Bubbly Business Cycles

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Abstract

We develop a quantitative model of the financial accelerator with asset bubbles. In this model, business cycles are driven by both, productivity and bubble shocks. As usual, productivity shocks capture changes in technology. Bubble shocks capture changes in investor sentiment that affect the valuation of firms and lead to the appearance and collapse of pyramid schemes in financial markets. Investor optimism raises the net worth of firms leading to an expansion in credit and output. Conversely, when investors become pessimistic, the net worth of firms falls and this leads to a contraction in credit and output. We calibrate the model with US data and use it to explore the relative importance of both types of shocks in recent US macroeconomic history.

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Wealth has fluctuated significantly in recent US macroeconomic history. Figure 1 documents this by plotting the ratio of US household and non-profit net worth to GDP between 1950 and 2009.[1] Between 1950 and 1989, household net worth seemed relatively stable around an average value of 3.2 times GDP. This is not to say that there were fluctuations during this period. In the 1970's, for instance, there was a significant drop in wealth approximately at the time of the first oil shock. But all in all, these fluctuations seemed mild and short lived and wealth seemed to exhibit a significant degree of mean reversion. Beginning in 1990, however, this behavior changed dramatically. First, the level of wealth increased substantially relative to the previous four decades: between 1989 and 2009, its average level was around 4 times GDP. Second, the volatility of wealth increased as well. Within this period, as Figure 1 shows, it twice experienced bursts of high growth followed by sharp declines. This greater volatility is well-captured by the standard deviation of net worth, which increased from 0.16 between 1950 and 1989 to 0.45 between 1989 and 2009.

This evidence suggests that something has changed in the US economy over the course of the past two decades. To the extent that an economy's wealth tracks the productive possibilities of its assets, we can look at the macroeconomic environment for clues of the underlying causes of such a change. Has economic growth accelerated and become more volatile in the recent past? Has the productivity of the capital stock increased during this period? To address these questions, we begin by expressing the total expected income generated by an economy's capital stock at time $t+1$ as the sum of two components: a measure of total expected dividends or capital income, which we denote by D_{t+1}, and; a measure of the capital's expected residual value, which we denote by V_{t+1}^R. It follows that

$$\rho_{t+1} \cdot V_t \equiv D_{t+1} + V_{t+1}^R, \tag{1}$$

where we decompose the total return into the product of V_t, the value of the capital stock at time t, and ρ_{t+1}, its expected unit return.

Equation (1) is useful because it instructs us where to look for sources of variation in V_t. And, as it turns out, it can be easily taken to the data once it has been rearranged into the following expression:

$$\frac{V_t}{Y_t} = \frac{\alpha_{t+1} \cdot (1 + g_{t+1})}{\rho_{t+1} - (1 - \delta)}, \tag{2}$$

where Y_t denotes total output, α_{t+1} denotes expected capital income (net of taxes) as a share of

[1] Data on household and non-profit net worth for the US was obtained from the Flow of Funds at the Federal Reserve. This series tracks the evolution of household assets and liabilities over time valued at market prices. To the extent that households are directly or indirectly the ultimate owners of the economy's entire capital stock, this series thus reflects the evolution of the market value of this capital stock over time. In reality, though, US households own some capital abroad and part of the US capital stock is in turn owned by foreigners. To account for this, we substract throughout the US net foreign asset position from the net worth series.

output, g_{t+1} denotes expected real output growth and δ denotes the rate of physical depreciation.[2] These variables are not directly observable and they can be obtained from standard sources.[3] The only exception is ρ_{t+1}, that is, the expected return to holding one unit of the economy's aggregate capital stock, including equipment, housing, etc... In practice, we construct it by taking into account the distribution of household wealth among different asset classes like equity, real estate, bonds and deposits.[4]

Once we have data on these variables, we can use Equation (2) to determine the extent to which they account for the evolution of wealth. It is remarkable to note that, despite the crudeness of the method, they do a good job on average for the sample period as a whole. Indeed, using Equation (2), we find that $\frac{V_t}{Y_t} \approx 3.2$. This is surprisingly close to the true average value of household net worth as depicted in Figure 1, which was equal to 3.5 times GDP for the entire period 1950-2009. But does Equation (2) account equally well for the level of net worth in each of the subsamples identified above? Here the answer is negative. If we consider the period 1950-1989, we find that $\frac{V_t}{Y_t} \approx 3$ and it still seems quite close to the observed value of 3.2. But if we consider instead the period 1990-2009, we find that $\frac{V_t}{Y_t} \approx 3.4$ while the observed value of the capital stock averaged 4 times GDP. This is consistent with the popular notion that the evolution of wealth since the late 1990s has not been led only by changes in fundamentals, but also by bubbles in the markets for key assets like equity and real estate.[5]

In order to further examine the merits of this idea, we can analyze the evolution of Equation (2) on a yearly basis.[6] Figure 2, in which the solid line tracks the observed value of household wealth whereas the dashed line tracks the estimated value of $\frac{V_t}{Y_t}$, displays the results of this exercise. The figure confirms our previous findings in terms of averages. From 1950 until the late 1980s,

[2] Equation (1) follows from dividing Equation (2) by output and assuming that $V_{t+1}^R = (1-\delta) \cdot V_t$. Note that this last step implictly assumes that the unit price of capital remains constant between t and $t+1$. This assumption has little effect on our numerical calculations below.

[3] Throughout, we obtain the series for α_{t+1} and g_{t+1} from the BEA database whereas δ is obtained from Rios-Rull (1996). We adjust α to take into account the self employed as in Gollin (2002), and we use its value net of capital taxation calculated from OECD data as in Mendoza et al. (1994).

[4] To construct ρ_{t+1}, we use the actual shares of equity and real estate in the household balance sheet. We compute the return to equity by using the S&P 500 price and dividend series as in Jagannathan et. al. (2000). We assign to the rest of the portfolio a return equal to the real interest rate, calculated from the 3-month Tbill nominal interest rate. This is because we have not at this point been able to compute a series of returns on real estate, which would require data on rents and depreciation that we do not yet have.

[5] In fact, the behavior of net worth during this period largely mirrored the behavior of stock and real estate prices. The first rise and fall of wealth depicted in Figure 1, which occured between 1995 and 2000, coincided exactly with the rise and fall of the stock market. The second rise and fall of wealth, which took place between 2002 and 2008, coincided largely with the rise and fall of real estate prices although the evolution of equity prices played an important role as well.

[6] This calculation is not straightforward because we have only the realizations, and not the expectations, of the different variables. This is especially problematic for the calculation of ρ, since observed returns tend to be highly volatile. The methodology of Jagannathan et al. (2000) that we use to compute the return on equity partially takes care of this concern by replacing capital gains with a measure of dividend growth. As for the real interest rate, we deal with this concern by taking the moving average over a 10 year window.

the estimated value of $\frac{V_t}{Y_t}$ and the actual value of household wealth both seem stable and similar to one another. Starting in the early 1990s, however, they diverge as household wealth increases substantially above the estimated value of $\frac{V_t}{Y_t}$. This difference is maintained until the end of the sample, even though wealth experiences two episodes of rapid growth with subsequent collapses while the estimated value of $\frac{V_t}{Y_t}$ seems to rise gradually over time. Figure 2 depicts this clearly through the dotted line at the bottom of the figure, which is labeled "Estimated Bubble" and is calculated simply as the difference between the solid and dashed lines. This line remains close to zero between 1950 and the early 1990s. Between 1994 and 2008, however, it experiences two episodes of rapid growth during which it crosses the one-GDP threshold before quickly falling again. This suggests, once again, that it is difficult to account for the large movements in wealth of the past two decades by invoking changes in macroeconomic variables alone.[7]

At the very least, the previous evidence indicates that bubbles may have played an important role in recent macroeconomic history.[8] And yet, despite this evidence and despite widespread talk of bubbles in recent years, macroeconomic models of bubbles remain scarce. Those models that do exist, moreover, are qualitative in nature and are difficult to integrate into the state-of-the-art framework used for macroeconomic analysis. This paper fills this gap by developing a quantitative model of the financial accelerator with bubbles. In this model, business cycles are driven by two types of shocks: fundamental shocks that affect technology; and bubble shocks that lead to the appearance and collapse of pyramid schemes in financial markets. Our objective is to explore the business cycles that arise in the model and relate them to actual data. In particular, we plan to calibrate the model with data from industrialized economies and use it to explore the relative importance of both types of shocks in recent macroeconomic history.

The model developed here builds upon previous work by Martin and Ventura (2011a, 2011b). The first of these papers develops the basic theoretical framework that we use here, and it also provides a full characterization of bubbly equilibria in a simple model of financial frictions. The second of these papers uses some of these equilibria to re-interpret the origin of the recent financial

[7] Incidentally, this presumption is consistent with the conclusions of detailed studies that have analyzed equity and real estate markets separately. LeRoy (2004), for instance, documented that the growth in the value of US equity between 1995 and 2000 far exceeded the growth of corporate earnings and of dividends. He also considered and discarded other popular explanations for the rise in equity values, most notably those based on demographics and on the valuation of intangible capital. In a similar vein, Shiller (2005) analyzed and also discarded popular explanations for the run-up in home prices during the early 2000s, which were based on the evolution of demographics, the interest rate and construction costs. Today, these explanations seem even less plausible because they should also be consistent with the ensuing collapse in prices.

[8] We are cautious in our interpretation of the evidence because we recognize that it admits other explanations. One possibility, for instance, is that people make mistakes. If individuals consistently over- or under-predict the rate of return to capital, the observed value of the capital stock will differ from its estimated value. Although this possibility has led to interesting macroeconomic research in recent years, we choose instead to explore the macroeconomic implications of asset bubbles.

crisis and the theoretical merits of alternative policy proposals to deal with it. Here we generalize the model substantially to study its quantitative implications. The general model cannot be analyzed using the simple analytical methods used in the two previous papers. Instead, we compute approximations to equilibria using the sparse-grid collocation methods proposed by Krueger and Kubler (2004).

The framework used by this paper is closely related to the recent body of work that has studied the effects of bubbles in the presence of financial frictions: (i) Caballero and Krishnamurthy (2006) and Farhi and Tirole (2011) show that bubbles can be a useful source of liquidity;[9] (ii) Kocherlakota (2009), Martin and Ventura (2011a) and Miao and Wang (2011) show that bubbles can also raise collateral or net worth; and (iii) Ventura (2011) shows that bubbles can lower the cost of capital. Unlike these papers, we explicitly introduce bubbles as shocks to asset-prices into an otherwise standard business-cycle model with financial frictions.[10] Finally, there are two papers that have used rational bubbles to interpret recent macroeconomic developments more directly: Kraay and Ventura (2007) use a model of bubbles and capital flows to study the origin and effects of global imbalances, while Martin and Ventura (2011a) use a model of bubbles and the financial accelerator to interpret the 2007-08 financial crisis and its effects. Our model is also related to the vast work on macroeconomic models with financial frictions, in which asset prices play an important role in determining the level of financial intermediation and economic activity.[11] Our theory differs from these models in that asset prices are not only a channel through which traditional or fundamental shocks are transmitted, but they are also the source of shocks themselves.

