

What Can Rational Investors Do About Excessive Volatility?*

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Abstract

We determine and analyze the trading strategy that would allow an investor to take advantage of the excessive stock price volatility that has been documented in the empirical literature on asset pricing. We construct a general equilibrium model where stock prices are excessively volatile because there are two classes of agents and one class places too much trust in a public signal and, as a result, change their expectations too often, sometimes being excessively optimistic, sometimes being excessively pessimistic. We analyze the trading strategy of the rational investors who is not overconfident about the signal. While rational risk-arbitrageurs benefit from trading on their belief that the market is being foolish, when doing so they must hedge future fluctuations in the market's foolishness. We find that fixed-income instruments can be used for the purpose of hedging. Thus, our analysis illustrates that risk arbitrage cannot be based on just a current price divergence; it must include also a protection against trading risk. We find that the presence of a few rational traders is not sufficient to eliminate the effect of overconfident investors on excess volatility. Overconfident investors of this kind may survive for a long time before being driven out of the market by rational investors.

1 Introduction

As Shiller (1981) and LeRoy and Porter (1981) have pointed out, it may very well be the case that “stock prices move too much to be justified by changes in subsequent dividends.”¹ The volatility of stock prices, if it is excessive relative to the volatility of fundamentals, may be an indication that the financial market is not information efficient. If so, there must exist a trading strategy that allows a rational, intertemporally optimizing investor (a “risk arbitrageur”) to take advantage of this inefficiency. The main goal of the present paper is to calculate and understand that strategy.²

To accomplish the above goal, we develop a model that is consistent with the view expressed by Thaler (1999, p. 17) that:

“In their enlightenment, economists will routinely incorporate as much “behavior” into their models as they observe in the real world.”

In the model we construct, some investors are non-Bayesian in the sense that that they give too much credence to public signals (that is, they are overconfident) just as in Scheinkman and Xiong (2003).³ We refer to “excessive volatility” as a situation in which, for the given utility functions of agents, the level of volatility is larger than it would be under rational Bayesian learning. Because some investors in our model are overconfident, they change their minds too often about economic prospects, and this is the source of excessive volatility. Of course, it is well-known that complete irrationality in the manner of positive “feedback traders” à la De Long, Shleifer, Summers, and Waldmann (1990a,b, 1991) can amplify the volatility of stock prices. The added volatility creates “noise-trader risk” for rational arbitrageurs, thereby creating a limit to arbitrage. However, feedback traders may not be the best representation of irrational behavior as they constitute excessively easy game for rational investors. Furthermore, models of feedback trading do not discuss the budget constraint of the feedback traders, and therefore, leave unclear the origin of the gains that the rational arbitrageurs would make at their expense. For these reasons, we prefer to model our irrational traders as being intertemporal optimizers, even if they are non-Bayesian in their learning.

¹A controversy about excess volatility has been going on since the publication of Shiller (1981) and the matter is not fully settled today. The empirical method of Shiller has been criticized. Flavin (1983) and Kleidon (1986) have pointed out that stock prices and dividends could not be detrended by a deterministic trend based on realized returns, as Shiller had done. Furthermore, if the process for prices and/or dividends is not stationary, the ergodic theorem does not apply and volatility, defined originally across the possible sample paths, cannot be measured over time. Even in the case of stationarity, a near-unit root may exist in the behavior of these two variables, causing the statistic to reject Shiller’s variance inequality in finite samples when it should not be rejected. Good methodological evaluations are provided by West (1988a,b) and Cochrane (1991). Generally speaking, as had been pointed out by LeRoy and Porter, variance-bound tests should not be implemented on the basis of the historical sequence of dividends taken at face value. The sequence must first be used to estimate the stochastic process of dividends. Stock prices must then be calculated from an extrapolation of the process to infinity before a variance bound can be placed on them. Mankiw, Romer, and Shapiro (1985, 1991) have improved upon Shiller’s tests and have basically reached the same conclusion as he had. Their method is applied to non U.S. markets by De Long and Becht (1998) and De Long and Grossman (1998) and they reach similar conclusions.

²The question being answered in our paper is the same as the one raised by Williams (1977) and Ziegler (2002) in a simpler setting in which the expected growth rate of dividends is constant (although unobserved) and in which there are fewer securities. In these two papers, the investor whose strategy one is studying is assumed to be of negligible weight in the market. For a comprehensive review of the literature on models with incomplete information, see Feldman (2005).

³In contrast to the model in Scheinkman and Xiong (2003), our model is of a general equilibrium economy, and rather than modeling agents as being risk neutral and then introducing short-sale constraints to limit the size of positions that agents take, we allow for short sales and agents who are risk averse, so that in our model it is risk aversion that induces agents to limit the size of their short positions.

Our main contribution is to derive and then analyze the optimal dynamic trading strategy of the rational investors in this model. There are two aspects to the portfolio strategy adopted by rational investors: First, these investors may not agree today with the market about its current estimate of the growth rate of dividends: when the rational investors are more optimistic than the market, they increase their investment in equity while decreasing their investment in bonds, because equity and bonds are positively correlated. Second, even when the two groups of investors happen to agree today, rational investors are aware that irrational investors will revise their estimate differently from the way their own estimate is revised. This second effect makes the rational investor hold fewer shares of equity than would be optimal in a market without excess volatility and to take a negative position in bonds (which would be zero in the absence of excess volatility). Overall, a rational risk-arbitrageur finds it beneficial to trade on his belief that the market is being foolish but when doing so, he must hedge future fluctuations in the market's foolishness. Thus, our analysis illustrates that risk arbitrage cannot be based on just a current price divergence; it must also be based on a model of irrational behavior and a prediction concerning the speed of convergence, and that the risk arbitrage must include a protection in case there is a deviation from that prediction.

Because the beliefs of others play a central role in the equilibrium, we can illustrate the role of "speculative behavior" in the sense of Harrison and Kreps (1978), when investors are risk averse rather than when they are risk neutral.

The profitability of the rational "risk arbitrage" strategy and the survival time of irrational investors are two sides of the same coin. We derive the speed of impoverishment of the irrational traders and the speed of enrichment of the rational ones. Previous work (Kogan, Ross, Wang, and Westerfield, 2003; Yan, 2004) have examined the survival of traders who are permanently overoptimistic or overpessimistic. Here, we study the survival of traders who are sometimes overoptimistic and sometimes overpessimistic, depending on the sequence of signals they have received. We also find that, in contrast to what is typically assumed in standard models of asset pricing in frictionless markets, in our model the presence of a few rational traders is not sufficient to eliminate the effect of overconfident investors on excess volatility, and that overconfident investors may survive for a long time before being driven out of the market by rational investors.

Some headway into the design of a portfolio strategy has already been made in past research on the logical linkage that exists between the phenomenon of excessive volatility and the predictability of stock returns. Campbell and Shiller (1988a,b) and Cochrane (2001, page 394 *ff*), have pointed out that the dividend-price ratio would be constant over time if dividends were unpredictable (specifically, if they followed a geometric Brownian walk) and expected returns were constant. Since the dividend-price ratio is changing, its changes must be predicting either future dividend changes or future changes in expected returns. This statement is true in any economic model, unless there are violations of the transversality conditions.⁴ Empirically, the dividend-price ratio hardly predicts subsequent dividends.

⁴There may be some violations of the terminal condition for stock prices, causing stock prices to deviate from the present discounted value of future dividends. A violation of the terminal condition means that investors pay a price today that reflects an expectation of a price at some distant date in the future that will not reflect the present discounted value of dividends. Such a deviation is called a "bubble". Recently, the theory of bubbles has endeavored to explain the process by which bubbles burst. Fluctuations in the probability of a bubble bursting can potentially generate volatility. But, it

It must, therefore, predict returns. But, if it predicts returns, it can serve as valuable information for a rational trader, or arbitrageur, entering the market. That aspect is present in our model below.

In the literature on excess volatility, there are at least two other kinds of models that have been considered. One class of models shows that Bayesian, rational learning alone can serve to develop theoretical models with volatility that matches the data, by assuming that investors do not know the true stochastic process of dividends. For instance, Barsky and De Long (1993) write that: “Major long-run swings in the U.S. stock market over the past century are broadly consistent with a model driven by changes in current and expected future dividends in which investors must estimate the time-varying *long-run* dividend growth rate” [our emphasis]. As investors do not know the expected rate of growth of dividends, prices are revised when they receive information about it. These price revisions go beyond the change in the current dividend because the current dividend also contains information about future dividends. A similar argument has been made by Timmermann (1993, 1996) and Bullard and Duffy (1998). Brennan and Xia (2001) calibrate a model in which a single type of investors populate the financial market and learn about the expected growth rate of dividends and, separately, about the expected growth rate of output. In that model, as in ours, the expected growth rate of dividends is unobservable and needs to be filtered, which then leads to an increase in the volatility of the stock price. They find that they can match all moments of stock returns. However, their model is not really “closed” since aggregate consumption is not set equal to aggregate dividends plus endowments. Needless to say, in the real world, consumption is not equal to dividends only. There is also labor income and, in any case, the real world is not a pure-exchange economy, since physical investment takes place. Another way to close the model is to account for labor income and physical investment.

A second class of models studying excess volatility focuses on the discount rate. Recall that in deriving their bounds, Shiller (1981) and LeRoy and Porter (1981) had made the assumption that discount rates, by which future dividends are discounted to obtain the current price, were constant into the future. The literature on the equity-premium puzzle has developed a number of models, such as habit formation models (see Constantinides (1990); Abel (1990); Campbell and Cochrane (1999)), in which the effective discount rate is strongly time varying even though the consumption stream remains very smooth. Using models of that kind, Menzly, Santos, and Veronesi (2004) have recently calibrated a model of the U.S. stock market in which the volatility of stock returns was *larger* than the one observed in the data. Also, Bansal and Yaron (2004) find that allowing for a small long-run predictable component in dividend growth rates can generate several observed asset-pricing phenomenon, including volatility of the market return.⁵

The balance of this paper covers the following material. In Section 2, we discuss our modeling choices against the background of the literature that we have just surveyed. In Section 3, we calculate the equilibrium. In Section 4, we discuss the impact of irrational traders on asset prices, return volatilities

is not clear whether a theory of bubbles can ever be developed without assuming the presence of some irrational investors anyway.

⁵Besides time-varying discount factors and learning about the dividend process, there exists at least one other theoretical reason for which stock prices may exhibit high volatility. Financial markets are vastly incomplete. The private valuation of nontraded risks may “rock the boat” of prices of traded securities, as in Citanna and Schmedders (2002). However, it is not clear what magnitude of stock market volatility could conceivably result from market incompleteness.

and risk premia and we figure out how many rational investors are needed to bring down volatility. In Section 5, we identify the main factors driving the portfolio strategy of the rational trader. In Section 6, we use our general setting to discuss the survival of irrational traders over time and the profits made at their expense by the rational ones. Section 7 contains the conclusion. All the cumbersome mathematical derivations are collected in appendixes.

2 Modeling choices and information structure

In our model below, we allow for the presence of irrational traders.⁶ It is well known that rationality entails two dimensions, which may not be completely independent of each other: rationality in information processing or learning (that is, application of Bayes' law) and rationality of decision making (that is, intertemporal optimality). While our irrational traders suffer from some learning disability, we want them to remain full-fledged intertemporal optimizers, so that welfare analysis and the analysis of gains and losses of the two categories of traders remain meaningful.

One way to achieve that goal has recently been proposed by Scheinkman and Xiong (2003). In their model of a "tree" economy, a stream of dividends is paid. Some aspect of the stochastic process of dividends is not observable by anyone. Risk neutral investors receive information in the form of the current dividend and some public signals. Rational agents are people who either know the true correlation between innovations in the signal and innovations in the unobserved variables or rationally learn about it from the information they receive. Irrational (they call them "overconfident") agents are people who steadfastly refuse to learn the value of this correlation. For instance, they insist on this correlation being a positive number when, in fact, it is zero. This causes them to give too much weight to the signals. Thus, when they receive a signal, they overreact to it, which then generates excessive stock price movements.⁷

Here, we consider a setting similar to that in Scheinkman and Xiong (2003) except that investors are risk averse (and are allowed to sell short) and only one group of agents is overconfident. More specifically, there are two categories of investors A (who are overconfident) and B (who are rational) with constant relative risk aversion. The risk aversion does not prevent investors from short selling but it prevents them from selling infinite amounts

We now describe the key features of our model. We adopt notation that is similar to the one used in the paper by Scheinkman and Xiong (2003).

⁶A recent paper by David (2004), developed concurrently with ours, uses non-Bayesian learning and contains a full-fledged calibration of securities price moments and differences of opinion in the financial market. However, like Brennan and Xia (2001), it is not closed. Also, in order to properly fit the rate of interest, David assumes that risk aversion is smaller than 1.

⁷Differences in the basic model or in some fixed model parameter was proposed earlier by Harris and Raviv (1993), Kandel and Pearson (1995), and Cecchetti, Lam, and Mark (2000) and used more recently by David (2004) to sustain heterogeneity of beliefs; under this approach, agents are non-Bayesian. Another modeling possibility is differences in priors, while agents remain Bayesian, as in Biais and Bossaerts (1998), Detemple and Murthy (1994), Basak (2004), Duffie, Garleanu, and Pedersen (2002), and Buraschi and Jiltsov (2002). A third, more sophisticated possibility that, however, includes noise traders, is to let agents receive private signal as in the vast "Noisy-Rational Expectations" originating from the work of Grossman and Stiglitz (1980), Hellwig (1980), and Wang (1993). In the case of private signals, agents also learn from price, a channel that is not present here.

2.1 Process for aggregate output

The dividend (output) paid by the aggregate economy at time t is equal to $\delta_t dt$. The true stochastic process for δ is:

$$\frac{d\delta_t}{\delta_t} = f_t dt + \sigma_\delta dZ_t^\delta, \quad (1)$$

where Z^δ is a Wiener under the effective probability measure, which governs empirical realizations of the process. The conditionally expected growth rate f_t of dividend is stochastic:

$$df_t = -\zeta (f_t - \bar{f}) dt + \sigma_f dZ_t^f; \quad \zeta > 0, \quad (2)$$

where Z^f is also a Wiener under the effective probability measure.

2.2 Information structure and filtering

The growth rate of dividends f is not observed by any agent. All categories of investors must estimate, or filter out, the current value of f and its future behavior. They do that from the observation of the current dividend and the observation of a public signal s , which has the following process:

$$ds_t = f_t dt + \sigma_s dZ_t^s, \quad (3)$$

where Z^s is a Wiener under the effective probability measure as well. All three Wiener processes are independent of each other (under any probability measure). Everyone knows that the drift of s at time t is equal to the drift f_t of the dividend process. So, the signal provides some long-run information about the drift of the dividend process, as does the dividend itself. That is the true reason for which the signal is informative.

Group A performs his/her filtering under the delusion that the signal s has correlation $\phi \in [0, 1]$ with f when, in fact, it is not correlated.⁸ Their “model” is:

$$ds_t = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^s, \quad (4)$$

while Group B is rational (and so knows that $\phi = 0$). So, now there are *two informative roles* played by the signal s : from the point of view of all people, the signal provides some information about the drift of the dividend process. But because of the assumed correlation in the eyes of irrational people, it also provides short-run and incorrect information about the current shock to the dividend growth rate. We can amplify or deamplify the second role relative to the first one by varying the parameter σ_s .

From filtering theory (see Lipster and Shiryaev (2001, Theorem 12.7, page 36)), the conditional expected values, \hat{f}^A and \hat{f}^B , of f according to individuals of group A (deluded; $\phi \neq 0$) and group B

⁸This is not just a prior, or it is an infinitely precise prior. They refuse to learn the true correlation.

