Optimal Sovereign Debt Default*

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Abstract

We determine optimal government default policies for a small open economy in which a domestic government can borrow internationally by issuing non-contingent debt contracts. Unlike earlier work, we consider optimal default policies under full government commitment and treat repayment of international debt as a decision variable. When government bond markets are incomplete, default can be optimal under commitment as it allows for increased international diversification of domestic output and consumption risk. In the absence of default costs, default optimally occurs very frequently and independently of the country's net foreign asset position. This drastically changes when a government default entails small but positive dead weight costs: default is then optimal only in response to disaster-like shocks to domestic output, or when a small adverse shock pushes international debt levels sufficiently close to the country's borrowing limit. Optimal default policies increase welfare significantly compared to a situation where default is ruled out by assumption, even for sizable default costs. For sufficiently low level of default costs the optimal default policies can approximately be replicated by issuing simple equity-like government bonds.

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1 Introduction

Sovereign debt crises are by no means rare events in history.[1] These crises and the subsequent debt defaults are widely believed to occur because governments

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[1] Over the past decade governments in Argentina, Uruguay, Moldova and the Dominican Republic partially defaulted on their debt, with the Argentinian default being in dollar terms the largest ever recorded in history. The governments in Greece, Ireland and Portugal have recently been forced to apply for foreign assistance.

are simply unwilling to honor initially promised payment streams and because there exist insufficient incentives making repayment optimal ex-post from the country's perspective. The weakness of ex-post incentives is thereby routinely attributed to 'sovereign immunity' which presumably protects governments from being sued in courts.[2] Viewed through this lens, the option of a sovereign to default is inefficient from an ex-ante welfare perspective, as anticipation of a possible default constrains international borrowing to suboptimally low levels.

In this paper we propose to interpret sovereign default events in a fundamentally different way. Instead of being the result of insufficient ex-post incentives in a situation without commitment, we propose interpreting sovereign defaults as an opportunity for more efficient international risk sharing in a situation where government debt is non-contingent. This interpretation has previously been advanced in Grossmann and Van Huyck (1988) who distinguish between 'excusable' and 'non-excusable' default, with the first being part of an ex-ante anticipated risk-sharing arrangement between the borrower and the lender, and the latter being the result of debt repudiation in the presence of weak ex-post incentives for repayment. Unlike Grossmann and Van Huyck, however, we consider a situation where the government possesses full commitment, thus discuss optimal borrowing and default from a purely normative perspective. And as we show, it has profound implications for the optimal default patterns. While in models with limited commitment the incentives to default are strongest in good times[3], the present model predicts default to be optimal in low output states.

The assumption of committed sovereigns is more plausible than generally recognized in the economics literature. First, as argued in Panizza, Sturzenegger and Zettelmeyer (2009), legal changes in a range of countries in the late 1970's and early 1980's eliminated the legal principle of 'sovereign immunity' when it comes to sovereign borrowing. Specifically, in the U.S. and the U.K. private parties can sue foreign governments in courts, if the complaint relates to a commercial activity, amongst which courts regularly count the issuance of sovereign bonds. Second, although there now exists a voluminous literature on potential mechanisms supporting sovereign debt in the absence of commitment, these mechanisms have received limited empirical support.[4] In

[2] Following this view, the main economic puzzle is then to explain how government debt can exist at all, if debt repudiation is an option available to sovereign debtors. An important literature, starting with the classic paper by Eaton and Gersovitz (1981) and ranging all the way to recent contribution by Broner, Martin and Ventura (2010), has examined this view.

[3] Grossmann and Van Huyck (1988), for example, state: 'the incentive to repudiate is largest in the good state' (p.1095). Recent work by Mendoza and Yue (2008) overcomes this problem and generates counterycyclical default by incorporating the effects of sovereign default on the default of domestic firms and the availability of foreign imports as inputs into domestic production.

[4] In the words of Panizza et al. (2009): *'Almost three decades after Eaton and Gersovitz' pathbreaking contribution there still exists no fully satisfactory answer to how sovereign debt can exist in the first place. None of the default punishments that the classic theory of*

the light of these facts, it appears natural to deduce that governments can issue debt simply because they can in fact credibly commit to repay debt in some future states of the world, although they might actually choose not to repay in some states in which repayment turns out to be excessively costly.

To analyze the role of sovereign debt default as a vehicle for international risk shifting in a setting with a committed government, we construct a small open economy with production in which a domestic government can internationally borrow by issuing non-contingent bonds. The government can also accumulate international reserves by investing in (riskless) bonds issued by foreign lenders. The domestic economy is subject to shocks that affect the productivity of the domestic capital stock and the government can smooth the consumption implications of such shocks either via borrowing and lending in international capital markets or via defaulting on its debt. The paper is concerned with the question of which channel the government should rely on to smooth domestic consumption, and specifically with the question: when is it optimal to (partially) default on government debt in a setting with a fully committed government?

In a first step, we analytically show that in the absence of default costs, optimal government default decisions can implement the first best consumption allocation and achieve full domestic consumption smoothing. The level of default is then generally decreasing with aggregate productivity and (partial) debt default occurs frequently and for all but the best productivity realization. In the absence of default costs, allowing the government to choose whether or not to repay government debt is thus a way to achieve the same consumption allocation as in a setting with a complete set of contingent government debt instruments.

In a second step, we introduce default costs. These costs feature prominently in political discussions and we model them as a simple dead weight cost that is proportional to the size of the government debt default. We show how low levels for the default costs make it generally optimal for the government *not* to default following business cycle sized shocks. Only when the country's net foreign debt position approaches the maximum level implied by the (marginally binding) natural borrowing limits, is a sovereign default still optimal after an adverse shock. With positive default costs, the optimal default policy thus depends on whether or not the country is close to its maximally sustainable net foreign debt position.

Given that small amounts of default costs largely eliminate government debt default, we introduce economic 'disaster' risk into the aggregate productivity process, following Barro and Jin (2011). Default then reemerges as part of optimal government policy, following the occurrence of a disaster shock. This is the case even for sizable default costs and even when the country's net foreign asset position is far from its maximally sustainable level. It continues to be optimal, however, not to default following business cycle sized

sovereign debt has focused on appears to enjoy much empirical backing' (p. 692).

shocks to aggregate productivity, as long as the net foreign debt position is not too close to its maximal level.

Finally, we evaluate the utility consequences of using the government default option as a way to insure domestic consumption against aggregate productivity shocks, comparing it to a situation where the government is *assumed* to repay debt unconditionally. In the latter case the government can use international wealth adjustments only to smooth domestic consumption. We show that the consumption equivalent welfare gain from considering default is in the order of one to two percentage points of consumption each period, even when there are sizable dead weight cost associated with a government debt default. If the default costs are sufficiently low, a large share of this welfare increase can be captured if instead of defaulting, the government optimally issues a combination on non-defaultable bonds and equity-like bonds that do not repay when one of the economic disaster states materializes. We thereby assume that non-repayment on the equity bond generates the same dead weight costs as an outright default. For higher levels of the dead weight costs, we show that outright default dominates the issuance of a combination of non-defaultable and equity bonds.

Sims (2001) discusses insurance in the context of whether or not Mexico should dollarize its economy, showing that giving up the domestic currency allows for less insurance in the presence of non-contingent nominal debt because the government is deprived of using the price level as a shock absorber. Unlike in the work of Sims, who considers non-contingent nominal bonds, the present paper considers a setting with non-contingent real bonds and considers optimal outright default policies. In the light of Sims' discussion, one could interpret the setting analyzed in the present paper as one in which the government issues (non-contingent) nominal bonds but has given up control over monetary policy and the price level, e.g., via joining a monetary union. As we show, the default option then still provides the country with a mechanism to make bond repayments contingent.

Angeletos (2002) explores an alternative insurance channel in a closed economy setting, showing that a government can use the maturity structure of domestic government bonds to insure against domestic shocks. This is achieved by exploiting the fact that bond yields of different maturities react differently to domestic shocks. This channel is unavailable, however, in our small open economy setting: in the absence of domestic default, the domestic yield curve is identical to the foreign yield curve for risk free assets and thus also independent of domestic shocks.

Juessen, Linnemann and Schabert (2010) also analyze government default and the behavior of government bond premia. Considering a setting in which government behavior is characterized by simple rules, they show that multiple equilibria with different risk premia and default probabilities exist.

The present paper is structured as follows. Section 2 introduces the economic model and derives the optimal policy problem. It also determines an equivalent formulation of the optimal policy problem that facilitates numerical solution of optimal policies. Section 3 derives an analytical result for the case with no default costs and section 4 evaluates the effects of introducing default costs in a setting with business cycle sized shocks. We then introduce economic disaster shocks in section 5 and discuss their quantitative implications for optimal default policies. In section 6 we consider the welfare implications of using the default option and show how optimal default policies can approximately be implemented with a simple equity-like government bond instrument. In section 7 we discuss the effects of introducing long maturity bonds. A conclusion briefly summarizes. Technical material is contained in a series of appendices.

2 The Model

This section introduces a small open production economy and derives the government's optimal policy problem.