The paper is organized in four sections. Section one describes our model of bubbly business cycles. Section two then presents a series of experiments to develop intuitions about the mechanisms that drive business cycles in the model. Section three describes our attempt to use the model to interpret recent U.S macroeconomic history. Section four takes stock of what has been achieved here and sets an ambitious agenda for further research.

[9] There is, of course, a long tradition of papers that view fiat money as a bubble. Indeed, Samuelson (1958) adopted this interpretation. For a recent paper that also emphasizes the liquidity-enhancing role of fiat money in the presence of financial frictions, see Kiyotaki and Moore (2008).

[10] To the best of our knowledge, Aoki and Nikolov (2011) is the only other paper that tries to study the effects of bubbles in a quantitative macroeconomic model.

[11] Here we are referring to the huge macroeconomic literature on the financial accelerator that originated with the seminal contributions by Bernanke and Gertler (1989) and Kiyotaki and Moore (1997).

1 A model of bubbly business cycles

We describe next an economy inhabited by an infinite sequence of overlapping generations. All generations live $T + 1$ periods and have the same size. Each generation contains a continuum of identical households who supply labor, receive a wage and save part of it before retiring. After this, households live off their savings. We normalize the measure of living households to one, meaning that each generation of households has measure $(T + 1)^{-1}$.

Firms produce goods using a common technology. Firms differ, however, depending on who controls them. Each generation contains an infinitesimal measure of identical entrepreneurs who have an innate skill to identify and exploit good investment opportunities. Firms controlled by an entrepreneur are labeled E-firms, while firms not controlled by an entrepreneur are labeled as N-firms. We model the innate skill of entrepreneurs by assuming that the investment technology of E-firms is better than that of N-firms. In the absence of any friction, all investment would take place within E-firms and the equilibrium would be efficient.

But we assume that giving the entrepreneur the control of the firm allows him/her to appropriate the firm's assets, reducing the return to other owners/creditors. This inability to control the entrepreneur implies that E-firms maximize the entrepreneur's utility, while N-firms maximize their value as usual. This agency cost reduces the ability of E-firms to raise funds, making them unable to absorb all the savings in the economy. As a result, and despite their inferior investment technology, some investments take place in N-firms. This inefficiency creates the conditions for *bubbly* business cycles to exist.

Households and firms interact in labor and financial markets. We assume throughout that these markets are competitive and frictionless. Indeed, our Bubbly Business Cycle (BBC) model is nothing but a Real Business Cycle (RBC) model with an agency cost. This minimal departure is crucial, however, since it is needed to generate bubble-driven business cycles alongside the usual productivity-driven business cycles. More formally, we define s_t as the vector of all shocks in period t. As we shall see, in our theory this vector includes both productivity and bubble shocks. Then, we define s^t as a history of shocks until period t, i.e. $s^t = \{s_0, s_1, ..., s_t\}$; and we let S_t be the set of all possible histories, i.e. $s^t \in S_t$. We also define $\pi_{s^t s^{t+n}}$ as the probability of history s^{t+n} in period t and history s^t.

1.1 Households

Households supply labor and savings to firms in the labor and financial markets. The representative household of generation τ maximizes the following utility function:

$$U^H_{\tau,s^t} = \sum_{n=t}^{\tau+T} \sum_{s^n \in S_n} \pi_{s^t s^n} \cdot \beta^{n-t} \cdot \frac{\left(c^H_{\tau,s^n}\right)^{1-\gamma} - 1}{1-\gamma} \tag{3}$$

where c^H_{τ,s^t} is the consumption of the representative household of generation τ in period t given the history s^t; and $\beta \in (0,+\infty)$ and $\gamma \in (0,+\infty)$. Throughout, we refer to β and γ as the discount factor and the risk aversion. We do not allow these preference parameters to vary across generations. Hence, we do not include preference shocks in the vector s_t.

The single source of income for the representative household is labor. In particular, the household supplies one unit of labor until $\tau + T^R$ and then retires. We assume all active households have the same labor productivity, and define w_{s^t} as the common wage for one unit of labor in period t and history s^t. Naturally, the representative household would like to save part of this labor income to finance consumption during retirement. To do this, the household can buy and sell securities in competitive financial markets. Consider a security that is traded in period t in history s^t and delivers a payment equal to $a_{s^{t+1}}$ in period $t+1$ in history s^{t+1}. The price of this security is $q_{s^t s^{t+1}} \cdot a_{s^{t+1}}$, where $q_{s^t s^{t+1}}$ is the market price in period t and history s^t of one unit of output delivered in period $t+1$ in history s^{t+1}. With this notation at hand, we can write the budget constraint of the representative household as follows:

$$c^H_{\tau,s^t} = \begin{cases} w_{s^\tau} - \sum_{s^{t+1} \in S_{t+1}} q_{s^\tau s^{\tau+1}} \cdot a_{\tau,s^{\tau+1}} & \text{if } t = \tau \\ w_{s^t} + a_{\tau,s^t} - \sum_{s^{t+1} \in S_{t+1}} q_{s^t s^{t+1}} \cdot a_{\tau,s^{t+1}} & \text{if } \tau < t \leq \tau + T^R \\ a_{\tau,s^t} - \sum_{s^{t+1} \in S_{t+1}} q_{s^t s^{t+1}} \cdot a_{\tau,s^{t+1}} & \text{if } \tau + T^R < t < \tau + T \\ a_{\tau,s^{\tau+T}} & \text{if } t = \tau + T \end{cases} \tag{4}$$

Equation (4) says that in period t in history s^t the consumption of the representative household equals labor income plus the value of the securities purchased in period $t-1$ minus the value of the securities purchased in period t. It also says that: (i) the household starts with zero financial wealth, i.e. $a_{\tau,s^\tau} = 0$; and (ii) even though financial wealth can be negative in some periods and histories, all debts must be paid before the representative household dies, i.e. $a_{\tau,s^{\tau+T+1}} = 0$.

The representative household maximizes Equation (3) subject to Equation (4) and this implies:

$$\pi_{s^t s^{t+1}} \cdot \beta \cdot \left(\frac{c^H_{\tau,s^{t+1}}}{c^H_{\tau,s^t}}\right)^{-\gamma} = q_{s^t s^{t+1}} \tag{5}$$

Equation (5) says that, as usual, the ratio of marginal utilities of consumption equals the ratio of prices of output in all periods and histories.

This completes the description of the problem of the representative household of generation τ. For given wages and security prices, Equations (4), and (5) determine consumption and security holdings in all periods and histories. To simplify, we have stripped households of all labor-related decisions such as choosing education, employment or retirement age. Indeed, the only decisions that households make are financial ones. Given their income profile and their preferences, the main reason households go to financial intermediaries is to save part of their labor income while they work to sustain consumption after retirement. In addition, the rich array of securities offered by financial intermediaries also allows them to obtain some insurance against aggregate shocks that have disproportionate effects on their labor income.

1.2 Firms

Firms demand labor and savings from households in the labor and financial markets. As mentioned, there are two types of firm, depending on how control is assigned. N-firms are controlled by all their owners and do not suffer from agency problems, but they have low investment efficiency. E-firms are controlled by entrepreneurs and have high investment efficiency, but they suffer from agency problems. We describe each of these firms in turn.

1.2.1 N-firms

The typical N-firm invests to produce capital, and hires labor to combine it with capital and produce output. Let $k_{s^t}^N$ and $y_{s^t}^N$ be the stock of capital and output produced by the typical N-firm in period t and history s^t. To do this, this firm has access to the following technologies:

$$y_{s^t}^N = A_{s^t} \cdot \left(l_{s^t}^N\right)^{1-\alpha} \cdot \left(k_{s^t}^N\right)^{\alpha} \tag{6}$$

$$k_{s^{t+1}}^N = I_{s^t}^N + (1-\delta) \cdot k_{s^t}^N \tag{7}$$

where $l_{s^t}^N$ and $I_{s^t}^N$ are the employment and investment chosen by the N-firm in period t in history s^t. We refer to $A_{s^t}^Q$ as the productivity shock, and it is one of the elements of the vector s_t. We refer to $\delta \in (0, +\infty)$ as the depreciation rate. Equation (6) says that production can be well approximated by a standard Cobb-Douglas production, while Equation (7) says that the capital of the firm equals new investment plus used capital. For simplicity, we assume that used capital can always be converted back into output one-to-one, that is, we have that $I_{s^t}^N \in \left[-(1-\delta) \cdot k_{s^t}^N, +\infty\right)$.

As a result, the price of used capital is always one.

Let $V_{s^t}^N$ be the amount of financing that the N-firm obtains by issuing securities and selling them to the competitive financial intermediaries. Although the model admits other interpretations, we find it useful to think of $V_{s^t}^N$ as the value or market price of the N-firm and holders of these securities as the owners of the firm. This value depends on the employment and investment plan that is implemented by the firm. The owners have control of the firm and choose a plan that maximizes the firm's value:

$$V_{s^t}^N = \max_{\left\langle l_{s^{t+1}}^N,\, I_{s^{t+1}}^N, \right\rangle} \sum_{s^{t+1} \in S_{t+1}} q_{s^t s^{t+1}} \cdot \left(y_{s^{t+1}}^N - w_{s^{t+1}} \cdot l_{s^{t+1}}^N - I_{s^{t+1}}^N + V_{s^{t+1}}^N \right) \tag{8}$$

subject to Equations (6) and (7). Equation (8) simply says that the price of the N- firm in period t equals the discounted value of the dividend plus the price one period ahead. The dividend is the profit from sales minus any investment expenses.

A crucial aspect of the theory is the determination of equilibrium firm values. Recall that $V_{s^{t+1}}^N$ is the value of the N-firm after dividends have been distributed. At this point, the firm contains capital ready to produce the next period. It is commonplace then to assume that the market price of the firm coincides with the market price of this capital, which is called the fundamental. We do not impose this restriction and instead allow the firm to contain also bubbles. Then, the price of the firm consists of the price of its capital plus these bubbles:

$$V_{s^t}^N = k_{s^{t+1}}^N + b_{s^t}^N \tag{9}$$

where $b_{s^t}^N$ is the bubble of the representative N-firm. Equation (9) says that the market price of a firm equals the fundamental plus the bubble. Bubbles are pyramid schemes by which financing is obtained today because tomorrow the firm will be able to raise enough new financing to pay for today's financing. Obviously, not all bubbles are possible. We shall later characterize the set of stochastic processes for $b_{s^t}^N$ that are consistent with equilibrium and explore their macroeconomic implications.