(rational; $\phi = 0$) are respectively:^{9,10}

$$d\hat{f}_t^A = -\zeta \left(\hat{f}_t^A - \bar{f} \right) dt + \frac{\gamma^A}{\sigma_\delta^2} \left(\frac{d\delta}{\delta} - \hat{f}_t^A dt \right) + \frac{\phi \sigma_s \sigma_f + \gamma^A}{\sigma_s^2} \left(ds - \hat{f}_t^A dt \right), \quad (5)$$

$$d\hat{f}_t^B = -\zeta \left(\hat{f}_t^B - \bar{f} \right) dt + \frac{\gamma^B}{\sigma_\delta^2} \left(\frac{d\delta}{\delta} - \hat{f}_t^B dt \right) + \frac{\gamma^B}{\sigma_s^2} \left(ds - \hat{f}_t^B dt \right), \quad (6)$$

The number γ^A (γ^B) is the steady-state variance of f as estimated by group A (B).¹¹ These would normally be deterministic functions of time. But we assume, as did Scheinkman and Xiong, that there has been a sufficiently long period of learning for people of both groups to converge to their level of variance, irrespective of their prior, while, at the same time, A types have refused to use the same information to infer the correlation number, which is the exact degree to which they are being irrational.

For the purpose of obtaining a martingale, “static” formulation as done in Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987), we now rewrite these stochastic differential equations in terms of processes that are Brownian motions under subjective probability measures. Consider a two-dimensional process $W^B = (W_\delta^B, W_s^B)$ that is Brownian under the probability measure that reflects the expectations of Group B . By the definition of \hat{f}^B , we can then write:

$$\frac{d\delta_t}{\delta_t} = \hat{f}_t^B dt + \sigma_\delta dW_{\delta,t}^B, \quad (7)$$

$$ds_t = \hat{f}_t^B dt + \sigma_s dW_{s,t}^B, \quad (8)$$

A similar two-dimensional process $W^A = (W_\delta^A, W_s^A)$ that is Brownian under A 's probability measure could be defined and a similar substitution could be made to represent A 's expectations. The relationship between them is:

$$dW_{\delta,t}^B = dW_{\delta,t}^A - \frac{\hat{f}_t^B - \hat{f}_t^A}{\sigma_\delta} dt, \quad (9)$$

$$dW_{s,t}^B = dW_{s,t}^A - \frac{\hat{f}_t^B - \hat{f}_t^A}{\sigma_s} dt. \quad (10)$$

⁹Observe once again that output δ serves as a signal, which causes an update of the rate of growth of output, just as the signal s does.

¹⁰Under the effective probability measure, under which Z^s and Z^δ are Brownian motions, the stochastic differential equations for s and δ are given by Equations (1) and (4). These could be substituted into (5) and (6) to get a complete Markovian description of the process for $\{\delta, f, s, \hat{f}^A, \hat{f}^B\}$.

¹¹The steady-state variances of f as estimated by group A and group B are, respectively:

$$\gamma^A \triangleq \frac{\sqrt{\left(\zeta + \frac{\phi \sigma_f}{\sigma_s}\right)^2 + (1 - \phi^2) \sigma_f^2 \left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}\right)} - \left(\zeta + \frac{\phi \sigma_f}{\sigma_s}\right)}{\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}},$$

$$\gamma^B \triangleq \frac{\sqrt{\zeta^2 + \sigma_f^2 \left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}\right)} - \zeta}{\frac{1}{\sigma_s^2} + \frac{1}{\sigma_\delta^2}}.$$

As has been pointed out by Scheinkman and Xiong, γ^A decreases as ϕ rises, which is the reason that Group B is called overconfident. γ^A starts at the value γ^B when $\phi = 0$ and reaches $\gamma^A = 0$ when $\phi = 1$. The signal can lead Population A ultimately to complete (and foolish) unconditional certainty. The numerator of the diffusion of \hat{f}^A , $\phi \sigma_s \sigma_f + \gamma^A$, however, also starts from γ^B at $\phi = 0$ but then rises to the value $\sigma_s \sigma_f > \gamma^B$ at $\phi = 1$. Thus, the signal *increases* the conditional uncertainty that A face because of their own learning, compared to that faced by B .

Since the effective measure is not defined on either agent's σ -algebra, we can ignore it for the purpose of calculating the equilibrium. Instead, we use B 's probability measure as the reference measure. From Equations (9) and (10), we can determine that the change from B 's measure to A 's measure is given by:

$$\eta_t = \exp\left(-\frac{1}{2}\int_0^t \|\nu\|^2 dt - \int_0^t \nu_t^\top dW_t^B\right), \quad (11)$$

or

$$\frac{d\eta_t}{\eta_t} = -\nu_t^\top dW_t^B, \quad (12)$$

where

$$\nu_t = \left(\widehat{f}_t^B - \widehat{f}_t^A\right) \begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}. \quad (13)$$

Substituting (9) and (10) into (5) and (6) gives:¹²

$$\begin{aligned} d\widehat{f}_t^A &= \left[-\zeta\left(\widehat{f}_t^A - \bar{f}\right) + \left(\frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\right)\left(\widehat{f}_t^B - \widehat{f}_t^A\right)\right] dt \\ &\quad + \frac{\gamma^A}{\sigma_\delta^2}\sigma_\delta dW_{\delta,t}^B + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\sigma_s dW_{s,t}^B \end{aligned} \quad (14)$$

$$d\widehat{f}_t^B = -\zeta\left(\widehat{f}_t^B - \bar{f}\right) dt + \frac{\gamma^B}{\sigma_\delta} dW_{\delta,t}^B + \frac{\gamma^B}{\sigma_s} dW_{s,t}^B. \quad (15)$$

The Markovian system made of (7), (8), (14) and (15) completely characterizes the evolution of the economy in the eyes of population B . The viewpoint of population A will be handled by means of the change of measure η . For later reference, we also write the process for the difference of opinion $\widehat{g} \triangleq \widehat{f}^B - \widehat{f}^A$:

$$d\widehat{g}_t = -\left(\zeta + \frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\right)\widehat{g}_t dt + \frac{\gamma^B - \gamma^A}{\sigma_\delta} dW_{\delta,t}^B + \frac{\gamma^B - (\phi\sigma_s\sigma_f + \gamma^A)}{\sigma_s} dW_{s,t}^B. \quad (16)$$

When $\widehat{g} > 0$, Group B of investors is comparatively optimistic or Group A comparatively pessimistic. Also, \widehat{g} (or its absolute value) can be viewed as a measure of the dispersion of beliefs or opinions. Because $\gamma^B - (\phi\sigma_s\sigma_f + \gamma^A) < 0$, a positive realization of the signal increment $dW_{s,t}^B$ causes Population A to become more optimistic relative to where it was before.

The joint dynamics of the four state variables $\{\delta, \eta, \widehat{f}^B, \widehat{g} \triangleq \widehat{f}^B - \widehat{f}^A\}$ are provided by Equations (7), (12), (15) and (16). They are driven by only two Brownians, W_δ^B and W_s^B . This is because variable f is unobserved by anyone and is only a latent variable. It is important to keep in mind that the four

¹²The conditional variance of \widehat{f}^A is equal to:

$$\left[\phi\sigma_f + \frac{\gamma^A}{\sigma_s}\right]^2 + \left[\frac{\gamma^A}{\sigma_\delta}\right]^2 = -2\zeta\gamma^A + \sigma_f^2,$$

which is a monotonically increasing function of ϕ , rising from $-2\zeta\gamma^B + \sigma_f^2$ at $\phi = 0$ to σ_f^2 at $\phi = 1$. The irrational investor Population A changes their mind in a more volatile way than does Population B .

variables are not independent of each other. Since there are only two Brownians, the diffusion matrix of $\{\delta, \eta, \widehat{f}^B, \widehat{g}\}$ is a 4×2 matrix:

$$\begin{bmatrix} \delta\sigma_\delta > 0 & 0 \\ -\eta\frac{\widehat{g}}{\sigma_\delta} & -\eta\frac{\widehat{g}}{\sigma_s} \\ \frac{\gamma^B}{\sigma_\delta} > 0 & \frac{\gamma^B}{\sigma_s} > 0 \\ \frac{\gamma^B - \gamma^A}{\sigma_\delta} \geq 0 & \frac{\gamma^B - (\phi\sigma_s\sigma_f + \gamma^A)}{\sigma_s} < 0 \end{bmatrix}. \quad (17)$$

Evidently, δ and \widehat{f}^B are always positively correlated and the diffusion vector of η has the sign of \widehat{g} .¹³ In the special case of pure Bayesian learning, in which everyone is rational ($\phi = 0$) and differences in beliefs can arise only from differences in priors, \widehat{g} has zero diffusion and reverts to zero deterministically (see Equation (16)). Even in that case, as long as \widehat{g} has not reached the value 0, η fluctuates randomly as public signals (δ, s) are realized.

To summarize, there are two distinct effects of imperfect learning. The first effect is instantaneous: \widehat{g} has nonzero diffusion. The second effect is cumulative: since \widehat{g} is stochastic and conditions the diffusion of η , it implies that η has a diffusion that can take large positive or negative values.

3 Individual optimization and equilibrium

In this section, we first describe the optimization problem faced by each investor, and then, assuming complete financial markets, the equilibrium in this economy, which includes a characterization of the instantaneously riskless interest rate and the market price of risk. We conclude this section by explaining how the complete-markets equilibrium can be implemented via dynamic trading in long-lived securities.

3.1 Preferences of agents and their optimization problems

In this paper, we are interested in the interaction between two groups one of which is rational and the other one not. Differences in risk aversion and differences in the rate of impatience are not our main focus. So, we restrict our analysis to a situation in which both groups have power utility with the same risk aversion, $1 - \alpha$, and rate of impatience, ρ .

Assuming a complete financial market,¹⁴ the problem of population B is to maximize the expected utility from lifetime consumption:

$$\sup_c \mathbb{E}^B \int_0^\infty e^{-\rho t} \frac{1}{\alpha} (c_t^B)^\alpha dt, \quad (18)$$

subject to the static budget constraint:

$$\mathbb{E}^B \int_0^\infty \xi_t^B c_t^B dt = \bar{\theta}^B \mathbb{E}^B \int_0^\infty \xi_t^B \delta_t dt, \quad (19)$$

¹³ \widehat{g} is positively correlated with δ and \widehat{f}^B if $\left| \frac{\gamma^B - \gamma^A}{\sigma_\delta} \right| > \left| \frac{\gamma^B - (\phi\sigma_s\sigma_f + \gamma^A)}{\sigma_s} \right|$

¹⁴Details of the securities needed to complete the market are given in Section 3.4.

where ξ^B is the change of measure from agents B 's probability measure to the risk neutralized measure and $\bar{\theta}^B$ is the share of equity with which B is initially endowed. The first-order condition for consumption equates marginal utility to $\lambda^B \xi_t^B$, where λ^B is the Lagrange multiplier of the budget constraint (19):

$$e^{-\rho t} (c_t^B)^{\alpha-1} = \lambda^B \xi_t^B. \quad (20)$$

Population A is assumed to have the same utility function (with risk aversion $1 - \alpha$ and rate of time preference ρ) and an initial share $\bar{\theta}^A = 1 - \bar{\theta}^B$ of the equity, and thus, an analogous optimization problem. The only difference is that Population A uses a probability measure that is different from that of Population B . Thus, the problem of Population A is to maximize the expected utility from lifetime consumption:

$$\sup_c \mathbb{E}^A \int_0^\infty e^{-\rho t} \frac{1}{\alpha} (c_t^A)^\alpha dt, \quad (21)$$

subject to the static budget constraint:

$$\mathbb{E}^A \int_0^\infty \xi_t^A c_t^A dt = \bar{\theta}^A \mathbb{E}^A \int_0^\infty \xi_t^A \delta_t dt, \quad (22)$$

where ξ^A is the change of measure from agents A 's probability measure to the risk neutralized measure.¹⁵

Using B 's probability measure as the reference measure, the problem of A can be restated as:

$$\sup_c \mathbb{E}^B \int_0^\infty \eta_t \times e^{-\rho t} \frac{1}{\alpha} (c_t^A)^\alpha dt, \quad (23)$$

subject to the static budget constraint:

$$\mathbb{E}^B \int_0^\infty \xi_t^B c_t^A dt = \bar{\theta}^A \mathbb{E}^B \int_0^\infty \xi_t^B \delta_t dt. \quad (24)$$

The first-order condition for consumption in this case is

$$\eta_t \times e^{-\rho t} (c_t^A)^{\alpha-1} = \lambda^A \xi_t^B, \quad (25)$$

where λ^A is the Lagrange multiplier of the budget constraint (24).

3.2 Complete-market equilibrium

An equilibrium is a price system and a pair of consumption-portfolio processes such that:¹⁶ (i) investors choose their optimal consumption-portfolio strategies, given their perceived price processes; (ii) the perceived security price processes are consistent across investors; and (iii) commodity and securities markets clear.

¹⁵ ξ^A is the density that makes prices martingales under A 's probability measure. ξ^B is the density that makes prices martingales under B 's probability measure. For any event E : $\mathbb{E}^A [\xi^A \mathbf{1}_E] = \mathbb{E}^B [\eta \xi^A \mathbf{1}_E] = \mathbb{E}^B [\xi^B \mathbf{1}_E]$ which implies: $\xi^B = \eta \xi^A$. The martingale pricing density is defined relative to each agent's probability measure. But the risk neutral measure is the same in the end.

¹⁶David (2004) says that the fluctuating difference of measure η between the two groups makes the market "effectively incomplete". That is a matter of semantics. Analytically, the equilibrium can be obtained by complete-market methods.

The aggregate resource constraint (clearing of the commodity market) is written:

$$\left(\frac{\lambda^A \xi_t^B e^{\rho t}}{\eta_t}\right)^{\frac{1}{\alpha-1}} + \left(\lambda^B \xi_t^B e^{\rho t}\right)^{\frac{1}{\alpha-1}} = \delta_t. \quad (26)$$

Hence:

$$\xi_t^B e^{\rho t} = \left[\frac{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\delta_t} \right]^{1-\alpha}, \quad (27)$$

$$c_t^A = \delta_t \times \frac{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}, \quad (28)$$

$$c_t^B = \delta_t \times \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}. \quad (29)$$

The consumption-sharing rule is linear in δ because both groups have the same risk aversion. But the slope of the linear relation (the share of consumption allocated to each group) is stochastic because of the improper use of signal by individuals of Group A. Observe that the equilibrium value of ξ^B – the martingale pricing density under B 's probability – depends on η , the probability density of A relative to B . In writing his/her budget constraint based on ξ^B , B anticipates A 's beliefs. This reflects “higher-order expectations.”

Equations (27), (28) and (29) allow us to solve for the consumption of each group's consumption and the pricing measure as a function of the current value of the dividend, δ_t , and the current value of the change of measure, η_t .

Given the constant multipliers λ^A and λ^B , and given the exogenous process for δ , and η , we have now characterized the complete-market equilibrium. It would only remain to relate the Lagrange multipliers λ^A and λ^B to the initial endowments. This requires the calculation of the wealth of each group; see Equation (36).

Since Equations (27), (28), and (29) give the pricing measure and each group's consumption as a function of the current value of the dividend, δ_t and the current value of the change of measure between the two groups, η_t , we need to carry along four state variables in the Markovian recursive formulation: $\{\delta, \eta, \hat{f}^B, \hat{g} \triangleq \hat{f}^B - \hat{f}^A\}$.¹⁷

3.3 Rate of interest and prices of risk

The rate of interest and the pricing of risk in this equilibrium are implied in the value (27) of the pricing measure. Defining, as in Cox and Huang (1989), the rate of interest r on an instantaneous maturity

¹⁷Notice that the current signals s^A and s^B are not state variables. This is because the instantaneous information they provide about the growth rate of dividends is negligible next to the cumulative information already coded into \hat{f}^A and \hat{f}^B .

deposit and the market prices of risk in the eyes of Group $i = \{A, B\}$, denoted by the vector κ^i , as the drift and the diffusion, respectively, of the risk-neutralized measure for Population i , ξ_t^i :

$$\xi_t^i = \delta_0^{\alpha-1} \exp \left(- \int_0^t r dt - \frac{1}{2} \int_0^t \|\kappa^i\|^2 dt - \int_0^t (\kappa^i)^\top dW^i \right), \quad (30)$$

one can obtain these by applying Itô's lemma to (27). The interest rate and the market prices of risk are given in the next proposition.

