Optimal Regulation in the Presence of Reputation Concerns*

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Abstract

We study a market with free entry and exit of firms, who can produce a high quality output if making a costly, but efficient, initial unobservable investment. If there is no learning about this investment, there is an extreme 'lemons problem' where no firm invests. Learning introduces reputation incentives such that a fraction of entrants do invest. If the market operates with spot prices, simple regulation can enhance the role of reputation to induce investment, mitigating the 'lemons problem' and improving welfare.

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1 Introduction

In many market settings, the “lemons problem” (Akerlof (1970)) is an impediment to trade — if buyers are unable to verify the quality of the goods or services being provided by sellers and sellers of low quality goods are free to enter the market, then adverse selection can lead to market outcomes that are inefficient. If buyers have access to public signals of the quality of sellers’ goods or services upon which a seller’s reputation can be based, then sellers’ concern for their reputation is one mechanism through which the lemons problem is mitigated.[1] But does sellers’ concern for their reputation in markets subject to a lemons problem lead to allocations that are constrained efficient? Can regulation of markets subject to a lemons problem enhance the role of reputation in improving welfare? If so, what form should this regulation take?

These questions take on added urgency in the aftermath of the 2008 financial crisis. In 1963, former Federal Reserve chairman Alan Greenspan wrote ”Reputation, in an unregulated economy, is a major competitive tool...Left to their own devices, it is alleged, businessmen would attempt to sell unsafe food and drugs, fraudulent securities, and shoddy buildings...but it is in the self-interest of every businessman to have a reputation for honest dealings and a quality product”.[2] Forty five years later, in his remarks before the House of Representatives he declared ”Those of us who have looked to the self-interest of lending institutions to protect shareholders’ equity, myself included, are in a state of shocked disbelief”.[3] So, does regulation substitute or complement reputation forces?

In this paper we argue that, as a general matter, simple regulatory interventions in markets subject to a lemons problem can in fact enhance market learning to foster reputation incentives and improve welfare. We do so in a general equilibrium model in which the production of one good is subject to an endogenous lemons problem when traded in spot markets. We consider the possibilities for welfare enhancing regulation under various assumptions about the information available to the regula-

[1] For an excellent survey of the literature on this subject, see Part IV of the book by Mailath and Samuelson (2006).

[2] ”The Assault on Integrity”. The Objectivist Newsletter, August 1963. See also this statement from Goldman Sachs’ 2009 Annual Report “Our assets are our people, capital and reputation. If any of these is ever diminished, the last is the most difficult to restore.”

[3] New York Times, ”Greenspan Concedes Error on Regulation”, October 24th, 2008

tor. We find that even if a regulator has access to much less information than private agents, simple forms of regulation can leverage reputation in mitigating the lemons problem and thus enhance welfare.

In our model, consumers have utility over two final goods: a homogenous good that we term the *numeraire good* and a final good that we term the *experience good*. The experience good is produced aggregating a continuum of intermediate goods of uncertain quality as inputs. The lemon's problem occurs in the market in which producers of the experience good purchase these intermediate goods.[4] Individual producers of these intermediate goods are long-lived and have zero marginal cost of production at each moment in time up to a capacity constraint. Entering producers of the intermediate good decide whether to make a costly investment of the numeraire good to become high quality or to not make the investment and to enter with low quality. A unit of the intermediate good provided by a high quality producer has a positive marginal product if used in the production of the experience good while a unit provided by a low quality producer has a negative marginal product if used in the production of the experience good. Producers of intermediate goods exit for exogenous reasons at a fixed rate and can also choose to exit endogenously at a higher rate, if it is optimal for them to do so. A steady-state in this economy has ongoing entry and exit of intermediate good producers.

The socially optimal allocation has entry of high quality producers of intermediate goods, each making the required initial investment of the numeraire good, until the discounted present value of output of the marginal high quality intermediate producer (valued at consumers' marginal utility for the experience good) is equal to the required initial investment of the numeraire good. There are no low quality intermediate producers in this allocation as these producers have a negative marginal product. Under full information, this optimal allocation is also the equilibrium outcome in a market in which intermediate good producers are paid a price in a spot market equal to the marginal product of their current output valued at consumers' marginal utility for the experience good.

The lemons problem in this spot market arises when it is not possible to observe if individual producers of intermediate goods have made the investment required

[4]Our assumption that consumers consume the experience good as an aggregate of underlying intermediate goods of uncertain quality simplifies the computation of equilibrium and allows us to construct a straightforward measure of social welfare.

to be high quality — it cannot be the case that all intermediate producers are paid a positive price for their output in equilibrium or else low quality producers will earn positive profits. In the absence of information about intermediate producers' quality, this lemons problem leads to no production of the experience good at all in equilibrium.

The lemons problem is mitigated if producers of the experience good have access to a public signal of each intermediate goods producer's quality that serves as the basis for that producer's reputation. In a spot market with such signals, at each moment of time, individual intermediate producers are paid the expected marginal product of their intermediate good valued at consumers' marginal utility for the experience good, where that expectation is based on the individual producer's current reputation. The reputation at entry of a given intermediate good producer depends on the equilibrium ratio of high to low quality entrants at the moment of entry. Intermediate good producers' reputations then evolve over time according to the stochastic structure of the public signals and the endogenous exit decisions of high and low quality producers.

In equilibrium, the ratio of high to low quality entrants must be consistent with an initial reputation at which the expected discounted payoff to entry for low quality entrants is zero while that for high quality entrants is sufficient to recoup these entrants' initial investment in quality. Likewise, because exit is also free, producers choose to cease production whenever the value of continuing with a given reputation falls to zero. In this sense, the severity of the lemons problem in this environment is endogenous. Depending on the stochastic structure and precision of the public signal, equilibrium either features very little entry of low quality producers of the intermediate good and high level of production of the experience good, similar to the outcome with full information, or entry dominated by low quality producers of the intermediate good and a low level of production of the experience good as in the outcome with no information about intermediate producers' quality.

In this paper, we characterize the steady-state equilibrium in this economy with producers of the experience good competing to purchase intermediate goods at spot market prices given a fixed stochastic signal structure that conveys some, but not complete, information about quality and we ask whether a regulator can, through the use of taxes and transfers, improve on the spot market equilibrium outcome.

We find that the extent to which a regulator can improve spot market outcomes depends on the information available to that regulator. We show that if the regulator can observe intermediate producers' reputations just as well as buyers in the market, then he can achieve an allocation resulting in welfare arbitrarily close to the informational unconstrained first best by committing to a scheme of taxes and subsidies that reward intermediate producers with good reputations more than the reward offered in the spot market. In this case, regulation can virtually eliminate the lemons problem with a combination of a fixed regulatory cost imposed each period on all active producers of intermediate goods per period of operation and a subsidy of intermediate producers' profits proportional to their reputation. We show that this simple regulatory scheme is sufficient to drive low quality producers of the intermediate good out of the market while, at the same time, offering high quality producers sufficient compensation for their initial investment in quality to achieve a high level of production of the experience good.

We interpret this finding as indicating that the lemons problem in this environment is a problem of commitment, not one of information — regulation is a means of implementing outcomes that could be achieved privately if buyers and sellers could commit to long-term contracts designed to solve the underlying incentive problem.

We next ask what if a regulator has access to limited information on activity in the market? What are the possibilities for welfare improvement here? We consider a case in which a regulator does not observe (or cannot verify) producers' reputations and instead can only observe the initial entry of intermediate goods producers. In this case, the regulator cannot use subsidies and taxes based on reputation but instead can only contemplate transfers to or from producers of the intermediate good at entry.[5]

A priori, it is not clear that imposition of a regulatory fixed entry cost on intermediate goods producers will enhance welfare relative to the unregulated spot market equilibrium outcome. A regulatory entry cost has two effects on the equilibrium allocation: it alters the equilibrium mix of high and low quality producers at entry and it alters the equilibrium level of production of the experience good. We show that for some public signal structures, an entry cost serves not only to increase the average quality of producers in the market but also serves to expand the equilibrium

[5] The regulator could also consider quantitative restrictions on entry. However, the set of allocations that can be achieved using quotas is just a subset of the set of allocations that can be achieved using entry costs, which may also involve an increase in production.

production of the experience good relative to the unregulated spot market outcome. In this case, regulatory entry costs have a double benefit of raising average quality and expanding trade and hence clearly improve welfare. For other signal structures, however, there is a tradeoff between these two objectives — a regulatory entry cost raises the average quality of producers in the market but reduces production of the experience good relative to the unregulated spot market equilibrium.

Our main qualitative result is that the optimal regulatory entry cost is positive even if the stochastic signal structure is such that the imposition of this entry cost comes at the expense of reducing the volume of trade. This result is based on the finding that, starting from zero entry costs, the increase in average quality of intermediate good producers that results from a small increase in entry costs has a first order impact on welfare while the reduction in production of the experience good has only a second order impact on welfare.

A technical contribution of the paper is the analytical derivation of the firms' value functions in continuous time when exit is an endogenous choice and firms know their type. We do this for three different processes of stochastic signals: one we term *bad news*, one we term *good news*, and Brownian motion. This result allows the analytical comparison between entry conditions for high and low quality firms, providing tractability in welfare comparisons across different regulation policies. Furthermore, since in this paper types are not assumed but are the result of investment decisions, we are able to obtain the reputation assigned to entrants and the extent of adverse selection in the market endogenously from entry conditions. Hence the strength of reputation incentives in the market are obtained in general equilibrium.

Literature Review

This paper is related to two strands of literature that to date have not been systematically connected: reputation and regulation. With respect to the reputation strand, there is a rich literature showing the effectiveness of reputation as a disciplining device provided by the market to discourage firms' opportunistic behavior, in particular the incentives to sell goods of poor quality. MacLeod (2007) recently provides a survey of the literature that analyzes the importance of reputational incentives in complementing contract enforcement to sustain discipline in markets. Mailath and Samuelson (2001) and Tadelis (1999 and 2002) interpret reputation as a valuable asset that modifies firms' actions. In these models firms differ in an unobservable exogenous type and they enter at an exogenous reputation to replace those that exogenously die. Relaxing this last assumption, Horner (2002), Bar-Isaac (2003) and Daley and Green (2010) introduce endogenous exit of firms, when these firms know their own type.

Our model departs from this literature in two relevant aspects. First, the unobservable types are not assumed exogenous but are endogenously determined by the decision of otherwise identical firms. Second, we introduce free entry to endogenize the reputation assigned to entrants, which determines in general equilibrium the level of adverse selection in a market. From a technical viewpoint, ours is the first paper that fully characterizes value functions with exit decisions in continuous time, when firms know their type and have the option to exit.[6] Other papers of reputation in continuous time are Board and Meyer-ter Vehn (2010) and Faingold and Sannikov (2011), however they do not consider entry and exit decisions.

With respect to the regulation strand, this paper contributes to Leland (1979), extended later by Shaked and Sutton (1981) and most notably to Shapiro (1983 and 1986), who introduce moral hazard and investment decisions in markets with asymmetric information. Our paper also complements von Weizsacker (1980), who discusses how barriers to entry may increase welfare once we consider economies of scale and differentiated products. More recently, Lizzeri (1999) and Albano and Lizzeri (2001) analyze the efficiency effects of certification intermediaries, but without making reference to reputation concerns while Garcia-Fontes and Hopenhayn (2000) focus on entry restrictions, while we allow for more general regulation possibilities and taxing schemes.

This paper extends the discussion raised by Prescott and Townsend (1984) and Arnott, Greenwald, and Stiglitz (1993) about whether or not, in a world with adverse selection and moral hazard, even if information imperfections cannot be corrected, government interventions can be Pareto improving. Here we show than in a model with endogenous adverse selection and learning, the market outcome with spot trade between producers and buyers is not constrained Pareto optimal. However this result does not arise from information asymmetry per se, but from the assumption that buyers and sellers are unable to commit to optimal prices, which are different than spot prices. Closely related to our paper, Klein and Leffler (1981, pp 168) find that ”mar-

[6]Prat and Alos-Ferrer (2010) solve similar value functions but without endogenous exit.

ket prices above the competitive price and the presence of nonsalvageable capital are means of enforcing quality promises." Our paper shows how a government can enhance this result even further with very simple taxes and subsidies, or even optimal entry costs. In this sense government intervention is Pareto improving only if the private sector cannot reproduce the commitment that a government can introduce with very simple taxes and subsidies.

Even when our paper proposes simple taxing schemes to replicate commitment, some papers have proposed ways for the market to achieve such a commitment. Boyd and Prescott (1986), show the relevance of large financial intermediaries as a way to allow for a welfare improving separating equilibrium. In our case, such an intermediary would allow an improvement on endogenous quality. Our model also gives rise to a justification for horizontal integration to cross subsidize divisions with different reputation, which is different than the reputational justification provided by Cai and Obara (2009), based on eliminating idiosyncratic shocks of individual markets and allowing firms for more sophisticated deviations.

In the next Section we describe the economy and characterize the spot market equilibrium for two extreme benchmarks: full information and no learning. In Section 3 we characterize the spot market equilibrium in steady state with imperfectly informative signals. In Section 4 we study the role of regulation in improving welfare relative to a spot market economy under two settings, one where the regulator can observe reputation, as the market does, and another where the regulator can only observe entry. In Section 5 we obtain analytically the value functions that characterize the solutions and provide a numerical illustration. In Section 6 we make some final remarks.

2 The Model

In this section we describe the economic environment, characterize the socially optimal allocation, and solve for the spot market equilibrium under two informational benchmarks: full information, in which the quality of the producers of the intermediate goods is fully observable, and no information, in which there are no signals of the quality of the intermediate good producers.

2.1 The Economy

Time is continuous and denoted by $t \in [0, \infty)$. At each time t, consumers in this economy derive utility from the consumption of two final goods: one which we term the *experience good* and one which we term the *numeraire good*. Let Y_t denote consumption of the experience good and N_t consumption of the numeraire good at t. Consumers' utility is given by

$$\int_t e^{-\hat{r}t} \left[U(Y_t) + N_t \right] dt \tag{1}$$

where $U' > 0$, $U'' < 0$ and $\hat{r}$ is the discount factor.

At each time t, there is an endowment of 1 unit of the numeraire good. This good is not storable. The experience good is produced with a constant returns to scale technology that uses produced intermediate goods as the only inputs.

At each point in time t, there is a stock of "trees" in the economy that yield a flow of the intermediate good as "fruit" at zero marginal cost. Each tree yields a flow of one unit of the intermediate good per unit time for as long as the tree remains active. Trees become inactive for exogenous reasons at a rate $\delta > 0$ per unit time and can also be rendered inactive to remove them from production. Trees that become inactive at t cannot be returned to production at later dates.

Trees can be one of two types, high quality (H) or low quality (L), depending on an initial investment made when the tree enters production (is planted). To plant a high quality tree at t, an investment of C units of the numeraire good is required at that moment. Low quality trees can be planted at zero cost at any moment. We refer to the planting of new trees as *entry*.

The quality of the tree yielding a flow of the intermediate good as fruit determines the expected productivity of those units of the intermediate good in use as an input to produce the experience good. One unit of the intermediate good from a high quality tree contributes $y(1) > 0$ units of output of the experience good at the margin, while one unit of fruit from a low quality tree yields $y(0) < 0$ units of output of the experience good at the margin.[7]

[7] The assumptions of zero marginal cost of production for the intermediate good and a negative marginal product of low quality intermediate goods are normalizations that simplify the exposition. Assuming positive production costs and positive marginal product of low quality intermediate goods with marginal product less than marginal cost delivers the same results but makes the equations less straightforward.

Let ϕ denote the public belief regarding the probability that a given tree is high quality. We refer to ϕ as the tree's *reputation*. The expected output of the experience good obtained from a unit of the intermediate good from a tree with reputation ϕ is denoted $y(\phi)$ and is given by the affine function

$$y(\phi) = \phi y(1) + (1 - \phi)y(0). \tag{2}$$

The resource constraint for the experience good is then given by

$$Y_t = y(1)m_t(1) + y(0)m_t(0), \tag{3}$$

where $m_t(1)$ is the measure of active high quality trees at t and $m_t(0)$ is the corresponding measure of active low quality trees.