Using this characterization of firm values, the optimal employment and investment policy of the representative N-firm is obtained as follows:

$$(1-\alpha) \cdot A_{s^t}^Q \cdot \left(\frac{k_{s^t}^N}{l_{s^t}^N} \right)^{\alpha} = w_{s^t} \tag{10}$$

$$\sum_{s^{t+1}\in S_{t+1}} q_{s^t s^{t+1}} \cdot \left(\alpha \cdot A^Q_{s^{t+1}} \cdot \left(\frac{(1-\alpha)\cdot A^Q_{s^{t+1}}}{w_{s^{t+1}}} \right)^{\frac{1-\alpha}{\alpha}} + 1 - \delta \right) = 1 \tag{11}$$

Equation (10) says that the N-firm hires workers until their marginal product of labor equals the wage. Equation (11) says that the N-firm invests until the marginal product of capital discounted at market prices equals the cost of capital, which is one. Equation (11) assumes that N-firms are active and this will indeed be the case in the equilibria that we consider below.

Equations (8), (9), (10) and (11) imply that:

$$b^N_{s^t} = \sum_{s^{t+1}\in S_{t+1}} q_{s^t s^{t+1}} \cdot b^N_{s^{t+1}} \tag{12}$$

We say that a firm is creating new bubbles if the discounted value of the bubble in period $t+1$ exceeds the value of the bubble in period t, i.e. $b^N_{s^t} < \sum_{s^{t+1}\in S_{t+1}} q_{s^t s^{t+1}} \cdot b^N_{s^{t+1}}$. Equation (12) says that bubble creation is not possible in an N-firm. That is, N-firms can run pyramid schemes but cannot start new ones. As we shall show next, it is the entrepreneurs who do the latter.

1.2.2 E-firms

Consider next the firm managed by the representative entrepreneur of generation τ. For brevity, we shall refer to this firm as the representative E-firm even though there is one such firm for every living generation.[12] Let k^E_{τ,s^t} and y^E_{τ,s^t} be the amount of capital and output produced by the representative E-firm in period t and history s^t. Then, we can write the technologies available to this firm as:

$$y^E_{\tau,s^t} = A^Q_{s^t} \cdot \left(l^E_{\tau,s^t}\right)^{1-\alpha} \cdot \left(k^E_{\tau,s^t}\right)^{\alpha} \tag{13}$$

$$k^E_{\tau,s^{t+1}} = A^K_{s^t} \cdot \left[I^E_{\tau,s^t} + (1-\delta)\cdot k^E_{\tau,s^t} \right] \tag{14}$$

where l^E_{τ,s^t} and I^E_{τ,s^t} are the employment and investment chosen by the E-firm in period t in history t; and $A^K_{s^t} \in (1, +\infty)$. A quick comparison of Equations (6)-(7) and (13)-(14) shows that E-firms are better at producing capital from both new investments and used capital. This is the advantage of giving the control of the firm to the entrepreneur.

The disadvantage of giving the control of the firm to the entrepreneur is that he/she will choose employment and investment policy so as to maximize his/her utility rather than the value of the firm. In itself, this would not be a problem if the entrepreneur could commit to make appropriate payments to the holders of any security that the firm issues. But we shall not allow this. Instead,

[12] That is, there is one representative E- firm for each generation of entrepreneurs. We describe one such firm and then we will aggregate appropriately.

we shall go to the other extreme and assume that the entrepreneur cannot commit any output and/or capital of the firm. This lack of commitment seems fatal, as it apparently impedes the entrepreneur to raise any financing to start a E-firm. Indeed, utility maximization implies that in the last period the entrepreneur appropriates all the output and capital of the firm. Knowing this, financial markets should not give financing to the E-firm in the previous period. But then, utility maximization implies that in the next-to-last period the entrepreneur also appropriates all the output and capital of the firm. Knowing this, financial markets should not finance the E-firm in the previous period. Iterating this argument backwards leads us to the conclusion that the entrepreneur cannot raise any financing. Since entrepreneurs have no financial wealth at the beginning of their lives, how can E-firms get started? The answer we provide here is new and quite simple: *by issuing or creating bubbles*! The idea is that the entrepreneur can also start a bubble and this is how it raises financing. We explain how this works next.

As everybody else, the representative entrepreneur of generation τ maximizes the following utility function:

$$U^E_{\tau,s^t} = \sum_{n=t}^{\tau+T} \sum_{s^n \in S_n} \pi_{s^t s^n} \cdot \beta^{n-t} \cdot \frac{\left(c^E_{\tau,s^n}\right)^{1-\gamma} - 1}{1-\gamma} \tag{15}$$

where c^E_{τ,s^t} is the consumption of the representative entrepreneur of generation τ in period t given history s^t. Unlike households, entrepreneurs do not work as employees and receive a wage, but instead own and manage a E-firm all their lives. We can write the budget constraint of the entrepreneur of the representative E-firm as follows:

$$c^E_{\tau,s^t} = \begin{cases} V^E_{\tau,s^\tau} - I^E_{\tau,s^\tau} & \text{if } t = \tau \\ y^E_{\tau,s^t} - w_{s^t} \cdot l^E_{\tau,s^t} - I^E_{\tau,s^t} + V^E_{\tau,s^t} - \hat{V}^E_{\tau,s^t} & \text{if } \tau < t \leq \tau + T \end{cases} \tag{16}$$

where V^E_{τ,s^τ} is the amount of financing that the entrepreneur can raise from financial markets in period t and history s^t; and $\hat{V}^E_{\tau,s^{t+1}}$ is the payments that the entrepreneur promises to these intermediaries in period $t+1$ and history s^{t+1}. Naturally, the following relationship must hold among these variables:

$$V^E_{\tau,s^t} = \sum_{s^{t+1} \in S_{t+1}} q_{s^t s^{t+1}} \cdot \hat{V}^E_{\tau,s^{t+1}} \tag{17}$$

Equation (16) says that, in the first period of life, the entrepreneur raises financing V^E_{τ,s^τ} from financial intermediaries and uses it to finance consumption and investment. After this, the entrepreneur uses the profits of the firm, $y^E_{\tau,s^t} - w_{s^t} \cdot l^E_{\tau,s^t}$, and any additional new financing $V^E_{\tau,s^t} - \hat{V}^E_{\tau,s^t}$ from financial intermediaries, to finance additional investment and consumption, $c^E_{\tau,s^t} + I^E_{\tau,s^t}$. In the last period, the entrepreneur sells its firm at price $V^E_{\tau,s^{\tau+T}}$, and the E-firm turns into an N-firm.

The entrepreneur's lack of commitment means that the entrepreneur empties the firm of any capital the last period of his/her life and, as a result, the value of the firm at this point equals its bubble. A standard backward-induction argument reveals that no payments will be made in earlier periods. As a result, financial markets will restrict the amount of financing given to the E-firm so as to ensure that:

$$\hat{V}^E_{\tau,s^t} \leq V^E_{\tau,s^t} \quad \text{and} \quad V^E_{\tau,s^{\tau+T}} = b^E_{\tau,s^{\tau+T}} \tag{18}$$

That is, any financing the E-firm receives today is backed by the prospects of rasing further financing tomorrow. This is therefore how the E-firms are started, the entrepreneur brings a bubble to the firm.

The entrepreneur chooses consumption, investment and employment so as to maximize Equation (15) subject to Equations (16), (17) and (18). Throughout, we shall consider only equilibria in which:

$$\hat{V}^E_{\tau,s^t} = V^E_{\tau,s^t} \;\text{ and }\; V^E_{\tau,s^t} = \sum_{s^{t+1}\in S_{t+1}} q_{s^t s^{t+1}} \cdot \sum_{s^{t+2}\in S_{t+2}} q_{s^{t+1} s^{t+2}} \cdot ... \sum_{s^{\tau+T}\in S_{\tau+T}} q_{s^{\tau+T-1} s^{\tau+T}} \cdot b^E_{\tau,s^{\tau+T}} \equiv b^E_{\tau,s^t} \tag{19}$$

Equation (19) says that the the optimal plan of the entrepreneur is credit constrained in all periods and histories. This will be the case in all the equilibria we consider below. In these equilibria, the ratio of marginal utilities of consumption does not equal the ratio of prices of output in all periods and histories:

$$\pi_{s^t s^{t+1}} \cdot \beta \cdot \left(\frac{c^E_{\tau,s^{t+1}}}{c^E_{\tau,s^t}}\right)^{-\gamma} < q_{s^t s^{t+1}} \tag{20}$$

Equation (20) says that, in all periods and histories, utility would be higher if the entrepreneur could raise more financing and increase current consumption. But this is not possible, since the entrepreneur cannot commit to make additional payments.

The optimal employment and investment plan of the E-firm can now be obtained as follows:

$$(1-\alpha) \cdot A^Q_{s^t} \cdot \left(\frac{k^E_{\tau,s^t}}{l^E_{\tau,s^t}}\right)^{\alpha} = w_{s^t} \tag{21}$$

$$\sum_{s^{t+1}\in S_{t+1}} \pi_{s^t s^{t+1}} \cdot \beta \cdot \left(\frac{c^E_{\tau,s^{t+1}}}{c^E_{\tau,s^t}}\right)^{-\gamma} \cdot \left(\alpha \cdot A^Q_{s^{t+1}} \cdot \left(\frac{(1-\alpha) \cdot A^Q_{s^{t+1}}}{w_{s^{t+1}}}\right)^{\frac{1-\alpha}{\alpha}} + 1 - \delta\right) = \frac{1}{A^K_{s^t}} \tag{22}$$

Equations (21) and (22) should be compared to Equations (10) and (11). Both types of firms hire labor and invest until the marginal product of labor and capital equal the wage and the cost of capital. This implies that the employment policy of E-firms coincide with that of N-firms. The investment policy of the E-firms differs however from that of the N-firms in two dimensions. First, the E-firm has a lower cost of capital, i.e. $\frac{1}{A^K_{s^t}}$ instead of one. Second, the marginal product of capital in different histories is discounted with the marginal utilities of the entrepreneur instead of market prices, i.e. $\pi_{s^t s^{t+1}} \cdot \beta \cdot \left(\frac{c^E_{\tau,s^{t+1}}}{c^E_{\tau,s^t}}\right)^{-\gamma}$ instead of $q_{s^t s^{t+1}}$. These two differences combined is what will allow both firms to coexist in the same economy. E-firms have lower cost of capital, but they also value less future returns. This is why Equations (11) and (22) are compatible in the equilibria we analyze here.

