

A Model of Job and Worker Flows^a

Nobuhiro Kiyotaki
London School of Economics

Ricardo Lagos
New York University

November 10, 2003

Abstract

We develop a model of gross job and worker flows and use it to study how the wages and employment status of individual workers evolve over time and how they are affected by aggregate labor market conditions. We also examine the effects that labor market institutions and public policy have on the gross flows, as well as on the resulting wage distribution, employment and aggregate output in the equilibrium. The model we propose also rationalizes various other features of labor markets. For example, why do displaced workers tend to experience a significant and persistent fall in wages? Why do workers stay unemployed when on-the-job-search is at least as effective as off-the-job-search? Why is it that good jobs are not only better paid, but often also more stable? From a theoretical point of view, we study the extent to which the competitive equilibrium achieves an efficient allocation of resources.

^aThis draft is preliminary and incomplete. Please do not circulate.

1 Introduction

Recent theoretical and empirical studies on gross job creation and destruction have changed the way we think about the labor market. We now view employment and unemployment as resulting from a large and continual job reallocation process, and analyze how changes in public policy and the economic environment affect these gross flows, which in turn affect unemployment. However, behind these gross job flows there are even larger worker flows, because the number of workers who quit, get displaced and get hired by each employer is at least as large as (and often significantly larger than) the net change of employment for each employer.

In this paper, we develop a model of gross job and worker flows and use it to study how the employment status and wages of individual workers evolve over time and how they are affected by aggregate labor market conditions. We also examine the effects that labor market institutions and public policy have on the gross flows, as well as on the resulting wage distribution, employment and aggregate output in the equilibrium. Our framework of worker flows also rationalizes various other features of labor markets. For example, why do displaced workers tend to experience a significant and persistent fall in wages? Why do workers stay unemployed when on-the-job-search is at least as effective as off-the-job-search? Why is it that good jobs are not only better paid, but often also more stable? Why do workers in good jobs often receive help from their employers to overcome personal problems (e.g. health problems) so that they can preserve their jobs? From a theoretical point of

view, we study the extent to which the competitive equilibrium achieves an efficient allocation.

2 The Model

Time is continuous and the horizon infinite. The economy is populated by a continuum of fixed and equal numbers of workers and employers. We normalize the size of each population to unity. Workers and employers are infinitely-lived and risk-neutral. They discount future utility at rate $r > 0$, and are ex-ante homogeneous in tastes and technology.

A worker meets a randomly chosen employer according to a Poisson process with arrival rate λ . An employer meets a random worker according to the same process. Upon meeting, the employer-worker pair randomly draws a production opportunity of productivity y , which represents the total net output each agent will produce while matched. (Thus the pair produces $2y$.) The random variable y takes one of N distinct values: $y_1; y_2; \dots; y_N$, where $0 < y_1 < y_2 < \dots < y_N$, and $y = y_i$ with probability $\frac{1}{N}$ for $i = 1; \dots; N$. For now, we assume y remains constant for the duration of the match.

Matched and unmatched agents meet potential partners at the same rate, so when an employer and a worker meet and draw a productive opportunity each of them may or may not already be matched with an old production partner. Each worker and employer can form at most one productive partnership simultaneously. The realization of the random variable y that an employer and worker draw when they first meet is observed without delay by

them as well as by their current partners. In fact, the productivity of the new potential match as well as the productivities of the existing matches are public information to all the agents involved, i.e. the worker and the employer who draw the new productivity and their existing partners if they have any. On the other hand, each agent's past history is private information, except for what is revealed by the current production match.

When a worker and an employer meet and find a new productive opportunity, the pair and their old partners (if they have any) determine whether or not the new match is formed (and consequently whether or not the existing matches are destroyed) as well as the once-and-for-all side payments that each party pays or receives, through a bargaining protocol which we will describe shortly. Utility is assumed to be transferable among all the agents involved in a meeting. There is no outside court to enforce any formal contract, so any effective contract must be self-enforcing among the parties involved. If the parties who made contact decide to form a new partnership, they leave their existing partners who then become unmatched. In addition to these endogenous terminations, we assume any match is subject to exogenous separation with a Poisson process separation rate \pm .

We use n_{it} to denote the measure of matches of productivity y_i and n_{0t} to denote the measure of unmatched employers or workers at date t . Let ζ_{ij}^k be the probability that a worker with current productivity y_i and an employer with current productivity y_j form a new match of productivity y_k , given that they draw an opportunity to produce y_k at time t . (Hereafter, we will suppress the time subindex when no confusion arises.) The measure of

workers in each state evolves according to:

$$\dot{n}_i = \sum_{j=0}^N \sum_{k=0}^N \lambda_{jk}^i n_j n_k - \sum_{j=0}^N \sum_{k=1}^N \lambda_{jk}^i n_j n_k + \sum_{j=1}^N \lambda_{ji}^k n_j - \delta n_i \quad (1)$$

$$\dot{n}_0 = \sum_{i=1}^N \sum_{j=1}^N \lambda_{ij}^k n_i n_j + \sum_{j=1}^N \lambda_{j0} n_j - \sum_{k=1}^N \lambda_{k0}^k n_0^2 - \delta n_0 \quad (2)$$

The first term on the right hand side of (1) is the flow of new matches of productivity y_i created by all types workers and employers. The second term is the total flow matches of productivity y_i destroyed endogenously when the worker or the employer leaves to form a new match. The last term is the flow of matches dissolved exogenously. On the right hand side of equation (2), the first term is the flow of workers who become unmatched when their employers decide to break the current match to form a new match with another worker. The second term is the flow of workers who become unmatched due to the exogenous dissolution of matches. The third term is the flow of new matches created by unmatched workers and employers. (The creation of a new match involving an unmatched agent and a matched agent does not affect the aggregate number of unmatched agents, since one previously unmatched agent becomes matched, while one previously matched agent loses the partner to become unmatched.)