2.1 Private Sector: Households and Firms

The household side of the domestic economy is described by a representative consumer with utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

where $\beta \in (0,1)$ denotes the discount factor and $u(c)$ the period utility function. The latter is assumed to be twice continuously differentiable, increasing in c and strictly concave, for all values of $c > \overline{c}$ where $\bar{c} \geq 0$ denotes the subsistence level for consumption. We shall assume that $u(c) = -\infty$ for all $c \leq \overline{c}$ and that Inada conditions hold, i.e., $\lim_{c \to \overline{c}^+} u'(c) = +\infty$ and $\lim_{c \to \infty} u'(c) = 0$.

The production side of the economy is described by a representative firm which produces consumption goods using the production function

$$y_t = z_t k_{t-1}^{\alpha},$$

where y_t denotes output in period t, k_{t-1} the capital stock from the previous period, $\alpha \in (0,1)$ the capital share, and $z_t > 0$ an exogenous stochastic productivity disturbance. Productivity shocks assume values from some finite set $Z = \left\{z^1, ..., z^N\right\}$ with $N \in \mathbb{N}$. The transition probabilities for productivity across periods are described by some measure $\pi(z'|z)$ for $z', z \in Z$. Firms are owned by households and must decide on the capital stock one period in advance, i.e., before future productivity is known. For simplicity we assume that capital depreciates fully after one period.

2.2 The Government

The government seeks to maximize the utility of the representative domestic household (1) and is fully committed to its plans. It can invest in riskless international bonds issued by foreign lenders, issue own non-contingent bonds, and decide on the repayment of its maturing bonds. Unless otherwise stated, we assume that the risk free interest rate r on international bonds satisfies $\frac{1}{1+r} = \beta$. Furthermore, we assume that all bonds are zero coupon bonds and have a maturity of one period. The effects of introducing also longer maturity domestic bonds are discussed in section 7.[5]

The government's holdings of international bonds in period t (which mature in period $t+1$) constitutes a long position and is denoted by $G_t^L \geq 0$. The own (potentially risky) bonds issued by the government in period t represents a short position and is denoted by $G_t^S \geq 0$. The government can use adjustments in the long and short positions to insure domestic consumption against domestic productivity shocks. In addition, it can decide in period t to (partially) default on the bonds maturing in period $t+1$. More formally, the default decision is described by a vector of default profiles

$$\Delta_t = (\delta_t^1, ..., \delta_t^N) \in [0,1]^N \,,$$

where $\delta_t^n \in [0,1]$ denotes the fraction of outstanding domestic bonds issued in period t that is *not* repaid in period $t+1$ when the bonds mature and when the productivity state is $z_{t+1} = z^n$. Default is thus state-contingent and an entry equal to zero indicates full repayment. Full repayment is typically *assumed* in much of the previous literature dealing with optimal fiscal policy under commitment and incomplete markets, e.g., Angeletos (2002), Buera and Nicolini (2004) or Marcet and Scott (2009). In the present setting repayment is treated as a choice variable.

Total repayment on maturing domestic bonds in period $t+1$ when productivity is equal to z_{t+1} is then given by

$$G_t^S \cdot (1 - (1-\lambda)\delta_t^{I(z_{t+1})}) \tag{2}$$

where $I(z_{t+1})$ denotes the index of the productivity shock.[6] The parameter $\lambda \geq 0$ captures the possibility that the government's default decision gives rise to dead weight costs, i.e., resource costs born by the borrower which do no accrue to the lender. Such costs plausibly arise because legally the debt contract specifies an unconditional payment that is independent of the productivity state. Therefore, even if the productivity state z_{t+1} is perfectly observable and the borrower and the lender fully understand the state-contingent nature of debt repayment, the lender has ex-post an incentive to sue the borrower in court for full payment whenever a default occurs in $t+1$. This is especially relevant if it is impossible for the lender to commit ex-ante not to go to court, a situation that appears relevant, given that is typically possible to sell government debt in secondary markets to other investors. The specification in equation (2) assumes that the judicial costs associated with warding off repayment claims in court (or any other costs associated with default decisions) are

[5] The fact that we consider only a single maturity for the international bond is without loss of generality. Since foreign interest rates are independent of domestic conditions, the government cannot use the maturity structure of foreign bonds to insure against domestic shocks.

[6] $I(z_{t+1}) = n$ if and only if $z_{t+1} = z^n$.

equal to $\lambda \in [0,1]$ times the default amount. This follows the specifications in Zame (1993) and Dubey, Geanakoplos and Shubik (2005) who previously introduced proportional default costs in a general equilibrium model with incomplete markets.[7] For the case with $\lambda = 0$, a default is costless for the country, while for the polar case $\lambda = 1$, a default never pays because the resources saved via a default are exactly equal to the judicial costs associated with not having fully repaid the bond to lenders.

We can now define the amount of resources available to the domestic government at the beginning of the period, i.e., before issuing new debt and making investment decisions on international bonds, but after (partial) repayment of maturing bonds.[8] We refer to these resources as beginning-of-period wealth and define them as

$$w_t \equiv z_t k_{t-1}^{\alpha} + G_{t-1}^{L} - G_{t-1}^{S} \cdot (1 - (1-\lambda)\delta_{t-1}^{I(z_t)})$$

Beginning-of-period wealth will serve as a useful state variable when computing optimal government policies later on. The government can raise additional resources in period t by issuing own government bonds. It can then use the resulting funds to invest in international riskless bonds, to invest in the domestic capital stock, and to finance domestic consumption. The economy's budget constraint is thus given by

$$c_t + k_t + \frac{1}{1+r} G_t^L = w_t + \frac{1}{1+R(z_t,\Delta_t)} G_t^S$$

where $\frac{1}{1+r}$ denotes the price of the risk-free international bond and $\frac{1}{1+R(z_t,\Delta_t)}$ the price of the domestic bond. The real interest rate $R(z_t, \Delta_t)$ of the domestic bond depends on the default profile Δ_t chosen by the government and on the current productivity state, as it may affect the likelihood of entering different states tomorrow. Due to the small open economy assumption, the government takes the pricing function $R(\cdot,\cdot)$ as given in its optimization problem. Assuming risk-neutral international lenders, no-arbitrage implies that the pricing function for domestic bonds is given by

$$\frac{1}{1+R(z_t,\Delta_t)} = \frac{1}{1+r}\sum_{n=1}^{N}(1-\delta_t^n)\cdot \pi(z^n|z_t)$$

so that the expected return on the domestic bond is equal to the return on the riskless international bond.

[7] Default costs in our setting represent a resource cost, while the general equilibrium literature models default cost as a direct utility cost, which enters separably into the borrower's utility function. While it is difficult to interpret legal costs as a direct utility cost, we conjecture that imposing a direct utility cost instead of a resource cost would deliver very similar default implications.

[8] Below we do not distinguish between the government budget and the household budget, instead consider the economy wide resources that are available. This implicitly assumes that the government can costlessly transfer resources between these two budgets, e.g., via lump sum taxes.

2.3 Alternative Interpretation of Default Costs

[to be added]

2.4 The Government's Optimal Policy Problem

We are now in a position to formulate the government's optimal policy problem (Ramsey allocation problem):

$$\max_{\{G_t^L \geq 0, G_t^S \geq 0, \Delta_t \in [0,1]^N, k_t \geq 0, c_t \geq \bar{c}\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{3a}$$

$$s.t.: c_t = w_t - k_t + \frac{G_t^S}{1 + R(z_t, \Delta_t)} - \frac{G_t^L}{1+r} \tag{3b}$$

$$w_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z \tag{3c}$$

$$w_0, z_0 : given$$

We have added the natural borrowing limits (3c) so as to prevent explosive debt dynamics (Ponzi schemes). In our numerical application we choose state-contingent values for the natural borrowing limits (NBLs) so that these constraints are just marginally binding. This is required because for beginning-of-period wealth levels below these marginally binding NLBs there exist no policies that are consistent with non-explosive debt dynamics along all contingencies, as we prove in appendix A.4. The marginally binding NBLs that we impose thus represent the laxest constraints on beginning-of-period wealth levels that are consistent with existence of policies that imply non-explosive debt dynamics. Appendix A.4 shows how one can compute the marginally binding NBLs and that they are unique. Finally, we assume that the initial condition satisfys $w_0 \geq NBL(z_0)$.

While intuitive, the formulation of the optimization problem (3) has a number of unattractive features. First, the price of the domestic government bond depends on the chosen default profile, so that constraint (3b) fails to be linear in the government's choice variables. It is thus unclear whether problem (3) is concave. Second, the inequality constraints for G_t^L, G_t^S and especially those for Δ_t are difficult to handle computationally, as they will be occasionally binding.[9] Moreover, the optimal default policies Δ_t turn out to be discontinuous. For these reasons, we derive in the next section an equivalent formulation of the problem that can be shown to be concave, that features fewer occasionally binding inequality constraints, and gives rise to continuos optimal policy functions.

2.5 Equivalent Formulation of the Government Problem

We now formulate an alternative optimal policy problem with a different asset structure than in problem (3) and thereafter show that it is equivalent to the

[9] The fact that marginal utility increases without bound as $c_t \to \bar{c}$ and that marginal productivity of capital increases without bound as $k_t \to 0$ will insure interior solutions for these two choice variables, allowing to ignore the inequality constraints for these variables when computing numerical solutions.

original problem (3).