1.3 Aggregation and markets

There are two key markets in this economy, the labor and financial markets. Both of them are competitive and frictionless. In the labor market, households supply labor and firms demand it. In the financial market, households supply savings and firms demand it. We describe each of these markets in turn.

1.3.1 The labor market

In the labor market, we determine the equilibrium wage by crossing the supply and demand. Since each generation contains $(T+1)^{-1}$ workers and there are T^R+1 generations that have not retired yet, we have that the labor supply, i.e. $l^S_{s^t}$, is constant and equal to:

$$l^S_{s^t} = \frac{T^R+1}{T+1} \equiv l \tag{23}$$

To determine the labor demand, i.e. $l^D_{s^t}$, we sum the demand of all firms using Equations (10) and (21) in period t and history s^t:

$$l^D_{s^t} = \left(\frac{(1-\alpha)\cdot A^Q_{s^t}}{w_{s^t}}\right)^{\frac{1}{\alpha}} \cdot k_{s^t} \tag{24}$$

where k_{s^t} is the aggregate capital stock, that is, the sum of all the capital contained in N-firms and E-firms. Equating labor demand and supply, we find that the wage in period t and history s^t is given by:

$$w_{s^t} = (1-\alpha)\cdot A^Q_{s^t} \cdot \left(\frac{k_{s^t}}{l}\right)^{\alpha} \tag{25}$$

Equation (25) says that the wage equals the marginal product of capital when evaluated at the average capital-labor ratio of the economy.

Let y_{s^t} be aggregate output in period t and history s^t. Not surprisingly, it follows that:

$$y_{s^t} = A^Q_{s^t} \cdot l^{1-\alpha} \cdot k^{\alpha}_{s^t} \tag{26}$$

Equation (26) simply says that aggregate production can be described by the same Cobb-Douglas production function that each firm in the economy has.

1.3.2 The financial market

In the financial market, we determine equilibrium security prices also by crossing demand and supply. Let $a^D_{s^t}$ be the aggregate household demand in period t and history s^t for the security that pays one unit of output in period $t+1$ and history s^{t+1}. Then, we have that:

$$a^D_{s^t s^{t+1}} = \sum_{\tau=t-T+1}^{t} a_{\tau,s^{t+1}} \tag{27}$$

Equation (27) says that this aggregate demand is the sum of all the holdings of this security by living generations of households, except for the oldest generation which does not save. Recall that we have already calculated these demands as a function of wages and security prices, in Equations (4), and (5).

Let $a^S_{s^t}$ be the aggregate firm supply in period t and history s^t for the security that pays one unit of output in period $t+1$ and history s^{t+1}. Then, we have that:

$$a^S_{s^t s^{t+1}} = \left(\alpha \cdot A^Q_{s^{t+1}} \cdot l^{1-\alpha} \cdot k^{\alpha-1}_{s^{t+1}} + 1 - \delta\right) \cdot \left(k_{s^{t+1}} - \sum_{\tau=t+2-T}^{t} k^E_{\tau,s^{t+1}}\right) + b_{s^{t+1}} + \sum_{\tau=t+2-T}^{t} b^E_{\tau,s^{t+1}} \tag{28}$$

where $b_{s^{t+1}}$ is the sum of all the bubbles of N-firms in period $t+1$ and history s^{t+1}. This contains all the bubbles than these firms contained in period t and history s^t plus the bubbles of E-firms that turn into N-firms in period $t+1$ and history s^{t+1}. It follows then from Equation (12) that this aggregate bubble must satisfy the following:

$$b_{s^t} = \sum_{s^{t+1} \in S_{t+1}} q_{s^t s^{t+1}} \cdot \left(b_{s^{t+1}} - b^E_{t+1-T,s^{t+1}}\right) \tag{29}$$

Now, we can describe Equation (28) as saying that the supply of securities by firms contains four terms that can be easily interpreted. The first three terms is the supply of securities by N-firms. In period $t+1$ and history s^{t+1}, N-firms cam commit to pay the holders of their securities the dividend plus the value of the securities that these firms will issue one-period ahead. The dividend consists of profits from sales, i.e.[13]

$$\alpha \cdot A^{Q}_{s^{t+1}} \cdot l^{1-\alpha} \cdot k^{\alpha-1}_{s^{t+1}} \cdot \left(k_{s^{t+1}} - \sum_{\tau=t+2-T}^{t} k^{E}_{\tau,s^{t+1}} \right);$$

minus any investment expense, i.e.

$$\left(k_{s^{t+2}} - \sum_{\tau=t+2-T}^{t+1} k^{E}_{\tau,s^{t+2}} \right) - (1-\delta) \cdot \left(k_{s^{t+1}} - \sum_{\tau=t+2-T}^{t} k^{E}_{\tau,s^{t+1}} \right);$$

while the value of securities issued one-period ahead consist of the capital plus the bubbles, i.e.

$$\left(k_{s^{t+2}} - \sum_{\tau=t+2-T}^{t+1} k^{E}_{\tau,s^{t+2}} \right) + b_{s^{t+1}}.$$

Adding these components, we obtain the first three terms of Equation (28). The last term of this Equation contains the securities issued by E-firms. Since these firms cannot commit to make any payment to the holders of those securities, they are backed up only by bubbles.

Equating demand and supply, we can now write the market-clearing condition in period t and history s^t for the security that pays in period $t+1$ and history s^{t+1} as follows:

$$\sum_{\tau=t+1-T}^{t} a_{\tau,s^{t+1}} = \left(\alpha \cdot A^{Q}_{s^{t+1}} \cdot l^{1-\alpha} \cdot k^{\alpha-1}_{s^{t+1}} + 1 - \delta \right) \cdot \left(k_{s^{t+1}} - \sum_{\tau=t+2-T}^{t} k^{E}_{\tau,s^{t+1}} \right) + b_{s^{t+1}} + \sum_{\tau=t+2-T}^{t} b^{E}_{\tau,s^{t+1}} \tag{30}$$

There is one such condition for each price $q_{s^t s^{t+1}}$ and, in period t and history s^t, there is one such price for any possible history s^{t+1} in period $t+1$. Equation (30) is quite intuitive and says that firms can promise payments to households that equal the dividends of N-firms plus the proceeds from any bubble or pyramid scheme that the firm runs. The only friction in this economy is that E-firms cannot commit to pay dividend. In the absence of bubbles, these firms would not exist and our economy would be nothing but an overlapping-generations version of the standard RBC model such as that of Rios-Rull (1996). But bubbles do exist in the equilibria of this world as we show next.

1.4 Equilibrium

A competitive equilibrium of this economy consists of a set of prices and quantities such that households and firms maximize and markets clear. As it is usual in models of bubbles there might

[13] Note that the capital in N-firms equals the aggregate stock of capital minus the part of it that is in E-firms, i.e. $k_{s^{t+1}} - \sum_{\tau=t+2-T}^{t} k^{E}_{\tau,s^{t+1}}$.

be many such equilibria, each of them implying a different stochastic process for the bubbles. In particular, we need to specify the behavior of existing bubbles, i.e. b_{s^t}, and newly created bubbles, i.e. b^E_{t-T,s^t}. Any stochastic process for these variables that satisfies Equation (29) could potentially be part of a competitive equilibrium.

2 Developing Intuitions

We have developed a model of bubbly business cycles that can be simulated numerically. Before doing so, however, it is useful to build intuitions by using a simplified version of the model. In particular, we modify the model of Section 1 along two dimensions: (i) we assume that $T = 1$, so that agents live for two periods, and; (ii) we assume that $\beta \to \infty$, so that individuals care only about consumption during old age. As we now show, these two simplifications allow us to solve the model in closed form and to illustrate the interactions between bubbles and economic fundamentals.

Under these assumptions, households work and save when young and consume off their savings when old. By solving the individual optimization problem subject to their budget constraints, it can be shown that the optimal consumption for household of generation t is given by

$$c_{t,s^t} = 0 \quad \text{and} \quad c_{t,s^{t+1}} = \frac{\left(q_{s^t,s^{t+1}}\right)^{-\frac{1}{\gamma}} \left(\pi_{s^t,s^{t+1}}\right)^{-\frac{1}{\gamma}}}{\sum\limits_{s^{t+1\prime} \in S_{t+1}} \left(q_{s^t,s^{t+1\prime}}\right)^{1-\frac{1}{\gamma}} \left(\pi_{s^t,s^{t+1\prime}}\right)^{-\frac{1}{\gamma}}} \cdot w_{s^t}, \tag{31}$$

Equation (31) has a very natural interpretation. Since $\beta \to \infty$, households postpone all of their consumption to their old age. Since $\gamma > 0$, however, they use asset markets to smooth this consumption across different histories.

As for firms, nothing fundamental changes in this simplified version of the model, except that $E-$firms are under the control of their founding entrepreneur for only one period. Recall that all such firms start with zero capital and bubbles and accumulate both types of assets as time passes. N-firms, as before, do not create any bubbles and so Equation (12) must hold. E-firms start with no bubble and so any expected overvaluation is pure bubble creation:

$$b^E_{t,s^t} = \sum_{s^{t+1} \in S_{t+1}} q_{s^t s^{t+1}} \cdot b^E_{t,s^{t+1}}, \tag{32}$$

where $b^E_{t,s^{t+1}}$ denotes total bubble creation by E-firms of generation t in history s^{t+1}. Since b_{s^t}

denotes the aggregate bubble, it follows that

$$b_{s^t} = \sum_{s^{t+1} \in S_{t+1}} q_{s^t s^{t+1}} \cdot \left(b_{s^{t+1}} - b^E_{t,s^{t+1}} \right). \tag{33}$$

Remember that a key aspect of the model is that bubble creation allows $E-$firms to borrow and finance their investments. This implies that $I^E_{t,s^t} = b^E_{t,s^t}$. We assume throughout, however, that the existing bubble and bubble creation are never large enough to eliminate investment in $N-$firms. Formally,

$$(1-\alpha) \cdot A^Q_{s^t} \cdot k^\alpha_{s^t} \cdot l^{1-\alpha} > k_{s^{t+1}} + b_{s^t} + b^E_{t,s^t}, \tag{34}$$

for all histories $s^t \in S_t$. Equation (34) requires the economy's total savings to be higher than the total value of assets of $N-$firms plus total investment by $E-$firms: this guarantees that, in all histories, there is some investment in $N-$firms.