Before describing the competitive matching equilibrium with bargaining, we solve the social planner's problem. The planner chooses $\lambda_{ij}^k \in [0; 1]$ to maximize the discounted value of aggregate output:

$$\int_0^{\infty} e^{-\rho t} \sum_{i=1}^N y_i n_i dt$$

subject to the flow constraints (1) and (2), and initial conditions for n_0 and n_i for $i = 1, \dots, N$. Letting $\lambda_{i,t}$ be the shadow price of a match with productivity y_i at date t , the Hamiltonian is

$$H = \sum_{i=1}^N 2y_i n_{i,t} \pm \sum_{i=1}^N (\lambda_{i,t} - \lambda_{i,t-1}) n_{i,t} + \sum_{i=0}^N \sum_{j=0}^N \sum_{k=1}^K n_i n_j \frac{1}{4} \zeta_{ij}^k (\lambda_{i,t} + \lambda_{j,t} - \lambda_{i,t-1} - \lambda_{j,t-1});$$

The optimality conditions are:

$$\zeta_{ij}^k = \begin{cases} 1 & \text{if } \lambda_{i,t} + \lambda_{j,t} > \lambda_{i,t-1} + \lambda_{j,t-1} \\ \in [0, 1] & \text{if } \lambda_{i,t} + \lambda_{j,t} = \lambda_{i,t-1} + \lambda_{j,t-1} \\ 0 & \text{if } \lambda_{i,t} + \lambda_{j,t} < \lambda_{i,t-1} + \lambda_{j,t-1} \end{cases} \quad (3)$$

together with

$$\begin{aligned} r_{\lambda_{i,t} - \lambda_{i,t-1}} &= 2y_i n_{i,t} \pm (\lambda_{i,t} - \lambda_{i,t-1}) n_{i,t} + \\ &+ \sum_{j=0}^N \sum_{k=1}^K n_j \frac{1}{4} \zeta_{ij}^k + \zeta_{ji}^k (\lambda_{i,t} + \lambda_{j,t} - \lambda_{i,t-1} - \lambda_{j,t-1}); \\ r_{\lambda_{j,t} - \lambda_{j,t-1}} &= \sum_{i=0}^N \sum_{k=1}^K n_i \frac{1}{4} \zeta_{ij}^k + \zeta_{j0}^k (\lambda_{i,t} + \lambda_{j,t} - \lambda_{i,t-1} - \lambda_{j,t-1}); \end{aligned}$$

and (1) and (2), for a given initial condition for n_0 and n_i at date 0. According to (3), to achieve the optimal allocation the planner specifies that a type i worker and type j employer should form a new match of productivity y_k for sure, if and only if the sum of the shadow prices of the new match and the unmatched worker and employee (which the new match would generate) exceeds the sum of the shadow prices of the existing matches of productivity y_i and y_j . From (3) we also learn that $\zeta_{ij}^k = \zeta_{ji}^k$, possibly except for the case of randomized strategies. Intuitively, there is no inherent asymmetry between a worker and an employer, so the planner treats them symmetrically in the

optimal allocation. These observations allow us to summarize the optimality conditions as:

$$r_{s_i i} - s_i = 2y_i i \pm (s_i i - s_0) + 2 \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \eta_j \frac{1}{4} \max_{0 \leq z_{ij}^k \leq 1} z_{ij}^k (s_k + s_0 i - s_i i - s_j) \quad (4)$$

$$r_{s_0 i} - s_0 = 2 \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \eta_j \frac{1}{4} \max_{0 \leq z_{0j}^k \leq 1} z_{0j}^k (s_k i - s_j): \quad (5)$$

3 Competitive Matching Equilibrium

In this section we characterize the competitive matching equilibrium with the following bargaining procedure. When an agent draws an opportunity to produce with a new partner, with probability a half, she makes take-it-or-leave-it offers to her new potential partner and her old partner (if she has one) about production and side payments. She can rank these two offers, by making her offer to the old partner contingent on her offer to the new potential partner being rejected. With another probability half, her old partner (if she has one) and new potential partner simultaneously make take-it-or-leave-it offers to her. After these offers are made, the recipient of the offers chooses which one to accept. We also specify that matched agents split the surplus symmetrically as long as neither agent encounters a production opportunity with another potential partner.¹

Because a worker and an employer who form a match are inherently

¹Alternatively, we can think of the matched pair without an outside production opportunity as being involved in continual negotiations by which the expected value of side payments net out to be zero.

symmetric, hereafter we restrict our attention to symmetric equilibria in which workers and employers are treated symmetrically and are distinguished only by the productivity of their current match (or unmatched state). We will refer to a match of productivity y_i as a “type i match”, and call a worker or an employer in a type i match a “type i agent”. Let V_i be the value of expected discounted utility of a type i agent (either a worker or employer), and let V_0 be the value of an unmatched agent. Let X_{ij}^k be the value that a type i agent offers to a type j agent in order to form a new match of productivity y_k . Specifically, X_{ij}^k includes the value of the new match plus the net side payment type j agent receives. Three qualitatively different types of meetings can result from the random matching process: (i) an unmatched employer and an unmatched worker meet and draw a production opportunity, (ii) an matched agent and an unmatched agent meet and draw a production opportunity, and (iii) a matched employer and a matched worker meet and draw a production opportunity. We begin by describing the equilibrium outcome of the bargaining for each of these three types of meetings, taking V_i and V_0 as given. Later, we will analyze how these values are determined in equilibrium.

(i). **An unmatched employer meets an unmatched worker.**

Suppose an unemployed worker and an employer with a vacancy draw an opportunity for each to produce y_k . Since both are unmatched, the outside option to each agent is V_0 . This case is illustrated in Figure 1, where we have named the two agents involved in this meeting A and B .

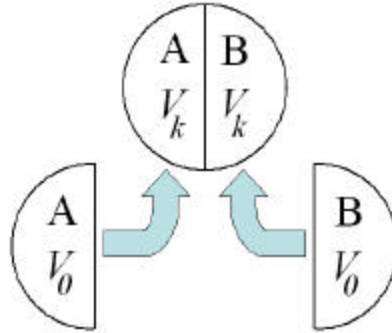


Figure 1: An unmatched employer meets an unmatched worker.

The bargaining unfolds as follows:

Subgame 1. With probability a half, the employer makes a take-it-or-leave-it offer X_{00}^k to the worker in order to maximize her own utility (which minimizes his partner's utility) subject to the constraint that his partner will accept. Then $X_{00}^k = V_0$, and the offer is accepted by the partner.

Subgame 2. With the same probability, the worker makes an offer $X_{00}^k = V_0$ to the employer which is again accepted.

Let v_j be the expected payoff to agent $j = A; B$ and i_j be her expected gain. For this case we have $v_A = v_B = \frac{1}{2}V_0 + \frac{1}{2}(2V_k - V_0) = V_k$, and

$$i_A = i_B = V_k - V_0. \quad (6)$$

In this symmetric situation the expected value of the side payment is zero, and both unmatched agents enjoy the same capital gains to becoming matched.

(ii). An matched agent meets an unmatched agent.

Suppose agent B, who is currently in a match of productivity y_i with

agent A, meets agent C –who is unmatched– and they draw a productive opportunity y_k . This situation is illustrated in Figure 2.

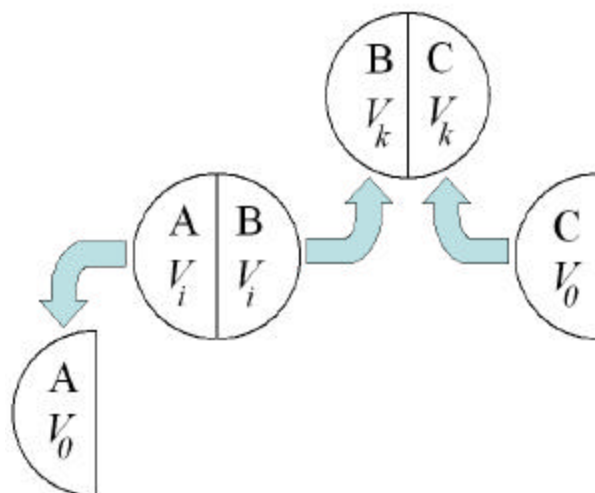


Figure 2: A matched agent meets an unmatched agent.