Specifically, we assume that there exist N Arrow securities and a single riskless bond in which the country can go either long or short. The vector of Arrow security holdings is denoted by $a \in \mathbb{R}^N$ and the n-th Arrow security pays one unit of output tomorrow if productivity state z^n materializes. The associated price vector is denoted by $p \in \mathbb{R}^N$. Given the risk-neutrality of international lenders, the price of the n-th Arrow security in period t is

$$p_t(z^n) = \frac{1}{1+r}\pi(z^n|z_t). \tag{4}$$

Letting b denote the country's holdings of riskless bonds, beginning-of-period wealth for this asset structure is then given by

$$\widetilde{w}_t \equiv z_t\widetilde{k}_{t-1}^{\alpha} + b_{t-1} + (1-\lambda)a_{t-1}(z_t) \tag{5}$$

where $a_{t-1}(z_t)$ denotes the amount of Arrow securities purchased for state z_t, $\widetilde{k}_{t-1}$ capital invested in the previous period, and $\lambda \geq 0$ is the parameter capturing potential default costs in the original problem (3).

Next, consider the following alternative optimization problem:

$$\max_{\{b_t, a_t \geq 0, \widetilde{k}_t \geq 0, \widetilde{c}_t \geq \bar{c}\}} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(\widetilde{c}_t) \tag{6a}$$

$$s.t. \;\; \forall t : \widetilde{c}_t = \widetilde{w}_t - \widetilde{k}_t - \frac{1}{1+r}b_t - p_t \cdot a_t \tag{6b}$$

$$\widetilde{w}_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z$$

$$\widetilde{w}_0 = w_0, z_0 \quad given.$$

Problem (6) has the same concave objective function as problem (3), but the constraint (6b) is now linear in the choices, so that first order conditions (FOCs) are necessary and sufficient. The FOCs can be found in appendix A.1. Furthermore, problem (6) reveals that the optimization problem has a recursive structure with the state in period t being described by the vector $(z_t, \widetilde{w}_t)$, allowing us to express optimal policy functions as a function of these two state variables only. Finally, the relevant inequality constraints are given by $a_t \geq 0$and the marginally binding natural borrowing limits.[10]

We now show that if a consumption path $\{c_t\}_{t=0}^{\infty}$ is feasible in problem (3), then is it also feasible in problem (6), and vice versa, i.e., the two different asset structures allow to implement the same set of consumption paths. One can thus use the solution to problem (6), which is easier to compute, to derive the asset structure and default profiles implementing the same consumption path in the original problem (3).

[10] As before, the Inada conditions on utility and the fact that marginal productivity of capital increases without bound as $k_t \to 0$ will insure interior solutions for c_t and k_t, allowing to ignore the inequality constraints for these variables when computing numerical solutions.

Consider some state contingent beginning-of-period wealth profile w_t arising from some combination of bond holdings, default decisions and capital investment $(G_{t-1}^L,\ G_{t-1}^S,\ \Delta_{t-1},\ k_{t-1})$ in problem (3). We now show that one can generate the same state contingent beginning-of-period wealth profile $\widetilde{w}_t = w_t$ in problem (6) by choosing $\widetilde{k}_{t-1} = k_{t-1}$ and by choosing an appropriate investment profile (a_{t-1}, b_{t-1}). Moreover, the funds required to purchase (a_{t-1}, b_{t-1}) are the same as those required to purchase $\left(G_{t-1}^L, G_{t-1}^S\right)$ when the default profile is Δ_{t-1}. With the costs of financial investments being the same in both problems, identical physical investments, and identical beginning of period wealth profiles, it then follows from constraints (3b) and (6b) that the implied consumption paths are also the same in both problems, establishing the equivalence between the two problems.

To keep notation as simple as possible we establish the previous claim for the case with 2 productivity states only. The extension to N states is relatively straightforward. Consider the following state contingent initial wealth profile

$$\begin{pmatrix} w_t(z^1) \\ w_t(z^2) \end{pmatrix} = \begin{pmatrix} z^1 k_{t-1}^\alpha + G_{t-1}^L - G_{t-1}^S(1-(1-\lambda)\delta_{t-1}^1) \\ z^2 k_{t-1}^\alpha + G_{t-1}^L - G_{t-1}^S(1-(1-\lambda)\delta_{t-1}^2) \end{pmatrix}.$$

One can replicate this beginning-of-period wealth profile in problem (6) by choosing $\widetilde{k}_{t-1} = k_{t-1}$ and by choosing the portfolio

$$b_{t-1} = G_{t-1}^L - G_{t-1}^S, \tag{7}$$

$$a_{t-1} = \begin{pmatrix} G_{t-1}^S \delta^1 \\ G_{t-1}^S \delta^2 \end{pmatrix} \tag{8}$$

The previous equations show that b in problem (6) has an interpretation as the net foreign asset position in problem (3) and that a in problem (6) can be interpreted as the state contingent default on outstanding own bonds. We will make use of this interpretation in the latter part of the paper. The funds f_{t-1} required for (G_{t-1}^L, G_{t-1}^S) under the default profile $(\delta_{t-1}^1, \delta_{t-1}^2)$ are given by

$$f_{t-1} = \frac{1}{1+r} G_{t-1}^L - \frac{1}{1 + R(z_{t-1}, (\delta_{t-1}^1, \delta_{t-1}^2))} G_{t-1}^S$$

where the interest rate satisfies

$$\frac{1}{1 + R(z_{t-1}, (\delta_{t-1}^1, \delta_{t-1}^2))} = \frac{1}{1+r} \left((1-\delta_{t-1}^1)\pi(z^1|z_{t-1}) + (1-\delta_{t-1}^2)\pi(z^2|z_{t-1})\right).$$

The funds $\widetilde{f}_{t-1}$ required to purchase (b_{t-1}, a_{t-1}) are

$$\widetilde{f}_{t-1} = \frac{1}{1+r}(G_{t-1}^L - G_{t-1}^S) + \frac{1}{1+r}\left(\delta_{t-1}^1 \pi(z^1\,|z_{t-1}) + \delta_{t-1}^2 \pi(z^2\,|z_{t-1})\right) G_{t-1}^S,$$

where we used the price of the Arrow security in (4). As is easy to see $\widetilde{f}_{t-1} = f_t$, as claimed.

Finally, note that we need to impose the restriction $a \geq 0$ on problem (6), as otherwise it would follow from equation (8) that one could implement a consumption path in problem (6) that cannot be implemented in problem (3) with values of δ^i satisfying $\delta^i \in [0,1]$ for all i. This completes the equivalence proof.

3 Zero Default Costs: Analytical Results

In the absence of default costs, the solution to problem (6) can be analytically determined. The following proposition summarizes the main finding. The proof can be found in appendix A.2.

Proposition 1 *Without default costs* ($\lambda = 0$) *the solution to problem (6) involves constant consumption equal to*

$$c = (1-\beta)(\Pi(z_0) + \widetilde{w}_0) \tag{9}$$

where $\Pi(\cdot)$ *denotes the maximized expected profits from future production, defined as*

$$\Pi(z_t) \equiv E_t \left[\sum_{j=0}^{\infty} \beta^j \left(-k^*(z_{t+j}) + \beta z_{t+j+1} \left(k^*(z_{t+j}) \right)^{\alpha} \right) \right]$$

with

$$k^*(z_t) = \left(\alpha \beta E(z_{t+1}|z_t) \right)^{\frac{1}{1-\alpha}} \tag{10}$$

denoting the profit maximizing capital level. For any period t*, the optimal default level satisfies*

$$a_t(z_t) \propto -\left(\Pi(z_t) + z_t \left(k^*(z_{t-1}) \right)^{\alpha} \right) \tag{11}$$

The proposition shows that in the absence of default costs, it is optimal to fully smooth consumption. The option of partial repayment thus allows for complete insurance of domestic production risk, as would be the case in a complete market setting. Equation (11) thereby reveals that default must occur frequently and for virtually all productivity realizations.[11] Default thereby insures the country against two components: first, against (adverse) news regarding the expected profitability of future investments, as captured by $\Pi(z_t)$; and second, against low output due to a low realization of current productivity, as captured by $z_t \left(k^*(z_{t-1}) \right)^{\alpha}$. If expected future profits commove positively with current productivity, e.g. if z_t is a persistent process, or in the special case with iid productivity shocks, where expected future profits are independent of current productivity, it follows from equation (11) that optimal default levels are inversely related to the current level of productivity. Default is then optimal whenever z_t falls short of its highest possible value and the optimal size of default is increasing in the amount by which productivity falls short if its highest possible level.

[11] Default is not required for states z_t achieving the maximal value for $\Pi(z_t) + z_t (k^*(z_{t-1}))^{\alpha}$ across all $z_t \in Z$. For such states default can be set equal to zero, otherwise default levels are strictly positive.

4 Optimal Default Policies with Default Costs

The previous section abstracted from potential dead weight costs associated with a government debt default decision. The trade-off between insuring consumption via default or via (international) wealth accumulation/decumulation is then resolved fully in favor of using the default option. As is clear from equation (5), however, it becomes optimal to rely exclusively on self-insurance via international wealth adjustments, i.e., to set $a_t \equiv 0$, if the dead weight costs from default become sufficiently high, e.g., if $\lambda \geq 1$. To evaluate how the trade-off between default and self-insurance is resolved for intermediate levels of default costs, we now consider a quantitative setup with business cycle shocks to aggregate productivity. As we show below, fairly low levels of default costs then make it optimal to almost exclusive rely on self-insurance through international reserve adjustments. Only when the country's net foreign asset position is sufficiently close to the (marginally binding) natural borrowing limits, will it be optimal to default on government debt.