Taking this into account, the law of motion of aggregate capital can be expressed as:

$$k_{s^{t+1}} = (1-\alpha) \cdot A^Q_{s^t} \cdot k^\alpha_{s^t} \cdot l^{1-\alpha} - b_{s^t} + \left(A^K - 1\right) \cdot b^E_{t,s^t}. \tag{35}$$

In the absence of bubbles, this law of motion reflects that the entire savings of young households are used to finance capital accumulation in $N-$firms. In the presence of bubbles, however, this is no longer true. First, part of the economy's savings must be devoted to the purchase of the bubble, which competes with capital for existing savings. This "crowding out" effect, which is captured by the second term of Equation (35), reduces investment in $N-$firms and thus capital accumulation one-to-one. Second, a part of total savings is diverted towards investment in $E-$firms, which are attractive to outside investors as long as there is some bubble creation. This "reallocation" effect, which is captured by the last term of Equation (35), enhances capital accumulation. To understand this term, note that bubble creation b^E_{t,s^t} reallocates investment towards $E-$firms by a factor of 1, and each unit reallocated entails an efficiency gain of $A^K - 1$. Depending on whether $A^K \lesseqgtr 1$, the aggregate bubble may have a positive or a negative effect on capital accumulation.

An equilibrium of this economy is thus characterized by a sequence for k_{s^t}, b_{s^t} and b^E_{t,s^t} satisfying Equations (33), (35) and (34), where b^E_{t,s^t} is as in Equation (32). Although the three equations also contain the prices of Arrow-Debreu securities $q_{s^t s^{t+1}}$, these are themselves functions of the aggregate capital stock and bubble in this economy.

This provides a full characterization of the equilibrium of our simplified model, which can be expressed in closed form and is therefore useful to illustrate the effects of bubbles on the economy. To do so, we can think that the economy fluctuates between states in which $b_{s^t} = 0$ and states in which $b_{s^t} > 0$. We say that the economy is in the fundamental state if $b_{s^t} = 0$. We say instead that the economy is experiencing a bubbly episode if $b_{s^t} > 0$. A bubbly episode starts when the economy leaves the fundamental state and ends the first period in which the economy returns to the fundamental state. Note that bubble creation b^E_{t,s^t} may be positive in the fundamental state as long as there is some positive probability that a bubbly episode starts next period. Let $v_{s^t} \in \{F, B\}$ be a sunspot variable that determines the state of the economy. We refer to v_t as investor sentiment. The transition probabilities $\Pr(v_{s^{t+1}} = F \,|v_{s^t} = B)$ and $\Pr(v_{s^{t+1}} = B \,|v_{s^t} = F)$ could be a function of any endogenous or exogenous variable of the model, and could fluctuate randomly over time.

To simplify the discussion, we will focus on a class of examples in which the probabilities of an episode beginning and ending are constant, i.e. $\Pr(v_{s^{t+1}} = B \,|v_{s^t} = F) = q$ and $\Pr(v_{s^{t+1}} = B \,|v_{s^t} = F) = p$; bubble creation during a bubbly episode is a constant share of the aggregate bubble, i.e. $b^E_{t,s^t} = n \cdot b_{s^t}$ with $n > 0$. These examples nicely capture the notion of a shock to investor sentiment. In any given history, $E-$firms are able to borrow and invest only to the extent that they are expected to be overvalued in the future. These expectations are self-fulfilling and this allows us to interpret transitions between these two states as shocks to investor sentiment.

This class of examples lends itself to a very simple analytical characterization if we further assume that there is full depreciation, i.e. $\delta = 1$, and that agents face no fundamental uncertainty when making their consumption and savings decisions, i.e. shocks to $A^Q_{s^{t+1}}$ are known in advance. These simplifications enable us to make the model recursive through a simple transformation of variables. Define x_t as the bubble's share of wealth or savings, i.e. $x_{s^t} \equiv \dfrac{b_{s^t}}{(1-\alpha) \cdot A^Q_{s^t} \cdot l^{1-\alpha} \cdot k^{\alpha}_{i,s^t}}$. Then, we can rewrite Equation (33) as follows:

$$\frac{\displaystyle\sum_{s^{t+1} \in S_{t+1}} \pi_{s^t s^{t+1}} \cdot \left(\frac{\alpha}{1-\alpha} + x_{s^{t+1}}\right)^{-\gamma}}{\displaystyle\sum_{s^{t+1'} \in S_{t+1}} \pi_{s^t s^{t+1'}} \cdot \left(\frac{\alpha}{1-\alpha} + x_{s^{t+1'}}\right)^{-\gamma}} \cdot \frac{x_{s^{t+1}}}{x_{s^t}} = \frac{\frac{\alpha}{1-\alpha} \cdot (1+n)}{1 + (\ (A^K - 1) \cdot n - 1) \cdot x_{s^t}}, \tag{36}$$

where $x_{s^{t+1}} > 0$ if $v_{s^{t+1}} = B$ and $x_{s^{t+1}} = 0$ if $v_{s^{t+1}} = F$. Naturally, the derivation of Equation (36) assumes that Equation (34) holds. This condition can now be rewritten as follows:

$$x_{s^t} \leq \frac{1}{1+n} \equiv \bar{x}. \tag{37}$$

The interesting feature of this class of bubbly episodes is that the capital stock does not appear in Equations (36) and (37). Any path for x_{s^t} that that satisfies these equation in all histories is an equilibrium of the economy. Since $x_{s^t} = 0$ does this, we trivially have that such a path always exists. Of course, the interesting challenge is to characterize other possible paths. After doing so, we can then use Equation (35) to determine the associated paths for the capital stock, which is given by

$$k_{s^{t+1}} = \left[1 + \left((A^K - 1) \cdot n - 1\right) \cdot x_{s^t}\right] \cdot (1 - \alpha) \cdot A^Q_{s^t} \cdot k^{\alpha}_{s^t} \cdot l^{1-\alpha}. \tag{38}$$

As Equation (38) illustrates, these class of examples can generate two types of bubbly episodes. The first type is the conventional or contractionary bubbly episode, which require that $(A^K - 1) \cdot n < 1$. These episodes, which were emphasized by Tirole (1985), occur in economies where some investments are dynamically inefficient in the fundamental state. This condition ensures that bubbles have a negative effect on capital accumulation, as the reduction of investment in $N-$firms is not compensated by the increase in the average efficiency of investment. Bubbles lower the capital stock and output. The second type of bubbly episodes, which we shall focus on here, are the non-conventional or expansionary ones that require $(A^K - 1) \cdot n > 1$. These episodes, which are analyzed by Martin and Ventura (2011b) in a related model, arise in economies with financial frictions and they exist even if all investments are dynamically efficient in the fundamental state. This condition ensures that bubbles have a positive effect on capital accumulation, as the reduction of investment in $N-$firms is more than compensated by the increase in the average efficiency of investment. These bubbles increase the capital stock and output.

We can illustrate the economic effects of bubbles with a number of specific cases within this general class of examples. All of the examples consider an economy that is dynamically efficient in the fundamental equilibrium.[14] This implies, in particular, that the only admissible bubbly episodes are expansionary ones. We consider different such episodes below.

2.1 Example 1: deterministic economy

Consider first the case of a deterministic economy that is not exposed to technology shocks (i.e. $A^Q_{s^t} = \overline{A^Q}$) and that experiences a bubbly episode that never ends (i.e. $p = 0$). Figure 3 plots the steady-state values of k_{s^t}, b_{s^t}, c_{s^t} and the risk-free interest rate for this "bubbly" economy. For convenience, the dotted lines represent the steady-state values of the same variables when there is no bubble.

In the bubbly economy, a permanently high level of investor sentiment sustains current and future bubbles. Bubble creation in each period helps the economy to partially overcome the contracting friction between entrepreneurs and outside investors, enabling $E-$firms to borrow and raising efficient investment. Relative to the bubbleless steady-state, the bubbly steady-state dis-

[14] Table 1 in the Appendix describes the values of the coefficients that have been used for the numerical examples.

plays higher levels of the capital stock and consumption and a lower interest rate. In our particular example, a bubble that amounts to a bare 4% of output is able to sustain a six-fold increase in the steady-state levels of capital and consumption.

During a bubbly episode, the young as a whole are reducing their investments in capital in order to invest in bubbles. They do so voluntarily in the expectation that the revenues from selling these bubbles will exceed the foregone investment income. These revenues, in turn, are nothing but the reduction in the investments of the next generation of young minus the value of any new bubbles being created. It follows that bubbly episodes can only be possible if there exists a chain of investments that is expected to absorb resources from the economy, that is, a chain whose cost is expected to exceed the income it produces in all periods. Let us denote such a chain of investments as "dynamically inefficient".[15]

Does a "dynamically inefficient" chain of investments exist in the fundamental equilibrium of our economy? The answer is negative. In the fundamental equilibrium, all capital accumulation is undertaken by $N-$firms: thus, the only possibility for such a chain to exist is that the chain of all investments in the economy absorbs more resources than it produces, i.e. for total savings to be higher than total capital income. In our economy, total savings amount to $(1-\alpha)\cdot \overline{A^Q} \cdot l^{1-\alpha} \cdot k^{\alpha}_{s^{t+1}}$ whereas total capital income amounts to $\alpha \cdot \overline{A^Q} \cdot l^{1-\alpha} \cdot k^{\alpha}_{s^{t+1}}$. Consequently, dynamic inefficiency in the fundamental equilibrium requires that $\alpha < 0.5$, which is not satisfied in our numerical example. Nonetheless, our example illustrates that expansionary bubbly episodes are indeed possible in equilibrium. The reason is that these episodes raise the capital stock of the economy and reduce the return to capital accumulation, thereby pushing some investments towards dynamic inefficiency. In this sense, an expansionary episode of the type considered here generates its own demand by creating the inefficient investments that it will absorb.

2.2 Example 2: stochastic economy with deterministic bubble

This example analyzes an economy that experiences i.i.d. shocks to total factor productivity (i.e. $A^Q_{s^t} \in [A_L, A_H]$) and that experiences a deterministic bubbly episode with $p = 0$. Figure 4 plots the evolution of k_{s^t}, b_{s^t}, c_{s^t}, total credit to $E-$firms, and the risk-free interest rate for a sequence of shocks to $A^Q_{s^t}$. The figure shows that, as could be expected, the economy displays pro-cyclical behavior with respect to productivity shocks. High values of $A^Q_{s^t}$ raise contemporaneous output and consumption. By expanding overall savings and capital accumulation, high realizations of $A^Q_{s^t}$ also lower the interest rate, increasing bubble creation and enabling $E-$firms to expand their borrowing

[15] See Martin and Ventura (2011b) for a detailed discussion on the relationship between the existence of rational bubbles and dynamic inefficiency.

and investment. Thus, the presence of an aggregate bubble amplifies the effects of shocks to $A^Q_{s^t}$, exacerbating volatility.