The bargaining proceeds as follows:

Subgame 1. With probability a half, B makes a take-it-or-leave-it offer to A or C. This offer involves payoffs as well as a proposal to engage in joint production. If B was to offer (continued) joint production to A, he would offer A her minimum acceptable payoff, $X_{BA}^k = V_0$. A would accept the offer and B's payoff from continued production with A would be $2V_i - V_0$. Alternatively, if B offers joint production to C, then he will offer C a payoff equal to her minimum acceptable level, $X_{BC}^k = V_0$. C will accept the offer and B's payoff would be $2V_k - V_0$. So clearly, if $V_k > V_i$ then B offers C to produce together, she accepts, and the payoffs to A, B and C will be V_0 , $2V_k - V_0$, and V_0 respectively. Conversely, if $V_i > V_k$, then B offers A to

Suppose agent B and agent C meet and draw a productive opportunity y_k . The situation now is that B is currently in a match of productivity y_i with agent A, while C, is currently in a match of productivity y_j with agent D. This case is illustrated in Figure 3.

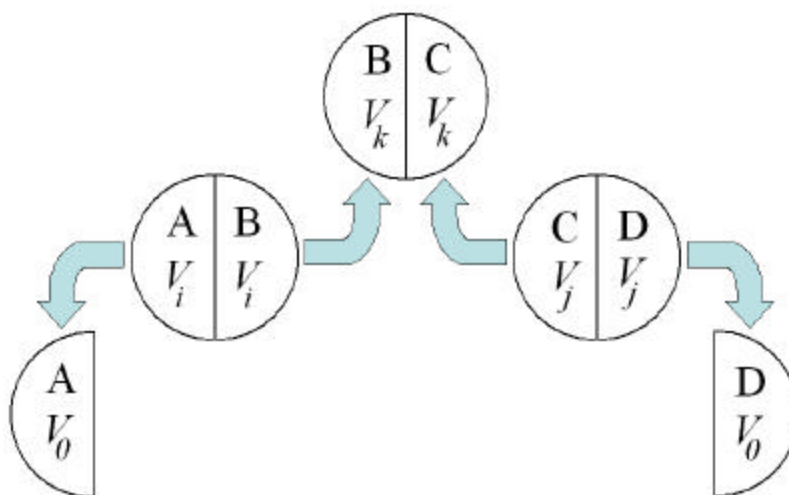


Figure 3: A matched employer meets a matched worker.

The bargaining procedure is as follows:

Subgame 1. With probability a half, A and C simultaneously make offers to B. C also makes a take-it-or-leave-it offer to his existing partner D, and this offer is contingent on his offer to B being rejected. C makes the smallest acceptable offer to D, and since D has no other productive opportunities, his proposed payoff to D is equal to the value of being unmatched, V_0 . The resulting payoff to C from continuing to match with D is $2V_j - V_0$, which constitutes the opportunity cost for C to form a new match. Thus the maximum C is willing to offer B is $2V_k - (2V_j - V_0)$. Because A's opportunity cost

of continuing to match is the value of being unmatched, V_0 , the maximum A is willing to offer B is $2V_i - V_0$. Since this valuation is positive, A will want to make sure that B finds her offer acceptable, and for this she must ensure that B's payoff is at least as large as V_0 . Therefore, A offers B's payoff to be $X_{AB}^i = \text{Max}\{V_0; \text{Min}[2V_i - V_0; 2V_k - (2V_j - V_0) + \epsilon]\}$ and C offers B's payoff to be $X_{CB}^i = \text{Min}[2V_k - (2V_j - V_0); 2V_i - V_0 + \epsilon]$ for an arbitrarily small positive ϵ . Then, B will accept C's offer to form the new match if and only if $2V_k - (2V_j - V_0) > 2V_i - V_0$, or $V_k + V_0 > V_i + V_j$, i.e., the sum of the values of the new match and the unmatched exceeds the sum of the values of the existing matches. The equilibrium payoffs are²

$$\begin{array}{l}
 \begin{array}{ccc}
 2 & & 3 \\
 & V_0 & \\
 \frac{6}{4} & \frac{2V_i - V_0}{2(V_k - V_j) + V_0} & \frac{7}{5} \\
 & & \text{if } V_i + V_j - V_0 < V_k
 \end{array} \\
 \begin{array}{ccc}
 2 & & 3 \\
 & V_0 & \\
 \frac{6}{4} & \frac{2(V_i + V_j - V_k) - V_0}{2(V_k - V_j) + V_0} & \frac{7}{5} \\
 & & \text{if } V_j < V_k < V_i + V_j - V_0
 \end{array} \\
 \begin{array}{ccc}
 2 & & 3 \\
 & V_0 & \\
 \frac{6}{4} & \frac{2V_i - V_0}{2V_j - V_0} & \frac{7}{5} \\
 & & \text{if } V_k < V_j:
 \end{array}
 \end{array}$$

If $V_k < V_i + V_j - V_0$, then A and B preserve their match and whether or not A may have to offer B a side-payment depends on whether the new potential match of B and C is better or worse than C's current match. If the new potential match is better (i.e. $V_j < V_k$), then C is willing to offer B as much as

²The first, second, third and fourth rows contain the payoffs to agents A, B, C and D respectively.

$V_0 + 2(V_k - V_j) > V_0$ to convince him to leave A, and therefore A has to "bid C away" by giving B a side-payment equal to C's valuation of B. However, if $V_k < V_j$, then C is willing to offer B no more than $V_0 + 2(V_k - V_j) < V_0$. But since B can always get V_0 on his own, in this case C's offer poses no threat to A who only has to transfer utility V_0 to B to convince him to preserve their current match.