4.1 Calibration

We now calibrate the model. A standard parameterization for quarterly productivity is given by a first order autoregressive process with quarterly persistence of 0.9 and a standard deviation of 0.5% for the quarterly innovation.[12] Since we use a yearly model, we annualize these values by choosing an annual persistence of technology equal to $(0.9)^4$ and use an annual standard deviation of the innovation of 1%. We then use Tauchen's (1986) procedure to discretize the shock process into a process with a high and a low productivity state. Normalizing average productivity to one, the resulting high productivity state is $z^h = 1.0133$ and the low productivity state $z^2 = 0.9868$. The procedure also yields the following transition matrix for the states

$$\pi = \begin{pmatrix} 0.8077 \, 0.1923 \\ 0.1923 \, 0.8077 \end{pmatrix}.$$

We set the capital share parameter in the production function to $\alpha = 0.34$. The annual discount factor is $\beta = 0.97$ and we consider households with a flow utility function given by

$$u(c) = \frac{(c - \bar{c})^{1-\sigma}}{1 - \sigma}$$

where $\bar{c} \geq 0$ denotes the subsistence level of consumption and σ parameterizes risk aversion. We choose $\sigma = 2$ and calibrate the subsistence level of consumption $\bar{c}$ such that in an economy where the government is forced to repay debt always, the marginally binding natural borrowing limit implies that the net foreign asset position of the country is not below -100% of average

[12] The quantitative results reported below are not very sensitive to the precise numbers used. A similar calibration is employed in Adam (2011).

GDP in any productivity state.[13] We thereby seek to capture the fact that industrialized countries do not appear to have net foreign asset positions below −100% of GDP, see figure 10 in Lane and Milesi-Ferretti (2007). Moreover, three out of the five industrialized countries approaching this boundary in the year 2004 later on faced fiscal solvency problems (Greece, Portugal and Iceland). It thus appears plausible to assume that countries cannot sustain higher external debt levels without running the risk of a government default.

Positive default costs and the small open economy assumption imply that the equilibrium outcomes are non-stationary, unless we choose $1+r < 1/\beta$. To insure that the equilibrium process is ergodic, we set the annual international interest rate five basis points below the rate implied by the inverse of the discount factor. Optimal default policies are rather robust to the precise number chosen.[14]

4.2 Evaluating the Effect of Default Costs

Figure 1 reports the optimal default policies for the next period as a function of the current (end-of period) net foreign asset position and the current productivity state.[15] Each row in the figure thereby corresponds to a different default cost parameterization (λ). To simplify the interpretation of results, the default policies and the net foreign asset positions are normalized by average GDP. The panels on the left thereby depicts the optimal default policy in the high productivity state (z^h) and the panels on the right policy for the low productivity state (z^l). Appendix A.3 explains how the optimal policies can be determined numerically.

The graphs shown in the first row of figure 1 report the outcome for the case when default costs are zero.[16] Specifically, they depict the optimal amount of default in the next period, when the future productivity state happens to be low (z^l). Note that there will never be default if the productivity state z^h realizes in the next period. Interestingly, the optimal amount of default is independent of the country's net foreign asset position and almost independent of the current state of productivity.[17] As is clear from proposition 1, these default policies fully insure future consumption against fluctuations

[13] Appendix A.4 explains how one can compute the marginally binding NBL for each productivity state. Average GDP is defined as the average output level associated with efficient investment, i.e., when $k_t = k^*(z_t)$ each period, and where we average over the ergodic distribution of the z process. For our parameterization this yields an average output level of 0.5661. Furthermore, the net foreign asset position of the country is independent of government policy at the marginally binding NBL, instead exlusivley determined by the desire to prevent debt from exploding, so that this measure can be used to calibrate the model. The resulting level for subsistence onsumption is $\bar{c} = 0.357$.

[14] We also experimented with larger gaps of 50 basis points.

[15] As explained in section 2.5, the net foreign asset position is given by the optimal value of b in the corresponding period.

[16] Since there exists a multiplicity of optimal default policies when $\lambda = 0$, the first row shows the outcome in the limiting case $\lambda \to 0$

[17] From equation (11) follows that the default in the next period does depend on the current state because the optimal investment $k^*(z_t)$ depends on the current productivity.

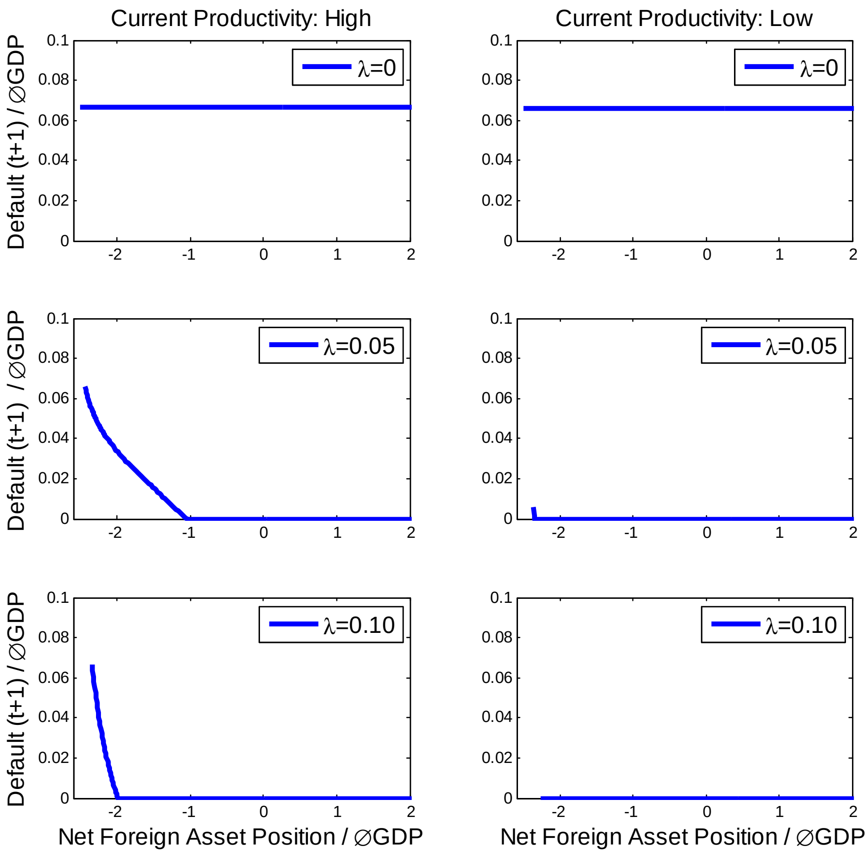

Figure 1: Optimal Default Policies: The Effect of Default Costs

in productivity.

The middle and bottom rows in figure 1 report the optimal default policies when default costs equal 5% and 10% of the defaulted amount, respectively. For large parts of the state space default then ceases to be optimal. Moreover, there is less default in the future if the current productivity state is low already. This is optimal because insurance against a future low state is more costly when current productivity state is low already, due to the persistence of productivity. Default continues to be optimal, however, if the net foreign asset position is sufficiently negative. Marginal utility of consumption is then very sensitive to further consumption fluctuations, because consumption approaches its subsistence level as the net foreign asset position approaches the limits implied by the (marginally binding) naturally borrowing limits.

Overall, figure 1 shows that moderate levels of default costs shift optimal policy strongly towards using adjustments in international wealth to insure domestic consumption. Only if the country's net foreign asset position approaches the borrowing limit will a government debt default still be optimal.

5 Optimal Default and Economic Disasters

The previous section showed that with moderate levels of default costs it becomes suboptimal to default on government debt, provided the country is not too close to its borrowing limit. In this section we evaluate whether this conclusion continues to be true for a setting with much larger economic shocks. This is motivated by the observation that countries occasionally experience very large negative shocks, as previously argued by Rietz (1988) and Barro (2006), and that such shocks tend to be associated with a government default.[18] To capture the possibility of large shocks, we augment the model by including disaster like shocks to aggregate productivity and then explore the quantitative implications of disaster risk on optimal government debt default decisions.

5.1 Calibrating Economic Disasters

To capture economic disasters we introduce two disaster sized productivity levels to our aggregate productivity process. We add two disaster states rather than a single one to capture the idea that the size of economic disasters is uncertain ex-ante. This will become important in section 6, when we discuss how well simple financial instruments can approximate optimal default policies.

We calibrate the disaster shocks to match the mean and variance of GDP disasters, as documented in Barro and Jin (2011). Using a sample of 157 GDP disasters, they report a mean reduction in GDP of 20.4% and a standard deviation of 12.64%. Assuming that it is equally likely to enter both disaster states, this yields the productivity states $z^d = 0.9224$ and $z^{dd} = 0.6696$.

[18] Barro (2006) and Gourio (2010) also allow for default on government bonds in disaster states. Since the focus of their analysis is different, they use exogenous probabilities and default rates.