In Figure 5 we also explore i.i.d. shocks to the productivity of investment (i.e. $A^K_{s^t} \in \left[A^K_L, A^K_H\right]$ for each s^t). In the absence of a bubble, these shocks would have no effects since $E-$firms would not exist. In the presence of a bubble, however, high values of $A^K_{s^t}$ lead to high levels of output and consumption with a one-period lag.

Besides their traditional effects on output and consumption, shocks to $A^Q_{s^t}$ also have a direct impact on the aggregate bubble. Changes in $A^Q_{s^t}$, for instance, affect the bubble in the same proportion as output. This is why, as can be seen in Equation (36), $A^Q_{s^t}$ does not affect the evolution of x_{s^t}. The same cannot be said of changes in $A^K_{s^t}$, which have an inverse effect on the growth rate of x_{s^t}. The reason is that, even though they have no contemporaneous effect on output, high realizations of $A^K_{s^t}$ lower the interest rate and this decreases the equilibrium growth rate of the aggregate bubble.

The main message of this example is that, even in the absence of bubble shocks, the mere presence of bubbles exacerbates volatility. This is because bubble creation increases intermediation and enables $E-$firms to expand their borrowing and investment. Consequently, the effect of technology shocks (to $A^Q_{s^t}$ or $A^K_{s^t}$) on the efficiency of investment is directly related to the amount of bubble creation and thus, in our example, to the size of the aggregate bubble. In the particular simulations of Figures 4 and 5, the variance of the capital stock and of consumption are many orders of magnitude larger than the variance of the underlying shocks technology shocks. But this is no longer true if we subject the economy to the same realization of shocks in the absence of any bubbles, present or future. In that case, the variance of the capital stock and of consumption are drastically reduced: they would be respectively lower than and approximately equal to the variance of shocks to $A^Q_{s^t}$. This reduction in the variance of capital and consumption would be even more dramatic with respect to shocks to the efficiency of investment $A^K_{s^t}$, which would have no effect whatsoever in the bubbleless economy because the investment of $E-$firms would equal zero in all histories.[16].

[16] The fact that

$$\frac{var(k)}{var(A^K_{s^t})} = 0$$

in the fundamental economy is a particular feature of our model, which arises because $E-$firms cannot obtain any funds if there is no bubble creation. Even if we allowed these firms to obtain resources from some other source, however, it would still be true that the existence of bubble creation would increase the effect of shocks to $A^K_{s^t}$ on the capital stock.

2.3 Example 3: bubbly business cycles

The previous example illustrated how bubbles can exacerbate volatility by amplifying the effects of productivity shocks. But bubbly episodes can also be a source of economic volatility in themselves. To see this, consider an economy that is not subject to any technology shocks, i.e. $A^K_{s^t} = \overline{A^K}$ and $A^Q_{s^t} = \overline{A^Q}$ for each s^t, but that is exposed to stochastic bubbly episodes. In particular, we want to think of the economy as oscillating between bubbly episodes and the fundamental state, i.e. as experiencing bubbly episodes, i.e. with $p > 0$ and $q > 0$.

Figure 6 simulates an economy that is exposed to this type of episodes. Besides the realization of the sunspot variable v_{s^t}, we allow for (i) shocks to x_{s^t} during the bubbly episode and (ii) shocks to $x^E_{s^t}$ in the fundamental state. The figure plots the evolution of capital, the aggregate bubble, consumption, total credit to $E-$firms, the riskless interest rate and security prices. A striking feature that immediately stands out is that the effects of investor sentiment shocks are potentially very large. The economy undergoes three full bubbly episodes, each lasting more than 20 periods, during which the bubble peaks at approximately 8% of wages. In each of these episodes, the capital stock, consumption and total credit to entrepreneur-run firms increases by 500% or more. This growth is particularly astounding for total credit to $E-$firms, which slows down to a trickle during fundamental states.

In order to get a better sense of the dynamics of a bubbly episode, Figure 7 isolates the first such episode of the previous simulations. During this bubbly episode, the economy's capital stock increases from approximately 2 to 12 in little more than 10 periods. Consumption shows an increase that is only slightly smaller in magnitude, whereas total credit to $E-$firms increases from practically zero to roughly 40% of output. This economic expansion is accompanied by a sharp reduction in the riskless interest rate, which follows the marginal product of capital. When the bubble bursts, there is a dramatic contraction that essentially undoes all of the previous growth in two periods.

Needless to say, the point of this example is not to make a realistic claim about the quantitative importance of investor sentiment shocks. It is nonetheless striking that these shocks can have such large equilibrium effects in an economy populated by rational agents. Since these agents are risk-averse, it might seem somewhat surprising that they are willing to hold a large aggregate bubble that may disappear overnight. But risk aversion actually makes the equilibrium bubble larger and this economy more volatile! Figure 8 illustrates this by subjecting two economies, which differ only on the degree of risk aversion of their agents, to the same bubbly episodes. When the coefficient of relative risk aversion is increased from 2 to 8, the variances of the capital stock and of consumption basically double. Interestingly, the figure also shows how – during the bubbly episodes– the increase in risk aversion raises the price of securities that deliver in the fundamental state, as would be expected given the greater demand for insurance.

2.4 Example 4: Stochastic economy and bubble

In this section, we use an economy that has both technological and bubble shocks to summarize the previous discussion (see Figure 9).

3 An interpretation of recent U.S. macroeconomic history

[UNDER CONSTRUCTION]

4 Concluding remarks

This paper has attempted to develop a quantitative model of the financial accelerator with bubbles. In this model, business cycles are driven by two types of shocks: fundamental shocks that affect technology; and bubble shocks that affect investor sentiment lead to the appearance and collapse of pyramid schemes in financial markets. Our objective has been to explore the business cycles that arise in the model and relate them to actual data. In the immediate future, we plan to calibrate the model with US data use it to explore the relative importance of both types of shocks in recent US macroeconomic history.

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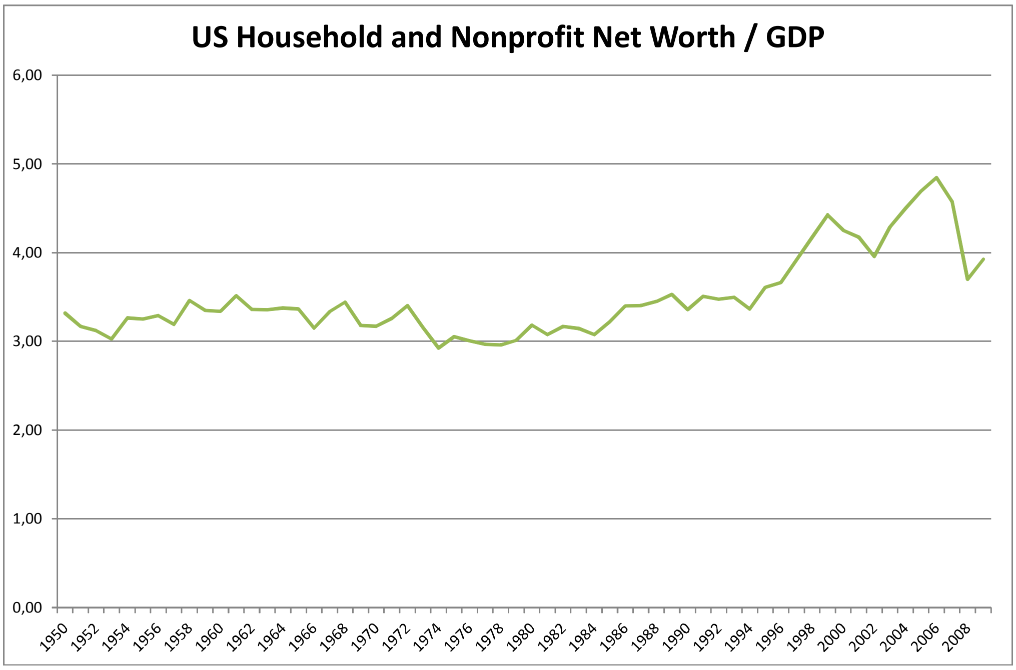
5 Appendix

Table 1: Parameter values for figures

Parameter	Description	Value	Shock
α	Capital Share	2/3	-
γ	Risk Aversion Coefficient	2	$\gamma' = 8$
A^Q	Total Factor Productivity	3	$A^{Q\prime} = [2.9841, 3.0159]$
A^K	Investment Productivity	3.75	$A^{K\prime} = [3.75, 3.8]$
$\overline{x}$	Initial bubble	0.02	$\overline{x}' = [0.02, 0.05]$
n	Growth Rate of Bubble	0.14	$n' = [0.0830, 0.1970]$
π_{LL}	Probability of remaining in low bubble state	0.85	-
π_{HH}	Probability of remaining in high bubble state	0.95	-


US Household and Nonprofit Net Worth / GDP
6,00
5,00
4,00
3,00
2,00
1,00
0,00
1950
1952
1954
1956
1958
1960
1962
1964
1966
1968
1970
1972
1974
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2002
2004
2006
2008

Figure 1

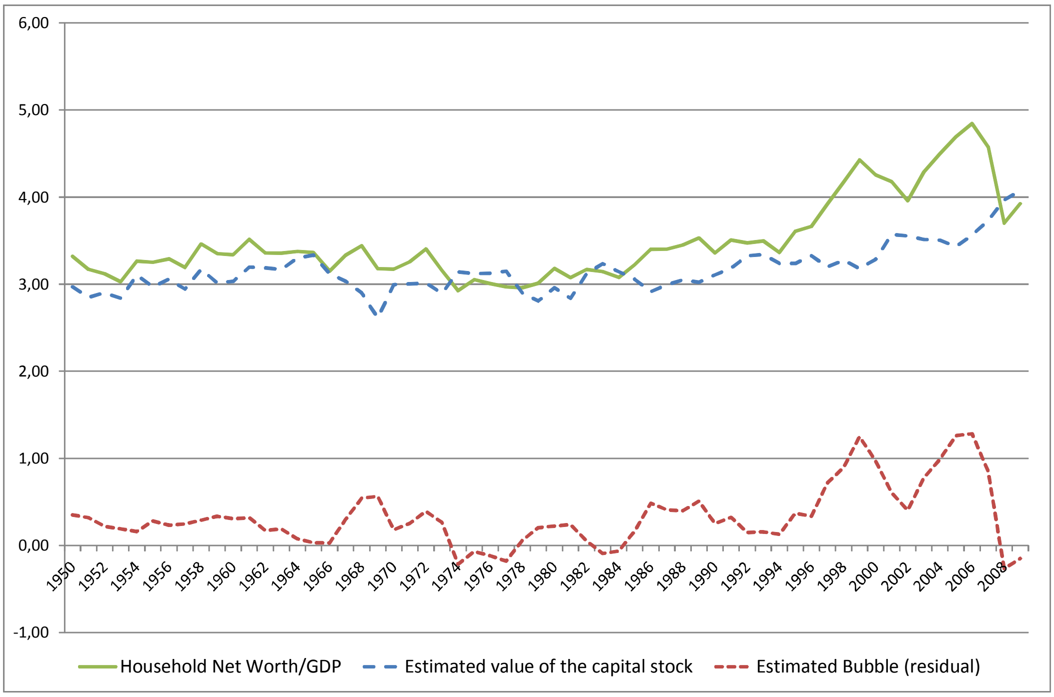
6,00
5,00
4,00
3,00
2,00
1,00
0,00
-1,00
1950
1952
1954
1956
1958
1960
1962
1964
1966
1968
1970
1972
1974
1976
1978
1980
1982
1984
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
Household Net Worth/GDP
Estimated value of the capital stock
Estimated Bubble (residual)