Subgame 2. With probability another half, B and D simultaneously make offers to C. B also makes an offer to his existing partner, A, and this offer is contingent on his offer to C being rejected. By an argument similar to the one used above, we conclude that B offers C's payoff to be $X_{BC}^i = \text{Min}[2V_k - (2V_i - V_0); 2V_j - V_0 + \epsilon]$ and D offers C's payoff to be $X_{DC}^i = \text{Max}\{V_0; \text{Min}[2V_j - V_0; 2V_k - (2V_i - V_0) + \epsilon]\}$ for an arbitrarily small positive ϵ . Here, $2V_j - V_0$ is the maximum D is willing to offer to C in order to continue matching, and $2V_k - (2V_i - V_0)$ is the maximum B is willing to offer to C in order to form a new match. Hence C will accept B's offer to form the new match for sure if and only if $2V_k - (2V_i - V_0) > 2V_j - V_0$, or $V_k + V_0 > V_i + V_j$, i.e., the sum of the values of the new match and the unmatched exceeds the sum of the values of the existing matches. The equilibrium payoffs are now

$$\begin{matrix} 2 & & 3 \\ & V_0 & \\ \frac{6}{4} & \frac{2(V_k - V_j) + V_0}{2V_j - V_0} & \frac{7}{5}, \text{ if } V_i + V_j - V_0 < V_k \\ & & \\ 2 & & 3 \\ & V_0 & \\ \frac{6}{4} & \frac{2V_i - V_0}{2V_k - 2V_i + V_0} & \frac{7}{5}, \text{ if } V_i < V_k < V_i + V_j - V_0 \\ & 2V_j - 2V_k + 2V_i - V_0 & \end{matrix}$$

$$\begin{array}{c} 2 \quad \quad \quad 3 \\ \quad \quad V_0 \\ \frac{6}{4} \begin{array}{c} 2V_i \quad i \quad V_0 \\ \quad \quad V_0 \\ \quad \quad 2V_j \quad i \quad V_0 \end{array} \quad \frac{7}{5}, \text{ if } V_k < V_i: \end{array}$$

So in the two possible sequences of bargaining (subgame 1 and subgame 2) we see that B and C abandon their old partners to form a new match for sure if and only if the sum of the value of the new match and the unmatched exceeds the sum of two existing matches. Without loss of generality, assume $V_j > V_i$. Then the expected equilibrium payoffs are given by:

$$\begin{array}{c} 2 \quad \quad \quad 3 \quad \quad \quad 2 \quad \quad \quad 3 \\ \quad \quad \quad V_0 \\ \frac{6}{4} \begin{array}{c} A \\ B \\ C \\ D \end{array} \quad \frac{7}{5} = \frac{6}{4} \begin{array}{c} V_k + V_i \quad i \quad V_j \\ V_k + V_j \quad i \quad V_i \\ V_0 \\ V_i \quad i \quad V_k + V_j \\ V_i + V_k \quad i \quad V_j \\ V_j + V_k \quad i \quad V_i \\ V_j \quad i \quad V_k + V_i \end{array} \quad \frac{7}{5}, \text{ if } V_i + V_j \quad i \quad V_0 < V_k \\ \\ \frac{6}{4} \begin{array}{c} A \\ B \\ C \\ D \end{array} \quad \frac{7}{5} = \frac{6}{4} \begin{array}{c} V_i \quad i \quad V_k + V_j \\ V_i + V_k \quad i \quad V_j \\ V_j + V_k \quad i \quad V_i \\ V_j \quad i \quad V_k + V_i \end{array} \quad \frac{7}{5}, \text{ if } V_j < V_k < V_i + V_j \quad i \quad V_0 \\ \\ \frac{6}{4} \begin{array}{c} A \\ B \\ C \\ D \end{array} \quad \frac{7}{5} = \frac{6}{4} \begin{array}{c} V_i \\ V_i \\ V_j + V_k \quad i \quad V_i \\ V_j \quad i \quad V_k + V_i \end{array} \quad \frac{7}{5}, \text{ if } V_i < V_k < V_j \\ \\ \frac{6}{4} \begin{array}{c} A \\ B \\ C \\ D \end{array} \quad \frac{7}{5} = \frac{6}{4} \begin{array}{c} V_i \\ V_i \\ V_j \\ V_j \end{array} \quad \frac{7}{5}, \text{ if } V_k < V_i: \end{array}$$

And the equilibrium expected gains are:

$$\begin{matrix} 2 & 3 \\ i & A \\ 6 & 7 \\ 4 & 5 \\ i & B \\ i & C \end{matrix} = \begin{matrix} 2 & 3 \\ i & (V_i \ i \ V_0) \\ 6 & 7 \\ 4 & 5 \\ i & (V_k \ i \ V_j) \\ i & (V_k \ i \ V_i) \end{matrix}, \text{ if } V_i + V_j \ i \ V_0 < V_k \quad (9)$$

$$\begin{matrix} 2 & 3 \\ i & D \\ 6 & 7 \\ 4 & 5 \\ i & A \\ i & B \\ i & C \end{matrix} = \begin{matrix} 2 & 3 \\ i & (V_j \ i \ V_0) \\ 6 & 7 \\ 4 & 5 \\ i & (V_k \ i \ V_j) \\ i & (V_k \ i \ V_i) \end{matrix}, \text{ if } V_i < V_k < V_i + V_j \ i \ V_0 \quad (10)$$

$$\begin{matrix} 2 & 3 \\ i & D \\ 6 & 7 \\ 4 & 5 \\ i & A \\ i & B \\ i & C \end{matrix} = \begin{matrix} 2 & 3 \\ i & (V_k \ i \ V_i) \\ 6 & 7 \\ 4 & 5 \\ & 0 \\ & 0 \\ & V_k \ i \ V_i \end{matrix}, \text{ if } V_i < V_k < V_j \quad (11)$$

$$\begin{matrix} 2 & 3 \\ i & D \\ 6 & 7 \\ 4 & 5 \\ i & A \\ i & B \\ i & C \\ i & D \end{matrix} = \begin{matrix} 2 & 3 \\ i & 3 \\ & 0 \\ & 0 \\ & 0 \\ & 0 \\ & 0 \\ & 0 \end{matrix}, \text{ if } V_k < V_i: \quad (12)$$

In (9), when B and C form a new match, the equilibrium expected side payment is such that the expected gains to each of them is equal to the capital gains to the new partner, instead of their own capital gain.³ In (10), although the existing matches continue, the old partner must on average pay her current partner his opportunity cost of giving up the option to form a new match. In (11), because the value of the new potential match is not as large as the value of existing match between C and D, A has no need to pay a side payment to B on average in order to persuade him to stay in the existing match. But in expectation, D still needs to pay a side payment to C in order to preserve their valuable match. In (12) the value of the new

³If B and C were to form a new match and there were no side payments, then B would gain $V_k \ i \ V_i$ and C would gain $V_k \ i \ V_j$, but the equilibrium side payments imply that these gains are swapped: B gains $V_k \ i \ V_j$ and C gains $V_k \ i \ V_i$. So when a new match is formed, the agent who is currently in the better match enjoys a larger capital gain.

potential match between B and C is so small that on average A does not have to make a side payment to B and D does not have to make a side payment to C.

We summarize the main features of the bargaining outcomes in Proposition 1. The proof follows from the previous discussion.

Proposition 1 For given value functions, the matching decisions and side payments are uniquely determined in the symmetric competitive matching equilibrium through the sequence of bilateral bargaining. Moreover,

(a) When two agents find an opportunity to form a new match, whether or not they form the new match abandoning their existing matches (if any) depends on whether or not the sum of the values of new match and the unmatched exceeds the total value of the existing matches.