Proposition 1 *In equilibrium, the instantaneous interest rate is*

$$\begin{aligned} r(\eta, \hat{f}^B, \hat{g}) &= \rho + (1 - \alpha) \hat{f}^B - \frac{1}{2} (1 - \alpha) (2 - \alpha) \sigma_\delta^2 - (1 - \alpha) \hat{g} \times \frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}} \\ &\quad - \frac{1}{2} \left(\frac{\alpha}{1 - \alpha} \right) \left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2} \right) \hat{g}^2 \times \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\right]^2}, \end{aligned} \quad (31)$$

and the market prices of risk in the eyes of Population B and A are:¹⁸

$$\kappa^B(\eta, \hat{g}) = \begin{bmatrix} (1 - \alpha) \sigma_\delta \\ 0 \end{bmatrix} + \hat{g} \times \frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}} \begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}, \quad (32)$$

$$\kappa^A(\eta, \hat{g}) = \kappa^B(\eta, \hat{g}) - \hat{g} \times \begin{bmatrix} \frac{1}{\sigma_\delta} \\ \frac{1}{\sigma_s} \end{bmatrix}. \quad (33)$$

Observe that the rate of interest is an increasing function of Population B 's expected rate of growth of the dividend \hat{f}^B . The rate of interest is influenced in a nonmonotonic and asymmetric way by the difference between beliefs, \hat{g} , as it is in David (2004). This is because \hat{g} contributes both to the averages of \hat{f}^A and \hat{f}^B , and also to the difference between them.

Under agreement ($\hat{g} = 0$), the prices of risk include a reward for output risk W_δ , but zero reward for signal risk W_s . As soon as there is disagreement, both populations of investors realize that the probability measure of the other population will fluctuate randomly. Hence, they start charging a risk premium for the vagaries of others.

It is noteworthy that neither the rate of interest nor the prices of risk depend directly on the parameter ϕ measuring irrationality. They depend on it indirectly via the current value of the probability difference η and the current value of the difference of opinion \hat{g} .

3.4 Securities-market implementation of the complete-market equilibrium

There are three Brownians in the economy. However, since the growth rate f is not observed, only two of the three variables that they drive $\{\delta, f, s\}$ are observable and can be used to define "states of nature" or

¹⁸The risk-neutral measures for Populations A and B differ only in the market prices of risk. That is, the instantaneously riskless interest rate perceived by all agents is the same, and so the difference in the risk neutral measures is purely a difference in the market prices of risk perceived by the two populations.

as a basis for writing the terms of a security's contract. Correspondingly, there are only two Wiener processes of consequence: W_δ^B and W_s^B . Therefore, three non linearly dependent securities are required to implement the equilibrium.

The choice of menu is largely arbitrary. Let there be a riskless, instantaneous bank deposit with a rate of interest r . "Equity" or total wealth pays the aggregate dividend δ . We introduce also a long-term bond of fixed maturity date T . That makes three securities, two of which are instantaneously risky.

Consider then the price of a security whose flow payoff at time t is δ_t . This security is the stock and its price, denoted by F , is also the total financial wealth of the economy. Its equilibrium price, using as reference measure the measure of B , can be obtained directly from the pricing measure (27):

$$\begin{aligned} F(\delta, \eta, \hat{f}^B, \hat{g}, t) &\triangleq \frac{1}{\xi_t^B} \mathbb{E}_{\delta, \eta, \hat{f}^B, \hat{g}}^B \int_t^\infty \xi_u^B \delta_u du. \\ &= \delta^{1-\alpha} \mathbb{E}_{\delta, \eta, \hat{f}^B, \hat{g}}^B \int_t^\infty e^{-\rho(u-t)} \left[\frac{\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}} \right]^{1-\alpha} \delta_u^\alpha du. \end{aligned} \quad (34)$$

Similarly, the price of a bond maturing at date T is:

$$\begin{aligned} P(\eta, \hat{f}^B, \hat{g}, t; T) &\triangleq \frac{1}{\xi_t^B} \mathbb{E}_{\delta, \eta, \hat{f}^B, \hat{g}}^B \xi_T^B. \\ &= \delta^{1-\alpha} e^{-\rho(T-t)} \mathbb{E}_{\delta, \eta, \hat{f}^B, \hat{g}}^B \left[\frac{\left(\frac{\eta_T}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}} \right]^{1-\alpha} \delta_T^{\alpha-1}. \end{aligned} \quad (35)$$

Using the same approach as above, we can compute the wealth of each individual agent. For instance, the price of a "security" whose flow payoff at time t is the consumption of group B investors given by (29) is:

$$\begin{aligned} F^B(\delta, \eta, \hat{f}^B, \hat{g}, t) &\triangleq \frac{1}{\xi_t^B} \mathbb{E}_{\delta, \eta, \hat{f}^B, \hat{g}}^B \int_t^\infty \xi_u^B c_u^B du \\ &= \frac{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\left[\frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\delta} \right]^{1-\alpha}} \mathbb{E}_{\delta, \eta, \hat{f}^B, \hat{g}}^B \int_t^\infty e^{-\rho(u-t)} \left[\frac{\left(\frac{\eta_u}{\lambda^A}\right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}}}{\delta_u} \right]^{-\alpha} du. \end{aligned} \quad (36)$$

We see that, to compute the expected values in (34), (35) and (36), we need the joint conditional distribution of η_u and δ_u , given $\delta_t, \eta_t, \hat{f}_t^A, \hat{g}_t$ at t . That joint distribution is not easy to obtain but its characteristic function $\mathbb{E}_{\delta, \eta, \hat{f}^B, \hat{g}}^B [(\delta_u)^\varepsilon (\eta_u)^\chi]; \varepsilon, \chi \in \mathbb{C}$ can be obtained in closed form. Guided by the functional form of the coefficients of the associated partial differential equation and following Heston (1993) and Kim and Omberg (1996) and undoubtedly others, we show in Appendix B that:

$$H(\delta, \eta, \hat{f}^B, \hat{g}, t; u, \varepsilon, \chi) \triangleq \mathbb{E}_{\delta, \eta, \hat{f}^B, \hat{g}}^B [(\delta_u)^\varepsilon (\eta_u)^\chi] = \delta^\varepsilon \eta^\chi \times H_f(\hat{f}^B, t, u; \varepsilon) \times H_g(\hat{g}, t, u; \varepsilon, \chi), \quad (37)$$

where functions $H_f(\hat{f}^B, t, u; \varepsilon) \triangleq \mathbb{E}_{\hat{f}^B}^B [(\delta_u)^\varepsilon]$ and H_g are defined explicitly by (B2) and (B5).¹⁹ Then, Appendix C provides two alternative methods to write the functions F , P and F^B explicitly.²⁰

4 The effect of irrationality on asset prices, return volatilities and risk premia

In this section, we study the effect of irrationality on the volatility of asset prices, and the risk premia. These are easily obtained by straightforward applications of Itô's lemma to the explicit expressions for the value of the stock market F and the value of bonds P and their derivatives. Generally, the diffusion vector of a security price is equal to the gradient of the price function premultiplying the diffusion matrix of state variables.

In order to illustrate the effect of irrationality on securities prices, we specify numerical values for the parameters of the model. Even though our objective is not to match the magnitude of particular moments in the data, we would like to work with parameter values that are reasonable. The parameter values that we specify are based on the estimation of models similar to ours undertaken in Brennan and Xia (2001) and Berrada (2004). In addition, we have set the volatility of the signal σ_s equal to 0.1. The particular values chosen are listed in Table 1.²¹

The dynamics of equilibrium prices are evidently driven by the state variables $\delta, \eta, \hat{f}^B, \hat{g}, t$. Among them, δ is only a scale variable multiplying the total value of the stock market and the wealth of each population and not affecting at all the price of a bond. The state variable η always appears in the ratio $\frac{\lambda^B \eta}{\lambda^A}$, incorporating Lagrange multipliers and the current probability measure difference between the two populations. It captures the relative Negishi weights of the two populations, with $\frac{\lambda^B}{\lambda^A}$ representing the initial (time-0) weights and the initial distribution of wealth and η representing the changes in the weights that have occurred as a result of the gains and losses accumulated by the irrational population because of its learning mistakes. When we do not vary that ratio, we set it equal to 1 to represent the situation in which they currently have roughly similar sizes.²² But we also vary the relative weights of the two populations because we wish to know how many rational investors are needed for the market to behave almost as it would under full rationality.

Each plot in the figures has two curves on it, with the dotted line representing the case where $\phi = 0$ and all agents are rational, and the dashed line representing the case where Population A is irrational, which corresponds to $\phi = 0.95$.

¹⁹Appendix B also contains technical conditions for the function H_g to be well defined.

²⁰As far as pricing is concerned, the limiting case in which $\theta^A \rightarrow 1$ and $\theta^B \rightarrow 0$ or vice versa (so that one population dominates the market), has some similarities with the model of a homogeneous-agent economy in Brennan and Xia (2001). The closeness of the Brennan and Xia model to these limiting cases means that their model serves as a useful pricing benchmark.

²¹Observe that the range of parameter values that can be considered is restricted by the need to satisfy the condition for the existence of a solution; this limits, in particular, the range of values for risk aversion that can be considered. Because of this constraint, the risk aversion we consider is somewhat too low to account for the equity premium (see below).

²²If we vary the parameter ϕ , we adjust this ratio in such a way that the time-0 lifetime budget constraints of the two populations still hold, with unchanged time-0 endowments of securities.

4.1 Average belief vs. dispersion of beliefs

In addition to δ and η , the two other state variables are the beliefs of Population B , \widehat{f}^B , and the difference of beliefs, \widehat{g} .²³ However, as we have seen, \widehat{g} contributes both to the average of \widehat{f}^A and \widehat{f}^B and also to the difference between them. For purposes of interpretation and exposition, it is clearer to define \widehat{f}^M , the population average belief about the expected rate of growth (where the weights are each population's share of consumption):

$$\widehat{f}^M \triangleq \frac{\widehat{f}^B \times \left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \widehat{f}^A \times \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}} = \widehat{f}^B - \widehat{g} \times \frac{\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}. \quad (38)$$

The rate of interest can be written:

$$\begin{aligned} r(\eta, \widehat{f}^B, \widehat{g}) &= \rho + (1-\alpha)\widehat{f}^M - \frac{1}{2}(1-\alpha)(2-\alpha)\sigma_\delta^2 \\ &\quad - \frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right)\widehat{g}^2 \times \frac{\left(\frac{1}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{1}{\lambda^B}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta}{\lambda^A}\right)^{\frac{1}{1-\alpha}}\right]^2}, \end{aligned} \quad (39)$$

and the effect of \widehat{g} , which appears in the last term is now purely quadratic and symmetric. In this way, \widehat{g} represents the effect of pure *dispersion* of beliefs in the population. If risk aversion is greater than 1, $\alpha < 0$, dispersion of beliefs increases the rate of interest above what it would be under homogeneous beliefs.²⁴

It is also conceivable to recognize average beliefs \widehat{f}^M and dispersion of beliefs \widehat{g} as the two drivers for the prices of risky securities. It is unfortunately not possible to define a concept of ‘‘average belief’’ in a manner that would be valid for all assets, specifically for assets of all maturities. The way in which beliefs compound over time and get discounted into prices via marginal utility would imply a different concept of average beliefs at different maturities.²⁵ The average beliefs \widehat{f}^M that we have defined in (38) applies only to the rate of interest, which is an instantaneous-maturity asset. Nonetheless, we use that concept below as a convenient, albeit only an approximate, interpretation device.

For these reasons, we wish to introduce a first change of state variables from $\{\delta, \eta, \widehat{f}^B, \widehat{g}\}$ to $\{\delta, \eta, \widehat{f}^M, \widehat{g}\}$. Moreover, while the exogenous state variables driving the equilibrium are undoubtedly $\{\delta, \eta, \widehat{f}^B, \widehat{g}\}$, it is also true that, from the point of view of Investor B , the only state variables he/she considers in his/her consumption-portfolio decision are $\{\delta, \lambda^B e^{\rho t} \xi^B, \widehat{f}^B$ or $\widehat{f}^M, \widehat{g}\}$.²⁶ Hence, as a way

²³The benchmark values of these state variables are set at: $\widehat{f}_0^B = \bar{f}$; $\widehat{g}_0 = 0$. The numerical values chosen are such that, under irrationality, \widehat{g} is positively correlated with δ but negatively correlated with \widehat{f}^B .

²⁴David (2004) assumed a risk aversion lower than 1, precisely in order to bring down the rate of interest. See our numerical illustrations below. Had we assumed a lifetime utility of the recursive, Epstein-Zin type, we could have distinguished risk aversion from elasticity of intertemporal substitution. It is likely that the condition for the rate of interest to be reduced (increased) by dispersion of beliefs would have hinged on the elasticity of substitution being higher (lower) than 1, not on the level of risk aversion.

²⁵This is related to the observation made by Allen, Morris, and Shin (2003) that, under risk aversion, market-average beliefs do not satisfy the law of iterated expectations.

²⁶See Equations (18), (19) and (20) above.

to prepare the ground for B 's portfolio decisions, we define:²⁷

$$\tilde{\xi}^B \triangleq \lambda^B e^{\rho t} \xi^B = \left[\frac{\left(\frac{\lambda^B \eta}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + 1}{\delta_t} \right]^{1-\alpha}, \quad (40)$$

and new pricing functions:

$$\tilde{F}(\delta, \tilde{\xi}^B, \hat{f}^M, \hat{g}, t) \triangleq F(\delta, \eta, \hat{f}^B, \hat{g}, t), \quad (41)$$

$$\tilde{P}(\delta, \tilde{\xi}^B, \hat{f}^M, \hat{g}, T, t) \triangleq P(\eta, \hat{f}^B, \hat{g}, T, t), \quad (42)$$

$$\tilde{F}^B(\delta, \tilde{\xi}^B, \hat{f}^M, \hat{g}, t) \triangleq F^B(\delta, \eta, \hat{f}^B, \hat{g}, t). \quad (43)$$

4.2 Prices

As has become clear from the previous subsection, there are two aspects to the dynamics of equilibrium prices, $\tilde{F}(\delta, \tilde{\xi}^B, \hat{f}^M, \hat{g}, t)$ and $\tilde{P}(\delta, \tilde{\xi}^B, \hat{f}^M, \hat{g}, T, t)$, under diversity of beliefs and irrationality. First, the rational group of investors may not agree today with the irrational ones about its estimate of the *current* rate of growth of dividend: $\hat{g} \neq 0$. Second, even when the two groups of investors happen to agree today ($\hat{g} = 0$), all investors know that they will revise their *future* estimates of the growth rate. In particular, the rational investors are conscious of the fact that irrational investors will revise their estimate in a manner that differs from his/hers, so that they know that they will not agree tomorrow.

Figure 1 plots the interest rate, r , the price of equity, F , and the bond price, P , against the four state variables. It illustrates that, in all cases, price levels are reduced by the presence of irrational traders (compare levels of dashed curves to those of dotted curves). This is because irrational traders add “noise” for which all traders require a risk premium.

Holding \hat{g} at zero ($\hat{f}^B = \hat{f}^A = \hat{f}^M$), we vary average belief \hat{f}^M . In this economy, the price of bonds decreases with an increase in B 's expectation of future growth of either population (see the graph in the third row and third column of Figure 1). In other words, the yields of bonds increase with the increase in the short rate, which arises from higher expected growth. This is because higher growth of dividends implies lower marginal utility of future consumption. In the case of equity (see the graph in the third column of the second row of Figure 1), the same effect is present and, when risk aversion is greater than 1 (as is our case), that effect dominates the effect of increased future dividends. The ratio of the price of equity to current dividends also drops with an increase in average belief of future growth. That is one way in which our model differs from that of Brennan and Xia (2001). When Brennan and Xia increase their investors' estimate of expected dividend growth, they keep constant their investors' estimate of expected aggregate consumption growth. The ratio of the price of equity to current dividend rises. When increasing their investors' estimate of expected aggregate consumption while keeping expected

²⁷ $(\tilde{\xi}^B)^{\frac{1}{\alpha-1}} = c^B \leq \delta$.

dividend growth constant, the price of equity drops. In our model, consumption equals dividend. When investors become more optimistic about future dividend growth, the price-dividend ratio drops – the second effect dominates – if risk aversion is greater than 1.