Figure 2

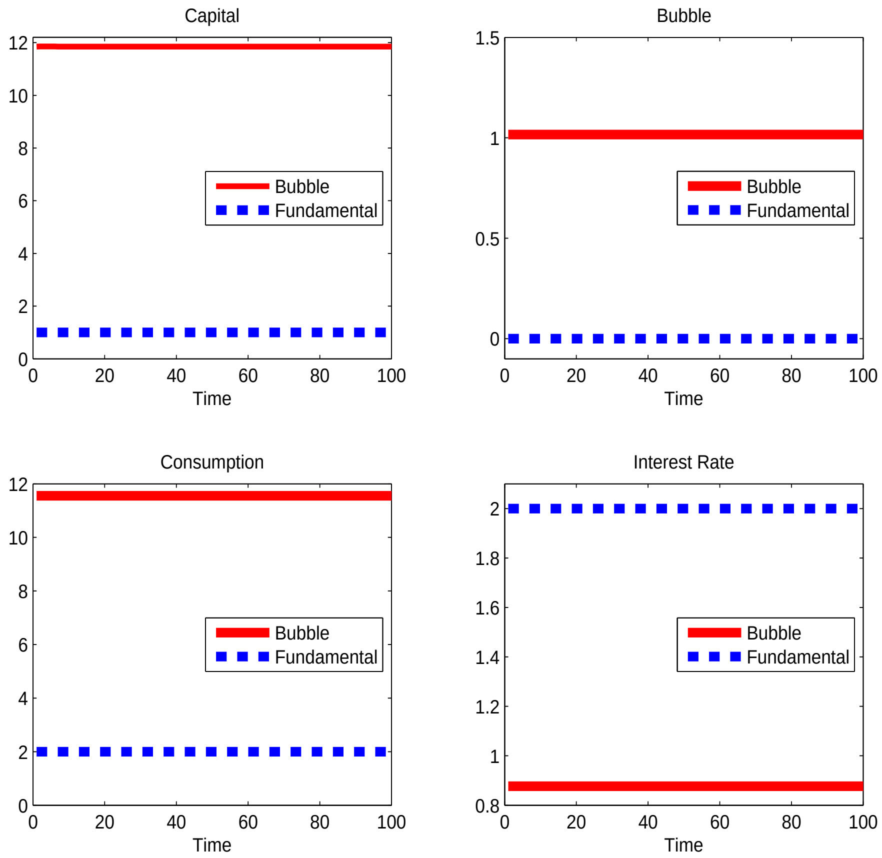

Figure 3: Deterministic steady state, with and without bubble

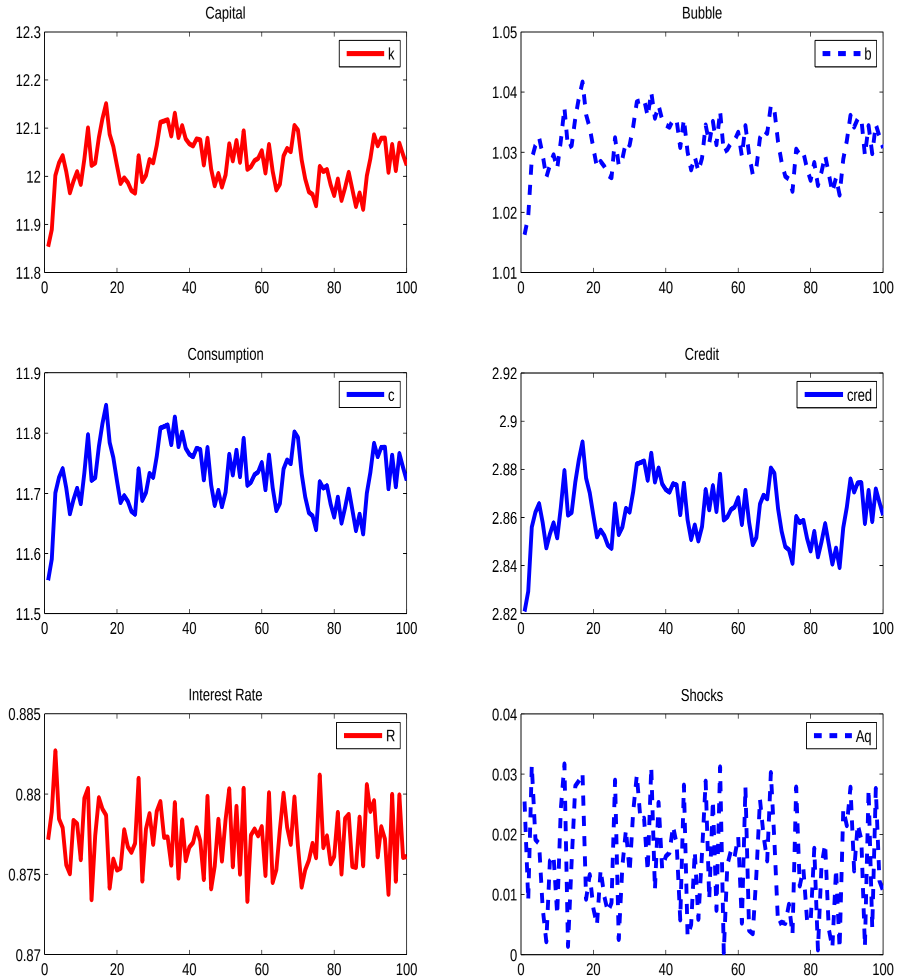

Figure 4: Deterministic bubble and shocks to A^Q.