We denote the flow of new trees entering at t by $m_t^e \geq 0$. The fraction of these entrants who invest to become high quality is denoted $\phi_t^e \in [0, 1]$. The corresponding resource constraint for the numeraire good is

$$N_t = 1 - C\phi_t^e m_t^e, \tag{4}$$

where $C\phi_t^e m_t^e$ are the resources invested in planting high quality trees at t.

A tree entered into production of the intermediate good at t starts production with reputation ϕ_t^e. As long as this tree is active, it generates signals that evolve over time at a stochastic rate dS_t. We refer to the removal of active trees from production as *exit*. We denote the rate of exit of a tree of quality $i = \{L, H\}$, reputation ϕ_t and signal dS_t at t by $\omega_t^i(\phi_t, dS_t) \in [\delta, \infty)$, where $\delta > 0$ is the exogenous rate of exit.

The evolution of the stocks of high and low quality trees over time specified in an allocation is required to be consistent with the initial distribution of reputations across trees, the dynamic evolution of those reputations for high and low quality trees, and the entry and exit rates specified in the allocation. To be concrete, we assume that at each t, there is a measure of reputations across high quality trees $\nu_{Ht}(\phi)$ and across low quality trees $\nu_{Lt}(\phi)$. These measures evolve over time as implied by the dynamics of reputation specified by Bayes Rule given the stochastic signal structure dS_t and the exit rates $\omega_t^i(\phi_t, dS_t)$.

Considering that each tree with reputation ϕ generates an individual signal dS_t at time t, we define the rate of exit of trees with reputation ϕ at time t as

$$\varpi_t^i(\phi) = \int_{dS_t} [\omega_t^i(\phi, dS_t)\nu_{it}(\phi, dS_t)]d(dS_t)$$

where $\nu_{it}(\phi, dS_t)$ is the measure of trees of quality $i \in \{L, H\}$ with reputation ϕ that generates a signal dS_t at moment t.

For an allocation to be feasible, we must have

$$m_t(1) = \int_\phi d\nu_{Ht}(\phi), \tag{5}$$

$$m_t(0) = \int_\phi d\nu_{Lt}(\phi), \tag{6}$$

$$dm_t(1) = [\phi_t^e m_t^e - \int_\phi d\varpi_t^H(\phi)]dt, \tag{7}$$

and

$$dm_t(0) = [(1 - \phi_t^e)m_t^e - \int_\phi d\varpi_t^L(\phi)]dt. \tag{8}$$

Note that at $t = 0$ we assume that is is feasible to have an atom of entry allowing for an immediate increase in either $m_0(1)$ or $m_0(0)$ or both.

An *allocation* in this environment is a sequence of consumption of the experience and numeraire good for the representative household $\{Y_t, N_t\}$, rates of entry of trees and initial reputations for entrants $\{m_t^e, \phi_t^e\}$, exit rates $\{\varpi_t^i(\phi)\}$ and reputational distribution $\{\nu_{it}(\phi)\}$ for $i = \{L, H\}$, and corresponding measures of active high and low quality trees $\{m_t(1), m_t(0)\}$. An allocation is *feasible* if it satisfies the final good resource constraints (3) and (4) and the constraints on the evolution of the stocks of high and low quality trees (5)-(8).

2.2 Signal Structures and Reputation

In what follows, we consider five signal structures on which reputation can be based, which we term *full information*, *no information*, *bad news*, *good news*, and *Brownian motion*. We define these signal structures here.

Under *full information*, there is an immediate, perfect signal of agents' quality so that the reputation of a high quality tree jumps to $\phi = 1$ immediately upon entry while that for a low quality tree jumps to $\phi = 0$ immediately upon entry.

Under *no information*, there are no signals so that the reputation of a tree entered into production with reputation ϕ_t^e evolves over time only if the exit rates for different quality trees differ.

In the *bad news* case, if the tree is of low quality, a signal that reveals that quality arrives at rate $\lambda > 0$. No such signal can arrive if the tree is high quality.

In the *good news* case, the assumption is reversed — if the tree is of high quality, a signal that reveals that quality arrives at rate $\lambda > 0$. No such signal can arrive if the tree is low quality.

Finally, in the *Brownian Motion* case, signals about tree's quality arrive continuously. Specifically,

$$dS_t = \mu_i dt + \sigma dZ_t, \tag{9}$$

where $i = \{L, H\}$, S_t is a Brownian motion with drifts that depend on the tree's type $\mu_H > \mu_L$ and the noise σ is the same for both types.

We interpret the signals in the bad news, good news, and Brownian motion cases as public signals of the quality of each tree. These signals might be interpreted as ratings in some widely published guide derived from either specialized testing or noisy surveys of past customers' experiences with the intermediate good obtained from each tree. Under this interpretation, past buyers of the intermediate good from a particular tree have more precise information about that tree's quality from their past consumption experience, but this experience is not fully revealed by a survey.

Alternatively, one might interpret the signals as reflecting a noisy outcome of production of the experience good with the intermediate output supplied by a particular tree. Then, we can interpret S_t as the cumulative output of the experience good from the fruit of a particular tree from the time that the tree was originally planted. Since there is a continuum of trees, it is possible to construct noisy production processes that fulfill the resource constraint for the experience good in equation (3) for all our signaling structures.

2.3 A Spot Market Equilibrium

We now consider the equilibrium allocation in a market in which the owners of trees sell the intermediate goods obtained as fruit from their trees to producers of the experience good at each time t at a spot market price $p_t(\phi)$ that depends on the reputation of the tree. We assume that this spot market price is equal to the expected value of the marginal product for the intermediate good, with expectations based on the reputation of the tree. This expected value of the marginal product has two components: the relative price of the experience good with respect to the numeraire good and the expected marginal product of the intermediate good from a tree with a given reputation, $y(\phi)$ from equation (2).

We assume the experience and numeraire final goods are transacted at spot prices in each moment t. We denote this relative price by P_t. In equilibrium, this price of the experience good relative to the numeraire good is given by the marginal utility of the experience good:

$$P_t = U'(Y_t). \tag{10}$$

Then, the *spot market price* at t in units of the numeraire good, for a unit of the intermediate good from a tree that is believed to be of high quality with probability ϕ is given by

$$p_t(\phi) = y(\phi)P_t. \tag{11}$$

Given that trees produce a flow of one unit of the intermediate good as fruit at zero marginal cost, the spot market prices $p_t(\phi)$ also correspond to the flow of profits from an active tree with reputation ϕ at t. Given a specified signal structure and exit rates for high and low quality trees with reputation ϕ, the owner of an active tree of quality $i = \{L, H\}$ expects a discounted present value of profits at spot market prices denoted by $W_{it}(\phi)$ and given by

$$W_{it}(\phi_t) = \max\left\{0, \pi_t(\phi_t)dt + (1 - rdt)\mathbf{E}_t\left(W_{i,t+dt}(\phi_{t+dt})\right)|i, \phi_t\right\}. \tag{12}$$

In a spot market equilibrium, we require that entry for both high and low quality trees have non-positive profits, i.e.:

$$W_{Ht}(\phi_t^e) - C \leq 0, \tag{13}$$

with equality if $\phi_t^e m_t^e > 0$, and

$$W_{Lt}(\phi_t^e) \leq 0, \tag{14}$$

with equality if $(1 - \phi_t^e)m_t^e > 0$.

Also, as required in equation (12) active trees continue in production if they have positive profits and exit if they have negative profits, that is, $\omega_t^i(\phi) = \delta$ if

$$W_{it}(\phi) > 0, \tag{15}$$

and $\omega_t^i(\phi) = 1$ if $W_{it}(\phi) < 0$ for $i = \{L, H\}$.

A *spot market equilibrium* is a feasible allocation together with prices $\{P_t, p_t(\phi)\}$ consistent with (10) and (11), value functions $\{W_{it}(\phi_t)\}$ consistent with (12) and the specified exit rates that satisfy the optimality conditions on entry and exit from (13)-(15).

A *steady-state spot market equilibrium* is a spot market equilibrium in which all prices and quantities are constant over time.

In the next two subsections, we solve for the steady-state spot market equilibrium under two extreme informational benchmarks: full information and no information. We show that under full information, the socially optimal allocation can be implemented as a spot market equilibrium while under no information, there is no production of the experience good in a steady-state spot market equilibrium.

2.4 Full Information Benchmark

We now show that the socially optimal allocation can be implemented as a spot market equilibrium outcome.

It is straightforward to characterize the socially optimal allocation in the full information case. We have that the measure of reputation across trees has mass $m_t(0)$ on $\phi = 0$ and $m_t(1)$ on $\phi = 1$, with no trees with intermediate reputations. The evolution of the stocks of trees (7) and (8) is given by

$$dm_t(1) = [\phi_t^e m_t^e - \omega_t^H(1)m_t(1)]dt, \tag{16}$$

and

$$dm_t(0) = [(1 - \phi_t^e)m_t^e - \omega_t^L(0)m_t(0)]dt, \tag{17}$$

since $\varpi_t^H(1) = \omega_t^H(1)m_t(1)$ and $\varpi_t^L(0) = \omega_t^L(0)m_t(0)$ in this extreme case with a perfect and immediate signal about the quality of the tree.

Clearly, since the output of a tree known to be low quality is expected to subtract from production of the experience good ($y(0) < 0$), it is optimal to set $\omega_t^L(0) = 1$ and $\phi_t^e = 1$. Likewise, since an existing tree known to be of high quality can contribute $y(1)$ to production of the experience good at zero cost as long as it continues in production, it is optimal to set $\omega_t^H(1) = \delta$, its minimum value. These results then characterize the optimal exit decisions.

Now consider the optimal level of entry of high quality trees. The marginal social cost, in terms of utility, of creating a new tree at t with probability $\phi_t^e = 1$ of being high quality is given by C while the marginal benefit is given by

$$\int_{s\geq 0} e^{-rs}U'(Y_{t+s})y(1)ds,$$

where $r = \hat{r} + \delta$ is the effective discount rate taking into account the exogenous exit rate δ. An allocation with constant consumption of the experience good is optimal at level $Y_t = \bar{Y}$ where

$$y(1)U'(\bar{Y}) = Cr. \tag{18}$$

Therefore, there is an optimal stock of high quality trees in steady-state determined by equation (3), $\bar{m}(1) = \bar{Y}/y(1)$.

The optimal choice of entry m_t^e is a dynamic choice. Because utility is quasi-linear, if $y(1)m_0(1)$ is less than this optimal level $\bar{Y}$, the regulator creates an atom of new high quality trees at $t = 0$ to attain the optimal stock $\bar{m}(1)$ of high quality trees immediately. If $y(1)m_0(1)$ exceeds this optimal level, the regulator creates no new trees until the stock of existing high quality trees has depreciated down to this level at rate δ. Once this optimal stock of high quality intermediate goods trees is attained, the regulator chooses a flow of new trees $m^e = \delta\bar{m}(1)$ to maintain the stock at a constant level.

The value function associated with a high quality tree in the socially optimal allocation is

$$W_{Ht}(1) = \int_{s\geq 0} e^{-rs}y(1)U'(Y_{t+s})ds > 0$$

while that associated with operating a low quality tree is

$$W_{Lt}(0) = 0.$$

This last result follows from the assumption that $y(0) < 0$, so it is always optimal to remove a low quality tree from production as rapidly as possible. Note also that $W_{Ht}(1) = C$ whenever there is positive entry and $W_{Ht}(1) < C$ in the transition to steady-state from above when there is no entry.

Clearly, the spot market prices implement the value functions above and hence the optimal allocation, characterized by entry of high quality trees, $\phi^e = 1$, and high production of the experience good, $\bar{Y}$.

2.5 No Information Benchmark

In contrast to the full information case, in the extreme case of non-observable investment and no signals from which to learn, the adverse selection problem associated with free entry of low quality trees is so severe that there is no production of the experience good in steady-state.

This result follows from the observation that it is impossible to offer high quality producers of the intermediate good a positive price for their good without attracting unbounded entry of low quality trees. With no dependence of the public signal on the quality of the tree, reputation for high and low quality trees will not change over time if both types of trees have the same exit rates $\omega_t^i(\phi)$. Likewise, both types of trees will have the same exit rates if reputation does not evolve because, if reputation does not evolve, then they both expect the same profits, that is, $W_{Ht}(\phi) = W_{Lt}(\phi)$. Of course, this equality of value functions means that it is impossible to satisfy the entry condition for high quality trees (13) as an equality (with positive entry of high quality trees) without violating the entry condition (14) for low quality trees. As a result, there can be no positive production of the experience good once the initial stock of high quality trees dies out. Thus, the optimal steady-state allocation with no information has $Y = 0$.

3 Reputation with Imperfectly Informative Signals

In this section, we establish a procedure to construct steady-state spot market equilibrium allocations and prices when we assume that public signals about each intermediate goods producing tree are revealed over time as long as the tree continues in operation, so that it is possible for experience goods producers to learn over time whether a given tree has invested in quality or not upon entry. We focus on three specific stochastic processes for the signals: the bad news case, the good news case, and Brownian motion.

We solve for the steady-state spot market equilibrium for these three signal structures by solving explicitly for the value functions (describing the discounted expected value of profits) of high and low quality trees. Before we delve into the specifics of the solution to the model under these different signal structures, it is useful to consider the basic features of equilibrium allocations when the economy has reached a steady-state. Once these basic features are laid out intuitively, it will be clear what technical results are needed to give more rigorous foundations to our analysis.

In a steady-state, the allocation, prices, and value functions are constant over time. To keep the notation simple, we suppress the time subscript. If $p(\phi)$ is the steady-state spot market price for intermediate goods based on reputation and Y is the steady-state production of the experience good, we find it useful to define prices $q(\phi)$ normalized by the marginal utility of the experience good

$$q(\phi) = p(\phi)/U'(Y),$$

so $q(\phi) = y(\phi)$ in a spot market with no regulations. Likewise, it is useful to define value functions for high and low quality trees $V_H(\phi)$ and $V_L(\phi)$ normalized by the marginal utility of the experience good, for $i = \{L, H\}$

$$V_i(\phi) = W_i(\phi)/U'(Y).$$

Assume that reputation is updated using Bayes' rule, both based on signals and continuation decisions. Using the normalized spot prices $q(\phi)$ and value functions $V_H(\phi)$ and $V_L(\phi)$ with certain basic properties, we characterize the steady-state allocations implemented by those spot prices and in Section 5, we show that, for the bad news, good news, and Brownian motion cases, these allocations generate value functions with the assumed properties.

It is important to highlight at this point that there may be equilibria with different allocations, but only based on specific and arbitrary off-equilibrium beliefs. For example, if producers of the experience good believe that high quality trees always exit, regardless of their reputation, then the market would cease to exist. This type of equilibrium satisfies the Cho and Kreps (1987) intuitive criterion because the only difference between the two types of trees is their productivity, and then spot prices only depend on reputation, so that arbitrary extreme beliefs have exactly the same effect on the price for both types of tree.

The equilibrium allocations described in the next Proposition are based on the natural restriction that reputation contingent on continuation is non-decreasing in the reputation prior. In other words continuation should constitute a positive signal about the tree's type. However, there still be multiplicity when imposing such a monotonicity on beliefs. Loosely, at relatively low reputation levels, high quality trees' continuation depends on what buyers believe about their continuation strategy. Hence different equilibria may be sustained by different off-equilibrium beliefs, even when restricting attention to monotonic beliefs non-decreasing in the reputation prior.

Bar-Isaac (2003) shows that in discrete time there is a unique limit of equilibria among a class of equilibria that satisfies the restriction of monotonicity on beliefs. The unique limit is obtained by imposing an arbitrarily high upper bound on the exit probability. This bound allows to eliminate in the limit the off-equilibrium beliefs that generate the multiplicity.