Our vector of possible productivity realizations thus takes the form $Z = \{z^h, z^l, z^d, z^{dd}\}$ where the parameterization of the business cycle states (z^h, z^l) is the same as in the previous section. The state transition matrix for the shock process is given by

$$\pi = \begin{pmatrix} 0.7770 & 0.1850 & 0.019 & 0.019 \\ 0.1850 & 0.7770 & 0.019 & 0.019 \\ 0.1429 & 0.1429 & 0.3571 & 0.3571 \\ 0.1429 & 0.1429 & 0.3571 & 0.3571 \end{pmatrix},$$

The transition probability from the business cycle states into the disaster states is chosen so as to match the unconditional disaster probability of 0.038, as reported in Barro and Jin (2011). We thereby assume that it is equally likely to reach both disaster states. The persistence of the disaster states is set to match the average duration of GDP disasters, which equals 3.5 years, see Barro and Jin. Finally, the transition probabilities of the business cycle states are adjusted to reflect the presence of disaster risk.

Since the presence of disaster risk strongly affects the marginally binding NBLs (they become much tighter and potentially require even positive net foreign asset positions in all states), we recalibrate the subsistence level for consumption $\bar{c}$. As in section 4 before, we choose $\bar{c}$ such that in an economy where bonds must be repaid always, the economy can sustain a maximum net foreign asset of -100% of average GDP in the business cycle states (z^h, z^l).[19] Choosing tighter limits does not affect the shape of the optimal default policies but only shifts the policies reported in the next subsection 'further to the right'.

5.2 Optimal Default with Disasters: Quantitative Analysis

Figure 2 reports the optimal default policies for the economy with disaster shocks. Each panel in the figure corresponds to a different productivity state today and reports the intended amount of default in tomorrow's states z^l, z^d and z^{dd} as a function of the country's net foreign asset position today.[20] We thereby assume that the dead weight costs of default equal 10% of the defaulted amount, corresponding to the default cost value used for computing the lowest row in figure 1.

Figure 2 shows that it is virtually never optimal to default in the low business cycle state (z^l), unless the net foreign debt position is very close to its maximally sustainable level, similar to section 4 where we considered business cycle shocks only. Furthermore, for a wide range of net foreign asset positions, it is optimal to default if the economy makes a transition from a business cycle state to a disaster state, see the top panels in the figure. Default is optimal for a transition to the severe disaster state (z^{dd}), even when the country's net foreign asset position is positive before the disaster. Overall, the optimal amount of default is increasing as the country's net foreign asset

[19] This yields an adjusted value of $\bar{c} = 0.198$.

[20] Recall that default is never optimal if z^h realizes in the next period.

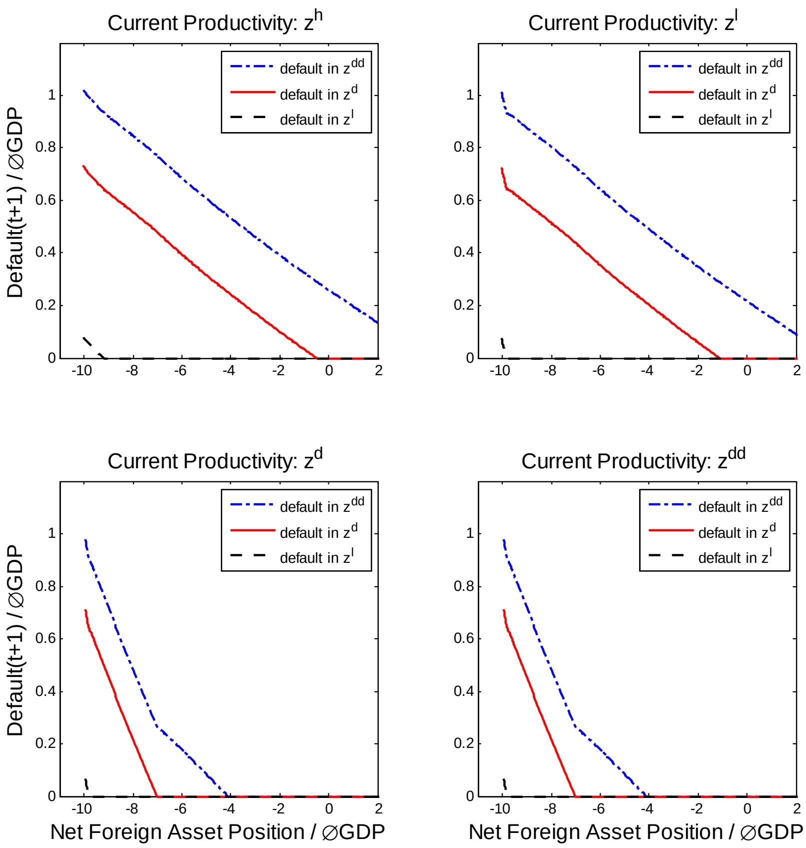

Figure 2: Optimal Default Policies with Disaster States ($\lambda = 0.1$)

position worsens. Yet, once the economy is in a disaster state, a further default in the event that the economy remains in the disaster state is optimal only if the net foreign asset position is very low, see the bottom panels of figure 2. Since the likelihood of staying in a disaster state is quite high, choosing not to repay if the disaster persists would have very high effects on interest rate costs ex-ante. As a result, serial default in case of a persistent disaster will not necessarily be part of the optimal default policy.

The overall shape of the optimal default policies is fairly robust to assuming different values for the default costs λ. Larger costs shift the default policies towards the left, i.e., default occurs only for more negative net foreign asset positions. However, higher costs also tighten the maximally sustainable net foreign asset positions, thereby reducing the range of net foreign asset positions over which default occurs. Lower cost have the opposite effect, i.e., they induce a rightward shift and allow to sustain more negative net foreign asset positions.

Figure 3 reports a typical sample path for the net foreign asset position and the amount of default implied by optimal policy for $\lambda = 0.1$. We start the path at a zero net foreign asset position and each model period corresponds to one year. The figure shows that it is optimal to improve the net foreign asset position when the economy is in the business cycle states, with faster improvements in the high state. This is the case even though the international risk free rate is 5 basis points below the inverse of the domestic discount factor. A transition to a disaster state leads to a default provided the economy's net foreign asset position is not too high (unlike in year 16). Also, following a disaster, the net foreign asset position deteriorates whenever the disaster persists for more than one period (see for example year 40), otherwise the net foreign asset position is largely unaffected or improves even slightly (see year 85). Overall, the net foreign asset dynamics are characterized by rapid deteriorations during persistent disaster periods and gradual improvements during normal times.

6 Welfare Analysis and Approximate Implementation

This section determines the welfare effects of letting the government choose whether or not to repay its debt compared to a situation where repayment is simply forced upon the government (or assumed) in each state. Furthermore, we study the approximate implementation of optimal default policies via a combination of equity-like bonds and non-defaultable bonds.

6.1 Welfare Comparison

We now compare the welfare gains associated with optimal default policies to a setting in which repayment of bonds is required to occur in all states. We base our welfare comparison on the model with disaster states from section 5 and consider a broad range of default costs. We evaluate the utility consequences in terms of welfare equivalent consumption changes over the first 500 years. Specifically, letting c_t^1 denote the optimal state contingent consumption path

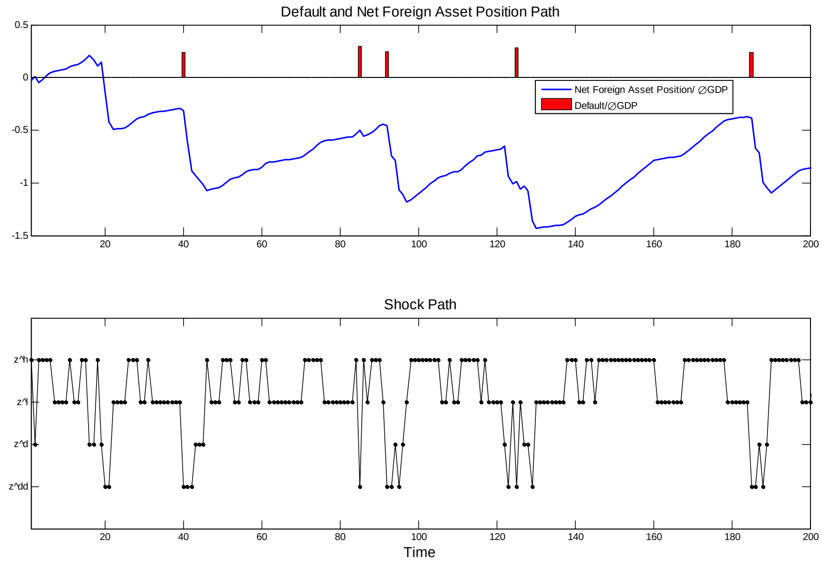

Figure 3: The Evolution of Net Foreign Assets and Default under Optimal Policy ($\lambda = 0.1$)