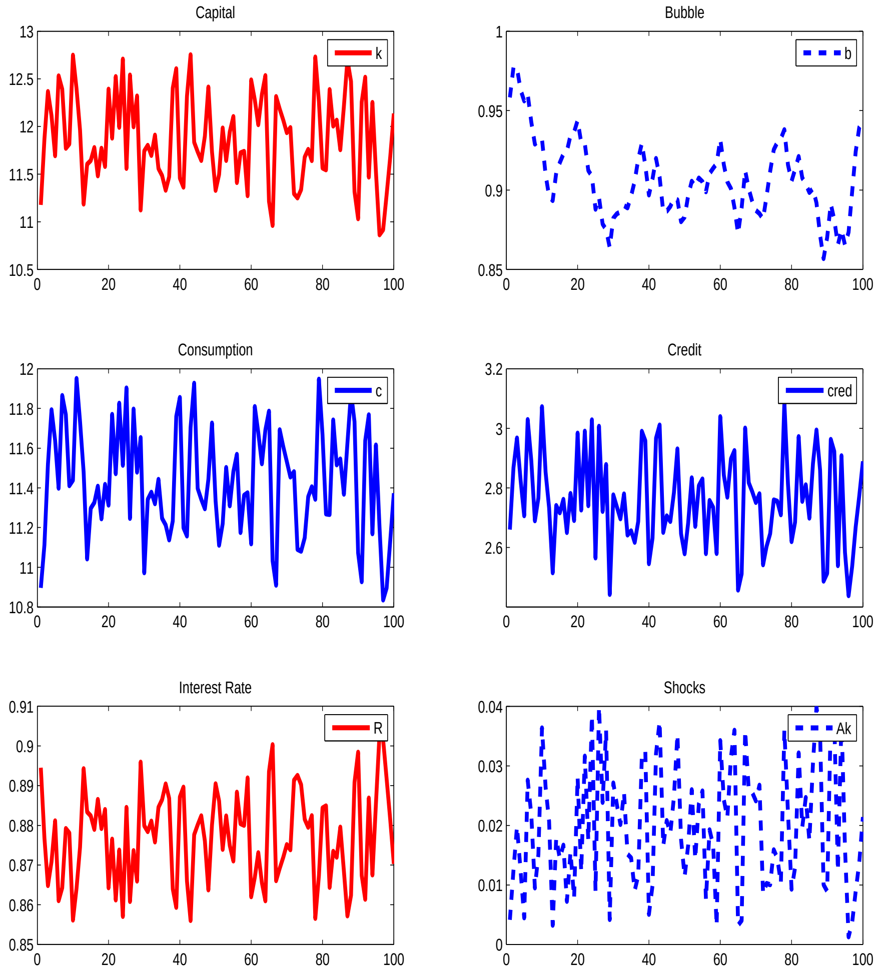

Figure 5: Deterministic bubble and shocks to A^K

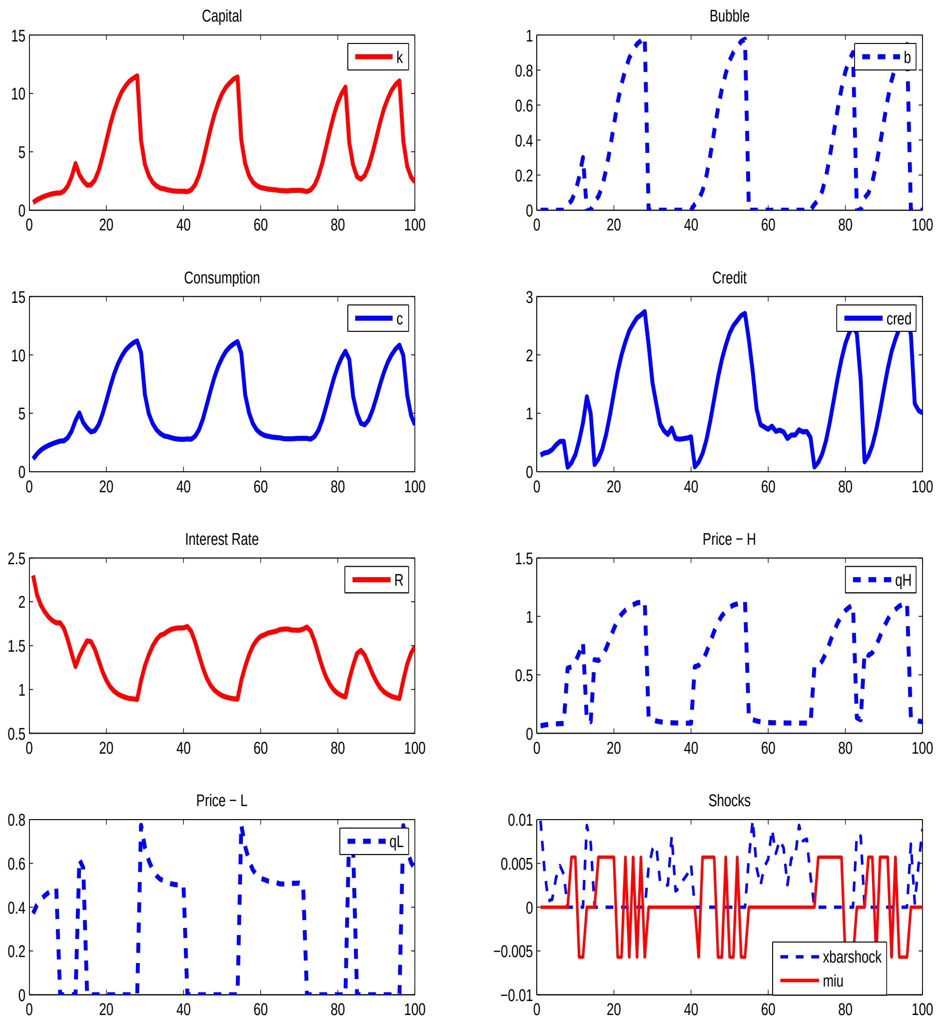

Figure 6: Bubble shocks

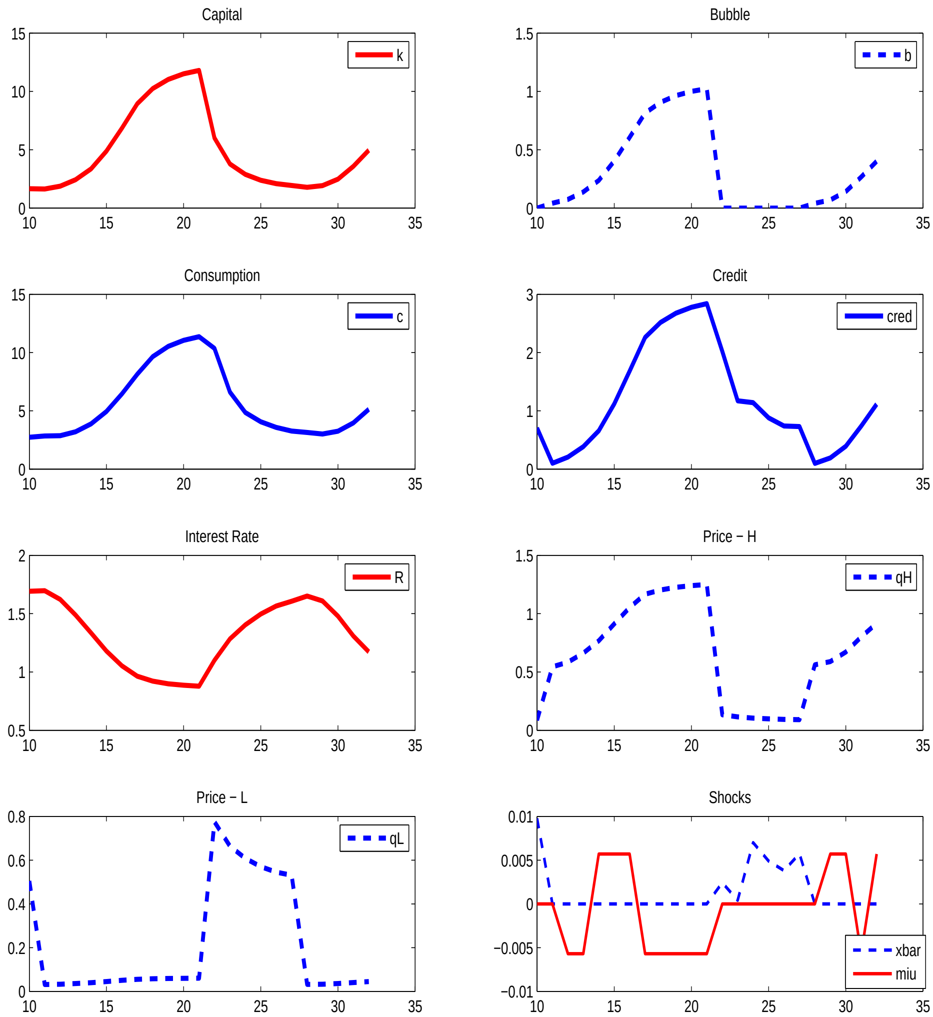

Figure 7: A bubbly event.

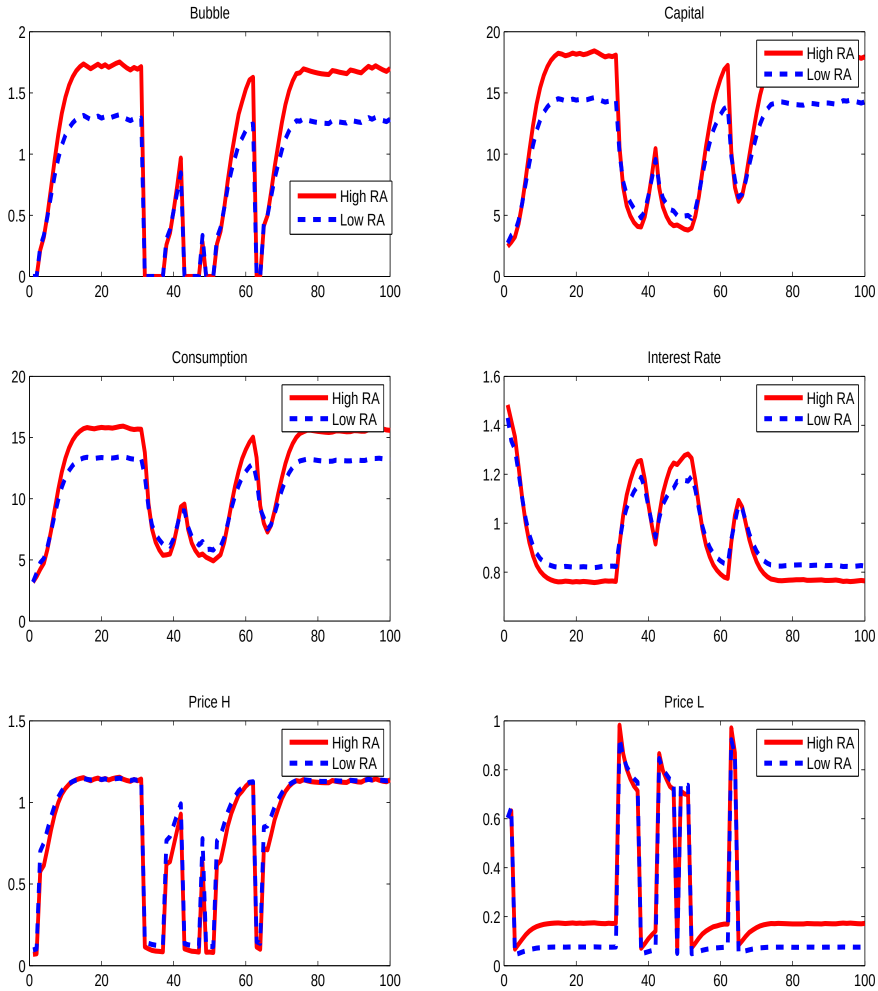

Figure 8: Bubble shocks and risk aversion.

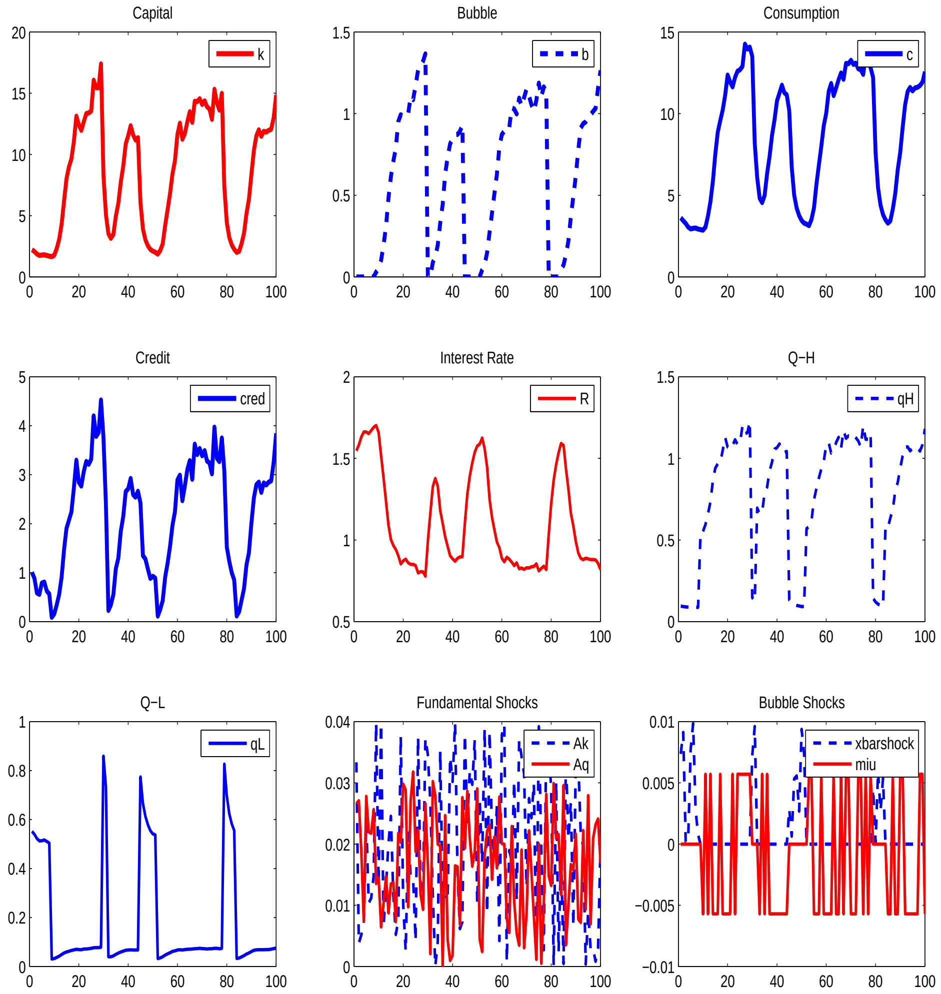

Figure 9: All shocks