Varying now the dispersion of beliefs (fourth column of Figure 1), we observe that the price functions for equity and the bond are symmetric under rational, that is, pure Bayesian, learning (dotted line), where disagreement can arise only from differences in priors. However, they are no longer symmetric when Population A is irrational. As mentioned earlier, this is because our concept of dispersion of beliefs \hat{g} is not strictly applicable to all maturities.

Next, we vary the relative weight of the two populations. This is shown in the second column of Figure 1, where we have not placed on the x -axis the variable $\tilde{\xi}^B$ itself but $\left(\tilde{\xi}^B\right)^{\frac{1}{\alpha-1}} = c^B$, the consumption of Population B .²⁸ When the weight of the rational Population B is very high, the prices of all securities are at their highest level, which is, of course, equal to the level reached when both populations are rational. As the weight of the irrational population increases, all prices drop. The price of the bond drops monotonically, with a graph that has positive curvature. Total wealth (that is, the stock price), which at first drops, ultimately rises again to reach a level when Population A is alone (100% consumption share), which is below the level we had when Population B was alone.

These were variations against the current value of ξ (or η) but, of course, they incorporate the future evolution of both variables. As we saw, this reflects the fact that investors of each group forecast the expectations of the investors of the other group. The prospect of others becoming more or less optimistic in the future than they are today plays a role in today's valuation. Harrison and Kreps (1978) define “speculative behavior” as follows:

“We say that investors exhibit *speculative behavior* if the right to resell a stock makes them willing to pay more for it than they would if obliged to hold it forever.”

Harrison and Kreps provide an example of a speculative-behavior equilibrium in which investors are risk neutral and are subject to a no-short-sale constraint. When a stock is overvalued in their eyes, they cannot sell it short. They go along with the overvaluation and hold the stock positively because there is a chance that a category of investors will grow to be very optimistic in the future and will want to buy the stock. Scheinkman and Xiong (2003) is an example of a dynamic equilibrium of that type. The option to resell adds value to the stock over and above the value it has in the eyes of the current owners. So, the stock has a “fundamental value” equal to what the current owners think of it and the rest is speculative value. Their definition is applicable because, in the risk neutral setting, only one category of investors holds the stock at any one time.

In our setting, investors are risk averse. Both categories of investors generally hold the stock simultaneously. Can we say still that speculative behavior increases today's valuation over and above what it would be if each category of investors had to value the anticipated dividend stream based on

²⁸This is just a change of scale to ease the display of the graph. The relationship is monotonically decreasing.

its own expectations process? The key issue is the proper definition of fundamental value. We could think of defining it by means of a mental experiment in which we would freeze η at its current level. It is easy to show, by substitution into Equations (34), (35) and (36), that all prices would then be equal to those prevailing when Population B is alone.²⁹ As we know, these represent the highest levels of prices for any value of η . While the option to resell to (or to repurchase from) fools (Population A), no doubt, contributes to the utility level of Population B , it only reduces the market values of equity and bonds and the wealth of both groups. This is because the option is “exercised” to maximize utility, not to maximize value. The two goals coincide only under risk neutrality. It is seen, therefore, that the definition and the results of Harrison and Kreps apply only to a very special case.

4.3 Return volatility and correlation

As we saw, the *levels* of security prices are reduced when the “irrationality parameter” ϕ is taken from $\phi = 0$ to $\phi = .95$. However, the main effect of irrationality of Group A on the *volatility* of asset prices arises from the greatly increased volatility of \hat{g} and, because of it, also from the volatility of η (or $\tilde{\xi}^B$). The diffusion of the securities’ prices is equal to the gradient of each price function premultiplying the diffusion of state variables.

Figure 2 plots the volatility of stock returns, the volatility of bond returns, and the correlation between stock and bond returns. Each quantity is plotted against four variables: the level of aggregate endowment (dividends), δ , the share of aggregate endowment consumed by Group B , c^B , which indicates the relative weight of the two populations, average beliefs, \hat{f}^M and dispersion in beliefs, \hat{g} .

From Figure 2, we see that irrational investors create “noise” that increases the volatility of both risky assets—the stock and the bond. The volatility of equity returns increases from about 18%, under rationality and agreement ($\hat{g} = 0$), to about 25% with irrationality and agreement. It increases without bounds when there is disagreement ($\hat{g} \neq 0$). The volatility of bond returns increases from 26% to 30%, respectively, and then also without bounds. The values produced by the model for the volatility of bond returns (and interest rates) are regrettably too high to fit real-world data.³⁰

The plots in the last row of Figure 2 show that the presence of overconfident investors, as well as the disagreement between investor groups, increase the correlation between stock and bond returns because, as we pointed out, the prices of the two assets move in the same direction when expectations fluctuate. The correlation goes from 0.84 under rationality to 0.86 under irrationality but agreement ($\hat{g} = 0$) and it approaches 1 as disagreement is introduced ($\hat{g} \neq 0$).

To determine how the level of volatility in the market varies with the relative weight of Groups A and B , we plot in the second column of Figure 2 the volatility of the stock market return, the bond return and their correlation as a function of Population B ’s consumption. This shows that the excess

²⁹That is only because we have used the probability measure of Group B as reference measure. Had we done the reverse, we would have found fundamental values equal to those prevailing when Population A is alone. The fundamental value is in the eyes of the beholder.

³⁰With a risk aversion smaller than 1, David (2004) was able to match the volatility of interest rates much better. Alternatively, if one wanted to match interest-rate volatility, one could introduce habit formation.

volatility in the market increases with a decrease in the relative weight of Population B . That is, when the relative weight of Population B is 100%, the volatility of the stock market is 18%. That is, of course, the same volatility as in the benchmark case with no irrationality ($\phi = 0$). As the relative wealth of Population B decreases, the volatility of stock returns also increases toward 35%, which is the volatility level when all agents are irrational. This plot shows that it is not enough to have just a few rational investors to get the volatility down to the level warranted by fundamentals.

5 The optimal portfolio of a rational investor B

In this section, we first study the wealth of the investor and then analyze the portfolio strategy of the rational investor.

5.1 The wealth of Population B

Figure 3 illustrates the variations of the wealth of Population B relative to the four state variables. As was the case for total wealth, or equity, the effect of a rise in average anticipated growth (\hat{f}^M) on the wealth of Population B is to decrease wealth; this is illustrated in the third plot of Figure 3. To prepare for the forthcoming section on portfolio choice, let us note that this means that a positive random shock in \hat{f}^M is “good news” for Population B (as well as for Population A).³¹ That is, when making up its portfolio, Population B will seek to protect itself against future *negative* shocks to \hat{f}^M .

Not surprisingly given that we have defined $\hat{g} \triangleq \hat{f}^B - \hat{f}^A$, implying that $\hat{g} > 0$ represents a situation in which Population B is more optimistic than the average about future growth, the wealth of Group B is a decreasing function of \hat{g} . A positive random shock in \hat{g} is also “good news” for Population B (but not for Population A). That is, when making up its portfolio, Population B will seek to protect itself against future *negative* shocks to \hat{g} .

The second plot of Figure 3 shows that varying $\tilde{\xi}^B$, and with it the relative weight of the two populations, the wealth of Population B rises monotonically with its equilibrium consumption c^B , with a graph that has positive curvature.

The various plots reveal the signs of the gradient of Population B 's wealth with respect to the four state variables.

³¹Denote by $J(F^B, Y)$ the value function of the investor's lifetime utility under a Merton-like dynamic-programming formulation of the investor's optimization problem, where F^B denotes the investor's wealth and Y any state variable. Then one can show, by applying the Cox-Huang transformation, that

$$F_Y^B = -\frac{J_{F^B Y}}{J_{F^B F^B}}. \quad (44)$$

From the concavity of the utility function, we know that $J_{F^B F^B} < 0$ so that F_Y^B has the same sign as $J_{F^B Y}$. Merton (1971) defines as “good news” a positive shock to a state variable Y such that $J_{F^B Y} < 0$, because that implies that a positive shock to Y induces a decrease in marginal utility and a rise in current consumption. Lower current financial wealth is associated with higher current consumption.

5.2 The portfolio composition of Population B

To solve the portfolio-choice problem of Agent B , we apply the risk-sensitivity method³² of Cox and Huang (1989) to the expression for the wealth of Investor B , which is given in (36). All that is needed for this purpose are the price functions $\tilde{F}(\delta, \tilde{\xi}^B, \hat{f}^M, \hat{g}, t)$, $\tilde{P}(\delta, \tilde{\xi}^B, \hat{f}^M, \hat{g}, T, t)$, $\tilde{F}^B(\delta, \tilde{\xi}^B, \hat{f}^M, \hat{g}, t)$ and the 4×2 diffusion matrix of the state variables:

$$\begin{bmatrix} \theta_F^B & \theta_P^B \end{bmatrix} \times \begin{bmatrix} \text{gradient of } \tilde{F} \\ \text{gradient of } \tilde{P} \end{bmatrix} \times \begin{bmatrix} \text{diffusion} & \\ \text{matrix} & \text{of} \\ \text{four} & \text{state} \\ \text{varia-} & \text{bles} \end{bmatrix} = \begin{bmatrix} \text{gradient of } \tilde{F}^B \end{bmatrix} \times \begin{bmatrix} \text{diffusion} & \\ \text{matrix} & \text{of} \\ \text{four} & \text{state} \\ \text{varia-} & \text{bles} \end{bmatrix}$$

In this way, we can obtain directly the total portfolio $(\Theta^B)^\top = [\theta_F^B \quad \theta_P^B]$ of investor B . These are the fractions of the outstanding total supply of securities held by Investor B .³³

Figure 4 gives Population B 's total portfolio holding (expressed as a percentage of wealth). There are two plots, with the one on the left giving the position in equity and the one on the right giving the position in bonds. In both the plots the variable on the x -axis is the dispersion in beliefs, \hat{g} . In the case of rationality of Population A and agreement ($\hat{g} = 0$), there is nothing to distinguish the two populations. Hence both are 100% invested in equity and 0% in bonds. Continuing with the case of rationality (dotted line) but introducing some current disagreement, we see that Population B continues to be 100% invested in equity but, when it is optimistic ($\hat{g} > 0$), it shorts the bond (and invests the proceeds in the short-term deposit). This is because there is then no uncertainty about future estimates of expected growth; the way in which these will differ is deterministic.

In the case of irrationality (dashed line), the plots in the first row of Figure 4 show that the rational investor B holds a number of shares of equity that is smaller than it would be in a rational market, unless he/she is extremely optimistic. Because the two risky securities are positively correlated (see Figure 2), the investor compensates the smaller weight in equity with a slightly larger weight placed on bonds.

To better understand these results, we now decompose the portfolio according to motives. Figure 5 gives Population B 's total portfolio expressed as the sum of four components. There are two rows of plots: the first row gives the investment in equity and the second the investment in bonds. There are four columns of plots in each row. The first column gives the static (mean-variance or myopic) investment in stocks and in bonds; the second column gives the investment in stocks and bonds in order to hedge against changes in δ ; the third column gives the investment in stocks and bonds in order to hedge against changes in average beliefs, \hat{f}^M ; and, the last column gives the investment in stocks and bonds that serve as a hedge against changes in the dispersion in beliefs, \hat{g} . In all the plots the variable on the x -axis is the dispersion in beliefs, \hat{g} . We describe each of these components below.

The first component of the overall portfolio is the static (i.e., Markowitz) portfolio, which is based only on current expected stock returns and risk. As Cox and Huang have shown, this portfolio is

³²That method is based on the ‘‘martingale representation theorem’’.

³³Over the two populations, these fractions sum to 1 for equity and to 0 for the bond so that the equilibrium holdings of Population A follow immediately from those of B .

equivalent to the portfolio that hedges pricing-measure or $\tilde{\xi}^B$ risk.³⁴ The second portfolio component hedges the investor against the part of dividend risk that is not already impounded in the pricing measure. That component could equivalently be viewed as a residual hedge against changes in η , the change of probability measure that captures the cumulative disagreement between the two populations. The third portfolio hedges the investor against future revisions in the market's average estimate of expected dividend growth (\hat{f}^M). The fourth and final component hedges him/her against future changes in the dispersion of beliefs (\hat{g}). Each of these four component corresponds to an element of the gradient of \tilde{F}^B .

Proposition 2 *The portfolio holdings of Investor B in the stock and the bond are given by:*

$$\begin{bmatrix} \theta_F^B & \theta_P^B \end{bmatrix} = \begin{bmatrix} \text{gradient of } \tilde{F}^B \end{bmatrix} \times \begin{bmatrix} \text{diffusion} \\ \text{matrix} & \text{of} \\ \text{four} & \text{state} \\ \text{varia-} & \text{bles} \end{bmatrix} \times \Sigma^{-1},$$

where

$$\Sigma \triangleq \begin{bmatrix} \text{gradient of } \tilde{F} \\ \text{gradient of } \tilde{P} \end{bmatrix} \times \begin{bmatrix} \text{diffusion} \\ \text{matrix} & \text{of} \\ \text{four} & \text{state} \\ \text{varia-} & \text{bles} \end{bmatrix}$$

is the diffusion matrix of the risky securities' dollars returns.

We now discuss the variations in the four components. But first, we take a brief look at the case of rational behavior, which will serve as a benchmark.

We have already mentioned the behavior of the system under rationality. When $\phi = 0$, the dispersion of beliefs \hat{g} , if nonzero because of differences of priors, is non-stochastic and the cumulative probability ratio η fluctuates randomly with a diffusion dictated by the value of \hat{g} . As a result, B 's portfolio components are very simple. For the reason we just gave, the hedge against changes in \hat{g} (last column of Figure 5, dotted lines) is always zero while the hedge against δ (or η) risk (second column) is zero when $\hat{g} = 0$ and changes sign with \hat{g} . The hedge against revisions in the average estimate of expected dividend growth (\hat{f}^M) contains no equity (see third column). It is entirely made up of the bond because the bond (see Figure 1) is exposed to \hat{f}^M risk exclusively³⁵ whereas equity is exposed to both \hat{f}^M risk and δ risk. As we saw, an increase in \hat{f}^M is "good news". The hedge contains the bond positively because a rise in \hat{f}^M will cause the price of the bond to drop, thereby generating an offsetting negative return. In the static portfolio, the equity weight hovers around 100% depending on the current optimism or pessimism of B , while the bond weight is negative. Since there is positive demand for the bond as a \hat{f}^M hedge, the hedging pressure brings down the equilibrium expected return and causes a negative holding as part of the static portfolio.

Under irrationality, the static portfolio (first column of Figure 5, dashed lines) puts a larger weight on equity and a smaller weight on the bond, than under rationality. This is made possible by the

³⁴It for that reason that we found it useful to introduce the change of variable and unknown function (43).

³⁵To be precise, it is also slightly sensitive to \hat{g} when $\hat{g} \neq 0$, but there is then no \hat{g} risk.

increased correlation between equity and bond return (recall Figure 2). The phenomenon is amplified when Population B is optimistic.

Although we have not mentioned it, an increase in δ reduces the value of B 's wealth and raises and is “good news”.³⁶ Figure 2 has revealed that an increase in δ increases the value of equity but reduces the value of the bond. The hedge is, therefore, to hold equity negatively and to hold the bond positively.³⁷

As we verify in the third column of Figure 5, the irrationality of Population A , which increases the volatility of average expectations, amplifies the bond's role – already identified in the case of rationality – as a device used by both populations to hedge future changes in \hat{f}^M . Equity held for the purpose remains almost equal to zero.

If the market has irrational agents while investor B is rational, even when investors happen to agree today about the expected rate of growth of dividends, they know that they will almost certainly disagree tomorrow. The fourth portfolio component is made of B 's equity and bond shares held for purposes of hedging future disagreement between himself/herself and the market. This component of the portfolio is displayed in the last column of Figure 5. The intuitive reason for which the hedge against future disagreement consists of a negative rather than a positive position in the equity and bond is as follows. We found that $F_{\hat{g}}^B < 0$ implying that an increase in \hat{g} is “good news”. In a neighborhood of $\hat{g} = 0$, the prices of the equity and the bond both decrease with an increase of \hat{g} . Recall, however, from the diffusion matrix (17) that the reactions of \hat{g} to the two shocks W_{δ}^B and W_s^B are of opposite signs. The hedge is a play on that sign difference in the diffusion coefficients.