However this refinement cannot be used when signals are unbounded, such as is the case under Brownian motion. As we will show next, exit strategies should compensate potentially negative unbounded signals, and hence exit cannot be bounded. However, with a continuum and unbounded signal structure, the potential multiplicity problem only arises exactly at reputation $\phi = 0$, which implies that effectively the next Proposition describes a limit unique equilibrium among the class of equilibria with monotonic beliefs non-decreasing in the reputation prior.

Proposition 1 *Steady-state spot market equilibrium.*

Assume normalized spot market prices $q(\phi) = y(\phi)$ and corresponding normalized value functions $V_H(\phi)$ and $V_L(\phi)$ that satisfy the following two conditions:

(a) $V_i(\phi)$ are both continuous and non-decreasing on $\phi \in (0,1)$, with $V_i(\epsilon) = 0$ for some $\epsilon > 0$ and $\lim_{\phi \to 1} V_i(\phi) > 0$.

(b) Let $\bar{\phi}$ be the largest value of ϕ such that $V_L(\bar{\phi}) = 0$. We refer to $\bar{\phi}$ as the exit threshold. Assume $V_H(\phi) > V_L(\phi)$ for all $\phi \in [\bar{\phi}, 1)$ and $V_H(1) = V_L(1)$.

The steady-state normalized spot market equilibrium implemented by $q(\phi)$ is characterized by the following four results:

(i) There is entry of some low quality trees, ($\phi^e < 1$),

(ii) Reputations of all active trees in steady state are in the interval $[\bar{\phi}, 1]$. High quality trees always strive to remain active, i.e. $\omega_H(\phi) = \delta$. Low quality trees randomize exit with a probability $\omega_L(\phi) \in [\delta, 1]$, proportional to the signal dS_t, such that their reputation, if they remain active, does not fall below the exit threshold $\bar{\phi}$.

(iii) The steady-state equilibrium entry reputation equals the exit threshold: $\phi^e = \bar{\phi}$

(iv) The steady-state equilibrium level of production of the experience good Y satisfies

$$V_H(\phi^e) U'(Y) = C. \tag{19}$$

Proof To prove (i) observe that $\phi^e = 1$ is an absorbing reputation, hence $V_H(1) = V_L(1)$ and thus we cannot satisfy the incentive constraints (13) and (14) at $\phi^e = 1$.[8]

To prove (ii), note first that $\bar{\phi} \in (0,1)$ is well-defined, which follows from properties (a) of the value functions. From property (b), we have $V_H(\phi) > 0$ on $[\bar{\phi}, 1)$ while $V_L(\phi) > 0$ on $(\bar{\phi}, 1)$ and equal to zero at $\bar{\phi}$.

Suppose both low and high quality trees only exit exogenously, this is $\omega^H(\phi, dS) = \omega^L(\phi, dS) = \delta$ regardless of their signal dS. In equilibrium, ϕ evolves following a

[8]Note that we do not require continuity of the value functions at $\phi = 1$. $V_L(\phi)$ is not continuous at 1 in the bad news case.

Bayes' rule purely based on signals for all reputation levels. Denote this "naive" reputation updating $\hat{\phi}$.

$$\hat{\phi}_t = Pr(H|S_t) = \frac{\phi^e Pr(S_t|H)}{\phi^e Pr(S_t|H) + (1-\phi^e) Pr(S_t|L)}.$$

The continuity of updating and $q(\phi)$ increasing in ϕ imply that $V_L(\hat{\phi}) < V_L(\bar{\phi}) = 0$ for all $\hat{\phi} < \bar{\phi}$. In this case, given equation (12), the best reaction of low quality trees is to exit as soon as signals push their reputation below $\bar{\phi}$ (this is, $\omega^L(\phi, dS) = 1$ if the update is $\hat{\phi}_t(\phi^e, S_t)$ such that $\hat{\phi} < \bar{\phi}$, or which is the same $Pr(Cont_t|S_t, L) = 0$). Assuming that high quality trees in the same situation choose to exit less likely (just to guarantee the assumption of belief monotonicity), the Bayesian updating that also considers exit decisions is

$$\begin{aligned} \phi_t &= Pr(H|S_t, Cont_t) \\ &= \frac{\phi^e Pr(S_t|H) Pr(Cont_t|S_t, H)}{\phi^e Pr(S_t|H) Pr(Cont_t|S_t, H) + (1-\phi^e) Pr(S_t|L) Pr(Cont_t|S_t, L)}. \end{aligned}$$

However, when the signal S_t pushes the reputation of a tree $\hat{\phi}_t$ below $\bar{\phi}$ and low quality trees exit for sure (i.e., $Pr(Cont_t|S_t, L) = 0$), the updating rule implies that reputation jumps from $\hat{\phi}_t$ to 1 instantaneously. In this case low quality trees always prefer to continue since they enjoy a perfect reputation instantaneously. As discussed earlier, this does not constitute an equilibrium.

Hence, low quality trees with reputation $\phi < \bar{\phi}$ exit randomly at a rate $\omega^L(\phi, dS)$, exactly proportional to the signal realization dS, such that reputation jumps from $\phi < \bar{\phi}$ to $\bar{\phi}$ instantaneously following continuation, and then $V_L(\phi) = V_L(\bar{\phi}) = 0$ for all $\phi < \bar{\phi}$.[9] Zero value functions for $\phi < \bar{\phi}$ imply that low quality trees are indeed willing to randomize continuation in the range of reputation $\phi \in (0, \bar{\phi}]$. Exactly at $\phi = 0$, reputation is an absorbing state, regardless of the signals or exit decisions. Since both high and low quality trees have incentives to exit, their decisions ultimately depend on off-the-equilibrium beliefs.[10]

[9] When signals follow a Brownian motion, $Pr(S_t|i)$ is unbounded. As $Pr(S_t|H) \to 0$ with respect to $Pr(S_t|L)$, we require a continuation strategy $Pr(Cont_t|S_t, L) \to 0$ to maintain reputation at $\bar{\phi}$, then the exit rate cannot be bounded as in Bar-Isaac (2003).

[10] It is possible to bound reputation away from $\phi = 0$ introducing an arbitrarily small probability that low quality trees become high quality. As in Mailath and Samuelson (2001), this technical assumption eliminates the potential discontinuity of strategies at exactly $\phi = 0$.

From property (b), $V_H(\phi) = V_H(\bar{\phi}) > 0$ for all $\phi \in (0, \bar{\phi}]$, which implies that high quality trees always want to continue (this is, $\omega^H(\phi, dS) = \delta$ for all ϕ and all dS). These continuation strategies in equilibrium imply that effectively no tree in the market will hold a reputation below $\bar{\phi}$. Any tree that is pushed by signals to a region $\phi < \bar{\phi}$ and still continues, experiences an instantaneous jump of reputation up to $\bar{\phi}$.

Result (iii) follows from the condition (14) that low quality trees earn zero profits in equilibrium and the previous result that the only reputation sustainable with such a value is $\bar{\phi}$.

Result (iv) follows from taking the difference between (13) and (14) evaluated as equalities, given $V_L(\phi^e) = V_L(\bar{\phi}) = 0$. Q.E.D.

These four results are very useful in helping us characterize a steady state spot market equilibrium with imperfectly informative signals. From result (i), we have that it is impossible to attain the full information first best. We show below, however, that if the regulator observes reputations and can modify payments $q(\phi)$ based on those reputations, then it is possible to implement allocations in the steady-state that are arbitrarily close in terms of welfare to the full information allocation.

Results $(ii - iv)$ provide us with a procedure to construct steady-state spot market equilibrium allocations in a fairly simple manner as follows:

Take prices $q(\phi)$ as given. First solve for the exit threshold $\bar{\phi}$ by computing $V_L(\phi)$ for $\phi \geq \bar{\phi}$ and finding the fixed point at which $V_L(\bar{\phi}) = 0$. To do this, one must compute the evolution of reputations for low quality trees using Bayes Rule taking the no exit decisions for $\phi > \bar{\phi}$ as given. We derive the analytical solution for these value functions in our three informational cases in Section 5 below.

Once one has found the exit threshold $\bar{\phi}$, one can solve for $V_H(\phi)$ for $\phi \geq \bar{\phi}$ in a similar manner. At this point, one can verify that these value functions have the required properties for proposition 1. We have the entry reputation $\phi^e = \bar{\phi}$ and solve for steady-state output Y from (19).

Given $\bar{\phi}$, ϕ^e, and Y, one then constructs the rest of the steady-state allocation by finding the entry rate m^e that will induce a steady state distribution of active high and low quality trees $m(1)$ and $m(0)$ needed to produce Y. The steady-state value of N is then found from the resource constraint for the numeraire good.

We now use this construction procedure to show that the lemons problem that arises with imperfectly informative signals leads to a reduction in the output of the experience good relative to the full information steady-state.

Proposition 2 *Comparison of equilibrium outcome with full-information benchmark.*

The steady-state level of experience good output when signals about trees' quality are not perfectly informative is lower than that in the full information benchmark. That is $Y < \bar{Y}$.

Proof Regardless of the information structure $V_H(1) = y(1)/r$. Using (18) from the full information benchmark, this implies that the first best level of experience good production is given by $V_H(1)U'(\bar{Y}) = C$.

With imperfectly informative signals, from result (iii) in Proposition 1, the fraction of high quality trees that enter is equal to the lowest level of reputation sustained by the market, this is $\phi^e = \bar{\phi}$. From result (iv) in Proposition 1, the output of the experience good is given by $V_H(\bar{\phi})U'(Y) = C$. Since $V_H(\bar{\phi}) < V_H(1)$, then $Y < \bar{Y}$. Q.E.D.

As we see from this proof, although reputation mitigates the lemons problem and allows for some positive production of the experience good (relative to the no-information benchmark), the need for high quality trees to endure lower profits after entry as they accumulate a good reputation constrains efficient production.

4 Regulation with Imperfectly Informative Signals

We now use this procedure for constructing the steady-state spot market equilibria to study the role of regulation in improving welfare relative to a market economy with reputation based on imperfectly informative signals. We examine the extent to which regulation can improve on welfare in the spot market outcome with reputation under alternative assumptions about what a regulator can observe. We begin with the assumption that the regulator can condition taxes and transfers on reputations, and hence control the equilibrium normalized prices $q(\phi)$. We then relax this assumption and consider whether a regulator can improve welfare if the regulator cannot alter the spot market prices (so $q(\phi) = y(\phi)$) and instead can only charge regulatory entry costs F rebated lump-sum to consumers.

4.1 Regulation with Policies Conditioned on Reputation

In the case that the regulator can condition payments on reputation, we show that simple linear transfers of the form

$$q(\phi) = b(y(\phi) - a)$$

for given a and b are sufficient to implement an allocation that achieves steady-state welfare arbitrarily close to that achieved in the full information first best allocation.

We show this result by construction adapting the constructive procedure used in Section 3. In Section 3, we presented a constructive procedure for finding the steady-state spot market equilibrium and in section 5 we solve explicitly for the value functions for the bad news, good news, and Brownian information cases. Our constructive procedure and the method we use in Section 5 to solve for the normalized value functions corresponding to given normalized prices $\{q(\phi)\}$ are both valid under the assumptions that these prices $q(\phi)$ are linear and increasing in ϕ. We now use this procedure to establish our result that a regulator who can use taxes and transfers to alter the slope and intercept of $q(\phi)$ can implement a steady-state spot market equilibrium allocation with regulation achieving welfare arbitrarily close to the first best.

Let $\bar{Y}$ and $\bar{N}$ denote the full information optimal steady-state levels of consumption of the experience and numeraire good. We then have the following proposition

Proposition 3 *Optimal regulation with policies based on reputation.*

A regulator who is able to make policies based on reputation to implement normalized prices of the form

$$q(\phi) = b(y(\phi) - a)$$

can implement a steady-state allocation with $Y = \bar{Y}$ and $N = \bar{N} - \epsilon$ for any $\epsilon > 0$.

Proof To prove this proposition, first observe that a regulator can, with a choice of $b = 1$ and a arbitrarily close to $y(1)$ ensure that the solution to the equation $V_L(\bar{\phi}) = 0$ occurs at $\bar{\phi}$ arbitrarily close to 1. Second observe that the regulator does not alter the solution for $\bar{\phi}$ by choosing an alternative value of $b > 0$. This follows because the reputation of active trees remains in the interval $[\bar{\phi}, 1]$ and hence multiplying the rewards to reputations by alternative values of b simply scales the value function $V_L(\phi)$ on this interval by b, leaving the exit threshold unchanged. Third, observe that multiplying the rewards to reputations by alternative values of b also scales the value function $V_H(\phi)$ on the interval $[\bar{\phi}, 1]$ by b. Since $V_H(\bar{\phi}) > 0$ when $b = 1$, the regulator has complete control over the value of $V_H(\bar{\phi})$ with the appropriate choice of b.

To prove our proposition, have the regulator choose b so that (19) is satisfied at $\bar{Y}$. The corresponding value of $\phi^e m^e$ needed to produce $\bar{Y}$ is slightly higher than in the full-information, first-best case because there is a small fraction (less that ϕ^e) of low quality active trees in steady-state detracting from the output of the experience good. Specifically, under full information, the steady-state measure of high quality trees is $m(1) = \bar{Y}/y(1)$, so the rate of entry of high quality trees is $\delta\bar{Y}/y(1)$ and the associated consumption of the numeraire good is $\bar{N} = 1 - C\delta\bar{Y}/y(1)$. In the equilibrium with regulation described here, because a fraction $(1 - \bar{\phi})$ of entering trees are low quality in steady-state, there is a fraction of all active trees that is low quality, where this fraction is positive but bounded above by $1 - \bar{\phi}$. Denote the equilibrium steady-state ratio of low to high quality trees by $\bar{m}(0)/\bar{m}(1)$. From the resource constraint for the experience good (3) in steady-state, to produce output $\bar{Y}$ there must be a stock of $\bar{m}(1)$ high quality trees given by

$$\bar{m}(1) = \frac{\bar{Y}}{y(1) + y(0)\bar{m}(0)/\bar{m}(1)},$$

and a steady-state entry rate of high quality trees of $\phi^e m^e = \delta\bar{m}(1)$ is required to maintain that production. As a result of this required elevated rate of entry of high quality trees, $N = 1 - C\delta\bar{m}(1)$ is slightly below $\bar{N}$ as more of the numeraire good needs to be spent on planting high quality trees. The gap between N and $\bar{N}$ can be made arbitrarily small by choosing a to set $\bar{\phi}$ as close to one as is required to drive $\bar{m}(0)/\bar{m}(1)$ sufficiently close to zero. Q.E.D.

We interpret this proposition as indicating that the lemon's problem in this economy is one of commitment rather than one of information. The lemon's problem arises because the competitive market prices based on the spot gains to trade between a buyer and a seller do not offer sufficient rewards to reputation to ensure high quality. There is a welfare gain to be achieved here if buyers are able to commit to pay prices that reward good reputation or punish poor reputation.

In some environments, it may be possible to achieve such commitment through long-term contracts between buyers and sellers. If a contract between a buyer and seller with prices based on reputation can be enforced, then the two parties can, with an appropriate choice of parameters a and b, design an incentive contract the ensures that the vast majority of sellers entering into the contract do indeed make the investment to be high quality. Here we interpret the relationships between the buyers and sellers of intermediate goods as one-shot or short lived and hence long-term contracts are not feasible. In this case, taxation is a substitute for missing private capabilities to commit.

Remark on the regulator observing reputations: Note that if the experience good is sold as a final good in the market, then it is not strictly necessary for a regulator to observe reputations of intermediate producers. The regulatory scheme proposed here can be implemented simply with a fixed operating cost imposed on active intermediate good producers of ab per unit of time, combined with a subsidy to final sales of the experience good of $(b-1)$. This subsidy will alter the spot market normalized price of the intermediate goods as required in the proposition. Here though we have not introduced additional margins that might be distorted by such a tax and subsidy scheme and hence we consider a regulator with more limited instruments in the next section.

Remark on non-Markov transfers: One can also see immediately that our assumption that transfers are based on reputation rather than the full history of signals for each tree is restrictive. With transfers conditioned on reputation, the standard result that a reputation of $\phi = 1$ is an absorbing state implies that $V_L(1) = V_H(1)$ so it is impossible to have only high quality trees enter ($\phi^e = 1$). We can get arbitrarily close to having only high quality trees enter, but not all the way there.