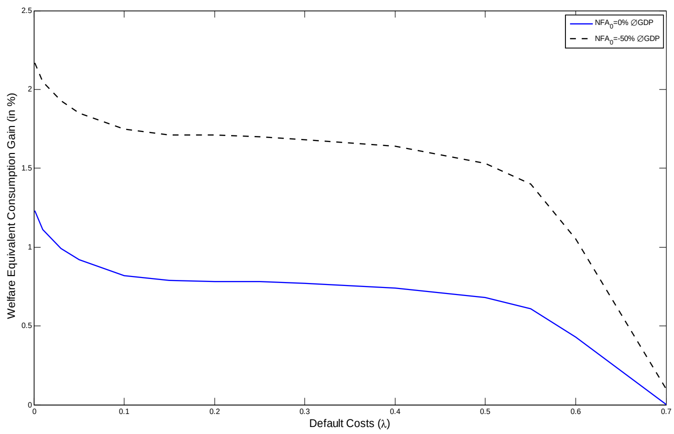

Figure 4: Welfare Gains from Using the Default Option

in the no-default economy and c_t^2 the corresponding consumption path with (costly) default, we report for each level of default costs the welfare equivalent consumption change ω solving

$$E_0\left[\sum_{t=0}^{500}\beta^t\frac{((c_t^1(1+\omega)-\bar{c}))^{1-\gamma}}{1-\gamma}\right]=E_0\left[\sum_{t=0}^{500}\beta^t\frac{(c_t^2-\bar{c})^{1-\gamma}}{1-\gamma}\right]$$

where the expectations are evaluated by averaging over 10000 sample paths. To highlight the effects of the country's initial international wealth position, we consider two scenarios, one where the initial net foreign asset position is zero and one where it equals -50% of average GDP.[21] The outcome of this procedure is reported in figure 4. It shows that the welfare gains amount to 1-2% of consumption each period for a broad range of default costs. The welfare gains are surprisingly robust to the level of the default costs, instead are more sensitive to the initial net foreign asset position. Yet, for default costs $\lambda \geq 0.5$ the welfare gains from default decrease steeply. This has to do with the fact that for such high levels of the default costs it becomes suboptimal to insure against a future disaster state when the economy is already in a disaster, independently of the country's net foreign asset position. This is shown in the lower panel of figure 5 which reports the optimal default policies when $\lambda = 0.7$. With these default costs, the government receives only 0.3 units of consumption for each unit of default. Since the likelihood of a specific disaster state (either z^d or z^{dd}) to re-occur is 0.3571, the cost of using the default option for any of these states is $0.3571/(1+r) > 0.3$. Therefore, use of the default option is dominated by using the unconditional bond to transfer resources into a future disaster state. Repayment therefore optimally occurs in all future states, once the economy has hit a disaster state. As a result, the borrowing constraints tighten significantly[22] in the disaster states at this level of default costs and the required amount of insurance in the business cycle state (z^l, z^h) increases strongly as the net foreign asset position deteriorates.

6.2 Approximate Implementation

We now consider a setting where the government issues two kinds of financial instruments: a simple non-contingent bond that repays in all future contingencies, as well as an equity-like bond that repays one unit of consumption in normal times (z^h, z^l), but zero when a the disaster occurs (either z^d or z^{dd}). The fact that there is only one instrument but two disaster states implies that the government bond market is still far from complete, so that it is unclear to what extent the welfare gains from outright default could approximately be captured by this simple contingent bond structure. To make the setting with a contingent bond comparable to the setting with outright default analyzed

[21] More precisely, we set the initial value of $(1-\lambda)a_{-1}+b_{-1}$ equal to these values and set period zero output equal to $(k^*(1))^\alpha$ in both economies and choose $z_1 = z^h$.

[22] They reach the levels applying in the economy with non-defaultable bonds.

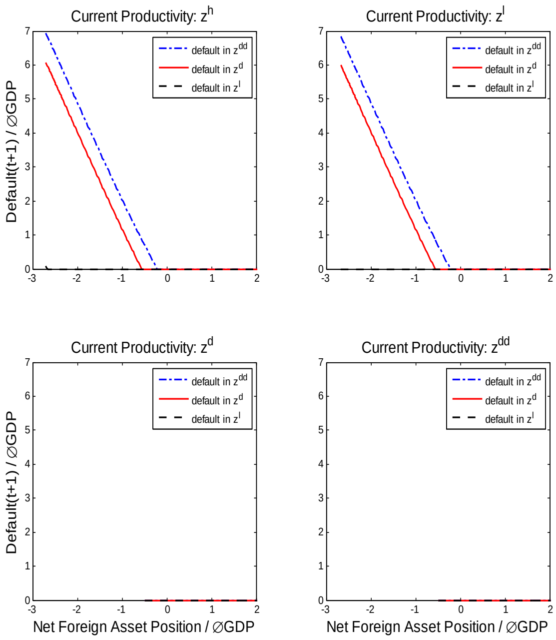

Figure 5: Optimal Default Policy with Disaster States and High Default Cost ($\lambda = 0.7$)

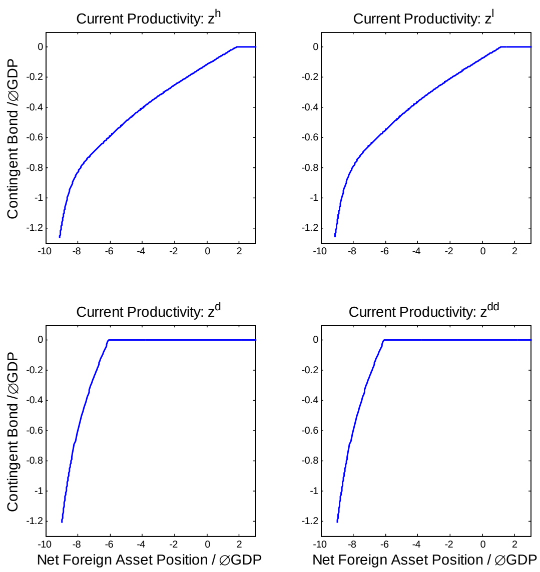

Figure 6: Optimal Equity Bond Issuance ($\lambda = 0.1$)

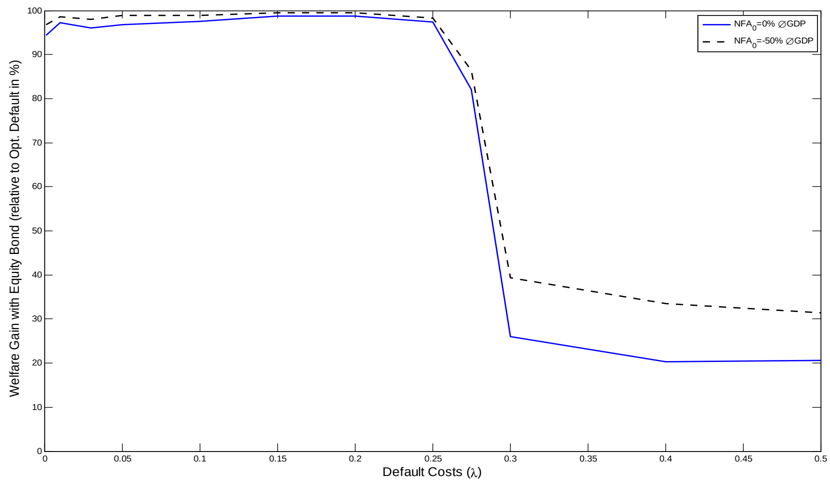

Figure 7: Welfare Implications of Approximate Implementation

in the previous section, we assume that the government must pay a cost λ per unit of equity bond issued in case a disaster state actually materializes. And to facilitate comparison to the results reported in section 5, we set $\lambda = 0.1$.

The optimal issuance of equity bonds is reported in figure 6. The figure shows that the equity bond policies are approximately a convex combination of the (negative of the) default policies for states z^d and z^{dd} shown in figure 2. The figure reveals that the government optimally issues the equity bond before an economic disaster actually happens and continues to issue such bonds while being in a disaster only if the net foreign asset position is sufficiently negative.

Figure 7 reports how well the two available financial instruments allow to capture the welfare gains induced by optimal default policies, as reported in figure 4. Specifically, the figure depicts the share of the welfare increase that can be realized with the considered simple asset structure. It shows that for default cost up to about $\lambda = 0.25$ there is virtually no difference between relying on optimal default policies or using the considered simple assets. This holds independently of the initial net foreign asset position. Yet, for sufficiently high levels of λ the simple asset structure cannot capture the achievable welfare gains from optimal default. Whenever $1 - \lambda$ exceeds the combined persistence of the disaster states (z^d and z^{dd}), it becomes suboptimal to issue the equity bond if the economy is already in a disaster. As discussed in section 6.1, it is then optimal to issue non-defaultable bonds only. This tightens the (marginally binding) borrowing limits significantly in the disaster states and decreases the opportunities for risk sharing, when compared to a setting with optimal default policies, where one can insure against disaster states individually.

7 Long Maturities and Optimal Bond Repurchase Programs

We now discuss the effects of introducing domestic bonds with longer maturity.[23] Long bonds can offer an advantage over one period bonds, as considered in the previous part of the paper, if the market value of long bonds reacts to domestic conditions in a way that allows the government to insure against domestic shocks. It would be desirable, for example, if the market value of outstanding long bonds decreases following a disaster shock. This allows the government to repurchase the outstanding stock of debt at a lower price, thereby realizing a capital gain that lowers the overall debt burden. Unlike in Angeletos (2002), capital gains will not materialize unless the government plans not to repay fully the long bonds in (at least some contingency) in the future. The depreciation of the market value, thus, can only be induced via the anticipation of default in the future when long bonds mature.