(b) Through the side payment, the expected net gain to the agent who forms a new match is equal to the capital gains of the new partner (instead of his own capital gains).

In the equilibrium, the agents expected payoffs satisfy the following Bellman equations:

$$rV_i - V_i = y_i + \sum_{j=0}^n \sum_{k=1}^K n_j \frac{1}{4} A_{ij}^k V_{k,i} - V_i + s_{ji}^k + 1 - i A_{ij}^k s_{ij}^k$$

$$i \sum_{j=0}^n \sum_{k=1}^K n_j \frac{1}{4} A_{ij}^k (V_i - V_0) + 1 - i A_{ij}^k s_{ij}^k - i (V_i - V_0)$$

for $i = 1; \dots; N$, and

$$rV_0 + V_0 = \sum_{j=0}^N \sum_{k=1}^K n_j \frac{1}{2} \hat{A}_{0j}^k (V_k + V_0 + s_{j0}^k) :$$

Here we are using s_{ij}^k to denote the net expected side payment that the agent in the type i match who met an agent in a type j match offers her to convince her to form a new match with productivity k . Note that $s_{ji}^k = -s_{ij}^k$. Also, we let b_{ij}^k be the side payment that type i agent's old partner offers him to persuade him to stay in the old match. Type i agent's choice of whether or not to form a new match with type j agent is represented by $\hat{A}_{ij}^k \in [0; 1]$. Type i agent's value function also depends upon his old partner's choice, which is represented by b_{ij}^k .

A competitive matching equilibrium with bargaining is a set of value functions, side payments and match formation decisions $(V_i; s_{ij}^k; \hat{A}_{ij}^k)_{i,j=0,k=1}^N$ together with a population distribution of partnerships $(n_i)_{i=0}^N$ such that: (i) Each agent with the opportunity to make an offer chooses how much side payment to offer to her potential partners, and the recipient of the offer chooses whether to accept or reject, in order to maximize her expected discounted utility, taking the strategies of the other agents and the population distribution of partnerships as given; (ii) The population distribution and the strategies of the other agents are equilibrium distribution and strategies. In what follows, we concentrate on a steady state equilibrium in which the population distribution and the strategies are constant over time.

From part (a) of Proposition 1 we know that $\hat{A}_{ij}^k = \hat{A}_{ji}^k$, and that $\hat{A}_{ij}^k = 1$ if $V_k + V_0 > V_i + V_j$, $\hat{A}_{ij}^k = 0$ if $V_k + V_0 < V_i + V_j$, and $\hat{A}_{ij}^k \in [0; 1]$ if

$V_k + V_0 = V_i + V_j$. And from part (b) of Proposition 1 we know that if $A_{ij}^k = 1$, then $V_k - V_i + s_{ji}^k = V_k - V_j$. Also using the fact that $s_{ij}^k = s_{ji}^k$ and $A_{ij}^k = A_{ji}^k$ in a symmetric equilibrium, the value functions reduce to

$$rV_i = y_i + \sum_{j=0}^N \sum_{k=1}^N n_j \frac{1}{4} \max_{0 \leq A_{ij}^k \leq 1} A_{ij}^k (V_k + V_0 - V_i - V_j) - \sum_{j=0}^N (V_i - V_0)$$

$$rV_0 = \sum_{j=0}^N \sum_{k=1}^N n_j \frac{1}{4} \max_{0 \leq A_{0j}^k \leq 1} A_{0j}^k (V_k - V_j):$$

Let us define the value of a match to the pair, $v_i^c = 2V_i$ for $i = 0; 1; \dots; N$.

Then we find the value of the match to the pair satisfies:

$$r_{v_i^c} = 2y_i - \sum_{j=0}^N \sum_{k=1}^N n_j \frac{1}{4} \max_{0 \leq A_{ij}^k \leq 1} A_{ij}^k (v_k^c + v_0^c - v_i^c - v_j^c) \quad (13)$$

$$r_{v_0^c} = \sum_{j=0}^N \sum_{k=1}^N n_j \frac{1}{4} \max_{0 \leq A_{0j}^k \leq 1} A_{0j}^k (v_k^c - v_j^c): \quad (14)$$

The competitive matching equilibrium can be summarized by a list $(v_i^c; A_{ij}^k; n_i)$ for $i; j = 0; \dots; N$ and $k = 1; \dots; N$ that satisfies (13), (14), and the laws of motion (1) and (2). Notice that the equilibrium value of the match to the pair satisfies very similar conditions to the ones that the shadow price of the match must satisfy for a social optimum. In fact, conditions (13) and (14) would be identical to (4) and (5), were it not for the fact that in the optimality conditions there is a "2" in front of the contact rate θ . This difference is due to a search (or match-formation) externality: in the decentralized economy, an individual agent does not take into account the impact that her decisions to form and destroy matches have on the arrival of opportunities

of the other agents. Although the arrival rate of any new opportunity is constant here, the arrival rate of a new opportunity with a particular type of agent is proportional to the measure of agents of that type. Also, whether or not a new match is formed depends not only on the quality of the new potential match, but also on the types of the existing matches. Therefore, the relevant meeting rate is quadratic, because the total number of contacts between type i agents and type j agents is equal to $\alpha n_i n_j$.⁴ We summarize these results as follows:

Proposition 2 The competitive matching equilibrium is similar to the planner's economy, except that it does not take into account the search externality due to the quadratic matching technology.

In the next section we use a special case of the general model to illustrate the main properties of the payoffs and allocations in the competitive matching equilibrium and the planner's solution.

4 A Special Case

Suppose $N = 2$; then the flow conditions (1) and (2) reduce to

$$n_2 = \alpha \frac{1}{4} n_0^2 + 2n_0 n_1 + n_1^2 \quad (1)$$

$$n_1 = \alpha \left(1 - \frac{1}{4}\right) n_0^2 + 2\alpha \frac{1}{4} n_0 n_1 + 2\alpha \frac{1}{4} n_1^2 \quad (2)$$

$$n_0 = \alpha (n_1 + n_2) + \alpha \frac{1}{4} n_1^2 + \alpha n_0^2$$

⁴Mortensen (1982) shows that "mating models" in which an agent's decisions affect other agents' meeting probabilities typically fail to achieve the socially optimal allocation due to a search externality.

Here we have already recognized that $\hat{A}_{0j}^2 = 1$ for $j = 0, 1$ and that $\hat{A}_{i2}^k = 0$ for $i = 0, 1, 2$ and $k = 1, 2$. To simplify notation, we are letting $\hat{A} = \hat{A}_{11}^2$ and $\frac{1}{4} = \frac{1}{4}_2$. The following lemma characterizes the steady state distribution of matches taking as given the separation decision \hat{A} .