In contrast, if we allowed the regulator to make transfers based on the full history of signals of quality associated with each tree, then, for a wide range of stochastic signal structures, the regulator could implement an allocation with $\phi^e = 1$. This result follows if the distribution of signal histories for low and high quality trees differs sufficiently such that over time, arbitrarily precise statistical tests of tree quality can be performed given long-enough realized signal histories. A transfer scheme that backloads payments to trees and conditions them on this statistical test of signal histories can then reward the investment of a high quality tree and (with an entry cost $F > 0$) and, at the same time, deter entry by low quality trees by leaving them with strictly negative expected profits upon entry. This is not possible with transfers that are Markov in reputation because buyers ignore further signals of quality once $\phi = 1$.

4.2 Regulation with Policies not Conditioned on Reputation

We now relax the assumption that the regulator has access to the same information as private agents and hence can condition transfers on reputation. Instead, we consider the alternative social planning problem in which the regulator takes as given that transfers to active trees are given by the spot market prices $q(\phi) = y(\phi)$ and the regulator can only use fixed regulatory entry costs F (rebated lump-sum to consumers) to influence steady-state welfare. We show below that for the three information structures that we consider, it is always possible for the regulator to improve on the steady-state spot market outcome with no regulation by choosing a strictly positive fixed regulatory entry cost $F > 0$.

We saw above that when a regulator has the ability to condition policies on reputation, he can raise the average quality of trees at entry and the overall level of production of the experience good arbitrarily close to their levels in the full information first best steady-state outcome by distorting the spot market prices. In contrast, when the regulator cannot condition transfers on reputation and instead has access only to fixed regulatory entry costs, he cannot independently control both average quality and the scale of production. This is because now we restrict the regulator to have access just to a single regulatory instrument. We now extend our procedure to compute steady-state spot market equilibria to allow for fixed regulatory entry costs.

Proposition 4 *Steady-state spot market equilibrium with positive regulatory entry costs.*

Assume that there exists a feasible steady state allocation implemented as a steady-state spot market equilibrium by prices $q(\phi)$ strictly increasing in reputation and corresponding value functions $V_H(\phi)$ and $V_L(\phi)$ that satisfy the conditions in Proposition 1.

Assume there are fixed regulatory entry costs $F > 0$. Results (i), (ii) from Proposition 1 still hold, while results (iii) and (iv) change to,

(iii') The steady-state equilibrium entry reputation ϕ^e satisfies

$$\frac{V_L(\phi^e)}{V_H(\phi^e)} = \frac{F}{C + F}. \tag{20}$$

(iv') *The steady-state equilibrium level of production of the experience good* Y *satisfies*

$$(V_H(\phi^e) - V_L(\phi^e))\, U'(Y) = C. \tag{21}$$

Proof The proof follows trivially from the entry conditions with fixed regulatory entry costs F.

$$\begin{aligned} W_L(\phi^e) \equiv V_L(\phi^e) U'(Y) &= F \\ W_H(\phi^e) \equiv V_H(\phi^e) U'(Y) &= C + F. \end{aligned}$$

Result (i) from Proposition 1 does not change because $\phi^e = 1$ is still an absorbing reputation, independent of F. Result (ii) from Proposition 1 does not change because fixed entry costs only affect entry conditions and not exit conditions. Results (iii') and (iv') follow trivially from taking the ratio and difference of the entry conditions with fixed regulatory entry costs. Q.E.D.

From result (iii') in Proposition 4, we see that fixed entry costs determine the reputation of entrants ϕ^e through the ratio of the value functions $V_L(\phi^e)/V_H(\phi^e)$ and then, from result (iv') in that proposition, the steady-state scale of production of the experience good Y through the difference of the value functions $V_H(\phi^e) - V_L(\phi^e)$. Thus, the impact of a fixed entry cost F on average quality at entry and the scale of production Y depends on the properties of the ratio and the difference of the value functions $V_i(\phi)$ as ϕ changes under different information structures. We show that, depending on the information structure, an increase in the fixed cost F can increase both entry quality and production of the experience good Y (bad news), or it can increase entry quality and decrease production of Y (good news), or a combination of these two, first increasing entry quality and production and then further increasing quality but decreasing production (Brownian motion). The quantitative solution to the social planning problem in these different cases will clearly depend on the specific shape of these value functions. We show as a qualitative result, however, that in all of these cases, the optimal fixed regulatory entry cost is strictly positive.

We now present a characterization of the value functions $V_L(\phi)$ and $V_H(\phi)$ for the bad news, good news, and Brownian motion cases with profits of trees given as a linear function of reputation, which applies to the spot market since spot market prices

$q(\phi) = y(\phi)$ are linear in reputation. We derive analytically the value functions for all these cases, and prove these properties in Section 5.

Proposition 5 *Properties of value functions with linear profits and bad news.*

- *Conditions* (a) *and* (b) *of Proposition 1 hold.*
- *The ratio* $\frac{V_L(\phi)}{V_H(\phi)}$ *is increasing in* ϕ *for* $\phi > \bar{\phi}$.
- *The difference* $V_H(\phi) - V_L(\phi)$ *is increasing in* ϕ *for* $\phi > \bar{\phi}$.

Proposition 6 *Properties of value functions with linear profits and good news.*

- *Conditions* (a) *and* (b) *of Proposition 1 hold.*
- *The ratio* $\frac{V_L(\phi)}{V_H(\phi)}$ *is increasing in* ϕ *for* $\phi > \bar{\phi}$.
- *The difference* $V_H(\phi) - V_L(\phi)$ *is decreasing in* ϕ *for* $\phi > \bar{\phi}$.
- $\frac{\partial V_H(\bar{\phi})}{\partial \phi} = \frac{\partial V_L(\bar{\phi})}{\partial \phi} = 0.$

Proposition 7 *Properties of value functions with linear profits and Brownian motion.*

- *Conditions* (a) *and* (b) *of Proposition 1 hold.*
- *The ratio* $\frac{V_L(\phi)}{V_H(\phi)}$ *is increasing in* ϕ *for* $\phi > \bar{\phi}$.
- *The difference* $V_H(\phi) - V_L(\phi)$ *is increasing from* $\bar{\phi}$ *to* ϕ^* *and decreasing from* ϕ^* *to 1, where* ϕ^* *is characterized by* $\frac{\partial V_H(\phi^*)}{\partial \phi} = \frac{\partial V_L(\phi^*)}{\partial \phi}$.
- $\frac{\partial V_H(\bar{\phi})}{\partial \phi} = \frac{\partial V_L(\bar{\phi})}{\partial \phi} = 0.$

These propositions imply that in all three cases, bad news, good news, and Brownian motion, with linear profits, the value functions satisfy the conditions (a) and (b) in Proposition 1 we need for our construction of the unique steady-state spot market equilibrium. Also, in all three of these cases with different stochastic signal structures, we have that the ratio $V_L(\phi)/V_H(\phi)$ rises monotonically from zero to some positive number (1 in the case of good news and Brownian motion, something less than one in the case of bad news). Thus, from result (iii') in Proposition 4, a regulator can implement any entry reputation $\phi^e \geq \bar{\phi}$ desired with an appropriate choice of $F \geq 0$. From result (iv') in Proposition 4, however, we see that the regulator does not have independent control of the scale of production of the experience good Y — a choice of F pins down ϕ^e and thus also pins down Y. Thus, a regulator potentially faces a conflict in choosing F between increasing average quality at entry ϕ^e and encouraging production of the experience good Y. Recall from proposition 2 that the steady-state spot market equilibrium has production of the experience good below that in the full information first best, so the regulator would like to improve both average quality at entry and output of the experience good if possible.

The trade-off that a regulator faces between the average quality of entrants and the equilibrium level of production of the experience good depends on the details of the signal process. From Proposition 5, we see that in the case of bad news, the regulator does not face a direct conflict between the objectives of increasing quality at entry ϕ^e and increasing production of the experience good Y. This follows from the result that both the ratio $V_L(\phi)/V_H(\phi)$ and the difference $V_H(\phi) - V_L(\phi)$ are increasing in ϕ for $\phi > \bar{\phi}$. Thus, a regulator who increases F increases ϕ^e and Y simultaneously. In this case the regulator wants to increase F to drive ϕ^e arbitrarily close to 1. (Note that in the bad news case $\lim_{\phi^e \to 1} V_L(\phi^e)/V_H(\phi^e) < 1$ so that this is achieved with a finite value of F.) From Proposition 5, we have that Y increases as ϕ^e increases and from Proposition 2 we have that Y remains below $\bar{Y}$ for all values of ϕ^e. Hence, in the bad news case, a policy of increasing the entry cost F to drive the average quality of entrants ϕ^e towards one is always welfare improving.

From Proposition 6, we see that in the case of good news, the regulator does face a direct conflict between the objectives of increasing quality ϕ^e and increasing production of the experience good Y. This follows from the result that the ratio $V_L(\phi)/V_H(\phi)$ is increasing and the difference $V_H(\phi) - V_L(\phi)$ is decreasing in ϕ for $\phi > \bar{\phi}$. Thus, a regulator who increases $F > 0$ increases ϕ^e but reduces Y simultaneously.

Likewise, from Proposition 7, we see that in the case of Brownian motion, the regulator may face a direct conflict between the objectives of increasing quality at entry ϕ^e and increasing production of the experience good Y. If $\bar{\phi} < \phi^*$, then $V_L(\phi)/V_H(\phi)$ and the difference $V_H(\phi) - V_L(\phi)$ are increasing in ϕ for $\phi \in (\bar{\phi}, \phi^*)$ while $V_L(\phi)/V_H(\phi)$ is increasing and the difference $V_H(\phi) - V_L(\phi)$ is decreasing in ϕ for $\phi \in (\phi^*, 1)$.

Still, our main qualitative result is that for all three information structures and all parameters values, positive entry costs $F > 0$, improves welfare relative to the steady-state spot market outcome. We establish this result in the next two propositions. We first show that an increase in the fraction of trees that invest at entry in steady state ϕ^e increases the average quality of high quality trees in steady-state. We use this result in the second proposition to establish that a higher steady-state average quality of trees at entry (ϕ^e) allows for a given level of steady-state production of the experience good with a smaller flow of investment of the numeraire good.

Proposition 8 *The average quality of trees in steady-state increases with the quality at entry.*

Assume there is a constant flow of m^e trees entering the market. The aggregate output of the experience good Y obtained in steady-state increases with the fraction of high quality trees that enter, ϕ^e.

Proof Assume, without loss of generality, that $m^e = 1$. First it is important to note that, if exit were only exogenous, then this would be trivial since we would have in steady-state that the portion of high quality trees was always ϕ^e and that of low quality trees $(1-\phi^e)$ and hence output would be $Y = y(\phi^e)/\delta$ which is clearly increasing in ϕ^e. With endogenous exit of low quality trees, however, we have that the steady-state fraction of high quality trees is greater than ϕ^e and is endogenous to ϕ^e.

The strategy of the proof is to show the proposition holds when the interest rate is $\hat{r} = 0$, and then extend the argument for $\hat{r} > 0$.

Assume $\hat{r} = 0$. If we have a constant flow of 1 tree per instant enter with ϕ^e of that flow being high quality trees and $(1 - \phi^e)$ being low quality trees, then steady-state output is given by

$$\phi^e V_H(\phi^e) + (1 - \phi^e) V_L(\phi^e).$$

Why is this the right measure of steady-state output? This is because, when the interest rate is zero, then the expected discounted value of profits (output since marginal cost is zero) for all trees at entry is equal to the integral across the cross section of profits in the steady-state. What about endogenous exit? The computation of the value function $V_L(\phi)$ takes the impact of endogenous exit on the cross-section of output into account.

A sufficient condition for the steady state output to be increasing in ϕ^e is that $V_H(\phi^e) \geq V_L(\phi^e)$ and $V_H'(\phi^e)$ and $V_L'(\phi^e)$ both greater than or equal to zero. These are the conditions (a) and (b) assumed in Proposition 1. We prove in Section 5 that these conditions hold for the signal structures and the payoff functions we impose.

When the interest rate is positive, then we no longer have the argument that expected discounted value of profits for high and low quality trees is equal to the integral across the cross section of profits in steady-state because of discounting by the interest rate. Instead, to compute steady-state output of the experience good, we must construct new value functions $\tilde{V}_H$ and $\tilde{V}_L$ based on the discounted expected value of profits with an interest rate of zero, where, in the case of low quality trees, the distribution over future values of ϕ is computed under the assumption that these trees exit at the threshold $\bar{\phi}$ at the rate required by Bayes Rule (the equilibrium rate) in equilibrium.

In this case, the steady-state output is given by

$$\phi^e \tilde{V}_H(\phi^e) + (1 - \phi^e)\tilde{V}_L(\phi^e),$$

which is increasing in ϕ^e when the two sufficient conditions $\tilde{V}_H(\phi^e) \geq \tilde{V}_L(\phi^e)$ and $\tilde{V}_H'(\phi^e)$ and $\tilde{V}_L'(\phi^e)$ both greater than or equal to zero, hold. We show this is the case when the value functions $V_H(\phi^e)$ and $V_L(\phi^e)$ fulfill conditions (a) and (b) in Proposition 1, when $\hat{r} > 0$.

We know in general, regardless of the signal structure, that

$$V_i(\phi) = \max\left\{0, y(\phi)dt + (1 - rdt)\mathbf{E}\left(V_i(\phi')|i, \phi\right)\right\},$$

and we want to characterize the properties of value functions with $r = \delta$ (i.e., $\hat{r} = 0$) for $\phi \in [\bar{\phi}, 1]$.

$$\tilde{V}_i(\phi) = y(\phi)dt + (1 - \delta dt)\mathbf{E}\left(\tilde{V}_i(\phi')|i, \phi\right).$$

We can use $V_i(\phi)$ as an initial guess for $\tilde{V}_i(\phi)$. In the first iteration it is clear that $\tilde{V}_i(\phi) > V_i(\phi)$ for all ϕ. The reason is that, by construction $V_i(\phi) > 0$ for all ϕ and $\mathbf{E}\left(\tilde{V}_i(\phi')|i, \phi\right)$ is discounted at a lower rate, $\delta < r$. Since we're maintaining the evolution of ϕ fixed in this exercise, iterating over an increasing sequence of value functions, generate that $\tilde{V}_i(\phi)$ is increasing in ϕ. This is the first sufficient condition, $\tilde{V}_i'(\phi) \geq 0$ for $i \in \{L, H\}$.

The second sufficient condition ($\tilde{V}_H(\phi) \geq \tilde{V}_L(\phi)$) is also fulfilled because, regardless of r, the difference between L and H comes from the construction of $\mathbf{E}\left(\tilde{V}_i(\phi')|i, \phi\right)$. For high quality trees, reputation is a submartingale. For low quality trees reputation is a supermartingale. At $\bar{\phi}$, the expected future reputation of high quality trees is higher than that of low quality trees.

$$\mathbf{E}\left(\phi'|H, \bar{\phi}\right) > \mathbf{E}\left(\phi'|L, \bar{\phi}\right) \qquad for\ all\ \phi$$

where the sign of the inequality does not depend on the exiting of low quality trees.

Since $\tilde{V}_i(\phi')$ is an increasing function of ϕ', the second sufficient condition $\tilde{V}_H(\phi) > \tilde{V}_L(\phi)$ for all $\phi \in [\bar{\phi}, 1]$ is also satisfied. Q.E.D.

In the next proposition we show that for all three signal structures, the solution to the social planning problem has $F > 0$ regardless of parameters. This result is immediate in the case of bad news and, in the good news and Brownian motion cases, follows from the observation that an increase in fixed entry cost F, when evaluated at $F = 0$, has a first order impact on quality ϕ^e and a second order impact on the volume of production of experience goods Y.

Proposition 9 *Optimal regulation when only entry costs are available.*

With either bad news, good news, or Brownian motion signals, the solution to the social planning problem with transfers given by spot market prices $q(\phi) = y(\phi)$ has fixed entry cost $F > 0$.