Issuing long bonds will offer an advantage against outright default on maturing bonds, whenever the dead-weight costs associated with repurchasing bonds at a devaluated market price is lower than the dead weight costs of an outright default on maturing bonds today. If both costs are identical, i.e., if the capital gains on long bonds resulting from default in the future induce the same costs as a default on maturing bonds, then there will be of no additional value associated with issuing long bonds. Yet, if the repurchase of long bonds at low prices fails to produce dead-weight costs, then the government could fully insure domestic consumption, i.e., achieve the first best allocation, independently of the costs associated with an outright default on maturing bonds. The optimal bond issuance strategy will then have the feature that the government issues each period long bonds that (partially) default at maturity, depending on the productivity realization tomorrow. The default at maturity needs to be calibrated such that the capital gains realized tomorrow fully insure domestic consumption against domestic productivity shocks, i.e., satisfies the proportionality restriction (11). Tomorrow, the government could then repurchase the existing stock of long bonds and issue a new long bonds with a new contingent repayment profile. In this way outright default on maturing bonds never occurs.

8 Conclusions

In a setting with incomplete government bond markets, debt default is part of the optimal government policy under commitment. The choice whether or not to repay maturing debt allows for increased international risk sharing and significantly relaxes the net foreign debt positions that a country can sustain. Moreover, it considerably increases welfare, even when default costs are sizable. Default in low productivity states can be part of a country's optimal policy in a setting with full commitment, especially if the net foreign asset position is close to the level implied by the country's (marginally binding)

[23] Introducing also longer maturities for the risk-free foreign debt has no consequences for the outcomes.

natural borrowing limits.

A Appendix

A.1 First Order Equilibrium Conditions

This appendix derives the first order conditions for problem (6). We first rewrite the problem replacing beginning-of-period wealth by components (see definition (5)):

$$\max_{\left\{b_t, a_t \geq 0, \widetilde{k}_t \geq 0, \widetilde{c}_t \geq \bar{c}\right\}} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(\widetilde{c}_t)$$

$$s.t. \ \ \forall t: \widetilde{c}_t = z_t \widetilde{k}_{t-1}^{\alpha} + b_{t-1} + (1-\lambda) a_{t-1}(z_t) - \widetilde{k}_t - \frac{1}{1+r} b_t - p_t \cdot a_t$$

$$z_{t+1} \widetilde{k}_t^{\alpha} + b_t + (1-\lambda) a_t(z_{t+1}) \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z$$

$$\widetilde{w}_0 = w_0, z_0 \quad given,$$

Next, we formulate the Lagrangian and let η_t denote the multiplier on the budget constraint in period t, $\nu_t^{z^n}$ the multiplier for the short-selling constraint on the Arrow security that pays off in state z^n in $t+1$, and ω_{t+1} the multiplier associated with the natural borrowing limits. We drop the inequality constraints for $\widetilde{k}_t$ and $\widetilde{c}_t$, as the Inada conditions guarantee an interior solution for these variables. Differentiating the Lagrangian with respect to the choice variables one obtains

$$\begin{aligned}
\widetilde{c}_t: &\quad u'(\widetilde{c}_t) - \eta_t = 0 \\
b_t: &\quad -\eta_t \frac{1}{1+r} + \beta E_t \eta_{t+1} + \beta E_t \omega_{t+1} = 0 \\
a_t(z^n): &\quad -\eta_t p_t(z^n) + \beta \pi(z^n|z^t) \eta_{t+1}(z^n)(1-\lambda) \\
&\quad + \nu_t^{z^n} + \beta \pi(z^n|z^t) \omega_{t+1}(z^n)(1-\lambda) = 0 \quad \forall n \in N \\
\widetilde{k}_t: &\quad -\eta_t + \alpha \widetilde{k}_t^{\alpha-1} \beta E_t \eta_{t+1} z_{t+1} + \alpha \widetilde{k}_t^{\alpha-1} \beta E_t \omega_{t+1} z_{t+1} = 0
\end{aligned}$$

Using the FOC for consumption to replace η_t in the last three FOCs, one obtains Euler equations for the bond holdings, the Arrow securities and capital investment:

$$Bond: -u'(\widetilde{c}_t) \frac{1}{1+r} + \beta E_t u'(\widetilde{c}_{t+1}) + \beta E_t \omega_{t+1} = 0 \tag{12a}$$

$$\begin{aligned}
Arrow: &-u'(\widetilde{c}_t) p_t(z^n) + \beta \pi(z^n|z_t) u'(\widetilde{c}_{t+1}(z^n))(1-\lambda) \\
&+ \nu_t^{z^n} + \beta \pi(z^n|z^t) \omega_{t+1}(z^n)(1-\lambda) = 0 \quad \forall n \in N
\end{aligned} \tag{12b}$$

$$Capital: -u'(\widetilde{c}_t) + \alpha \widetilde{k}_t^{\alpha-1} \beta E_t u'(\widetilde{c}_{t+1}) z_{t+1} + \alpha \widetilde{k}_t^{\alpha-1} \beta E_t \omega_{t+1} z_{t+1} = 0 \tag{12c}$$

In addition, the Kuhn-Tucker FOCs include the following complementarity conditions:

$$0 \leq a_t(z^n) \quad \perp \quad \nu_t^{z^n} \geq 0 \ \ \forall n \in N \tag{12d}$$

$$0 \leq z^n \widetilde{k}_t^{\alpha} + b_t + (1-\lambda) a_t^{z^n} - NBL(z^n) \quad \perp \quad \omega_{t+1}(z^n)) \geq 0 \ \forall n \in N. \tag{12e}$$

Combined with the budget constraint, the Euler equations and the complementarity conditions constitute the optimality conditions for problem (6).

A.2 Proof of Proposition 1

We first show that the proposed consumption solution (9) satisfies the budget constraint, that the inequality constraints $a \geq 0$ are not binding, and that the NBLs are also not binding. Thereafter, we show that the remaining first order conditions of problem (6), as derived in appendix A.1, also hold.

We start by showing that the portfolio implementing (9) in period $t = 1$ is consistent with the flow budget constraint and $a \geq 0$. The result for subsequent periods follows by induction. In period $t = 1$ with productivity state z^n, beginning-of-period wealth under the optimal capital investment strategy (10) is

$$\widetilde{w}_1^n \equiv z^n \left(k^*(z_0)\right)^\alpha + b_0 + a_0(z^n) \tag{13}$$

To insure that consumption can stay constant from $t = 1$ onwards we need again

$$c = (1-\beta)(\Pi(z^n) + \widetilde{w}_1^n) \tag{14}$$

for all possible productivity realizations $n = 1, ..N$. This provides N conditions that can be used to determine the $N + 1$ variables b_0 and $a_0(z^n)$ for $n = 1, ..., N$. We also have the condition $a_0(z^n) \geq 0$ for all n and by choosing $\min_n a_0(z^n) = 0$, we get one more condition that allows to pin down a unique portfolio (b_0, a_0). Note that the inequality constraints on a do not bind for the portfolio choice, as we have one degree of freedom, implying that the multipliers $v_1^{z_n}$ in appendix A.1 are all zero. It remains to show that the portfolio achieving (14) is feasible given the initial wealth $\widetilde{w}_0$. Using (13) to substitute $\widetilde{w}_1^n$ in equation (14) we get

$$c = (1-\beta)(\Pi(z^n) + z^n \left(k^*(z_0)\right)^\alpha + b_0 + a_0(z^n)) \forall n = 1, ...N.$$

Combining with (9) we get

$$\Pi(z^n) + z^n \left(k^*(z_0)\right)^\alpha + b_0 + a_0(z^n) = \Pi(z_0) + \widetilde{w}_0$$

Multiplying the previous equation with $\pi(z^n|z_0)$ and summing over all n one obtains

$$E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + b_0 + \sum_{n=1}^{N} \pi(z^n|z_0) a_0(z^n) = \Pi(z_0) + \widetilde{w}_0.$$

Using $\Pi(z_0) = -k^*(z_0) + \beta E_0 \left[z_1 \left(k^*(z_{t+j})\right)^\alpha\right] + \beta E_0 \left[\Pi(z_1)\right]$ and (4) the previous equation delivers

$$(1-\beta) E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + b_0 + (1+r) p_0 a_0 = -k^*(z_0) + \widetilde{w}_0$$

Using $\beta = 1/(1+r)$ this can be written as

$$\begin{aligned} &(1-\beta) E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] \\ &+ \frac{1-\beta}{\beta} p_0 a_0 + (1-\beta) b_0 + \frac{1}{1+r} b_0 + p_0 a_0 = -k^*(z_0) + \widetilde{w}_0 \end{aligned} \tag{15}$$

From (14) follows that the first terms in the previous equation are equal to

$$(1-\beta)E_0\left[\Pi(z_1)+z_1\left(k^*(z_0)\right)^{\alpha}\right]+\frac{1-\beta}{\beta}p_0a_0+(1-\beta)b_0=c$$

so that (15) is just the budget equation for period zero. The portfolio giving rise to (14) in $t=1$ thus satisfies the budget constraint of period zero. The results for $t\geq 1$ follow by induction.