Lemma 1 A unique steady state distribution of workers exists for any given $\hat{A} \in [0, 1]$. The number of unemployed workers, n_0 , solves

$$r n_0^2 + (1 - n_0) \left[(r + 2\frac{1}{4}n_0)^2 + \hat{A} \frac{1}{4} \right] 2(1 - n_0) + (1 + \frac{1}{4}) n_0^2 = 0:$$

The number of workers employed in matches with productivity y_1 is $n_1 = \frac{2(1 - n_0) + (1 + \frac{1}{4})n_0^2}{r + 2\frac{1}{4}n_0}$, and the number of workers employed in matches with productivity y_2 is $n_2 = 1 - n_0 - n_1$.

Proof. See the Appendix.

In a stationary equilibrium the value functions satisfy:

$$rV_2 = y_2 + (V_2 - V_0)$$

$$rV_1 = y_1 + (V_1 - V_0) + \frac{1}{4}n_0(V_2 - V_1) + \frac{1}{4}\hat{A}(V_2 + V_0 - 2V_1)$$

$$rV_0 = \frac{1}{4}(V_2 - V_0) + (1 - \frac{1}{4})(V_1 - V_0) + \frac{1}{4}(V_2 - V_1):$$

From Proposition 1 we know that $\hat{A} = 1$ with certainty if and only if $V_2 + V_0 - 2V_1 > 0$. We can use the Bellman equations to eliminate the value functions and write this inequality as

$$\frac{y_2}{y_1} > 2 + \frac{\frac{1}{4}n_1 + (1 - \frac{1}{4})n_0}{r + \frac{1}{4}(n_0 + \frac{1}{4}n_1)} \quad (15)$$

where n_0 and n_1 are the steady state numbers of matches characterized in Lemma 1. Since the right hand side of (15) is bounded, it is clear that $\hat{A} = 1$ with certainty for $y_2=y_1$ large enough. In these cases, the agents involved will destroy two middle-productivity matches in order to form a single high-productivity match whenever the opportunity arises. Perhaps more surprisingly, notice that there is always some $\epsilon > 0$ such that $\hat{A} = 1$ for all $y_2=y_1 > 2 \epsilon$. That is, there may be instances in which two middle-productivity matches are destroyed to form a single high-productivity match even if this entails a reduction in current output. To compute a stationary equilibrium, let $n_i(\hat{A})$ denote the steady state number of matches of productivity y_i as characterized in Lemma 1. Then define the best-response map $\phi(\hat{A}) = \frac{y_2}{y_1} + \frac{\epsilon[\frac{1}{4}n_1(\hat{A}) + (1-\frac{1}{4})n_0(\hat{A})]}{r + \epsilon + \epsilon[n_0(\hat{A}) + \frac{1}{4}n_1(\hat{A})]}$. From this we see that $\hat{A} = 1$ is an equilibrium if $\phi(1) > 0$, $\hat{A} = 0$ is an equilibrium if $\phi(0) < 0$ and $\hat{A}^* \in [0; 1]$ is an equilibrium if $\phi(\hat{A}^*) = 0$.⁵

Given (15), Proposition 2 tells us that the social planner chooses to destroy a pair of matches of productivity y_1 to create a single match of productivity y_2 if and only if

$$\frac{y_2}{y_1} > 2 \epsilon \frac{2\epsilon[\frac{1}{4}n_1 + (1-\frac{1}{4})n_0]}{r + \epsilon + 2\epsilon(n_0 + \frac{1}{4}n_1)}, \quad (16)$$

with n_0 and n_1 given by Lemma 1. Notice that also here, there are instances in which the planner chooses to destroy two matches of productivity y_1 to

⁵The equilibrium map ϕ is continuous on $[0; 1]$, so there always exists a stationary equilibrium. However, the map is highly nonlinear, so we cannot provide weak parametric restrictions to guarantee uniqueness. We explored several reasonable parametrizations numerically and have always found that the stationary equilibrium is unique. We report several of these numerical exercises below.

create a single match of productivity y_2 at the cost of reducing current output. Both in the competitive equilibrium and in the planner's solution the basic logic for this result goes as follows. Although unmatched agents generate zero current output, they generate a positive expected discounted value of output. Hence for some parametrizations (e.g. $y_2=y_1$ slightly below 2), the planner may choose to reduce current output in order to maximize the present discounted value of output. From a static point of view, this may come as a surprise since unmatched agents are unproductive; but from the planner's dynamic perspective, unmatched agents are a valued input in the matching process that makes production possible. This intuition can be formalized by noticing that both (15) and (16) approach $y_2=y_1 > 2$ as r becomes large. The higher the degree of impatience, the less willing the planner is to trade off current for future production.

From (15) and (16) we also learn that being able to internalize the search externality makes the planner more willing to destroy middle matches. The reason is that the shadow value the planner assigns to a pair of unmatched agents is larger than in their value in the competitive equilibrium (because the planner also imputes as part of their return the fact that the unmatched pair helps other agents climb the productivity ladder). Consequently, the planner is relatively more willing to trade two matches of productivity y_1 for two agents in a match of productivity y_2 and two unmatched agents. Figure 4 illustrates the difference between the relevant destruction margins in the efficient and the competitive solutions. On the horizontal axis is r , a measure of impatience, and on the vertical axis $y_2=y_1$, the relevant measure

of inequality in instantaneous productivities. Notice that the $(n_0; n_1)$ pair that appears in (15) is identical to that in (16) and is independent of y_1 , y_2 and r . (See Lemma 1.) The solid lines with the higher and lower intercepts

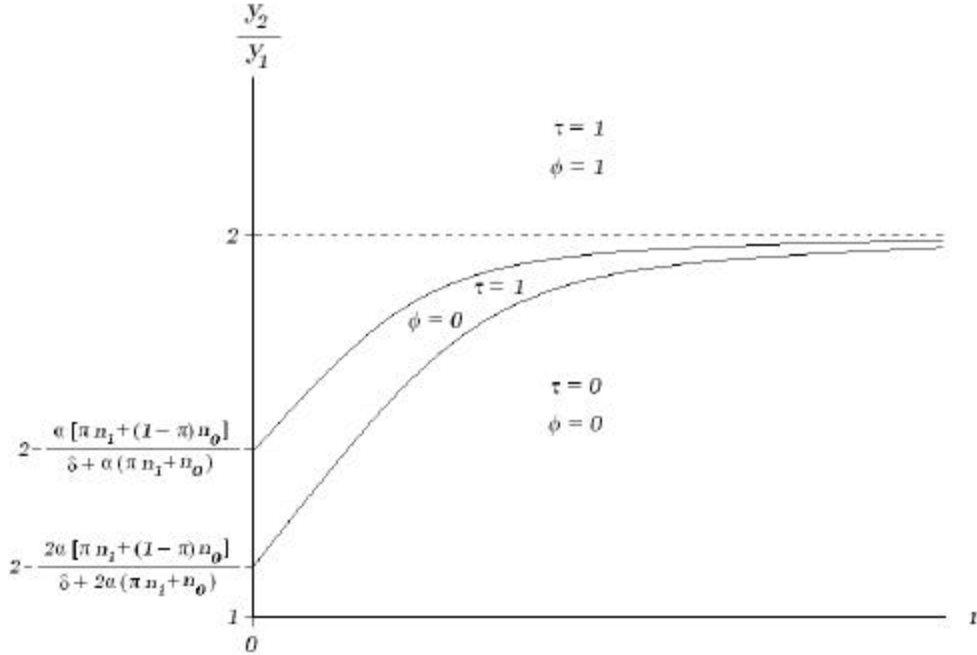


Figure 4: Destruction regions for the case with $N = 2$.