Proof We have already proved this proposition for the bad news case, where an increase in F increases both ϕ^e and Y. We now consider the good news and Brownian motion cases. We make use of the results in Propositions 6 and 7 that

$$\frac{\partial V_H(\bar{\phi})}{\partial \phi} = \frac{\partial V_L(\bar{\phi})}{\partial \phi} = 0.$$

We can use equation (20) to define an implicit function $F(\phi^e)$ mapping $\phi^e \in [\bar{\phi}, 1]$ to $F \in [0, \infty)$ to restate the social planning problem in this case as one of choosing ϕ^e directly. Likewise, we can use equation (21) to define an implicit function $Y(\phi^e)$. Evaluating $dY(\phi^e)/d\phi^e$ at $\phi^e = \bar{\phi}$, we find that this derivative is zero. Hence, in the good news and Brownian cases, the regulator can, at least initially, increase the average quality of entrants above $\bar{\phi}$ without diminishing aggregate production of the experience good Y.

However, as we show in Proposition 8, an increase in ϕ^e pushes up the production of Y, by increasing the average quality in the market $m(1)/m(0)$. In steady state, from equation (7) that show the evolution of high quality trees

$$\phi^e m^e = \int_\phi d\varpi^H(\phi)$$

Using the results that $\omega^H(\phi, dS) = \delta$ for all $\phi > 0$ and dS, and that $m(1) = \int_\phi d\nu_H(\phi)$ (equation 5),

$$\frac{m(1)}{m(0)} = \frac{m^e\phi^e}{\delta m(0)},$$

Also in steady state, from equation (8) that shows the evolution of low quality trees,

$$(1-\phi^e)m^e = \int_\phi d\varpi^L(\phi)$$

Evaluating the increase of ϕ^e at $\bar{\phi}$, exit rates of low quality firms do not change because $\frac{\partial V_L(\bar{\phi})}{\partial \phi} = 0$. This implies that a marginal increase in ϕ^e does not modify any $\omega^L(\phi, dS)$, having a first order effect in reducing $m(0) = \int_\phi d\nu_L(\phi)$ (equation 6).

Then, the average quality in the market $m(1)/m(0)$ grows more than the quality of entrants ϕ^e. This is,

$$\frac{\partial[m(1)/m(0)]}{\partial \phi^e}|_{\phi^e=\bar{\phi}} > 1$$

Evaluating the increase of ϕ^e at $\bar{\phi}$, Y does not change in equilibrium. To maintain Y constant we need a decline in m^e greater than the increase in ϕ^e to maintain $m(1)/m(0)$ constant, or which is the same a decrease in $m^e\phi^e$. This implies an increase in the consumption of the numeraire good from equation (4), while maintaining the consumption of the experience good Y, improving welfare. Q.E.D.

Remark on other regulatory tools: In the previous section we considered the extreme case in which the government can use taxes and subsidies conditioned on reputation, and can achieve an allocation arbitrarily close to the unconstrained first best. At the other extreme, this section considers the case in which the government is restricted to use only simple entry costs, still achieving a welfare improving allocation.

Naturally, the government can use other unconditional tools, additional to entry costs, to achieve better results. For example, the regulator can also impose operational subsidies. Loosely, on the one hand these subsidies are expected to compensate mostly high quality trees, since in expectation they live longer. On the other hand, these subsidies delay exit of low quality trees, since they prefer to wait longer to be lucky. Depending on this trade-off, operational subsidies can position the market in a situation where entry costs can increase welfare even further.

Closer to the previous section, it is also possible to consider policies that subsidize variables more likely to be experienced by high quality trees, such as age, or that punish variables more likely to be experienced by low quality trees, such as exit. Even when welfare improving, since these two variables are only imperfect signals of reputation, these tools are likely not as effective as taxes and subsidies that condition directly on reputation to improve welfare.

There is a wide array of policy combinations the regulator can use to improve welfare. More or less directly, all of them should aim at front-loading costs and back-loading subsidies. The effects of policies that affects per period payoffs can be obtained from the analytical solutions of the value functions we derived. The effects of lump sum transfers can be obtained from the interplay of entry conditions we characterize. In this paper we just consider two extremes of such policies, but we conjecture other combinations can be designed to achieve allocations that lie between these two extremes in terms of welfare.

5 Value Functions for Different Information Structures

Here we obtain analytical solutions for the value functions $V_i(\phi)$, under *bad news, good news, and Brownian Motion*, for general payment functions $\pi(\phi)$. We also show the properties described in Propositions 5-7 hold when the function $\pi(\phi)$ is linear in ϕ, as we assume is the case with spot prices where $\pi(\phi) = q(\phi) = y(\phi)$. In this Section, for notational simplicity we denote,

$$y(\phi) = a_1\phi - a_0$$

where $a_1 = y(1) - y(0) > 0$ and $a_0 = -y(0) > 0$.

5.1 Bad News

In this case $dS_t \in \{0, 1\}$, which means either there is a signal or not at each t. The bad news case is defined by $Pr(dS_t = 1|H) = 0$ and $Pr(dS_t = 1|L) = \lambda dt$, which means there is a positive Poisson arrival only for low quality trees. When a signal arrives the tree is revealed to be of low quality and hence the public belief about its quality drops to $\phi = 0$. With this reputation, the tree would never be able to sell its output at a non-negative price. Thus, following this event, it is optimal for the tree to cease production and exit as quickly as possible.

It is convenient to use a transformed variable $l = (1 - \phi)/\phi : [0, 1] \to (\infty, 0]$ to summarize the reputation level of a tree. The evolution of l is determined by,

$$\frac{dl_t}{dt} = \left[\frac{Pr(dS_t|L) - Pr(dS_t|H)}{Pr(dS_t|H)}\right] l_t.$$

When bad news arrive (i.e., $dS_t = 1$)

$$\frac{dl_t}{dt} = \left[\frac{\lambda dt - 0}{0}\right] l_t = \infty,$$

and reputation jumps immediately to $l = \infty$. Since $\phi = \frac{1}{1+l}$, this means reputation drops immediately to $\phi = 0$.

While there are no news (i.e., $dS_t = 0$), reputation increases. From the Poisson distribution, the probability that a high quality tree does not generate news during t periods is $e^{-\lambda t}$. Then, after t periods of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_t = 0|L)}{Pr(S_t = 0|H)}\right] l_0 = \left[\frac{e^{-\lambda t}}{1}\right] l_0 = e^{-\lambda t} l_0,$$

where $l_0 = \frac{1-\phi^e}{\phi^e}$. This means l_t is decreasing (reputation is increasing) over time at a rate $\lambda \in [0, \infty)$. While there are no news, the evolution of reputation for trees with high and low quality is the same. After bad news, a tree exits and obtains zero thereafter. Then, the value functions for both types only differ in their discount factor.

Proposition 10 *Value functions for general profit functions and bad news*

A value function for a low quality tree with reputation l, for a general $\pi(l)$, is

$$\hat{V}_L(l) = \int_{s=0}^{\infty} e^{-(r+\lambda)s}\pi(e^{-\lambda s}l)ds$$

and the value function for a high quality tree with reputation l is

$$\hat{V}_H(l) = \int_{s=0}^{\infty} e^{-rs}\pi(e^{-\lambda s}l)ds$$

Solving explicitly the integrals for the case of linear payoffs and no marginal costs, $\pi(\phi) = y(\phi) = a_1\phi - a_0$ (hence $\pi(l) = \frac{a_1}{1+l} - a_0$),

$$\hat{V}_L(l) = \frac{1}{r+\lambda}\left[a_1\Upsilon_{m_{r+\lambda}}(-l) - a_0\right], \tag{22}$$

$$\hat{V}_H(l) = \frac{1}{r}\left[a_1\Upsilon_{m_r}(-l) - a_0\right], \tag{23}$$

where $\Upsilon_m(-l) =_2 F_1(1, m; m+1, -l)$ is an hypergeometric function, and

$$m_r = \frac{r}{\lambda} > 0 \qquad \texttt{and} \qquad m_{r+\lambda} = \frac{r+\lambda}{\lambda} = 1 + m_r.$$

The hypergeometric function has well defined properties when $m > 0$. In particular, it is monotonically increasing in ϕ (from 0 to 1) and monotonically decreasing in m.

$$\Upsilon_m\left(-\frac{1-\phi}{\phi}\right) : [0,1] \to [0,1] \qquad \texttt{and} \qquad \frac{\partial\Upsilon_m(\cdot)}{\partial m} < 0.$$

Now we denote $V_i(\phi) = \hat{V}_i(l)$ for all ϕ and $i \in \{L, H\}$. Since $\lim_{\phi\to 0} V_L(\phi) = -\frac{a_0}{r+\lambda} < 0$ with no exit, there is a $\phi = \bar{\phi}$ such that $V_L(\bar{\phi}) = 0$. Hence $\bar{\phi}$ is the highest reputation at which low quality trees are indifferent between exiting or not. As discussed above, exiting strategies imply that in equilibrium, no tree has a reputation below $\bar{\phi}$. Value functions in the range $[\bar{\phi}, 1]$ are

$$\begin{aligned} V_L(\phi) &: [\bar{\phi}, 1] \to [0, \frac{a_1 - a_0}{r + \lambda}] \\ V_H(\phi) &: [\bar{\phi}, 1] \to [V_H(\bar{\phi}), \frac{a_1 - a_0}{r}], \end{aligned}$$

where $V_H(\bar{\phi}) = \frac{1}{r}\left[a_1 \Upsilon_{m_r}\left(-\frac{1-\bar{\phi}}{\bar{\phi}}\right) - a_0\right] > 0$ (since $m_r < m_{r+\lambda}$).

The properties of these value functions (directly from the properties of hypergeometric functions) are summarized in Proposition 5.

5.2 Good News

In this case $Pr(dS_t = 1|H) = \lambda dt$ and $Pr(dS_t = 1|L) = 0$. When a signal arrives the tree is revealed to be of high quality and hence the public belief ϕ regarding this tree jumps up to $\phi = 1$. After good news the tree maintains a reputation of $\phi = 1$ until it exits exogenously.

Again, we use the variable $l = (1 - \phi)/\phi$. When good news arrive (i.e., $dS_t = 1$)

$$\frac{dl}{dt} = \left[\frac{0 - \lambda dt}{\lambda dt}\right] l_t = -l_t,$$

and reputation jumps immediately to $l = 0$, or $\phi = 1$.

While there are no news (i.e., $dS_t = 0$), reputation decreases. After t periods of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_t = 0|L)}{Pr(S_t = 0|H)}\right] l_0 = \left[\frac{1}{e^{-\lambda t}}\right] l_0 = e^{\lambda t} l_0,$$

which means l_t is increasing (reputation is decreasing) over time at a rate λ.

Denoting $\pi(l(1))$ the payoffs for a tree with $\phi = 1$, the value function for a tree that has experienced good news is,

$$V(l(1)) = \frac{\pi(l(1))}{r}.$$

There is a key difference between good news and bad news. Under bad news, reputation only increases, which means exit never occurs, unless a bad signal is revealed.

Under good news, reputation continuously decrease and low quality trees that hit $\bar{\phi}$ will exit at a rate λ such that reputation never drifts below $\bar{\phi}$.

Proposition 11 *Value functions for general profit functions and good news*

The value function for a low quality tree with reputation l is

$$\hat{V}_L(l) = \int_{s=0}^{T(l)} e^{-rs}\pi(e^{\lambda s}l)ds \tag{24}$$

The value function for a high quality tree with reputation l is

$$\hat{V}_H(l) = \int_{s=0}^{T(l)} e^{-(r+\lambda)s}\left[\pi(e^{\lambda s}l) + \lambda\frac{\pi(l(1))}{r}\right]ds + \int_{s=T(l)}^{\infty} e^{-(r+\lambda)(s-T(l))}\lambda\frac{\pi(l(1))}{r}ds \tag{25}$$

where $T(l)$ is the time required for l to increase up to $\bar{l} = \frac{1-\bar{\phi}}{\bar{\phi}}$.

$$T(l) = \frac{log(\bar{l}/l)}{\lambda} > 0. \tag{26}$$

In the case of linear payoffs and no marginal costs, the reputation at which low quality trees are willing to exit is given by $\pi(\bar{l}) = \frac{a_1}{1+\bar{l}} - a_0 = 0$. In this case, $\bar{l}$ is given by the reputation above which profits are negative. Then $\bar{l} = \frac{a_1-a_0}{a_0}$ and $T(l)$ is given following equation (26). Value functions are,

$$\begin{aligned}\hat{V}_L(l) &= \frac{1}{r}\left[a_1\left(1-\Upsilon_{m_r}\left(-\frac{1}{l}\right)\right)-a_0\right] \\ &\quad -\frac{e^{-rT(l)}}{r}\left[a_1\left(1-\Upsilon_{m_r}\left(-\frac{a_0}{a_1-a_0}\right)\right)-a_0\right],\end{aligned} \tag{27}$$

$$\begin{aligned}\hat{V}_H(l) &= \frac{1}{r+\lambda}\left[a_1\left(1-\Upsilon_{m_{r+\lambda}}\left(-\frac{1}{l}\right)\right)-a_0+\lambda\frac{a_1-a_0}{r}\right] \\ &\quad -\frac{e^{-(r+\lambda)T(l)}}{r+\lambda}\left[a_1\left(1-\Upsilon_{m_{r+\lambda}}\left(-\frac{a_0}{a_1-a_0}\right)\right)-a_0\right].\end{aligned} \tag{28}$$

Now we denote $V_i(\phi) = \hat{V}_i(l)$ for all ϕ and $i \in \{L, H\}$. Since $T(l(1)) = \infty$, using the previously discussed properties of the hypergeometric functions,

$$\begin{aligned} V_L(\phi) &: [\bar{\phi}, 1] \to [0, \frac{a_1 - a_0}{r}], \\ V_H(\phi) &: [\bar{\phi}, 1] \to [\frac{\lambda}{r+\lambda}\frac{a_1 - a_0}{r}, \frac{a_1 - a_0}{r}]. \end{aligned}$$

The properties of these value functions (also direct consequences of hypergeometric functions properties) are summarized in Proposition 6.

5.3 Brownian Motion

Assume now the signal process follows a Brownian motion

$$dS_t = \mu_i dt + \sigma dZ_t$$

where $i = \{L, H\}$, drifts depend on the tree's type $\mu_H > \mu_L$ and the noise σ is the same for both types.

The following Proposition shows that reputation, both for high and low quality trees, also follows a Brownian motion process when based purely on signals. As discussed in Proposition 1, given the equilibrium exit rates, this is also the updating rule for all $\phi > \bar{\phi}$, while the updating for all $\phi < \bar{\phi}$ follows an immediate jump up to $\bar{\phi}$. The proof is in Appendix A.1.

Proposition 12 *Reputation process based on Brownian motion signals.*

The reputation process high quality trees expect is a submartingale

$$d\phi_t^H = \frac{\lambda^2(\phi_t)}{\phi_t}dt + \lambda(\phi_t)dZ_t, \tag{29}$$

and the reputation process low quality trees expect is a supermartingale

$$d\phi_t^L = -\frac{\lambda^2(\phi_t)}{(1-\phi_t)}dt + \lambda(\phi_t)dZ_t, \tag{30}$$

where $\lambda(\phi_t) = \phi_t(1-\phi_t)\zeta$ and $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ is the signal to noise ratio.

Four clear properties arise from inspecting equations (29) and (30). First, high quality trees expect a positive drift in their evolution of reputation while low quality trees expect a negative drift. Second, when reputation ϕ_t is either 0 or 1, drifts and volatilities are zero, which means at those points reputation do not change, both for high and low quality trees. Third, reputation varies more at intermediate levels of ϕ_t, and volatilities are larger. Finally, the drift for high quality trees is higher than for low quality trees for bad reputations and lower for good reputations, since ϕ_t is in the denominator of the drift for high quality trees, while $(1-\phi_t)$ is in the denominator of the drift for low quality trees.