We have demonstrated that a rational risk-arbitrageur finds it beneficial to trade on his/her belief that the market is being foolish. When doing so, however, he/she must hedge future fluctuations in the market's foolishness. This illustrates the general idea that risk arbitrage cannot just be based on a current price divergence. It must also be based on a model of irrational behavior and a prediction concerning the speed of convergence. The risk arbitrage must include a protection in case there is a deviation from that prediction, a form of risk that David (2004) has called “trading risk”. We have found that, for the current form of heterogeneous beliefs, bonds are an essential accompaniment to hedge the trading risk of equity investment.

³⁶For a fixed $\tilde{\xi}^B$, an increase in δ is also an increase in η (see Equation (40)).

³⁷Note that, while $F_{\hat{g}}^B > 0$ when evaluated at $\hat{g} = 0$, it may not be positive for all values of \hat{g} . The reason for this is as follows. F^B is the present value of agent B 's future consumption, and future consumption is a function of future δ and η , where \hat{g} is the volatility of future η . If the covariance between δ and η were equal to zero, then the sign of $F_{\hat{g}}^B$ would be determined only by the curvature of c^B with respect to η (which is negative because $\alpha < 0$) and by the fact that the sign of the diffusion term of η would not matter. In other words, the function F^B would be symmetric, looking like a parabola with a maximum attained at $\hat{g} = 0$. However, the fact that $\text{cov}(\delta, \eta)$ is not equal to zero “shifts” this “parabola” to the right, so that $F_{\hat{g}}^B > 0$ when $\hat{g} = 0$.

6 Vindication: Profits of rational investors *vs.* survival of overconfident agents

We now return to the question we asked originally concerning the potential for gains that the excessive volatility creates for the rational investors who follow the portfolio strategy that we described. By asking whether rational risk arbitrageurs can take advantage of overconfident investors, we simultaneously ask whether they eliminate them from the economy very quickly, or whether overconfident investors can survive for some time. Survival of irrational traders is an issue that is the focus of recent papers by Berrada (2004), Kogan, Ross, Wang, and Westerfield (2003) and Yan (2004). Here, however, we consider a different kind of irrational agents, who change their mind too frequently.

One way to measure the survival of irrational agents in the economy is to study the evolution of the expected value of the share of total dividend that will be consumed by them under the objective probability measure.

The expected value of Population A's consumption-dividend ratio under the objective probability measure is:

$$\mathbb{E}^P \left[\frac{c_u^A}{\delta_u} \right] = \mathbb{E}^P \left(\frac{\lambda^B}{\lambda^A} \eta_u \right)^{\frac{1}{1-\alpha}} \left[1 + \left(\frac{\lambda^B}{\lambda^A} \eta_u \right)^{\frac{1}{1-\alpha}} \right]^{-1}. \quad (45)$$

To compute this expectation, we need conditional distribution of η_u , given $\eta_t, f_t, \hat{f}_t^A, \hat{f}_t^B$ at t . As in the previous section, we first obtain its characteristic function:

$$\mathbb{E}_{\eta, \hat{g}^A, \hat{g}^B}^P [\eta_u]^X = \eta^X \times H_P (\hat{g}^A, \hat{g}^B, t; \chi; u), \quad (46)$$

where $\hat{g}^A \triangleq \hat{f}^A - f$, $\hat{g}^B \triangleq \hat{f}^B - f$, and

$$\begin{aligned} H_P (\hat{g}^A, \hat{g}^B, t; \chi; u) &= \exp \left\{ A_P (\chi; u - t) + C^A (\chi; u - t) \times (\hat{g}^A)^2 + C^B (\chi; u - t) \times (\hat{g}^B)^2 \right. \\ &\quad \left. + 2C^{AB} (\chi; u - t) \times \hat{g}^A \times \hat{g}^B \right\}, \end{aligned} \quad (47)$$

for certain functions of time A_P, C^A, C^B , and C^{AB} that are given in Appendix C.

Then, by Fourier inversion, survival denoted by $S (\eta, \hat{g}^A, \hat{g}^B, t; u)$, is:

$$\begin{aligned} S (\eta, \hat{g}^A, \hat{g}^B, t; u) &\triangleq \mathbb{E}_{\eta, \hat{g}^A, \hat{g}^B}^P \left[\frac{c_u^A}{\delta_u} \right] \\ &= \int_0^\infty \left(\frac{\lambda^B}{\lambda^A} \eta_u \right)^{\frac{1}{1-\alpha}} \left[1 + \left(\frac{\lambda^B}{\lambda^A} \eta_u \right)^{\frac{1}{1-\alpha}} \right]^{-1} \\ &\quad \times \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\eta_u}{\eta} \right)^{-i\chi} H_P (\hat{g}^A, \hat{g}^B, t; i\chi; u) d\chi \right] \frac{d\eta_u}{\eta_u}. \end{aligned} \quad (48)$$

Figure 6 illustrates the case where irrational agents ($\phi = 0.95$) start out having 50% of the total wealth. We plot against future dates the expected percentage of the total dividends consumed by

Group A. The first conclusion is that, ultimately, irrational agents become extinct. But, the more interesting observation is that, in contrast to what is typically assumed in models of asset pricing, irrational agents do not lose their wealth right away. For instance, even after 200 years the overconfident agents consume 25% of the aggregate dividends.

7 Conclusion

Assuming that there is excess volatility in capital markets, our objective was to analyze the trading strategy that would allow an investor to take advantage of this excess volatility. To achieve our goal, we first constructed a general equilibrium model where stock prices are excessively volatile using the same device as in Scheinkman and Xiong (2003). That is, there are two classes of agents, and one class (irrational or overconfident) believes that the magnitude of the correlation between the innovations in the signal and innovations in the unobserved variable, the growth rate of dividends, is larger than it is actually. Consequently, when a signal is received, this class of agents overreacts to it, which then generates excessive stock price movements. We then analyzed the trading strategy of the rational investors who knows that the true magnitude of this correlation is zero.

Our analysis shows that the portfolio of rational investors consists of four components: a static (i.e., Markowitz) portfolio based only on *current* expected stock returns and risk (this is also a hedge against changes in the pricing measure), a portfolio that hedges the investor against the future changes in output that are not impounded in the change of the pricing measure, a portfolio that hedges the investor against *future* revisions in the market's expected dividend growth, and a portfolio that hedges against future *disagreement* in revisions of expected dividend growth. There are two aspects to the portfolio strategy of such investors: First, these investors may not agree today with the market about its current estimate of the growth rate of dividends: when the rational investors are more optimistic than the market, they increase their investment in equity. Second, even when the two groups of investors happen to agree today, rational investors are aware that irrational investors will revise their estimate in a manner that differs from theirs. This second effect makes the rational investor hold fewer shares of equity than would be optimal in a market without excess volatility and causes him/her to take a negative position in bonds (which would be zero in the absence of excess volatility).

In short, we find that rational risk-arbitrageurs find it beneficial to trade on their belief that the market is being foolish but when doing so, they must hedge future fluctuations in the market's foolishness. Thus, our analysis illustrates that risk arbitrage cannot be based just on a current price divergence; it must also be based on a model of irrational behavior and a prediction concerning the speed of convergence, and that the risk arbitrage must include a protection in case there is a deviation from that prediction.

We also find that, in contrast to what is typically assumed in standard models of asset pricing in frictionless markets, the presence of a few rational traders is not sufficient to eliminate the effect of overconfident investors on excess volatility and that overconfident investors may survive for a long time before being driven out of the market by rational investors.

A Proofs for propositions

Proof of Proposition 1

Taking into account (7) and (12) and applying Itô's lemma, we get:

$$\frac{d\delta_t^{\alpha-1}}{\delta_t^{\alpha-1}} = -(1-\alpha) \left(\widehat{f}_t^B dt + \sigma_\delta dW_{\delta,t}^B \right) + \frac{1}{2} (1-\alpha) (2-\alpha) \sigma_\delta^2 dt, \quad (\text{A1})$$

$$\frac{d \left(\frac{\eta_t}{\lambda^B} \right)^{\frac{1}{1-\alpha}}}{\left(\frac{\eta_t}{\lambda^B} \right)^{\frac{1}{1-\alpha}}} = \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \nu_t^\top \nu_t dt - \frac{1}{1-\alpha} \nu_t^\top dW_t^A, \quad (\text{A2})$$

$$\begin{aligned} \frac{d \left[\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}}{\left[\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}} &= \frac{1}{2} \frac{\alpha}{1-\alpha} \frac{\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}} \right]^2} \nu_t^\top \nu_t dt \\ &\quad - \frac{\left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}} \nu_t^\top dW_t^A. \end{aligned} \quad (\text{A3})$$

Then, from (27):

$$\begin{aligned} \frac{d\xi_t^B}{\xi_t^B} &= -\rho dt - (1-\alpha) \left(\widehat{f}_t^B dt + \sigma_\delta dW_{\delta,t}^A \right) + \frac{1}{2} (1-\alpha) (2-\alpha) \sigma_\delta^2 dt + \\ &\quad \frac{1}{2} \frac{\alpha}{1-\alpha} \frac{\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}} \right]^2} \nu_t^\top \nu_t dt - \frac{\left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}} \nu_t^\top dW_t^A \\ &\quad + (1-\alpha) \sigma_\delta \frac{\left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}}{\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} + \left(\frac{\eta_t}{\lambda^A} \right)^{\frac{1}{1-\alpha}}} \nu_t^\top \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt. \end{aligned} \quad (\text{A4})$$

Collecting drift and diffusion terms in $\frac{d\xi_t^A}{\xi_t^A}$ and taking into account (30), we obtain (31) and (32).

As η is a change from A 's measure to B 's measure, $\xi^B = \xi^A \times \eta$. Consequently, $\kappa^B = \kappa^A + \nu$ and we finally get (33).

B The characteristic function

We want to compute

$$H \left(\delta, \eta, \widehat{f}^B, \widehat{g}, t, u; \varepsilon, \chi \right) \triangleq \mathbb{E}_{\delta, \eta, \widehat{f}^B, \widehat{g}}^B \left[(\delta_u)^\varepsilon (\eta_u)^\chi \right].$$

This function satisfies the linear PDE:

$$0 \equiv \mathcal{D}H \left(\delta, \eta, \widehat{f}^B, \widehat{g}, t, u; \varepsilon, \chi \right) + \frac{\partial H}{\partial t} \left(\delta, \eta, \widehat{f}^B, \widehat{g}, t, u; \varepsilon, \chi \right), \quad (\text{B1})$$

with the initial condition $H(\delta, \eta, \widehat{f}^B, \widehat{g}, t, t; \varepsilon, \chi) = \delta^\varepsilon \eta^\chi$, and where \mathcal{D} is the differential generator of $(\delta_t, \eta_t, \widehat{f}_t^B, \widehat{g}_t, t)$ under the probability measure of group B . In doing that, we shall use for the following function:

$$\begin{aligned} H_f(\widehat{f}^B, t; u, \varepsilon) &\triangleq \mathbb{E}_{\delta, \widehat{f}^B}^B [(\delta_u)^\varepsilon] = \exp \left\{ \varepsilon \left[\bar{f}(u-t) + \frac{1}{\zeta} (\widehat{f}^B - \bar{f}) [1 - e^{-\zeta(u-t)}] \right] + \frac{1}{2} \varepsilon (\varepsilon - 1) \sigma_\delta^2 (u-t) \right. \\ &\quad \left. + \frac{\varepsilon^2 \gamma^B}{2\zeta^2} [1 - e^{-\zeta(u-t)}]^2 + \frac{\varepsilon^2 \sigma_f^2}{4\zeta^3} \left[2\zeta(u-t) - 2[1 - e^{-\zeta(u-t)}] - [1 - e^{-\zeta(u-t)}]^2 \right] \right\}. \end{aligned} \quad (\text{B2})$$

Spelling out (B1) we have:

$$\begin{aligned} 0 &\equiv \frac{\partial H}{\partial \delta} \delta \widehat{f}^B - \frac{\partial H}{\partial \widehat{f}^B} \zeta (\widehat{f}^B - \bar{f}) - \frac{\partial H}{\partial \widehat{g}} \widehat{g} \left(\zeta + \frac{\phi \sigma_s \sigma_f + \gamma^A}{\sigma_s^2} + \frac{\gamma^A}{\sigma_\delta^2} \right) \\ &\quad + \frac{1}{2} \frac{\partial^2 H}{\partial \delta^2} (\delta \sigma_\delta)^2 + \frac{1}{2} \frac{\partial^2 H}{\partial \eta^2} (\eta \widehat{g})^2 \left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2} \right) \\ &\quad + \frac{1}{2} \frac{\partial^2 H}{\partial \widehat{g}^2} \left[\left(\frac{\gamma^B - \gamma^A}{\sigma_\delta} \right)^2 + \left(\frac{\gamma^B - (\phi \sigma_s \sigma_f + \gamma^A)}{\sigma_s} \right)^2 \right] \\ &\quad + \frac{1}{2} \frac{\partial^2 H}{\partial (\widehat{f}^B)^2} \left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2} \right) (\gamma^B)^2 \\ &\quad - \frac{\partial^2 H}{\partial \delta \partial \eta} \delta \eta \widehat{g} + \frac{\partial^2 H}{\partial \delta \partial \widehat{g}} \delta (\gamma^B - \gamma^A) + \frac{\partial^2 H}{\partial \delta \partial \widehat{f}^B} \delta \gamma^B \\ &\quad - \frac{\partial^2 H}{\partial \eta \partial \widehat{g}} \eta \widehat{g} \left(\frac{\gamma^B - \gamma^A}{\sigma_\delta^2} + \frac{\gamma^B - (\phi \sigma_s \sigma_f + \gamma^A)}{\sigma_s^2} \right) \\ &\quad - \frac{\partial^2 H}{\partial \eta \partial \widehat{f}^B} \eta \widehat{g} \left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2} \right) \gamma^B \\ &\quad + \frac{\partial^2 H}{\partial \widehat{g} \partial \widehat{f}^B} \left(\frac{\gamma^B - \gamma^A}{\sigma_\delta^2} + \frac{\gamma^B - (\phi \sigma_s \sigma_f + \gamma^A)}{\sigma_s^2} \right) \gamma^B \\ &\quad + \frac{\partial H}{\partial t} \end{aligned} \quad (\text{B3})$$

The appropriate solution of this PDE is:

$$H(\delta, \eta, \widehat{f}^B, \widehat{g}, t, u; \varepsilon, \chi) \triangleq \delta^\varepsilon \eta^\chi \widetilde{H}(\widehat{f}^B, \widehat{g}, t, u; \varepsilon, \chi)$$

where:

$$\widetilde{H}(\widehat{f}^B, \widehat{g}, t, u; \varepsilon, \chi) = H_f(\widehat{f}^B, t, u; \varepsilon) \times H_g(\widehat{g}, t, u; \varepsilon, \chi), \quad (\text{B4})$$

$$H_g(\widehat{g}, t, u; \varepsilon, \chi) = \exp \{ A_0(u-t) + B_0(u-t) \times \widehat{g} + C(u-t) \times \widehat{g}^2 \} \quad (\text{B5})$$