are conditions (15) and (16) at equality respectively. As in the competitive economy, we know that $\chi_{0j}^2 = 1$ for $j = 0; 1$; that $\chi_{i2}^k = 0$ for $i = 0; 1; 2$ and $k = 1; 2$ and therefore we use χ to denote χ_{11}^2 , the only nontrivial decision.

Double breaches occur in the competitive equilibrium only for parametrizations that lie above the higher solid line. In contrast, the planner implements double breaches for parametrizations that lie above the lower solid line. For any given degree of impatience r , the competitive and the efficient

allocations coincide only if the low productivity differential $y_2=y_1$ is either large enough (i.e. above the higher solid line) or small enough (below the lower solid line). For intermediate values (i.e. those that lie between the two solid lines) the allocations differ: relative to the efficient benchmark, matches of productivity y_1 are too stable in the competitive economy.

It is possible to design policies that bring the competitive allocations in line with their efficient counterparts. For example, suppose every agent receives a payoff $b > 0$ while unmatched, and that this transfer is paid for by levying a tax T_i from every match of productivity i .⁶ The balanced-budget condition is $bn_0 = T_1n_1 + T_2n_2$. The Bellman equations for the competitive economy become

$$\begin{aligned} r\hat{V}_2 &= y_2 - T_2 + \beta(\hat{V}_2 - \hat{V}_0) \\ r\hat{V}_1 &= y_1 - T_1 + \beta(\hat{V}_1 - \hat{V}_0) + \beta n_0 \lambda (\hat{V}_2 - \hat{V}_1) + \beta n_1 \lambda A (\hat{V}_2 + \hat{V}_0 - 2\hat{V}_1) \\ r\hat{V}_0 &= b + \beta n_0 \lambda (\hat{V}_2 - \hat{V}_0) + (1 - \beta) (\hat{V}_1 - \hat{V}_0) + \beta n_1 \lambda (\hat{V}_2 - \hat{V}_1): \end{aligned}$$

Notice that for a given destruction decision \hat{A} , the stationary distribution of agents across states is still as described in Lemma 1. However, now $\hat{A} = 1$ with certainty if and only if $\hat{V}_2 + \hat{V}_0 - 2\hat{V}_1 > 0$, which can be rewritten as

$$\frac{y_2 - T_2 + b}{y_1 - T_1 + b} > 2 + \frac{\beta [n_1 + (1 - \beta)n_0]}{r + \beta + \beta(n_0 + n_1)}$$

⁶For the discussion of this section we will ignore the issue of exactly how a government may be able to collect taxes from agents in a random matching economy, as well as why the same government is unable to facilitate the matching process.

Now let $T_1 = T_2 = T$ and note that for a given T , the budget equation implies $b = \frac{n_1 + n_2}{n_0} T$, so the above condition becomes

$$\frac{y_2 - i \frac{T}{n_0}}{y_1 - i \frac{T}{n_0}} > 2 \frac{r + \pm + \frac{1}{2}(n_0 + \frac{1}{2}n_1)}{r + \pm + \frac{1}{2}(n_0 + \frac{1}{2}n_1)} \quad (17)$$

Observe that if we let $T = T^*$, where

$$T^* = \frac{\frac{1}{2}n_0 (r + \pm) [\frac{1}{2}n_1 + (1 - \frac{1}{2})n_0]}{[r + \pm + 2 \frac{1}{2}(n_0 + \frac{1}{2}n_1)] (r + \pm + \frac{1}{2}n_0)} y_1;$$

then (16) and (17) coincide. In other words, the compensation $b^* = \frac{n_1 + n_2}{n_0} T^*$ makes agents internalize the search externality in the competitive matching equilibrium and implements the same destruction decisions as the planner's. Quite intuitively, note that b^* approaches zero as either $r \rightarrow 1$ or $y_1 \rightarrow 0$.

The model has clear predictions regarding individual agents' employment histories and the various attributes of different types of jobs. For example, a job of productivity y_2 is not only better paid, but also more stable than a job of productivity y_1 . The first observation is immediate because $y_2 > y_1$ (and, in fact, also $V_2 > V_1$). The second follows from the fact that the expected time until a worker gets displaced is $\frac{1}{\pm}$ for a job of productivity y_2 and $\frac{1}{\pm + \frac{1}{2}(n_0 + \frac{1}{2}n_1)}$ for a job of productivity y_1 . Displacement from a job with productivity i is associated with a capital loss equal to $V_i - V_0$, and it takes workers some time to climb back up to a job of productivity equal or higher to the one they were displaced from. For example, suppose a worker is displaced from a job of productivity y_1 (i.e. his match is either hit by the exogenous destruction shock \pm , or his employer fires him in order to form a new match of productivity y_2 with another worker). The expected time it takes this worker

to find a job at least as good as the one he lost is $\frac{1}{\theta(n_0 + \lambda n_1)}$. Note that the degree of inequality (say as measured by $V_i \neq V_j$) as well as the shapes of the various hazard rates depend crucially on the separation decisions λ . Therefore, we can expect these variables to vary systematically across economies with different labor-market policies that affect this endogenous destruction margin.

We can also construct the theoretical counterparts to the usual empirical measures of job and worker flows. For example, let JC , JD and WR denote gross job creation, gross job destruction and gross worker reallocation respectively in the stationary equilibrium. Then we have

$$JC = \theta(n_0 + \lambda n_1) n_0$$

$$JD = \theta \lambda (n_0 + \lambda n_1) n_1 + \delta (n_1 + n_2)$$

$$WR = \theta n_0 n_0 + 2\theta n_0 n_1 \lambda + \theta n_1 n_0 \lambda + 2\theta n_1 n_1 \lambda + \delta (n_1 + n_2) :$$

Job creation includes all those unmatched employers who meet and start productive relationships with either unmatched or matched workers. Job destruction consists of all those filled jobs which become unfilled. This occurs every time an employed worker quits to form a better match with another employer and also when the match is destroyed for exogenous reasons. It can be verified that, naturally, $JC = JD = 0$ since the net employment change is zero in the steady state. Worker reallocation counts the number of workers who change state. In the first term are the number of unemployed workers who fill vacant jobs. In the second term are the unemployed workers who contact a filled job and get hired. The "2" multiplying this term accounts

for the change of state of the previously employed worker who gets displaced. The third term represents the number of previously employed workers who contact a vacant job and quit to form a more productive relationship. The fourth term accounts for the number of workers who are employed and quit to form a new match with an employer who was previously matched to another worker, as well as for the corresponding displaced workers. The number of workers who change state (i.e. become unemployed) for exogenous reasons are accounted for in the last term.