We can now write the ordinary differential equations that characterize the value functions for high and low quality trees. The discussion about the determination of these ODEs is in Appendix A.2.

$$r\rho V_L\left(\phi\right) = \rho\pi\left(\phi\right) - \phi^2\left(1-\phi\right)V_L'\left(\phi\right) + \frac{1}{2}\phi^2\left(1-\phi\right)^2 V_L''\left(\phi\right), \tag{31}$$

$$r\rho V_H\left(\phi\right) = \rho\pi\left(\phi\right) + \phi\left(1-\phi\right)^2 V_H'\left(\phi\right) + \frac{1}{2}\phi^2\left(1-\phi\right)^2 V_H''\left(\phi\right), \tag{32}$$

where

$$\rho = \frac{\sigma^2}{(\mu_H - \mu_L)^2}. \tag{33}$$

Solving these ODEs we can obtain the value functions for high and low quality trees. Imposing that these value functions are non-negative introduces endogenous exit, which regulates the reputation process. The discussion about the determination of these value functions is in Appendix A.3.

Proposition 13 *Value functions for general profit functions and Brownian motion*

The value function of low quality trees with reputation l is

$$\hat{V}_L\left(l\right) = K\left\{\int_0^1 \theta^{\gamma-1}\pi\left(\theta l\right)d\theta - \int_{\chi/l}^1 \theta^{-\gamma}\pi\left(\theta l\right)d\theta\right\}. \tag{34}$$

The value function of high quality trees with reputation l is

$$\hat{V}_H\left(l\right) = K\left\{\int_0^1 \theta^{\gamma-2}\pi\left(\theta l\right)d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1}\pi\left(\theta l\right)d\theta + \frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right\}, \tag{35}$$

where $\theta = l'/l$,

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} \qquad \textit{and} \qquad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2r\rho}}.$$

Boundary conditions (value matching and smooth-pasting for low and high types) must be satisfied at $\bar{l}$. These conditions jointly determine $\bar{l}$, χ and ψ:[11]

$$\hat{V}_L\left(\bar{l}\right) = \hat{V}_L'\left(\bar{l}\right) = \hat{V}_H'\left(\bar{l}\right) = 0.$$

Using the formal expressions of the value functions and derivatives (in the Appendix),

$$\begin{aligned} \hat{V}_L(\bar{l}) &= 0 \Rightarrow \int_{\chi/\bar{l}}^{1} \theta^{-\gamma} \pi\left(\theta\bar{l}\right) d\theta = \int_0^1 \theta^{\gamma-1} \pi\left(\theta\bar{l}\right) d\theta, \\ \bar{l}\hat{V}_L'(\bar{l}) &= 0 \Rightarrow (1-\gamma) \int_{\chi/\bar{l}}^{1} \theta^{-\gamma} \pi\left(\theta\bar{l}\right) d\theta = \gamma \int_0^1 \theta^{\gamma-1} \pi\left(\theta\bar{l}\right) d\theta. \end{aligned}$$

Combining the two conditions, we find the equation that pins down $\bar{l}$:

$$\int_0^1 \theta^{\gamma-1} \pi\left(\theta\bar{l}\right) d\theta = 0, \tag{36}$$

and the equation that pins down χ

$$\int_{\chi/\bar{l}}^{1} \theta^{-\gamma} \pi\left(\theta\bar{l}\right) d\theta = 0. \tag{37}$$

Finally, the condition that pins down ψ is

$$(1-\gamma) \int_0^1 \theta^{\gamma-2} \pi(\theta\bar{l}) d\theta = \gamma \left[\int_{\psi/\bar{l}}^{1} \theta^{-\gamma-1} \pi(\theta\bar{l}) d\theta - \frac{\pi(0)}{\gamma} \left(\frac{\psi}{\bar{l}}\right)^{-\gamma} \right]. \tag{38}$$

These expressions completely characterized value functions and the reputation at

[11]Value matching and smooth pasting conditions for low quality trees arise from optimal exiting decisions and the smooth pasting condition for high quality trees arises from the belief process that is reflecting at $\bar{\phi}$

which low quality trees exit. Proposition 7 lists the main properties of the value functions. The proof is in Appendix A.4.

5.4 Limits of Information

In this section we show that, regardless of the information structure, the steady-state spot market equilibrium converges to the benchmark without information as the precision of signals go to zero and converges to the benchmark with perfect information as the precision of signals go to infinity. Hence, as the effectiveness of learning improves, the equilibrium ranges from complete market shut down to the unconstrained first best.

Proposition 14 *(Limits of Information)*

In the three information structures considered (bad news, good news and Brownian motion), the spot market equilibrium converges to $Y = 0$ as the precision of signals go to zero and to the unconstrained first best $Y = \bar{Y}$, as the precision of signals go to infinity.

Proof We split the proof in two parts.

1) As the precision of signals go to zero.

In this case, to prove the steady-state spot market equilibrium converges to the benchmark without information ($Y \to 0$), it is enough to prove that $V_H(\phi) \to V_L(\phi)$ for all ϕ. This is because, from equation (21), $Y \to 0$ as $V_H(\phi^e) \to V_L(\phi^e)$.

In the bad and good news cases, the precision of signals is zero when $\lambda = 0$, hence there are no news about the true type of the firm. It is trivial to see, from Propositions 10 and 11, that $V_L(\phi) = V_H(\phi) = \pi(\phi)/r$ for all ϕ when $\lambda = 0$.

In the Brownian motion case, the precision of signals is zero when the signal to noise ratio $\frac{\mu_H - \mu_L}{\sigma} = 0$, and then $\rho = \infty$. From the ODEs 31 and 32, $V_H(\phi) = V_L(\phi) = \pi(\phi)/r$.

2) As the precision of signals go to infinity.

In this case, to prove that the steady-state spot market equilibrium converges to the unconstrained first best benchmark with perfect information ($Y \to \bar{Y}$), it is enough to prove that $V_H(\phi) \to V_H(1)$ and $V_L(\phi) \to 0$ for all $\phi > 0$, as precision goes to infinity. This is because, from Proposition 1, $V_H(\phi^e)U'(Y) = C$. As precision goes to infinity, low quality firms exit fast, and the reputation at entry does not matter to determine the average quality of firms in steady state (this is, $\frac{m(1)}{m(0)+m(1)} \to 1$ regardless of $\bar{\phi} > 0$).

In the bad and good news cases, the precision of signals is infinity when $\lambda = \infty$, hence news about the true type of the firm arrive immediately. In this case low quality firms spend almost no time with a reputation different than 0. From Propositions 10 and 11, it is straightforward to check that $V_H(\phi) = \pi(1)/r$ and $V_L(\phi) = 0$ for $\lambda = \infty$ and all $\phi > 0$. Even when $\bar{\phi} < 1$, since all low quality firms almost instantaneously leave the market when $\lambda \to \infty$, effectively $\frac{m(1)}{m(0)+m(1)} \to 1$ in the market.

In the Brownian motion case, the precision of signals is infinite when the signal to noise ratio $\frac{\mu_H - \mu_L}{\sigma} = \infty$. Then $\rho = 0$ and $\gamma = 1$. From evaluating equation (32) at l with $\rho = 0$, $V_H''(l) = 0$ for all l. Combining this result with equations (38) and the definition of $V_H'(l)$ in the Appendix, $V_H'(\bar{l}) = 0$. This implies that $V_H(\bar{\phi}) = V_H(1)$, and then the production of the experience good is $\bar{Y}$. Furthermore, even when $\bar{\phi} < 1$, since all low quality firms almost instantaneously leave the market, effectively $\frac{m(1)}{m(0)+m(1)} \to 1$ in the market. Q.E.D.

This result shows that more precise signals are welfare improving, since they move the equilibrium output closer to the unconstrained first best benchmark. The information precision also affects the effectiveness of regulatory policies.

Remark on the effectiveness of regulation: The effectiveness of regulation is non-monotonic on the precision of signals. When the precision goes to zero, entry costs do not increase Y much, since the difference between value functions is negligible. However, since the production of the experience good is very small, the marginal welfare gain can still be important. At the other extreme, when the precision goes to infinity, there is no much room for improvement in the market by regulating it, since its outcome is already close to the unconstrained first best. This implies that regulatory policies are more effective in improving the outcome of a market with spot-prices when the precision of signals is intermediate.

5.5 Numerical Exercise

In this section we provide a numerical illustration of the value functions derived above under linear profits for the bad news, good news and Brownian motion cases.

We assume $U(Y) = 2\sqrt{Y}$ (hence $U'(Y) = Y^{-0.5}$) and the following set of parameters: $C = 1$, $r = \hat{r} + \delta = 0.1$, $y(0) = -0.2$ and $y(1) = 0.6$ (hence $a_1 = 0.8$ and $a_0 = 0.2$).

The first best is characterized by $\phi^e = 1$ and

$$\bar{Y} = \left[\frac{y(1)}{rC}\right]^2 = 6^2 = 36.$$

5.5.1 Bad and Good News

We assume $\lambda = 0.1$ in both cases. With bad news, the value functions in Figure 1 illustrate the steady state spot market outcome, which is $\phi^e = \bar{\phi} = 0.16$ and

$$Y = \left[\frac{V_H(\bar{\phi})}{C}\right]^2 = 0.9^2 = 0.8.$$

With good news, the value functions in Figure 2 illustrate the steady-state spot market outcome, which is $\phi^e = \bar{\phi} = 0.25$ and

$$Y = \left[\frac{V_H(\bar{\phi})}{C}\right]^2 = 3^2 = 9.$$

When a market is characterized by good news, it achieves an allocation closer to the first best than a market characterized by bad news. Since quality and production under good news are higher, economies characterized by this signaling structure have an unequivocally higher welfare.

In the case of bad news, trees enter with a lower reputation $\phi^e = 0.16$. This is because reputation increases with time, allowing low quality trees to be able to reap some profits in the future and motivating them to start at lower reputation levels, suffering negative profits initially. While both high and low quality trees experience an always increasing reputation, low quality trees can generate bad news at a rate λ, case in which they are forced to exit. Since entry happens at the reputation with the smallest gap between value functions, production of the experience good Y is low.

Figure 1: Value Functions - Bad News

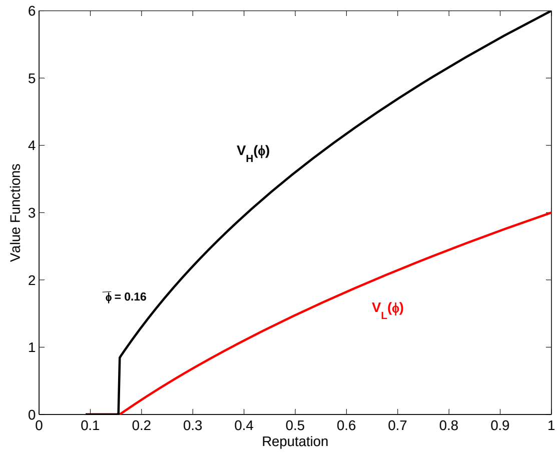
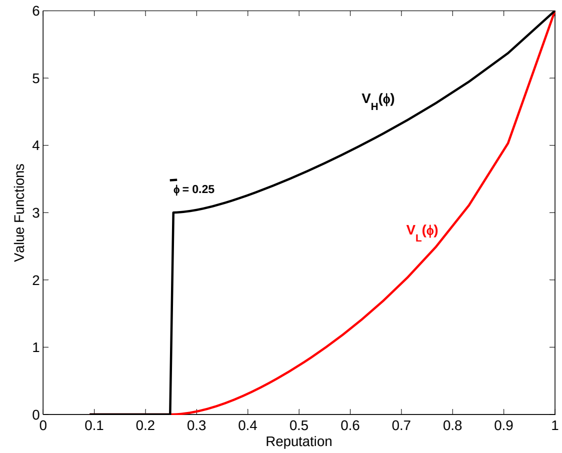


Figure 2: Value Functions-Good News



In the case of good news, trees enter with reputation $\phi^e = 0.25$ at which profits are zero. High quality trees operate at that reputation until they generate a good signal and jump to operate at reputation $\phi = 1$, just then obtaining positive profits. Low quality trees always operate at reputation $\phi = 0.25$ and randomize exiting at a rate λ, always obtaining zero profits. In an unregulated spot market economy, trees enter at the reputation with the highest gap between value functions, allowing for a production of the experience good Y higher than in the bad news case.

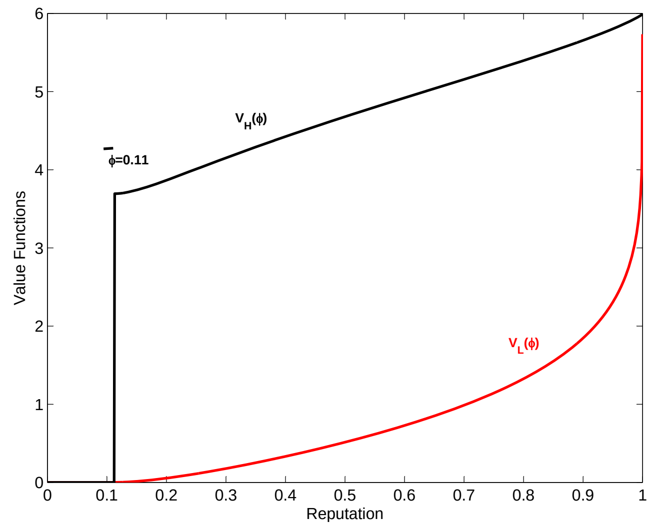
5.5.2 Brownian Motion

We assume $\sigma = 0.2$, $\mu_H - \mu_L = 0.2$ (hence $\rho = 1$ from equation 33). The value functions in Figure 3 illustrate the steady state spot market outcome, which is $\phi^e = \bar{\phi} = 0.11$ and

$$Y = \left[\frac{V_H(\bar{\phi})}{rC}\right]^2 = 3.7^2 = 14.$$

The results on this case cannot be compared with bad and good news cases so directly. However we can see that value functions combine those two cases. As in the bad news case, the gap between value functions increases at low reputation levels. As in the good news case, the gap between value functions decreases at high reputation levels, disappearing at $\phi = 1$.

Figure 3: Value Functions - Brownian motion



6 Conclusions

We have argued that the lemons problem in markets with imperfect signals about sellers' quality is a problem of commitment and not a problem of information — the lemons problem can be essentially eliminated if buyers can commit to offer sellers incentives strong enough to invest in high quality so as to improve their reputation. When a regulator can design taxes and subsidies contingent on sellers' reputation, a simple taxing scheme may provide the commitment required to mitigate the lemons problem.

Even if a regulator does not have the ability to tax or subsidize sellers contingent on their reputation, that regulator still has the ability to improve welfare by mitigating the lemons problem in a spot market equilibrium by imposing a positive fixed entry cost that is then rebated lump sum to household. However, the regulator faces a tradeoff between increasing the average quality of entering sellers and restricting the overall volume of production. We show, however, that this tradeoff is resolved in favor of increasing quality, at least for small entry costs.

Entry costs are typically criticized for reducing production and market size. The main logic is clearly exposed in Hopenhayn (1992): Higher entry costs must be compensated by higher aggregate prices, hence by less total output. This argument has been widely used by the economic literature - from supporting trade liberalization to explaining TFP differences across countries - and by international organisms in proposing policy reforms to underdeveloped countries. Still, as shown by Djankov et al. (2002), there is heavy regulation of entry for start up firms around the world, under the main justification of discouraging the entry of low quality firms. In this paper we provide a unifying framework to study the trade off that entry costs create between production and quality. Interestingly we show there is a range of entry costs that increase quality without reducing total output – sometimes also increasing total output – and we characterize the optimal level of entry costs that maximize welfare by enhancing market provided reputation incentives.

From a technical viewpoint we contribute in providing analytical solutions in continuous time for a model of reputation with free entry and exit of firms that know their type – since they know their own initial investments that determine their type. The explicit analytical solution allows a complete welfare comparison across different regulation policies. We also endogenize the initial reputation assigned to entrants in a market, since the 'lemons problem' is generated by an endogenous decision in general equilibrium of ex-ante identical firms.

