recall that

$$
\begin{aligned}
\Phi_{t+1} &= \int_{0}^{1} (a_{i,t+1} - \bar{a})^{2} di \\
\Psi_{t+1} &= \int_{0}^{1} (a_{i,t+1} - \bar{a})(e_{i,t+1} - \bar{e})\, di.
\end{aligned}
$$

A second order approximation to these two expressions is easily shown to provide

$$
\begin{aligned}
\hat{\Phi}_{t+1} &= h_{e}^{a} \int_{0}^{1} \hat{e}^{2} di + (h_{a}^{a})^{2} \hat{\Phi}_{t} + 2h_{a}^{a} h_{e}^{a} \hat{\Psi}_{t} + \left(2h_{a}^{a} h_{k}^{a} + (h_{k}^{a})^{2}\right) \hat{k}_{t} \\
&\quad + 2h_{z}^{a} (h_{a}^{a} + h_{k}^{a})\, \hat{z}_{t} \hat{k}_{t} + (h_{z}^{a})^{2} \hat{z}_{t}^{2} \\
\hat{\Psi}_{t+1} &= h_{e}^{a} \rho_{e} \int_{0}^{1} \hat{e}_{i,t}^{2} di + h_{a}^{a} \rho_{e} \hat{\Psi}_{t}.
\end{aligned}
$$

Hence all coefficients depends on either known model primitives or first order coefficients already determined. The solution therefore satisfies

$$
h_{\Phi}^{\Phi} = (h_{a}^{a})^{2};\; h_{\Psi}^{\Phi} = 2h_{a}^{a} h_{e}^{a};\; h_{\Phi}^{\Psi} = 0;\; h_{\Psi}^{\Psi} = h_{a}^{a} \rho_{e}.
$$

Finally, consider solving for the second order partials in σ. Note that the following are true

$$\begin{aligned} F^a_{\sigma\sigma} &= -g_{\sigma\sigma} - h^a_{\sigma\sigma} \\ F^c_{\sigma\sigma} &= \beta u_{cc} g_{\sigma\sigma} (r + 1 - \delta) - u_{cc} g_{\sigma\sigma} + \beta u_c r_k h^k_{\sigma\sigma}. \end{aligned}$$

Finally note from the aggregation constraint that the following restriction must hold

$$h^k_{\sigma\sigma} = h^a_{\sigma\sigma} + h^a_{ee} \int_0^1 \hat{e}^2_{i,t} di.$$

Hence there are three equations in the unknowns $g_{\sigma\sigma}$, $h^a_{\sigma\sigma}$ and $h^k_{\sigma\sigma}$ which can be readily solved.

The twelve cross partials (with the remaining 12 given by symmetry and again ignoring terms in the penalty function for simplicity) are given by

$$\begin{aligned} F^a_{\sigma k} &= -g_{\sigma k} - h^a_{\sigma k} \\ F^a_{\sigma z} &= -g_{\sigma z} - h^a_{\sigma z} \\ F^a_{\sigma a} &= -g_{\sigma a} - h^a_{\sigma a} \\ F^a_{\sigma e} &= -g_{\sigma e} - h^a_{\sigma e} \\ F^c_{\sigma k} &= \beta u_{cc} g_{\sigma k} (r + 1 - \delta) - u_{cc} g_{\sigma k} + \beta u_{cc} r_k h^k_{\sigma k} \\ F^c_{\sigma z} &= \beta u_{cc} g_{\sigma z} (r + 1 - \delta) - u_{cc} g_{\sigma z} + \beta u_{cc} r_k h^k_{\sigma z} \\ F^c_{\sigma a} &= \beta u_{cc} g_{\sigma a} (r + 1 - \delta) - u_{cc} g_{\sigma a} + \beta u_{cc} r_k h^k_{\sigma a} \\ F^c_{\sigma e} &= \beta u_{cc} g_{\sigma e} (r + 1 - \delta) - u_{cc} g_{\sigma e} + \beta u_{cc} r_k h^k_{\sigma e} \\ F^k_{\sigma k} &= h^a_{\sigma k} - h^k_{\sigma k} \\ F^k_{\sigma z} &= h^a_{\sigma z} - h^k_{\sigma z} \\ F^k_{\sigma a} &= h^a_{\sigma a} - h^k_{\sigma a} \\ F^k_{\sigma e} &= h^a_{\sigma e} - h^k_{\sigma e}. \end{aligned}$$

Once more this is a system of 12 equations in 12 unknowns that is linear and homogeneous. It follows that if there is a unique solution then it must have all these terms equal to zero. This therefore completes the solution of the model and also establishes the proof of theorem 1.

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Table 1: Aggregate Statistics

	Baseline Calibration				**Risky Economy**	
	Representative Agent		Heterogeneous Agent		Heterogeneous Agent	
Solution:	First Order	Second Order	First Order	Second Order	First Order	Second Order
Steady State Consumption	0.782	0.782	0.782	0.782	0.782	0.782
Prec. Saving (% of S.S.)	0.000	-0.011	0.000	-0.039	0.000	-0.039
Aggregate Consumption:						
Mean (% of S.S.)	0.000	-0.060	0.000	-1.154	0.000	-1.937
Standard Deviation	0.010	0.010	0.009	0.010	0.016	0.017
First order serial Corr	0.986	0.986	0.986	0.986	0.993	0.993
Capital Distribution:						
Mean Variance				8.767		12.826
Std Deviation of Variance				0.816		1.297
Welfare:						
Mean	-14.375	-14.371	-14.558	-14.920	-13.139	-14.001
Standard Deviation			4.279	4.379	4.626	5.228

Table 3: Euler Equation Errors

	Mean	Std Dev	Max	Min
Perturbation:				
RMSE	0.0005	0.0002	0.0009	0.0002
Value Function Iteration				
RMSE	0.0016	0.0007	0.0054	0.0005