Notice that the gross job and worker flows satisfy:

$$WR = JC + JD + \frac{1}{4} (n_0 + \hat{A}n_1) n_1$$

This relationship shows that in the model –as in the data– gross worker reallocation is larger than gross job reallocation, $JC + JD$. Instances of “replacement hiring” are behind this discrepancy, since job creation and destruction are unchanged when a firm hires a worker to replace him with an unemployed one. But also, in economies in which $\hat{A} > 0$, there is yet another reason for worker reallocation in excess of job reallocation, since when a matched employer and an employed worker decide to form a new match the worker reallocation count increases by 2 while job reallocation only increases by 1 (job creation is unchanged by this transition).⁷

⁷Several recent empirical studies argue that distinguishing between job and worker flows is essential for a complete characterization of aggregate labor-market dynamics. See Nagypál (2003) and Stewart (2002). Notice that in the macro labor model most commonly used, e.g. the one in Pissarides (2000), gross job and worker flows are one and the same by construction.

A Appendix

Proof of Lemma 1. Let $f(n_0) = \frac{2\pm(1-n_0)i + (1+\frac{1}{4})n_0^2}{\pm+2\frac{1}{4}n_0}$. Combining the $n_2 = 0$ and $n_0 = 0$ conditions we see that $n_1 = f(n_0)$. It can be shown that $f' < 0$ on $[0; 1]$, so to each $n_0 \in [0; 1]$ corresponds a unique n_1 . In addition, $f(n_0) \leq 0$ if $n_0 \geq \bar{n}_0$ and $f(n_0) > 0$ if $n_0 < \bar{n}_0$, where

$$\bar{n}_0 = \frac{\sqrt{\pm^2 + 2\pm(1+\frac{1}{4})i \pm}}{\pm(1+\frac{1}{4})} \quad \text{and} \quad \underline{n}_0 = \frac{\sqrt{(\pm+\frac{1}{4})^2 + \pm(1+\frac{1}{4})i (\pm+\frac{1}{4})}}{\pm(1+\frac{1}{4})};$$

with $0 < \underline{n}_0 < \bar{n}_0 < 1$. Let

$$G(n_0; \hat{A}) = \pm n_0^2 i \pm (1 - n_0) (\pm + 2\frac{1}{4}n_0)^2 i - \hat{A} \frac{2\pm(1-n_0)i + (1+\frac{1}{4})n_0^2}{\pm+2\frac{1}{4}n_0};$$

Substituting $n_1 = f(n_0)$ back into the $n_0 = 0$ delivers a single equation in n_0 which can be written as $G(n_0; \hat{A}) = 0$. Direct calculations reveal that $G(\bar{n}_0; \hat{A}) = \pm \bar{n}_0^2 i \pm (1 - \bar{n}_0) > 0$ for all $\hat{A} \in [0; 1]$. Also, $G(\underline{n}_0; \hat{A}) = \pm \underline{n}_0^2 i \pm (1 - \underline{n}_0) i - \hat{A} \frac{1}{4}$. Note that an increase in \hat{A} causes G to shift down uniformly. Therefore, to ensure that $G(\underline{n}_0; \hat{A}) < 0$ for all \hat{A} it suffices to guarantee that $G(\underline{n}_0; 0) < 0$. This condition can be written as

$$\pm > \frac{\pm \sqrt{(\pm+\frac{1}{4})^2 + \pm(1+\frac{1}{4})i (\pm+\frac{1}{4})}}{\pm(1+\frac{1}{4})} \frac{i}{\pm(1+\frac{1}{4})};$$

a parametric restriction that is always satisfied. Finally, note that

$$\frac{\partial G(n_0; \hat{A})}{\partial n_0} \Big|_{G(n_0; \hat{A})=0} > 0;$$

which together with the fact that $f' < 0$ implies that the steady state is unique whenever it exists. ■

References

- [1] Burdett, Kenneth and Dale T. Mortensen. "Wage Differentials, Employer Size and Unemployment," *International Economic Review* 39 (1998): 257-73.
- [2] Burdett, Kenneth, Ryoichi Imai and Randall Wright. "Unstable Relationships." *Frontiers of Macroeconomics*, forthcoming, 2003.
- [3] Diamond, Peter A. and Eric S. Maskin. "An Equilibrium Analysis of Search and Breach of Contract I: Steady States." *Bell Journal of Economics* 10(1) (September 1979): 282-316.
- [4] Diamond, Peter A. and Eric S. Maskin. "An Equilibrium Analysis of Search and Breach of Contract II: A Non-Steady State Example." *Journal of Economic Theory* 25(2) (October 1981): 165-195.
- [5] Diamond, Peter A. and Eric S. Maskin. "Externalities and Efficiency in a Model of Stochastic Job Matching." MIT mimeo, November 1981.
- [6] Eshel, Ilan and Avner Shaked. "Partnership." *Journal of Theoretical Biology*, 208 (2001): 457-474.
- [7] Mortensen, Dale T. "Property Rights and Efficiency in Mating, Racing, and Related Games." *American Economic Review* 72(5) (1982): 968-79.
- [8] Mortensen, Dale T. "The Matching Process as a Non-Cooperative/Bargaining Game." In *The Economics of Information and*

Uncertainty, ed. J. J. McCall. Chicago: University of Chicago Press, 1982.

- [9] Nagypál, Éva. "Worker Reallocation over the Business Cycle: The Importance of Job-to-Job Transitions." Northwestern University mimeo, November 2003.
- [10] Pissarides, Christopher A. *Equilibrium Unemployment Theory*. Cambridge: MIT Press, 2000 (second edition).
- [11] Postel-Vinay, F. and J.-M. Robin. "To Match Or Not To Match? Optimal Wage Policy with Endogenous Worker Search Intensity." *Review of Economic Dynamics*, forthcoming 2003.
- [12] Postel-Vinay, F. and J.-M. Robin. "Wage Dispersion with Worker and Employer Heterogeneity." *Econometrica* 70(6) (2002): 2295-350.
- [13] Postel-Vinay, F. and J.-M. Robin. "The Distribution of Earnings in an Equilibrium Search Model with State-Dependent Offers and Counter-Overs." *International Economic Review* 43(4) (2002): 989-1016.
- [14] Stewart, Jay. "Recent Trends in Job Stability and Job Security: Evidence from the March CPS." Bureau of Labor Statistics mimeo, February 2002.