An important next step in understanding optimal regulation in the presence of reputation concerns is considering moral hazard problems at each moment. We have assumed that quality is fixed as the result of a one-time investment decision. There is a large literature that examines outcomes when sellers must maintain ongoing investments to preserve quality.[12] We anticipate that our first main result will extend to this setting — the problem of moral hazard arises because buyers cannot commit to pay sellers prices contingent on reputation that are high enough to preserve the incentives to invest in quality. We conjecture then that a regulator with sufficient flexibility to design transfers contingent on reputation would be able to mitigate both the lemons problem and the moral hazard problem associated with investments to maintain quality. We are not able to derive these results formally as the required transfer schemes are likely to be non-linear in reputation and thus outside the scope of what we can solve at this time.

Another natural extension is to study mechanisms and institutions the market can endogenously create to reduce commitment problems and align learning and reputation compensations to improve welfare. Possible institutions are vertical integra-

[12]See for example Marvel and McCafferty (1984), Maksimovic and Titman (1991) and, more recently, Board and Meyer-ter Vehn (2010)

tion between experience good producers and intermediate good producers that relax informational problems and financial intermediaries or horizontal integration of intermediate goods producers that commit to cross subsidize members with different reputation (in the spirit of Biglaiser and Friedman (1994)).

Similarly, an alternative channel that markets can use to replicate positive entry costs is burning money at the moment of entry as a signal of investment. There are multiple equilibria introducing this possibility, all of them sustained by an implausible degree of coordination among producers of the intermediate good. Based on this required degree of coordination, but beyond the scope of this paper, we conjecture the only robust equilibrium, from an evolutionary perspective, is the one we characterize without money burning. Furthermore, money burning is an inefficient way to replace entry fees, unless that money goes back to the economy, as we assume the regulator does by making lump sum transfers of the entry fees to households.

Finally it is important to mention that most of the literature that studies the effects of costly certifications to enter into a market, focuses in the informational element of certificates as screening of the initial investment (see Lizzeri (1999) and Albano and Lizzeri (2001)). Our case is more extreme, and suggests that even if certification does not provide any additional information about the quality of new firms, it may still be welfare improving, just because it is costly to entrants.

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A Appendix

A.1 Proof Proposition 12

The activities of the two types of trees induce two different probability measures over the paths of the signal S_t. Fix a prior ϕ^e and assume exogenous exit. Then reputation evolves following the equation:

$$\phi_t = \frac{\phi^e Pr(S_t|H)}{\phi^e Pr(S_t|H) + (1-\phi^e) Pr(S_t|L)}$$

or

$$\phi_t = \frac{\phi^e \xi_t}{\phi^e \xi_t + (1-\phi^e)} \tag{39}$$

where ξ_t is the ratio between the likelihood that a path $S_s : s \in [0,t]$ arises from type H and the likelihood that it arises from type L. As in Faingold and Sannikov (2011), from Girsanov's Theorem, this ratio follows a Brownian motion characterized by $\mu_\xi = 0$ and $\sigma_\xi = \xi_t \zeta$,

$$d\xi_t = \xi_t \zeta dZ_s^L \tag{40}$$

where $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ and $dZ_s^L = \frac{dS_t - \mu_L dt}{\sigma}$ is a Brownian motion under the probability measure generated by type L.[13]

By Ito's formula,

$$d\phi = [\phi' \mu_\xi + \frac{1}{2}\phi'' \sigma_\xi^2]dt + \phi' \sigma_\xi dZ_s^L$$

$$d\phi_t = -\frac{1}{2}\frac{2\phi^{e2}(1-\phi^e)}{(\phi^e \xi_t + (1-\phi^e))^3}\xi_t^2 \zeta^2 dt + \frac{\phi^e(1-\phi^e)}{(\phi^e \xi_t + (1-\phi^e))^2}\xi_t \zeta dZ_s^L$$

and from equation (39) we can express it in terms of ϕ_t rather than ϕ^e

$$d\phi_t = -\phi_t^2(1-\phi_t)\zeta^2 dt + \phi_t(1-\phi_t)\zeta dZ_s^L$$

$$d\phi_t = \phi_t(1-\phi_t)\zeta[dZ_s^L - \phi_t \zeta dt]$$

replacing by the definition of dZ_s^L,

$$d\phi_t = \lambda(\phi_t) dZ_t^\phi \tag{41}$$

where $dZ_t^\phi = \frac{1}{\sigma}[dS_t - (\phi_t \mu_H + (1-\phi_t)\mu_L)dt]$ and

$$\lambda(\phi_t) = \phi_t(1-\phi_t)\frac{\mu_H - \mu_L}{\sigma} \tag{42}$$

[13] It is also possible to solve the problem defining $\xi_t = \frac{Pr(S_t|L)}{Pr(S_t|H)}$ such that $\phi_t = \frac{\phi^e}{\phi^e + (1-\phi^e)\xi_t}$, where $d\xi_t = \xi_t \zeta dZ_s^H$

Conversely, suppose that ϕ_t is a process that solves equation (41). Define ξ_t using equation (39),

$$d\xi_t = -\frac{1-\phi^e}{\phi^e}\frac{\phi_t}{1-\phi_t}$$

By applying Ito's formula again, ξ_t satisfies equation (40). This implies ξ_t is the ratio between the likelihood that a path $S_s : s \in [0,t]$ arises from type H and the likelihood it arises from type L. Hence ϕ_t is determined by Bayes rule.

Finally, consider that different types will have different paths, that in expectation will move their reputation. Replacing dS_t^i in dZ_t^ϕ in equation (41) for the two different types of trees, deliver equations (29) and (30).

A.2 Ordinary Differential Equations with Brownian motion

Here we obtain the differential equations that characterizes the values functions.

Proposition 15 *Define Ψ the space of progressively measurable processes ψ_t for all $t \geq 0$ with $E[\int_0^T \psi_t^2 dt] < \infty$ for all $0 < T < \infty$. A bounded process W_t^i for all $t \geq 0$ is the continuation value for type $i = \{H, L\}$ if and only if, for some process ψ_t^i in Ψ we have,*

$$dW_t^i = [rW_t^i - r\pi(\phi_t)]dt + \psi_t^i dZ_t \tag{43}$$

Proof The flow continuation value W_t^i for type i is the expected payoff at time t,

$$W_t^i = rE_t^i\left[\int_t^\infty e^{-r(s-t)}\pi(\phi_s)ds\right]$$

Denote U_t^i the discounted sum of payoffs for type i conditional on the public information available at time t,

$$U_t^i = rE_t^i\left[\int_0^\infty e^{-rs}\pi(\phi)ds\right] = \int_0^t e^{-rs}r\pi(\phi_s)ds + W_t^i \tag{44}$$

Since U_t^i is a martingale, by the Martingale Representation Theorem, there exist a process ψ_t^i in Ψ such that,

$$dU_t^i = e^{-rt}\psi_t^i dZ_t \tag{45}$$

Differentiating (44) with respect to time

$$dU_t^i = e^{-rt}r\pi(\phi_t)dt - re^{-rt}W_t^i dt + e^{-rt}dW_t^i \tag{46}$$

Combining (45) and (46), we can obtain (43). Q.E.D.

In a Markovian equilibrium, we know $W_t^i = V_i(\phi_t)$. Since this continuation value depends on the reputation, which follows a Brownian motion, using Ito's Lemma,

$$dV_i(\phi) = \left[\mu_{i,\phi} V_i'(\phi) + \frac{1}{2}\sigma_\phi^2 V_i''(\phi)\right] dt + \sigma_\phi V_i'(\phi) dZ \tag{47}$$

where $\mu_{H,\phi} = \frac{\lambda^2(\phi)}{\phi}$, $\mu_{L,\phi} = -\frac{\lambda^2(\phi)}{(1-\phi)}$ and $\sigma_\phi = \lambda(\phi)$ from Proposition 12.

Matching drifts of equations (43) and (47) for each type i, yields the linear second order differential equation that characterizes continuation values $V_H(\phi)$ and $V_L(\phi)$,

$$\frac{1}{2}\lambda^2(\phi)V_L''(\phi) - \frac{\lambda^2(\phi)}{(1-\phi)}V_L'(\phi) - rV_L(\phi) + \pi(\phi) = 0 \tag{48}$$

and

$$\frac{1}{2}\lambda^2(\phi)V_H''(\phi) + \frac{\lambda^2(\phi)}{\phi}V_H'(\phi) - rV_H(\phi) + \pi(\phi) = 0 \tag{49}$$

Using the definition for $\lambda(\phi)$ from equation (42) we can rewrite the second order differential equations as

$$\begin{aligned} r\rho V_L(\phi) &= \rho\pi(\phi) - \phi^2(1-\phi)V_L'(\phi) + \frac{1}{2}\phi^2(1-\phi)^2 V_L''(\phi) \\ r\rho V_H(\phi) &= \rho\pi(\phi) + \phi(1-\phi)^2 V_H'(\phi) + \frac{1}{2}\phi^2(1-\phi)^2 V_H''(\phi) \end{aligned}$$

where

$$\rho = \frac{\sigma^2}{(\mu_H - \mu_L)^2}$$

A.3 Proof Proposition 13

A.3.1 Solving the ODE's

Changing variables to $l = (1-\phi)/\phi$ and defining $\hat{V}(l) = V(\phi)$, the ODEs above can be written as

$$\begin{aligned} r\rho\hat{V}_L(l) &= \rho\pi(l) + l\hat{V}_L'(l) + \frac{1}{2}l^2\hat{V}_L''(l) \\ r\rho\hat{V}_H(l) &= \rho\pi(l) + \frac{1}{2}l^2\hat{V}_H''(l) \end{aligned}$$

a) Solving for $\hat{V}_L(l)$: We conjecture a solution of the form:

$$\hat{V}_L(l) = K\left[l^{-\gamma}\int_{\chi_1}^{l} l'^{\gamma}\frac{\pi(l')}{l'}dl' - l^{\gamma-1}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{\pi(l')}{l'}dl'\right]$$

for some parameters γ and K. With this, we have

$$\hat{V}_L'(l) = K\left[(-\gamma)\, l^{-\gamma-1}\int_{\chi_1}^{l} l'^{\gamma}\frac{\pi(l')}{l'}dl' - (\gamma-1)\, l^{\gamma-2}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{\pi(l')}{l'}dl'\right]$$

$$\begin{aligned}\hat{V}_L''(l) &= K\left[(-\gamma)(-\gamma-1)\, l^{-\gamma-2}\int_{\chi_1}^{l} l'^{\gamma}\frac{\pi(l')}{l'}dl' - (\gamma-1)(\gamma-2)\, l^{\gamma-3}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{\pi(l')}{l'}dl'\right] \\ &\quad + K(1-2\gamma)\frac{\pi(l)}{l^2}\end{aligned}$$

$$l\hat{V}_L'(l) + \frac{1}{2}l^2\hat{V}_L''(l) = \frac{\gamma(\gamma-1)}{2}\hat{V}_L(l) + K\left(\frac{1-2\gamma}{2}\right)\pi(l)$$

which solves the ODE when $2r\rho = \gamma(\gamma-1)$ and $K(1-2\gamma) = -2\rho$, or

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} \qquad \texttt{and} \qquad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2r\rho}}$$

Recall $\gamma(\rho) : [0,\infty] \to [1,\infty]$ and $K(\rho) > 0$. The parameters χ_1 and χ_2 will be determined later from boundary conditions.

b) Solving for $\hat{V}_H(l)$: Define: $\Delta_H(l) = \pi(0) - \hat{V}_H(l)$, $\bar{\pi}(l) = \pi(0) - \pi(l)$. Notice $\bar{\pi}(l)$ is increasing in l.

Rewriting the ODE for the high type as

$$\rho\Delta_H(l) = \rho\bar{\pi}(l) + \frac{1}{2}l^2\Delta_H''(l)$$

Proceeding as above we conjecture a solution of the form:

$$\Delta_H(l) = K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right]$$

for the same parameters γ and K defined previously. With this, we have

$$\begin{aligned}
\Delta_H'(l) &= K\left[(1-\gamma)l^{-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + \gamma l^{\gamma-1}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right] \\
\Delta_H''(l) &= K\left[-\gamma(1-\gamma)l^{-\gamma-1}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + \gamma(\gamma-1)l^{\gamma-2}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right] \\
&\quad + K(1-2\gamma)\frac{\bar{\pi}(l)}{l^2} \\
\frac{1}{2}l^2\Delta_H''(l) &= \frac{\gamma(\gamma-1)}{2}\Delta_H(l) + K\left(\frac{1-2\gamma}{2}\right)\pi(l)
\end{aligned}$$

that fulfill the ODE by construction with the parameters γ and K defined above. The parameters ψ_1 and ψ_2 will be determined later also from boundary conditions.

A.3.2 Dealing with the boundary conditions at $l = 0$

Notice that we need $\lim_{l\to 0}\hat{V}_L(l) = \lim_{l\to 0}\pi(l) = \pi(0)$, and $\lim_{l\to 0}\Delta_H(l) = \lim_{l\to 0}\bar{\pi}(l) = \lim_{l\to 0}\pi(l) - \pi(0) = 0$. The two limiting properties hold if and only if $\chi_1 = 0$ and $\psi_1 = 0$ (we then relabel $\chi_2 = \chi$ and $\psi_2 = \psi$).

We will proceed with the proof for the high type. The proof for the low type is related. Using Lipschitz continuity of $\bar{\pi}(l)$, assuming $\bar{\pi}(l) \leq \Lambda l$, and $\psi_2 \leq \infty$:

$$\begin{aligned}
\Delta_H(l) &= K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right] \\
&\leq \Lambda K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}dl'\right] \\
&= \Lambda K\left[l^{1-\gamma}\left(\frac{l^{\gamma}}{\gamma} - \frac{\psi_1^{\gamma}}{\gamma}\right) + l^{\gamma}\left(\frac{\psi_2^{1-\gamma}}{1-\gamma} - \frac{l^{1-\gamma}}{1-\gamma}\right)\right] \\
&= \Lambda K\left[l\left(\frac{1}{\gamma} - \frac{1}{1-\gamma}\right)\right] \\
&= \Lambda l
\end{aligned}$$

if and only if $\psi_1 = 0$ and assuming $\psi_2 = \infty$. Hence, $\lim_{l\to 0}\Delta_H(l) = 0$ if and only if $\psi_1 = 0$. A similar analysis delivers $\lim_{l\to 0}\hat{V}_L(l) = \pi(0)$ if and only if $\chi_1 = 0$

A.3.3 Simplifying Value Functions

Changing variables inside the integrals: $\theta = l'/l$, so $ld\theta = dl'$ and the limits of integration. We start from obtaining $V_H(l)$.

$$\Delta_H(l) = K\left\{\int_0^1 \theta^{\gamma-2}\bar{\pi}(\theta l)\,d\theta + \int_1^{\psi/l} \theta^{-\gamma-1}\bar{\pi}(\theta l)\,d\theta\right\}$$

Since $\bar{\pi}(\theta l) = \pi(0) - \pi(\theta l)$ and $\hat{V}_H(l) = \pi(0) - \Delta_H(l)$

$$\begin{aligned}
\hat{V}_H(l) &= \pi(0)\left(1 - K\int_0^1 \theta^{\gamma-2}d\theta - K\int_1^{\psi/l}\theta^{-\gamma-1}d\theta\right) \\
&\quad + K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\,d\theta + \int_1^{\psi/l}\theta^{-\gamma-1}\pi(\theta l)\,d\theta\right\} \\
\hat{V}_H(l) &= \pi(0)\left[\frac{K}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right] + K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\,d\theta + \int_1^{\psi/l}\theta^{-\gamma-1}\pi(\theta l)\,d\theta\right\}
\end{aligned}$$

Hence

$$\hat{V}_H(l) = K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\,d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1}\pi(\theta l)\,d\theta + \frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right\} \tag{50}$$

Similarly, the low type's value function can be written as

$$\hat{V}_L(l) = K\left\{\int_0^1 \theta^{\gamma-1}\pi(\theta l)\,d\theta - \int_{\chi/l}^1 \theta^{-\gamma}\pi(\theta l)\,d\theta\right\} \tag{51}$$