:where:

$$C(\chi; u-t) = \frac{2a(1 - e^{-q(u-t)})}{q-b + (q+b)e^{-q(u-t)}}, \quad (\text{B6})$$

$$a = \frac{1}{2}\chi(\chi-1)\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right) \quad (\text{B7})$$

$$b = -2\left[\zeta + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2} + \frac{\gamma^A}{\sigma_\delta^2} + \chi\left(\frac{\gamma^B - \gamma^A}{\sigma_\delta^2} + \frac{\gamma^B - (\phi\sigma_s\sigma_f + \gamma^A)}{\sigma_s^2}\right)\right] \quad (\text{B8})$$

$$c = 2 \times \left[\left(\frac{\gamma^B - \gamma^A}{\sigma_\delta}\right)^2 + \left(\frac{\gamma^B - (\phi\sigma_s\sigma_f + \gamma^A)}{\sigma_s}\right)^2 \right] \quad (\text{B9})$$

$$k = -\chi\left[1 + \frac{\gamma^B}{\zeta}\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right)\right] \quad (\text{B10})$$

$$l = \chi\frac{\gamma^B}{\zeta}\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right) \quad (\text{B11})$$

$$m = \left[(\gamma^B - \gamma^A) + \frac{\gamma^B}{\zeta}\left(\frac{\gamma^B - \gamma^A}{\sigma_\delta^2} + \frac{\gamma^B - (\phi\sigma_s\sigma_f + \gamma^A)}{\sigma_s^2}\right) \right] \quad (\text{B12})$$

$$n = -\frac{\gamma^B}{\zeta}\left(\frac{\gamma^B - \gamma^A}{\sigma_\delta^2} + \frac{\gamma^B - (\phi\sigma_s\sigma_f + \gamma^A)}{\sigma_s^2}\right) \quad (\text{B13})$$

$$q = \sqrt{b^2 - 4ac}. \quad (\text{B14})$$

$$B_0(\varepsilon, \chi; u-t) = \varepsilon B(\chi; u-t) \quad (\text{B15})$$

where³⁸

$$B(\chi; u-t) = \frac{2\left[\vartheta_1 + \vartheta_2 e^{-\frac{1}{2}q(u-t)} + \vartheta_3 e^{-q(u-t)} + \vartheta_4 e^{-\zeta(u-t)} + \vartheta_5 e^{(-q-\zeta)(u-t)}\right]}{q-b + (q+b)e^{-q(u-t)}}, \quad (\text{B16})$$

³⁸

$$\begin{aligned} \widehat{C}(u-t) &= \exp\left\{\frac{b}{2}(u-t) + c \int_t^u C(\tau-t) d\tau\right\} \\ &= \exp\left\{\frac{b}{2}(u-t) + \ln(2q) - \ln\left(q-b + (q+b)e^{-q(u-t)}\right) - \frac{b+q}{2}(u-t)\right\} \\ &= \frac{2qe^{-\frac{1}{2}q(u-t)}}{q-b + (q+b)e^{-q(u-t)}} \end{aligned}$$

and

$$\vartheta_1 = \frac{4am + k(q - b)}{q}, \quad (\text{B17})$$

$$\vartheta_2 = -2 \left(\frac{4am - bk}{q} + q \frac{4an - bl + 2l\zeta}{q^2 - 4\zeta^2} \right), \quad (\text{B18})$$

$$\vartheta_3 = \frac{4am - k(q + b)}{q}, \quad (\text{B19})$$

$$\vartheta_4 = \frac{4an + l(q - b)}{q - 2\zeta}, \quad (\text{B20})$$

$$\vartheta_5 = \frac{4an - l(q + b)}{q + 2\zeta}. \quad (\text{B21})$$

whereas:

$$A_0(\varepsilon, \chi; u - t) = A_1(\chi; u - t) + \varepsilon^2 A_2(\chi; u - t),$$

where

$$A_1(\chi; u - t) = \frac{1}{4} \left[2 \ln(2q) - 2 \ln \left(q - b + (q + b) e^{-q(u-t)} \right) - (b + q)(u - t) \right] \quad (\text{B22})$$

$$\begin{aligned} A_2(\chi; u - t) = & 2m \left[\vartheta_1 D_1(0; u - t) + \vartheta_2 D_1\left(\frac{q}{2}; u - t\right) + \vartheta_3 D_1(q; u - t) \right. \\ & \left. + \vartheta_3 D_1(\zeta; u - t) + \vartheta_5 D_1(q + \zeta; u - t) \right] \\ & + 2n \left[\vartheta_1 D_1(\zeta; u - t) + \vartheta_2 D_1\left(\frac{q}{2} + \zeta; u - t\right) + \vartheta_3 D_1(q + \zeta; u - t) \right. \\ & \left. + \vartheta_3 D_1(2\zeta; u - t) + \vartheta_5 D_1(q + 2\zeta; u - t) \right] \\ & + c \left[\vartheta_1^2 D_2(0; u - t) + \vartheta_2^2 D_2(q; u - t) + \vartheta_3^2 D_2(2q; u - t) + \vartheta_4^2 D_2(2\zeta; u - t) \right. \\ & \left. + \vartheta_5^2 D_2(2q + 2\zeta; u - t) + 2\vartheta_1 \vartheta_2 D_2\left(\frac{q}{2}; u - t\right) + 2\vartheta_1 \vartheta_3 D_2(q; u - t) \right. \\ & \left. + 2\vartheta_1 \vartheta_4 D_2(\zeta; u - t) + 2\vartheta_1 \vartheta_5 D_2(2\zeta; u - t) + 2\vartheta_2 \vartheta_3 D_2\left(\frac{3q}{2}; u - t\right) \right. \\ & \left. + 2\vartheta_2 \vartheta_4 D_2\left(\frac{q}{2} + \zeta; u - t\right) + 2\vartheta_2 \vartheta_5 D_2\left(\frac{3q}{2} + \zeta; u - t\right) + 2\vartheta_3 \vartheta_4 D_2(q + \zeta; u - t) \right. \\ & \left. + 2\vartheta_3 \vartheta_5 D_2(2q + \zeta; u - t) + 2\vartheta_4 \vartheta_5 D_2(q + 2\zeta; u - t) \right], \quad (\text{B23}) \end{aligned}$$

and

$$\begin{aligned}
D_1(p; u-t) &= \int_t^u \frac{e^{-p(\tau-t)}}{q-b+(q+b)e^{-q(\tau-t)}} d\tau \\
&= \begin{cases} \frac{q(u-t)-\ln(2q)+\ln(q-b+(q+b)e^{-q(u-t)})}{q(q-b)}, & p=0, \\ \frac{1}{p(q-b)} \left[\mathcal{H}\left(1, \frac{p}{q}, 1+\frac{p}{q}, -\frac{q+b}{q-b}\right) \right. \\ \left. -e^{-p(u-t)} \mathcal{H}\left(1, \frac{p}{q}, 1+\frac{p}{q}, -\frac{q+b}{q-b}e^{-q(u-t)}\right) \right], & p>0, \end{cases} \tag{B24}
\end{aligned}$$

$$D_2(p; u-t) = \frac{1}{q(q-b)} \left[\frac{1}{2q} - \frac{e^{-p(u-t)}}{q-b+(q+b)e^{-q(u-t)}} + (q-p) D_1(p; u-t) \right]. \tag{B25}$$

where \mathcal{H} is the hypergeometric function.

Proposition 3 *Function $H_g(\hat{g}, t, u; \varepsilon, \chi)$ is well-defined for $\chi \in [0, 1]$ and $u \geq t$.*

Proof. The radicand in Equation (B14) for q can be written as a quadratic trinomial of χ :

$$b^2 - 4ac = q_2\chi^2 + q_1\chi + q_0, \tag{B26}$$

where

$$q_2 = -\frac{4\sigma_f^2\phi^2}{\sigma_\delta^2}, \tag{B27}$$

$$q_1 = 8\phi\sigma_f \left(\frac{\phi\sigma_f}{\sigma_\delta^2} - \frac{\zeta}{\sigma_s} \right), \tag{B28}$$

$$q_0 = 4 \left[\left(\zeta + \frac{\phi\sigma_f}{\sigma_s} \right)^2 + (1-\phi^2) \left(\frac{\sigma_f^2}{\sigma_s^2} + \frac{\sigma_f^2}{\sigma_\delta^2} \right) \right]. \tag{B29}$$

As $q_2 \leq 0$, $q_0 > 0$, and

$$q_2 + q_1 + q_0 = 4 \left(\zeta^2 + \frac{\sigma_f^2}{\sigma_\delta^2} + \frac{\sigma_f^2}{\sigma_s^2} \right) > 0, \tag{B30}$$

then, when $\chi \in [0, 1]$, $b^2 - 4ac > 0$, and $q = \sqrt{b^2 - 4ac}$ is real and strictly positive.

Taking into account that $c \geq 0$, and for $\chi \in [0, 1]$, $a \leq 0$ and

$$b = -2 \left[\chi \sqrt{\zeta^2 + \frac{\sigma_f^2}{\sigma_\delta^2} + \frac{\sigma_f^2}{\sigma_s^2}} + (1-\chi) \sqrt{\left(\zeta + \frac{\phi\sigma_f}{\sigma_s} \right)^2 + (1-\phi^2) \left(\frac{\sigma_f^2}{\sigma_s^2} + \frac{\sigma_f^2}{\sigma_\delta^2} \right)} \right] < 0, \tag{B31}$$

we obtain that $q+b \geq 0$ and

$$q-b+(q+b)e^{-q(u-t)} \geq q-b > 0. \tag{B32}$$

Consequently, when $\chi \in [0, 1]$ and $u \geq t$, functions $C(\chi; u - t)$, and $B(\chi; u - t)$ are well-defined, integrals $A_1(\chi; u - t)$ and $A_2(\chi; u - t)$ are convergent and their closed-form expressions (B22) and (B23) are obtained correctly. ■

Note that we consider only $\chi \in [0, 1]$, because in Appendix C, we consider the values: $\chi = \frac{j}{1-\alpha}$, $j = 0, \dots, 1 - \alpha$, when $\alpha \in \mathbb{Z}$.

C The wealth and price functions

Knowing the characteristic function (37) from Appendix B, the securities market prices (34), (35) and (36) can be obtained by one of two methods. One is general. It is the inverse Fourier transform, for which the formulae are given below and which can be computed by means of the Fast Fourier Transform but for which we have no proof of convergence. By Fourier inversion, aggregate wealth is:

$$F(\delta, \eta, \hat{f}^B, \hat{g}, t) = \delta \frac{1}{\left[\left(\frac{\eta}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f(\hat{f}^A, t, u; \alpha) \quad (C1)$$

$$\times \left\{ \int_0^\infty \left[\left(\frac{\eta_u}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\eta_u}{\eta} \right)^{-i\chi} H_g(\hat{g}, t, u; \alpha, i\chi) d\chi \right] \frac{d\eta_u}{\eta_u} \right\} du.$$

Similarly, the price of a bond is:

$$P(\eta, \hat{f}^A, \hat{g}, t; T) = \frac{1}{\left[\left(\frac{\eta}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}} e^{-\rho(T-t)} \times H_f(\hat{f}^A, t, T; \alpha - 1) \quad (C2)$$

$$\times \left\{ \int_0^\infty \left[\left(\frac{\eta_u}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\eta_u}{\eta} \right)^{-i\chi} H_g(\hat{g}, t, T; \alpha - 1, i\chi) d\chi \right] \frac{d\eta_u}{\eta_u} \right\}.$$

The wealth of group B investors is:

$$F^B(\delta, \eta, \hat{f}^B, \hat{g}, t) = \delta \frac{\left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}}}{\left[\left(\frac{\eta}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f(\hat{f}^A, t, u; \alpha) \quad (C3)$$

$$\times \left\{ \int_0^{+\infty} \left[\left(\frac{\eta_u}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{-\alpha} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\eta_u}{\eta} \right)^{-i\chi} H_g(\hat{g}, t, u; \alpha, i\chi) d\chi \right] \frac{d\eta_u}{\eta_u} \right\} du.$$

The second method is applicable in the special case in which $1 - \alpha \in \mathbb{N}$ and $\alpha < 0$. Then the bracket $\left[\left(\frac{\eta_u}{\lambda^A} \right)^{\frac{1}{1-\alpha}} + \left(\frac{1}{\lambda^B} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}$ can be expanded into an exact finite series by virtue of the binomial formula.

The overall calculation is then greatly simplified. The equity price is equal to:

$$\begin{aligned}
F(\delta, \eta, \hat{f}^B, \hat{g}, t) &= \delta \frac{1}{\left[1 + \left(\frac{\lambda^B}{\lambda^A} \eta\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f(\hat{f}^B, t; \alpha; u) \\
&\quad \times \left\{ \sum_{j=0}^{1-\alpha} C_{1-\alpha}^j \left(\frac{\lambda^B}{\lambda^A} \eta\right)^{\frac{j}{1-\alpha}} H_g(\hat{g}, t; \alpha, \frac{j}{1-\alpha}; u) \right\} du.
\end{aligned} \tag{C4}$$

Similarly, the price of a bond is:

$$\begin{aligned}
P(\eta, \hat{f}^B, \hat{g}, t; T) &\triangleq \frac{1}{\left[1 + \left(\frac{\lambda^B}{\lambda^A} \eta\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} e^{-\rho(T-t)} \times H_f(\hat{f}^B, t; \alpha - 1; T) \\
&\quad \times \left\{ \sum_{j=0}^{1-\alpha} C_{1-\alpha}^j \left(\frac{\lambda^B}{\lambda^A} \eta\right)^{\frac{j}{1-\alpha}} H_g(\hat{g}, t; \alpha - 1, \frac{j}{1-\alpha}; T) \right\},
\end{aligned} \tag{C5}$$

and the wealth of Group B is:

$$\begin{aligned}
F^B(\delta, \eta, \hat{f}^B, \hat{g}, t) &= \delta \frac{1}{\left[1 + \left(\frac{\lambda^B}{\lambda^A} \eta\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha}} \int_t^\infty e^{-\rho(u-t)} \times H_f(\hat{f}^B, t; \alpha; u) \\
&\quad \times \left\{ \sum_{j=0}^{-\alpha} C_{-\alpha}^j \left(\frac{\lambda^B}{\lambda^A} \eta\right)^{\frac{j}{1-\alpha}} H_g(\hat{g}, t; \alpha, \frac{j}{1-\alpha}; u) \right\} du.
\end{aligned} \tag{C6}$$

Analogously, we can derive all other functions: gradients, second moments, portfolio holdings.

Proposition 4 *When α is integer, the growth conditions for the function F to be well-defined is:*

$$\alpha \bar{f} + \frac{1}{2} \alpha (\alpha - 1) \sigma_\delta^2 + \frac{\alpha^2 \sigma_f^2}{2 \zeta^2} < \rho. \tag{C7}$$

Proof. As for $u \geq t$, and $\chi \in [0, 1]$, $b(\chi) < 0$, $q - b > 0$ and

$$0 \leq \frac{q+b}{q-b} e^{-q(u-t)} < 1, \tag{C8}$$

we can take Taylor below

$$\frac{1}{q-b + (q+b) e^{-q(u-t)}} = \frac{1}{(q-b) \left(1 - \frac{q+b}{b-q} e^{-q(u-t)}\right)} = \frac{1}{q-b} \sum_{j=0}^{\infty} \left[\frac{q+b}{b-q} e^{-q(u-t)}\right]^j, \tag{C9}$$

and the series is uniformly convergent. So, we can interchange summation and integral operator in the expression for $A_1(\chi; u-t)$ and obtain:

$$\begin{aligned}
A_1(\chi; u-t) &= \frac{c}{2} \int_t^u C(\tau-t) d\tau \\
&= \frac{c}{2} \int_t^u \frac{2a(1-e^{-q(\tau-t)})}{q-b} \sum_{j=0}^{\infty} \left[\frac{q+b}{b-q} e^{-q(\tau-t)} \right]^j d\tau \\
&= \varrho_1 \times (u-t) + \sum_{j=0}^{\infty} \varrho_{1j} \times e^{-jq(u-t)}, \tag{C10}
\end{aligned}$$

where

$$\varrho_1 = \frac{ac}{q-b} \leq 0. \tag{C11}$$

Similarly,

$$\begin{aligned}
A_2(\chi; u-t) &= \int_t^u B(\tau-t) \left[B(\tau-t) \times \frac{c}{4} + m + ne^{-\zeta(\tau-t)} \right] d\tau \\
&= \varrho_2 \times (u-t) + \sum_{j=0}^{\infty} \left[\varrho_{2j} \times e^{-\frac{1}{2}jq(u-t)} + \tilde{\varrho}_{2j} \times e^{-(\frac{1}{2}jq+\zeta)(u-t)} \right], \tag{C12}
\end{aligned}$$

where

$$\begin{aligned}
\varrho_2 &= \frac{c\vartheta_1^2}{(q-b)^2} + \frac{2m\vartheta_1}{q-b} = \frac{4am^2 - 2bmk + ck^2}{q^2} \\
&= -\frac{2\phi^2\sigma_f^2\chi(1-\chi)}{q^2} \left(1 + \frac{\sigma_f^2}{\zeta^2\sigma_\delta^2} \right) \leq 0. \tag{C13}
\end{aligned}$$

So, the function $H_g(\hat{g}, t, u; \varepsilon, \chi)$ can be represented as:

$$\begin{aligned}
H_g(\hat{g}, t, u; \varepsilon, \chi) &= \exp \left\{ [\varrho_1(\chi) + \varepsilon^2\varrho_2(\chi)] \times (u-t) \right. \\
&\quad \left. + \sum_{j=0}^{\infty} \left[h_{1j}(\hat{g}; \varepsilon, \chi) e^{-\frac{1}{2}jq(\chi)(u-t)} + h_{2j}(\hat{g}; \varepsilon, \chi) e^{-(\frac{1}{2}jq(\chi)+\zeta)(u-t)} \right] \right\}. \tag{C14}
\end{aligned}$$

Similarly to the way it has been done in Brennan & Xia (Theorem 6)³⁹, we can prove that, when $\alpha \in \mathbb{Z}$, the growth condition for the general economy is:

$$\alpha\bar{f} + \frac{1}{2}\alpha(\alpha-1)\sigma_\delta^2 + \frac{\alpha^2\sigma_f^2}{2\zeta^2} + \max_{\chi \in \left\{ \frac{j}{1-\alpha} \right\}_{j=0}^{1-\alpha}} [\varrho_1(\chi) + \alpha^2\varrho_2(\chi)] < \rho. \tag{C15}$$

Finally, from (C11) and (C13)

$$\max_{\chi \in [0,1]} [\varrho_1(\chi) + \alpha^2\varrho_2(\chi)] = \varrho_1(0) + \alpha^2\varrho_2(0) = 0, \tag{C16}$$

and the growth condition (C15) turns into (C7). ■

³⁹In addition, using standard "epsilon-delta" reasoning, we should consider only finite sum in (C14).