It then follows from equation (14) that $\widetilde{w}_t$ is bounded, as $\Pi(z_t)$ is bounded, so that the process for beginning-of-period wealth does not involve explosive debt. The NBLs are thus not binding so that the multipliers $\omega_{t+1}=0$ for all t and all contingencies.

Using $v_t^{z_n}\equiv 0$, $\omega_{t+1}\equiv 0$, the fact that capital investment is given by (10) and that the Arrow security price is (4), the Euler conditions (12a) - (12c) then all hold when consumption is given by (9). This completes the proof.

A.3 Numerical Solution Approach

To compute recursive equilibria for Problem 6 we apply a global solution method as to account for the non-linear default policies in our model. As endogenous state variable we use beginning-of-period wealth, defined as above. Combined with exogenous productivity shocks we define our state space S to be

$$S=\left\{z^1\times\left[NBL(z^1),w_{max}\right],...,z^N\times\left[NBL(z^N),w_{max}\right]\right\}$$

where we set w_{max} such that in equilibrium optimal policies never imply wealth values above this threshold. The NBLs are set such they are marginally binding. How these values are derived is shown in Appendix A.4.

We want to describe equilibrium in terms of time-invariant policy functions that map the current state into current policies. Hence, we want to compute policies

$$\widetilde{f}:(z_t,w_t)\rightarrow\left(\{c_t,k_t,b_t,a_t\}\right),$$

where their values (approximately) satisfy the equilibrium conditions derived above. We use a time iteration algorithm where equilibrium policy functions are approximated iteratively. In a time iteration procedure, one takes tomorrow's policy (denoted by f^{next}) as given and solves for the optimal policy today (denoted by f) which in turn is used to update the guess for tomorrow's policy. Convergence is achieved once $||f-f^{next}||<\epsilon$ and we set $\widetilde{f}=f$. In each time iteration step we solve for optimal policies on a sufficient number of grid points distributed over the continuous part of the state space. Between grid points we use linear splines to interpolate tomorrow's consumption policy. Following Garcia and Zangwill (1981), we can transform the complementarity conditions of our first order equilibrium conditions into equations. To solve for a root of the resulting non-linear equation system at a particular grid point we use Ziena's Knitro, an optimization software that can be called from Matlab. For more details on the time iteration procedure and how one transforms complementarity conditions into equations, see for example, Brumm and Grill (2010). To come up with a starting guess for the consumption policy we use the fact that at the NBLs optimal consumption equals the subsistence level. We therefore guess a convex, monotonically increasing function g which satisfies $g(z^i, NBL(z^i)) = \bar{c}\ \forall i$ and use a reasonable value for $g(z^i, w_{max})$.

A.4 The Marginally Binding Natural Borrowing Limits (NBLs)

This appendix explains how we compute the state-contingent marginally binding NBLs that we use as lower bounds for the state space in our numerical solution approach. We also prove that the marginally binding NBLs are unique and that if beginning-of-period wealth ever falls short of them in some contingency, there is a positive probability that debt dynamics will violate *any* finite debt limit, independently of how lax it is chosen.

To simplify the exposition, we consider a setting with just two productivity levels ($N = 2$) and order these such that $z^1 > z^2$. We also surpress time subscripts for the moment. The extension to more productivity states is straightforward, as we explain below. Let $NBL(z^i)$ denote the marginally binding NBL in productivity state $i = 1, 2$. The marginally binding NLBs solve the following problem

$$\begin{aligned} NBL(z^i) = &\arg\max w(z^i)\ s.t. \\ &w'(z^j) \geq NBL(z^j) \text{ for } j = 1, 2 \end{aligned} \tag{16a}$$

for $i = 1, 2$, where $w(z^i)$ denotes beginning-of-period wealth in state z^i and $w'(z^j)$ the beginning-of-period wealth in the next period if the productivity state is z^j. Marginally binding NBLs can thus be interpreted as a set of state-contingent *minimum* beginning-of-period wealth levels, such that beginning-of-period wealth in all future states remains above the limits defined by the set of state-contingent marginally binding NBLs. As we show below, the fixed point problem defined by the system of optimization problems (16a) has a unique solution.

Using the budget constraint from the equivalent fromulation of the decision problem (6), beginning of period wealth in state z^i has the following uses

$$w(z^i) = c(z^i) + k(z^i) + \frac{1}{1+r} \cdot b(z^i) + p^2(z^i) \cdot a(z^i)$$

where $p^2(z^i)$ denotes the price of the Arrow security that pays one unit of the consumption good in state z^2.[24] Given the choices $\big(c(z^i), k(z^i), b(z^i), a(z^i)\big)$, future beginning of period wealth levels are

$$\begin{aligned} w'(z^1) &= z^1 k(z^i)^\alpha + b(z^i) \\ w'(z^2) &= z^2 k(z^i)^\alpha + b(z^i) + (1-\lambda) a(z^i) \end{aligned}$$

[24] Since $z^1 > z^2$, we can ignore the Arrow security for state z^1 as use of this security is either inefficient (if default costs are positive) or the asset is redundant (in the absence of default costs).

Since $c(z^i)$ does not affect future beginning-of-period wealth levels, it is optimal to choose the lowest possible level of consumption in (16a), i.e., $c(z^i) = \overline{c}$. The Lagrangian of the maximization problem (16a) can thus be written as

$$\begin{aligned} L = &\overline{c} + k(z^i) + \frac{1}{1+r} \cdot b(z^i) + p^2(z^i) \cdot a(z^i) \\ &+\lambda^1(z^i) \left(z^1 k(z^i)^\alpha + b(z^i) - NBL(z^1)\right) \\ &+\lambda^2(z^i) \left(z^2 k(z^i)^\alpha + b(z^i) + (1-\lambda) a(z^i) - NBL(z^2)\right) \end{aligned}$$

The first order conditions of this problem are[25]

$$\begin{aligned} 1 + \lambda^1(z^i)\alpha z^1 k(z^i)^{\alpha-1} + \lambda^2(z^i)\alpha z^2 k(z^i)^{\alpha-1} &= 0 \\ \frac{1}{1+r} + \lambda^1(z^i) + \lambda^2(z^i) &= 0 \\ p^2(z^i) + \lambda^2(z^i)(1-\lambda) &= 0 \end{aligned}$$

and imply after eliminating Lagrange multipliers in the first equation:

$$k^*(z^i) = \left(\frac{1}{\alpha \left(\frac{1}{1+r} z^1 + \frac{p^2(z^i)}{1-\lambda}(z^2 - z^1)\right)} \right)^{\frac{1}{\alpha-1}}$$

where a starred variable denotes the value of the optimal solution. The constraints then imply

$$\begin{aligned} b^*(z^i) &= NBL(z^1) - z^1 k^*(z^i)^\alpha \\ a^*(z^i) &= \frac{NBL(z^2) - z^2 k^*(z^i)^\alpha - b^*(z^i)}{(1-\lambda)} \end{aligned}$$

Since (k^*, b^*, a^*) are either independent of the marginally binding NBLs or linear functions thereof, the system of equations

$$NBL(z^i) = \overline{c} + k^*(z^i) + \frac{1}{1+r} \cdot b^*(z^i) + p^2(z^i) \cdot a^*(z^i)$$

for $i = 1, 2$ has a unique solutions for the marginally binding $NBL(z^i)$. An extension of this approach to more productivity states is straightfoward (though notationally more tedious) as the structure of the Lagrangian remains unchanged when additing more productivity states.

We now prove that violationg the marginally binding NBLs computed above in some contingency necessarily leads to unboundedly negative beginning-of-period wealth dynamics and violates any finite NBL, no matter how loosely it is set, with positive probability. Suppose that in some productivity state z^i

[25] Since the objective is linear and the constraint set convext, first order conditions are necessary and sufficient.

and some period t, beginning-of-period wealth falls short of the limits implied by the marginally binding NBL computed above, i.e.,

$$w_t(z^i) = NBL(z^i) - \varepsilon \tag{17}$$

for some $\varepsilon > 0$. We then prove below that for at least one contingency z^j in $t+1$ it must hold that

$$w_{t+1}(z^j) \leq NBL(z^j) - \varepsilon(1+r) \tag{18}$$

so that for this contingency the distance to the marginally binding NBL is increasing. Since the same reasoning applies also for future periods, and since the marginally binding NBLs assume finite values, this implies the existence of a contingency along which future wealth far in the future becomes unboundedly negative, implying that any finite natural borrowing limit will be violated with positive probability.

It remains to prove that if (17) holds in period t and contingency z^i, this implies that (18) holds for some contingency z^j in $t+1$. Suppose for contradiction that

$$w_{t+1}(z^h) > NBL(z^h) - \varepsilon(1+r) \tag{19}$$

for all h can be achieved. The cost minimizing way to satisfy the constraints (19) for all h is to choose $a^*(z^i), k^*(z^i)$ and $\widetilde{b} = b^*(z^i) - \gamma$ for some $\gamma < \varepsilon$.[26] This, however, is not a feasible choice because it would require beginning-of-period t wealth strictly larger than (17), which concludes the proof.

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[26] To verify this just solve the minimization problem (16a) where the constraints on future wealth are replaced by (19).

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