In reduced form

$$\begin{aligned}
\hat{V}_L(l) &= K[B_L(l) - A_L(l)] \quad \texttt{and} && (52)\\
\hat{V}_H(l) &= K[B_H(l) - A_H(l)] && (53)
\end{aligned}$$

where

$$B_L(l) = \int_0^1 \theta^{\gamma-1}\pi(\theta l)\,d\theta \qquad \texttt{and} \qquad A_L(l) = \int_{\chi/l}^1 \theta^{-\gamma}\pi(\theta l)\,d\theta$$

$$B_H(l) = \int_0^1 \theta^{\gamma-2}\pi(\theta l)\,d\theta \qquad \texttt{and} \qquad A_H(l) = \int_{\psi/l}^1 \theta^{-\gamma-1}\pi(\theta l)\,d\theta - \frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}$$

A.3.4 Derivatives

Taking derivatives of $\hat{V}_L(l)$ components and multiplying by l,

$$l\frac{\partial A_L(l)}{\partial l} = \int_{\chi/l}^{1} \theta^{-\gamma}\pi'(\theta l)\,\theta l d\theta - \left(\frac{\chi}{l}\right)^{-\gamma}\pi(\chi)\left(-\frac{\chi}{l^2}\right)l$$

Integrating the first term by parts

$$\begin{aligned}\int_{\chi/l}^{1} \theta^{1-\gamma}\pi'(\theta l)\,l d\theta &= \theta^{1-\gamma}\pi(\theta l)|_{\chi/l}^{1} - \int_{\chi/l}^{1}(1-\gamma)\theta^{-\gamma}\pi(\theta l)d\theta \\ &= \pi(l) - \left(\frac{\chi}{l}\right)^{1-\gamma}\pi(\chi) - (1-\gamma)\int_{\chi/l}^{1}\theta^{-\gamma}\pi(\theta l)d\theta\end{aligned}$$

Then

$$l\frac{\partial A_L(l)}{\partial l} = \pi(l) - (1-\gamma)\int_{\chi/l}^{1}\theta^{-\gamma}\pi(\theta l)d\theta = \pi(l) - (1-\gamma)A_L(l)$$

Similarly

$$\begin{aligned}l\frac{\partial A_H(l)}{\partial l} &= \pi(l) + \gamma\int_{\psi/l}^{1}\theta^{-\gamma-1}\pi(\theta l)d\theta - \frac{\pi(0)}{\gamma}(-\gamma)\left(\frac{\psi}{l}\right)^{-\gamma-1}\left(-\frac{\psi}{l^2}\right)l \\ &= \pi(l) + \gamma\int_{\psi/l}^{1}\theta^{-\gamma-1}\pi(\theta l)d\theta - \gamma\frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma} = \pi(l) + \gamma A_H(l)\end{aligned}$$

$$l\frac{\partial B_L(l)}{\partial l} = \pi(l) - \gamma\int_{0}^{1}\theta^{\gamma-1}\pi(\theta l)d\theta = \pi(l) - \gamma B_L(l)$$

$$l\frac{\partial B_H(l)}{\partial l} = \pi(l) - (\gamma-1)\int_{0}^{1}\theta^{\gamma-2}\pi(\theta l)d\theta = \pi(l) - (\gamma-1)B_H(l)$$

The derivatives can then be simplified as follows,

$$l\hat{V}_L'(l) = K[-\gamma B_L(l) + (1-\gamma)A_L(l)] \quad \texttt{and} \tag{54}$$

$$l\hat{V}_H'(l) = K[(1-\gamma)B_H(l) - \gamma A_H(l)] \tag{55}$$

A.4 Proof Proposition 7

First we prove the ratio of value functions $V_L(\phi)/V_H(\phi)$ is monotonically increasing with ϕ. Then we prove the difference between value functions $V_H(\phi) - V_L(\phi)$ is increasing for low reputation levels and decreasing for high reputation levels when the public signal follows a Brownian motion.

A.4.1 Increasing Ratio $V_L(\phi)/V_H(\phi)$

The ratio $\frac{V_L(\phi^e)}{V_H(\phi^e)}$ is an increasing function of ϕ^e, or which is the same $\frac{V_L(l_0)}{V_H(l_0)}$ is a decreasing function of l_0, that maps from $l_0 = [0, \bar{l}]$ to $[1, 0]$.

First, we define the domain and image of the function. The lowest possible reputation in the market is $\bar{l}$, where $\hat{V}_L(\bar{l}) = 0$ and $\hat{V}_H(\bar{l}) > 0$. We also know that $\hat{V}_L(1) = \hat{V}_H(1) > 0$. Finally, $0 < \hat{V}_L(l) < \hat{V}_H(l)$ for all other $l_0 \in [0, \bar{l})$. This implies $\frac{\hat{V}_L(l_0)}{\hat{V}_H(l_0)}$ is a mapping from $l_0 = [0, \bar{l}]$ to $[1, 0]$.

We show the ratio $\frac{\hat{V}_L(l)}{\hat{V}_H(l)}$ is monotonically decreasing in $l \in [0, \bar{l}]$. This is the case if

$$\frac{l\hat{V}_L'(l)}{\hat{V}_L(l)} < \frac{l\hat{V}_H'(l)}{\hat{V}_H(l)}$$

$$\frac{-\gamma B_L(l) - (\gamma - 1)A_L(l)}{B_L(l) - A_L(l)} < \frac{-(\gamma - 1)B_H(l) - \gamma A_H(l)}{B_H(l) - A_H(l)}$$

$$\frac{B_L(l) - A_L(l)}{B_H(l) - A_H(l)} < \frac{\gamma B_L(l) + (\gamma - 1)A_L(l)}{(\gamma - 1)B_H(l) + \gamma A_H(l)}$$

After some algebra, dropping the argument l, this condition implies,

$$B_H\left[(B_L - A_L) + (2\gamma - 1)A_L\right] > A_H\left[2\gamma(B_L - A_L) + (2\gamma - 1)A_L\right]$$

or

$$B_H\left[\left(1 - \gamma\frac{A_H}{B_H}\right)(B_L - A_L) + (2\gamma - 1)A_L\right] > A_H\left[\gamma(B_L - A_L) + (2\gamma - 1)A_L\right] \quad (56)$$

We show the left hand side of (56) is positive and the right hand side of (56) is negative for all $l \in [0, \bar{l}]$, hence the condition is always satisfied and the ratio of value functions decreasing in that range.

1. $B_H(l) > 0$ for all $l \in [0, \bar{l}]$

   First, we develop the integrals $B_L(l)$ and $B_H(l)$.
   Recall the profit function is linear in ϕ, ($y(\phi) = a_1\phi - a_0$) and $\phi = \frac{1}{1+l}$,

$$\pi(\theta l) = \frac{a_1}{1 + \theta l} - a_0$$

and consider the general solution to the following integral (see Abramowitz and Stegun (1972)),

$$\int \theta^m \left(\frac{a_1}{1+\theta l} - a_0 \right) d\theta = a_1 \theta^{m+1} \Phi(-\theta l, 1, m+1) - \frac{\theta^{m+1}}{m+1} a_0$$

where $\Phi(-\theta l, 1, m+1)$ is a Hurwitz Lerch zeta-function.

Applying this result to B_L,

$$B_L(l) = \int_0^1 \theta^{\gamma-1} \left(\frac{a_1}{1+\theta l} - a_0 \right) d\theta = \left[a_1 \theta^{\gamma} \Phi(-\theta l, 1, \gamma) - \frac{\theta^{\gamma}}{\gamma} a_0 \right]_0^1$$

$$B_L(l) = a_1 \Phi(-l, 1, \gamma) - \frac{a_0}{\gamma}$$

and similarly,

$$B_H = a_1 \Phi(-l, 1, \gamma - 1) - \frac{a_0}{\gamma - 1}$$

Our strategy is to prove first $B_L(l) > 0$ for all $l \in [0, \bar{l}]$ an then to prove $B_H(l) > B_L(l)$ for all $l \in [0, \bar{l}]$.

Important properties of Herwitz Lerch zeta functions for the parameters we are considering ($\gamma \geq 1$) are (see Laurincikas and Garunkstis (2003)):

- $\Phi(0, 1, \gamma) = \frac{1}{\gamma}$
- $\frac{\partial \Phi(-l,1,\gamma)}{\partial l} = \frac{1}{l} \left[\frac{1}{l+1} - \gamma \Phi(-l, 1, \gamma) \right] < 0$
- $(\gamma - 1)\Phi(-l, 1, \gamma - 1) > \gamma \Phi(-l, 1, \gamma)$

By construction, $B_L(\bar{l}) = 0$, hence $\Phi(\bar{l}, 1, \gamma) = \frac{a_0}{\gamma a_1}$. Given the properties above

$$B_L(l) : [0, \bar{l}] \to [\frac{a_1 - a_0}{\gamma}, 0]$$

Furthermore, $B_L(l)$ is monotonically decreasing in the range

$B_H(l) > B_L(l)$ for all $l \in [0, \bar{l}]$ if

$$\gamma(\gamma - 1)[\Phi(-l, 1, \gamma - 1) - \Phi(-l, 1, \gamma)] > \frac{a_0}{a_1}$$

Considering the third property above,

$$(\gamma - 1)\Phi(-l, 1, \gamma - 1) > \Phi(-l, 1, \gamma) + (\gamma - 1)\Phi(-l, 1, \gamma) > \frac{a_0}{\gamma a_1} + (\gamma - 1)\Phi(-l, 1, \gamma)$$

and hence, $B_H(l) > 0$ for all $l \in [0, \bar{l}]$

2. $A_H(l) < 0$ for all $l \in [0, \bar{l}]$

   We develop the integral $A_L(l)$ and $A_H(l)$ following the steps above.

$$A_L(l) = \int_{\chi/l}^{1} \theta^{-\gamma} \left(\frac{a_1}{1+\theta l} - a_0 \right) d\theta = \left[a_1 \theta^{1-\gamma} \Phi(-\theta l, 1, 1-\gamma) - \frac{\theta^{1-\gamma}}{1-\gamma} a_0 \right]_{\chi/l}^{1}$$

$$A_L(l) = a_1 \left[\Phi(-l, 1, 1-\gamma) - (\chi/l)^{1-\gamma} \Phi(-\chi, 1, 1-\gamma) \right] + \frac{a_0}{\gamma - 1} \left(1 - (\chi/l)^{1-\gamma} \right)$$

   and,

$$A_H(l) = a_1 \left[\Phi(-l, 1, -\gamma) - (\psi/l)^{-\gamma} \Phi(-\psi, 1, -\gamma) \right] + \frac{a_0}{\gamma} - \frac{a_1}{\gamma} (\psi/l)^{-\gamma}$$

   Consider $A_H(0) = A_H(\psi) = -\frac{a_1 - a_0}{\gamma} < 0$. We show that, if the function grows, the maximum is still negative. This is, we prove that $A_H(\hat{l}) < 0$ where $\hat{l} = argmax A_H(l)$ (hence $\frac{\partial A_H(l)}{\partial l}|_{l=\hat{l}} = 0$).

$$\frac{\partial A_H(l)}{\partial l} = \frac{a_1}{l} \left[\left(\frac{1}{1+l} + \gamma \Phi(-l, 1, -\gamma) \right) - \gamma (l/\psi)^{\gamma} \Phi(-\psi, 1, -\gamma) \right] - \frac{a_1}{l} (l/\psi)^{\gamma}$$

   The condition satisfied at $l \frac{\partial A_H(l)}{\partial l} = 0$ is,

$$\left[\Phi(-l, 1, -\gamma) - (\psi/l)^{-\gamma} \Phi(-\psi, 1, -\gamma) \right] = \frac{1}{\gamma} (l/\psi)^{\gamma} - \frac{1}{1+l}$$

   Evaluating $A_H(\hat{l})$ considering that condition,

$$a_1 \left[\frac{1}{\gamma} (l/\psi)^{\gamma} - \frac{1}{1+l} \right] + \frac{a_0}{\gamma - 1} \left(1 - (\chi/l)^{1-\gamma} \right) < 0$$

   since

$$\gamma a_1 \frac{1}{1+l} > a_0$$

   Hence, $A_H(l) < 0$ for all $l \in [0, \bar{l}]$

Finally, just for completeness, $A_L(0) = -\frac{a_1-a_0}{\gamma-1} < 0$, $A_L(\chi) = 0$ because $\chi/l = 1$ and $A_L(\bar{l}) = 0$ by construction. It can be further shown that $A_L(l) < 0$ for all $l \in (0, \chi)$ and $A_L(l) > 0$ for all $l \in (\chi, \bar{l})$.

3. $\gamma(B_L(l) - A_L(l)) + (2\gamma - 1)A_L(l) > 0$ for all $l \in [0, \bar{l}]$

   Recall $\gamma(B_L - A_L) + (2\gamma - 1)A_L = \gamma B_L + (\gamma - 1)A_L = -\frac{l\hat{V}_L'(l)}{K}$.

   By construction $\gamma B_L + (\gamma - 1)A_L = 0$ at $l = 0$ and $l = \bar{l}$.

   For $l \in (\chi, \bar{l})$, since $A_L(l) \geq 0$ and $B_L(l) > 0$, $\gamma B_L + (\gamma - 1)A_L > 0$. In particular, at $\bar{l}$, $A_L(\chi) = 0$ and $\gamma B_L(\chi) > 0$.

   As shown above, for $l \in [0, \chi]$, $B_L(l) > 0$ monotonically increasing and $A_L(l) < 0$ monotonically increasing. This implies $\gamma B_L + (\gamma - 1)A_L$ goes monotonically from 0 at $l = 0$ to $\gamma B_L(\bar{l}) > 0$, and hence positive in the whole range.

4. $\left[\left(1 - \gamma\frac{A_H(l)}{B_H(l)}\right)(B_L(l) - A_L(l)) + (2\gamma - 1)A_L(l)\right] > 0$ for all $l \in [0, \bar{l}]$

   First, recall $(\gamma - 1)B_H + \gamma A_H = -\frac{l\hat{V}_H'(l)}{K}$. Hence, as in the point above, $(\gamma - 1)B_H + \gamma A_H = 0$ at $l = 0$ and $l = \bar{l}$ by construction, which we can rewrite as $1 - \gamma\frac{A_H(0)}{B_H(0)} = 1 - \gamma\frac{A_H(\bar{l})}{B_H(\bar{l})} = \gamma$. Hence at these two extreme points, the term in the left hand side is 0, the same as the one in the right hand side.

   More generally $(\gamma - 1)B_H + \gamma A_H > 0$ (and then $1 < 1 - \gamma\frac{A_H(l)}{B_H(l)} < \gamma$). Since $A_L(\chi) = 0$, $\left(1 - \gamma\frac{A_H(l)}{B_H(l)}\right)B_L(l) > 0$. By the same monotonicity arguments above, $\left[\left(1 - \gamma\frac{A_H(l)}{B_H(l)}\right)(B_L(l) - A_L(l)) + (2\gamma - 1)A_L(l)\right] > 0$ for all $l \in [0, \bar{l}]$.

A.4.2 Non-monotonic Difference $V_H(\phi) - V_L(\phi)$

First, $\hat{V}_L'(\bar{\phi}) = \hat{V}_H'(\bar{\phi}) = 0$ by construction and $\hat{V}_L'(1) = \hat{V}_H'(1) = 0$, from the expressions above. Second $\hat{V}_L'(\phi)$ and $\hat{V}_H'(\phi)$ are positive for all $\phi \in (\bar{\phi}, 1)$. Third, these derivatives are single peaked and the reputation that maximizes $\hat{V}_H'(\phi)$ is lower than the reputation that maximizes $\hat{V}_L'(\phi)$. Finally, $\hat{V}_H''(\bar{\phi}) > \hat{V}_L''(\bar{\phi})$ and $\hat{V}_H''(1) < \hat{V}_L''(1)$, which means the two derivatives cross only one time, at ϕ^*. These properties arise from inspection of the derivatives of linear profits value functions and from properties of the hypergeometric functions that characterize them.