D The expected value of A 's consumption share

In this appendix, we wish to compute the following expected value:⁴⁰

$$H(\eta, \hat{g}^A, \hat{g}^B, t; u, \chi) \triangleq \mathbb{E}_{\eta, \hat{g}^A, \hat{g}^B}^P [\eta_u]^\chi.$$

Note that the expectation above is computed with respect to the true probability measure rather than the measure of B , as was done in the previous appendix, where we computed $\mathbb{E}_{\delta, \eta, \hat{f}^B, \hat{g}}^B [(\delta_u)^\varepsilon (\eta_u)^\chi]$.

We know that

$$\begin{aligned} d\hat{g}_t^A &= -\hat{g}_t^A \left(\zeta + \frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi \sigma_s \sigma_f + \gamma^A}{\sigma_s^2} \right) dt + \frac{\gamma^A}{\sigma_\delta} dZ_t^\delta + \frac{\phi \sigma_s \sigma_f + \gamma^A}{\sigma_s} dZ_t^s - \sigma_f dZ_t^f \\ &= -\psi^A \hat{g}_t^A dt + \frac{\gamma^A}{\sigma_\delta} dZ_t^\delta + \frac{\phi \sigma_s \sigma_f + \gamma^A}{\sigma_s} dZ_t^s - \sigma_f dZ_t^f \end{aligned} \quad (\text{D1})$$

$$\begin{aligned} d\hat{g}_t^B &= -\hat{g}_t^B \left(\zeta + \frac{\gamma^B}{\sigma_\delta^2} + \frac{\gamma^B}{\sigma_s^2} \right) dt + \frac{\gamma^B}{\sigma_\delta} dZ_t^\delta + \frac{\gamma^B}{\sigma_s} dZ_t^s - \sigma_f dZ_t^f \\ &= -\psi^B \hat{g}_t^B dt + \frac{\gamma^B}{\sigma_\delta} dZ_t^\delta + \frac{\gamma^B}{\sigma_s} dZ_t^s - \sigma_f dZ_t^f \end{aligned} \quad (\text{D2})$$

$$\frac{d\eta_t}{\eta_t} = (\hat{g}_t^B - \hat{g}_t^A) \hat{g}_t^B \left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2} \right) dt - (\hat{g}_t^B - \hat{g}_t^A) \left(\frac{1}{\sigma_\delta} dZ_t^\delta + \frac{1}{\sigma_s} dZ_t^s \right). \quad (\text{D3})$$

The function $H(\eta, \hat{g}^A, \hat{g}^B, t; u, \chi)$ satisfies the linear PDE:

$$0 \equiv \mathcal{D}H(\eta, \hat{g}^A, \hat{g}^B, t; u, \chi) + \frac{\partial H}{\partial t}(\eta, \hat{g}^A, \hat{g}^B, t; u, \chi), \quad (\text{D4})$$

⁴⁰In this appendix, $H(\cdot)$ should not be confused with the H function used in Appendix B; that is, we use the same character, but they denote different functions.

with the initial condition $H(\eta, \hat{g}^A, \hat{g}^B, t; \chi) = \eta^\chi$, where \mathcal{D} is the differential generator of $(\eta, \hat{g}^A, \hat{g}^B, t)$ under the true probability measure. Spelling out (D4) we have:

$$\begin{aligned}
0 \equiv & \frac{\partial H}{\partial \eta} \eta (\hat{g}_t^B - \hat{g}_t^A) \hat{g}_t^B \left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2} \right) - \frac{\partial H}{\partial \hat{g}^A} \psi^A \hat{g}^A - \frac{\partial H}{\partial \hat{g}^B} \psi^B \hat{g}^B \\
& + \frac{1}{2} \frac{\partial^2 H}{\partial \eta^2} [\eta (\hat{g}_t^B - \hat{g}_t^A)]^2 \left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2} \right) \\
& + \frac{1}{2} \frac{\partial^2 H}{\partial (\hat{g}^A)^2} \left(\frac{(\gamma^A)^2}{\sigma_\delta^2} + \frac{(\phi \sigma_s \sigma_f + \gamma^A)^2}{\sigma_s^2} + \sigma_f^2 \right) \\
& + \frac{1}{2} \frac{\partial^2 H}{\partial (\hat{g}^B)^2} \left(\frac{(\gamma^B)^2}{\sigma_\delta^2} + \frac{(\gamma^B)^2}{\sigma_s^2} + \sigma_f^2 \right) \\
& - \frac{\partial^2 H}{\partial \eta \partial \hat{g}^A} \eta (\hat{g}_t^B - \hat{g}_t^A) \left(\frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi \sigma_s \sigma_f + \gamma^A}{\sigma_s^2} \right) \\
& - \frac{\partial^2 H}{\partial \eta \partial \hat{g}^B} \eta (\hat{g}_t^B - \hat{g}_t^A) \left(\frac{\gamma^B}{\sigma_\delta^2} + \frac{\gamma^B}{\sigma_s^2} \right) \\
& + \frac{\partial^2 H}{\partial \hat{g}^A \partial \hat{g}^B} \left(\frac{\gamma^A \gamma^B}{\sigma_\delta^2} + \frac{(\phi \sigma_s \sigma_f + \gamma^A) \gamma^B}{\sigma_s^2} + \sigma_f^2 \right) + \frac{\partial H}{\partial t}.
\end{aligned} \tag{D5}$$

The appropriate solution of this PDE is

$$H(\eta, \hat{g}^A, \hat{g}^B, t; u, \chi) \triangleq \eta^\chi H_P(\hat{g}^A, \hat{g}^B, t; u, \chi)$$

where:

$$\begin{aligned}
H_P(\hat{g}^A, \hat{g}^B, t; \chi; u) = \\
\exp \left\{ A_P(u-t) + C^A(u-t) \times (\hat{g}^A)^2 + C^B(u-t) \times (\hat{g}^B)^2 + 2C^{AB}(u-t) \times \hat{g}^A \times \hat{g}^B \right\}, \tag{D6}
\end{aligned}$$

and:

$$\begin{aligned}
A_P(u-t) = & \int_t^u \left[C^A(\tau-t) \left(\frac{(\gamma^A)^2}{\sigma_\delta^2} + \frac{(\phi \sigma_s \sigma_f + \gamma^A)^2}{\sigma_s^2} + \sigma_f^2 \right) + C^B(\tau-t) \left(\frac{(\gamma^B)^2}{\sigma_\delta^2} + \frac{(\gamma^B)^2}{\sigma_s^2} + \sigma_f^2 \right) \right. \\
& \left. + 2C^{AB}(\tau-t) \left(\frac{\gamma^A \gamma^B}{\sigma_\delta^2} + \frac{(\phi \sigma_s \sigma_f + \gamma^A) \gamma^B}{\sigma_s^2} + \sigma_f^2 \right) \right] d\tau.
\end{aligned} \tag{D7}$$

The functions of time C^A, C^{AB} and C^B are defined as the elements of the matrix Z :

$$Z = \begin{pmatrix} C^A & C^{AB} \\ C^{AB} & C^B \end{pmatrix},$$

itself defined as follows. Let matrices $X(u-t)$ and $Y(u-t)$ be the unique solution of the linear Cauchy problem

$$\begin{cases} \dot{X} = Q^{11}X + Q^{12}Y, & X(0) = I, \\ \dot{Y} = Q^{21}X + Q^{22}Y, & Y(0) = 0, \end{cases} \tag{D8}$$

where I is the identity 2×2 matrix. Let:

$$Z(u-t) = Y(u-t)[X(u-t)]^{-1}. \quad (\text{D9})$$

The coefficients are:

$$Q^{21} = \begin{pmatrix} \frac{1}{2}\chi(\chi-1)\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right) & -\frac{1}{2}\chi^2\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right) \\ -\frac{1}{2}\chi^2\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right) & \frac{1}{2}\chi(\chi+1)\left(\frac{1}{\sigma_\delta^2} + \frac{1}{\sigma_s^2}\right) \end{pmatrix}, \quad (\text{D10})$$

$$Q^{11} = -(Q^{22})^\top = \begin{pmatrix} \psi^A - \chi\left(\frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\right) & \chi\left(\frac{\gamma^A}{\sigma_\delta^2} + \frac{\phi\sigma_s\sigma_f + \gamma^A}{\sigma_s^2}\right) \\ -\chi\left(\frac{\gamma^B}{\sigma_\delta^2} + \frac{\gamma^B}{\sigma_s^2}\right) & \psi^B + \chi\left(\frac{\gamma^B}{\sigma_\delta^2} + \frac{\gamma^B}{\sigma_s^2}\right) \end{pmatrix}, \quad (\text{D11})$$

$$Q^{12} = \begin{pmatrix} -2\left(\frac{(\gamma^A)^2}{\sigma_\delta^2} + \frac{(\phi\sigma_s\sigma_f + \gamma^A)^2}{\sigma_s^2} + \sigma_f^2\right) & -2\left(\frac{\gamma^A\gamma^B}{\sigma_\delta^2} + \frac{(\phi\sigma_s\sigma_f + \gamma^A)\gamma^B}{\sigma_s^2} + \sigma_f^2\right) \\ -2\left(\frac{\gamma^A\gamma^B}{\sigma_\delta^2} + \frac{(\phi\sigma_s\sigma_f + \gamma^A)\gamma^B}{\sigma_s^2} + \sigma_f^2\right) & -2\left(\frac{(\gamma^B)^2}{\sigma_\delta^2} + \frac{(\gamma^B)^2}{\sigma_s^2} + \sigma_f^2\right) \end{pmatrix}. \quad (\text{D12})$$

Table 1: Choice of parameter values

This table lists the particular choice of parameter values used for all the figures in the paper. These values are based on the estimation results reported in Brennan and Xia (2001) and Berrada (2004).

Name	Symbol	Value
<i>Parameters for aggregate endowment and the signal</i>		
Long-term average growth rate of aggregate endowment	\bar{f}	0.015
Volatility of expected growth rate of endowment	σ_f	0.02
Volatility of aggregate endowment	σ_δ	0.13
Mean reversion parameter	ζ	0.10
Volatility of the signal	σ_s	0.10
<i>Parameters for the agents</i>		
Agent A's correlation between signal and mean growth rate	ϕ	0.95
Agent B's correlation between signal and mean growth rate	—	0
Agent A's initial share of aggregate endowment	θ^A	0.50
Time-preference parameter for both agents	ρ	0.11
Relative risk aversion for both agents	$1 - \alpha$	3

Figure 1: Prices

This figure shows the interest rate, r , the price of equity, F , and the bond price, P , against the four state variables: the level of aggregate endowment (dividends), δ , the share of aggregate endowment consumed by Group B, c^B , average beliefs, \hat{f}^M , and dispersion in beliefs, \hat{g} . In these plots, we set $\hat{f}^A = \hat{f}^B = \bar{f}$. The parameter values are: $\lambda^A = \lambda^B$, $\eta = 1$, $\alpha = -2$, $\sigma_\delta = .13$, $\sigma_f = 0.02$, $\sigma_s = 0.1$, $\zeta = 0.1$, $\phi = 0.95$.

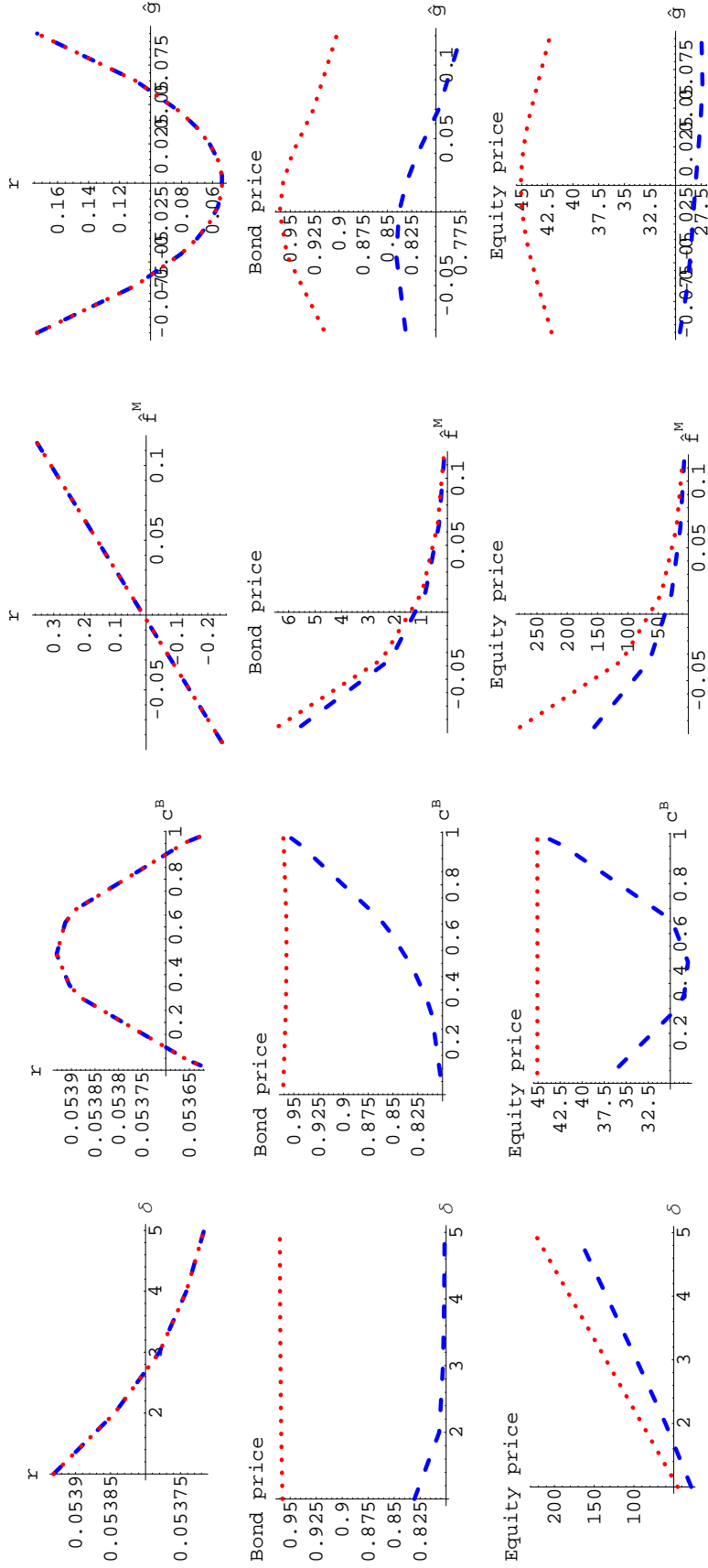


Figure 2: Volatilities and Correlations

This figure shows the volatility of stock returns, the volatility of bond returns, and the correlation between stock and bond returns. Each second moment is plotted against four variables: the level of aggregate endowment (dividends), δ , the share of aggregate endowment consumed by Group B, c^B , which indicates the relative weight of the two populations, average beliefs, \hat{f}^M , and dispersion in beliefs, \hat{g} . In these plots, we set $\hat{f}^A = \hat{f}^B = \bar{f}$. The parameter values are: $\lambda^A = \lambda^B$, $\eta = 1$, $\alpha = -2$, $\sigma_\delta = .13$, $\sigma_f = 0.02$, $\sigma_s = 0.1$, $\zeta = 0.1$, $\phi = 0.95$.

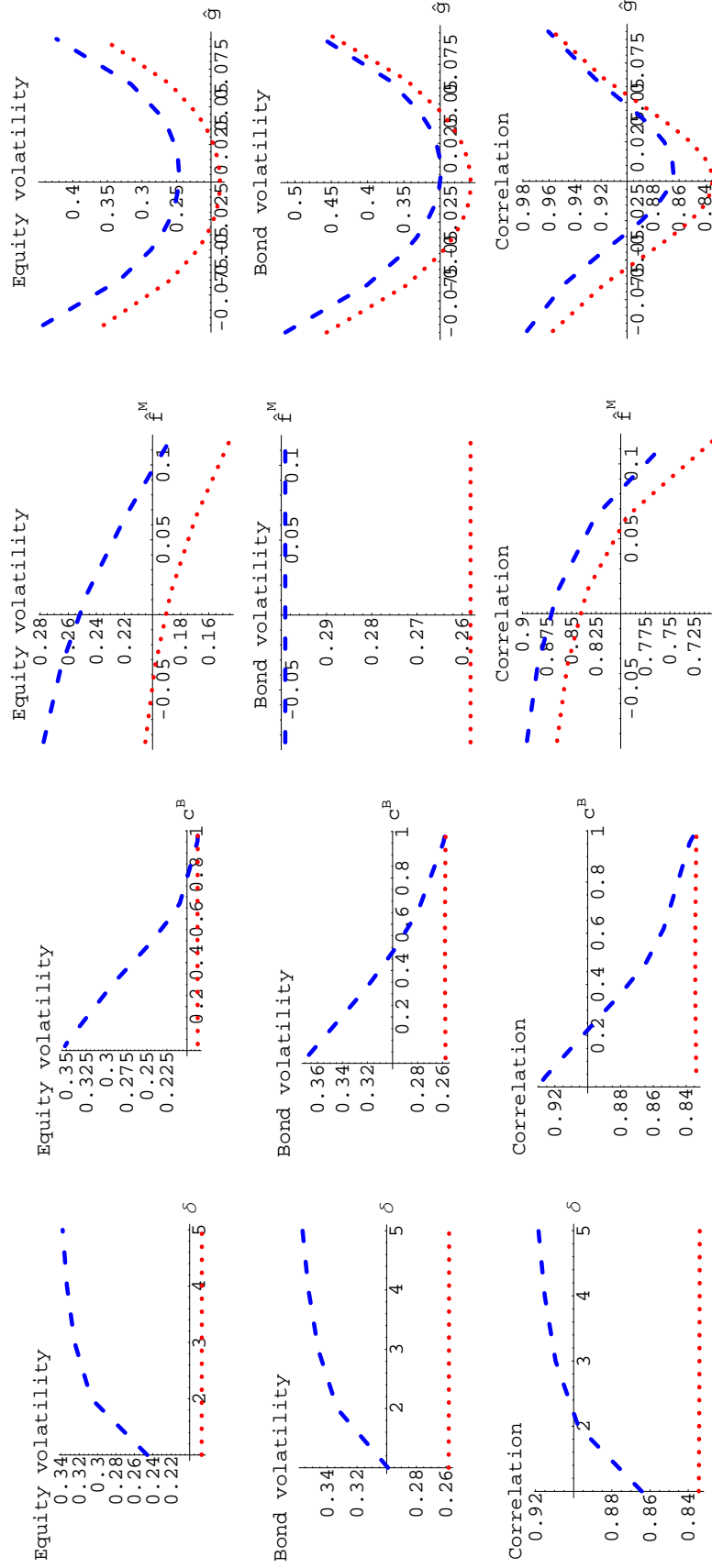


Figure 3: Wealth of Rational Investors (Population B)

This figure illustrates the variations of the wealth of Population B relative to the four state variables: the level of aggregate endowment (dividends), δ , the share of aggregate endowment consumed by Group B, c^B , which indicates the relative weight of the two populations, average beliefs, \bar{f}^M , and dispersion in beliefs, \hat{g} . In these plots, we set $\hat{f}^A = \bar{f}^B = \bar{f}$. The parameter values are: $\lambda^A = \lambda^B$, $\eta = 1$, $\alpha = -2$, $\sigma_\delta = .13$, $\sigma_f = 0.02$, $\sigma_s = 0.1$, $\zeta = 0.1$, $\phi = 0.95$.

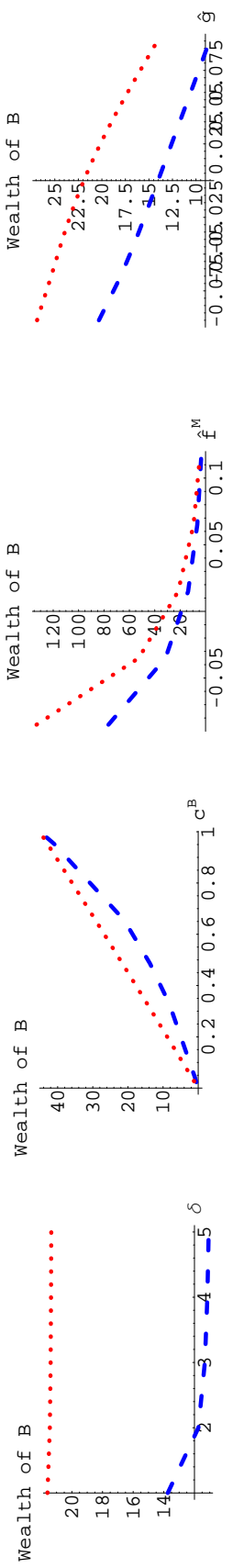


Figure 4: Total Portfolio Weights of Rational Investors (Population B)

This figure gives Population B 's total portfolio holding (expressed as a percentage of B 's wealth). In both the plots, the variable on the x -axis is the dispersion in beliefs, \hat{g} . There are two plots, with the one on the left giving the position in equity and the one on the right giving the position in bonds. In these plots, we set $\hat{f}^A = \hat{f}^B = \hat{f}$. The parameter values are: $\lambda^A = \lambda^B = \lambda$, $\eta = 1$, $\alpha = -2$, $\sigma_\delta = .13$, $\sigma_f = 0.02$, $\sigma_s = 0.1$, $\zeta = 0.1$, $\phi = 0.95$.

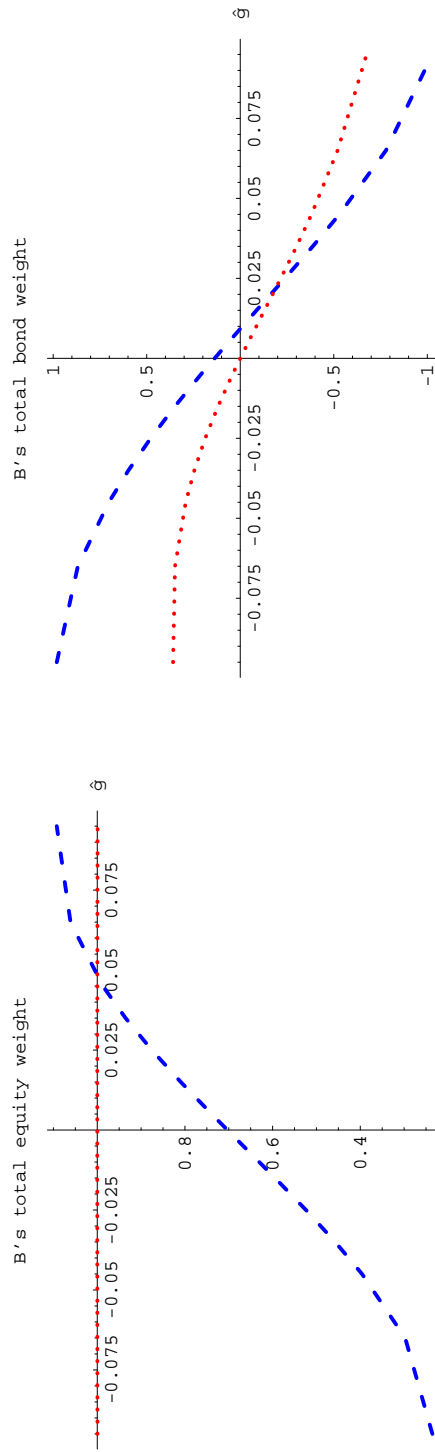


Figure 5: Decomposition of Portfolio Weights of Rational Investors (Population B)

This figure decomposes Population B 's portfolio holding (expressed as a percentage of B 's wealth). In all the plots, the variable on the x -axis is the dispersion in beliefs, \hat{g} . The first row of plots is for the holding in equity, while the second row is for the holding in bonds. The first column gives the static (mean-variance or myopic) investment; the second column gives the investment in order to hedge against changes in δ ; the third column gives the investment in order to hedge against changes in average beliefs, \hat{f}^M ; and, the last column gives the investment that serve as a hedge against changes in the dispersion in beliefs, \hat{g} . In these plots, we set $\hat{f}^A = \hat{f}^B = \bar{f}$. The parameter values are: $\lambda^A = \lambda^B$, $\eta = 1$, $\alpha = -2$, $\sigma_\delta = .13$, $\sigma_f = 0.02$, $\sigma_s = 0.1$, $\zeta = 0.1$, $\phi = 0.95$.

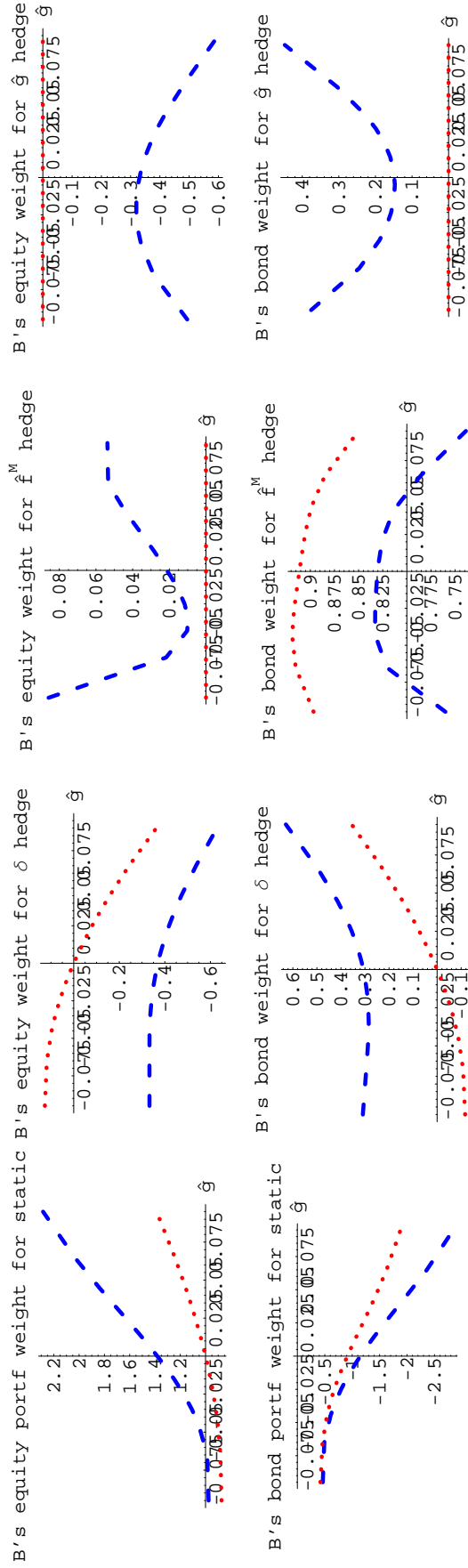
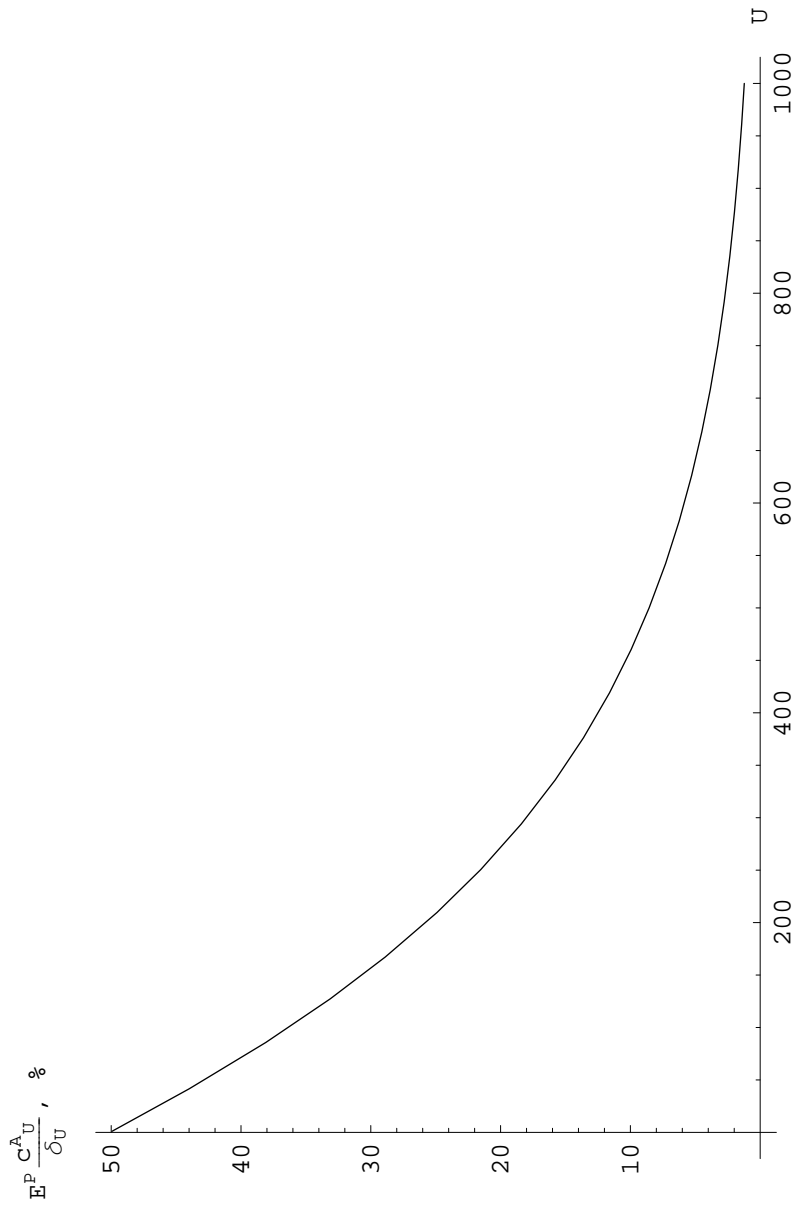


Figure 6: Survival of Population A

This figure shows the expected value of Population A's consumption share (percentage of the total dividends consumed by Group A) as a function of time measured in years, where current time is assumed to be 0 and the future time is denoted on the x -axis by u . We draw the figure for the case where $\hat{f}^A = \hat{f}^B = \bar{f}$. The parameter values are: $\lambda^A = \lambda^B$, $\eta = 1$, $\alpha = -2$, $\sigma_\delta = .13$, $\sigma_f = 0.02$, $\sigma_s = 0.1$, $\zeta = 0.1$, $\phi = 0.95$.



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