Investment Cycles and Sovereign Debt Overhang*

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Abstract

We characterize optimal taxation of foreign capital and optimal sovereign debt policy in a small open economy where the government cannot commit to policy and seeks to insure a risk averse domestic constituency. The expected tax on capital is shown to vary with the state of the economy, generating cyclicality in investment and debt in an environment where the first best capital stock is a constant. The government's lack of commitment induces a negative correlation between investment and the stock of government debt, a "debt overhang" effect. If the government discounts the future at a rate higher than the market, then capital oscillates indefinitely at a level strictly below the first best. Debt relief is never Pareto improving and cannot affect the long-run level of investment. Further, restricting the government to a balanced budget can eliminate the cyclical distortion of investment.

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1 Introduction

We study optimal taxation of foreign capital and optimal sovereign debt policy in a small open economy (SOE) where the government cannot commit to policy and seeks to insure a risk averse domestic constituency. The environment we consider includes two defining features of emerging markets, namely a lack of access to financial markets on the part of domestic agents and the inability of the government to commit to policy. We show that even when the government is benevolent, the expected tax on capital endogenously varies with the state of the economy and investment is distorted by more in recessions than in booms, generating cyclicality in investment and debt in an environment where the first best capital stock is a constant. The government's lack of commitment induces a negative correlation between investment and the stock of government debt, generating a "debt overhang" effect.

We show that if the government discounts the future at a rate higher than the world interest rate, then investment and the level of debt oscillates indefinitely. Investment remains strictly below the first best level even in the long run. Further, we show that imposing a balanced budget on the government may eliminate the amplification of shocks. That is, the government's access to international debt markets generates the volatility of capital.

In the model, the government implements fiscal policy on behalf of risk averse domestic agents (or a preferred sub-set of agents) who lack access to financial markets and do not own capital. Uncertainty is driven by an *i.i.d.* stochastic productivity process. The shock can be interpreted as a productivity shock or a terms of trade shock. This generates a risk that the domestic agents cannot insure. Risk neutral foreigners invest capital that is immobile for one period and has an opportunity cost given by the world interest rate. The government provides insurance to domestic agents by taxing or subsidizing foreign capitalists and trading a non-contingent bond with international financial markets. Since the expected marginal product of capital is independent of the shock's realization, the first-best capital stock is acyclical. This environment allows us to isolate the role of fiscal policy in generating investment and debt cycles.

If the government could commit, optimal fiscal policy (the Ramsey solution) does not distort capital in this economy (similar to Judd (1985) and Chamley (1986)). The combination of state contingent taxes and the bond is equivalent to the government having access to a complete set of state-contingent assets, as in Zhu (1992), Judd (1992) and Chari et al. (1994). Under commitment, full insurance is achieved while maintaining an expected foreign tax of zero. The government exploits the fact that capital is ex post inelastic and the risk-neutrality of foreign capitalists to transfer capital income across states. The ex ante elasticity of capital provides the necessary incentive to keep average tax payments at zero.

The result that there is no distortion of capital holds regardless of the government's discount rate as long as the government can commit.

What if the government cannot commit to its promised tax and debt plan? While the sunk nature of capital allows the government to insure domestic agents, it also tempts the government to renege on tax promises ex post. Similarly, a government may wish to default on its outstanding debt obligations. We show that the optimal taxation problem can be written as a constrained efficient contract between a risk-neutral foreigner (who can commit) and the government (who cannot commit). An efficient allocation is sustained by prescribing that if the government deviates on its tax policy or defaults on its debt obligations, foreign investment will drop to zero, and the country will remain in financial autarky thereafter.

An important feature of the optimal program is that when the government's participation constraints bind, capital following high income shocks is strictly greater than capital following low shocks, despite the shocks being *i.i.d.*. This cyclical variation in investment arises due to sovereign debt. The strongest temptation to deviate from the optimal plan arises after receiving the highest income shock. An optimal contract then accommodates such temptation by prescribing higher domestic consumption. However, consumption smoothing implies that it is optimal to increase future domestic consumption as well, a result that is achieved through a reduction in the stock of sovereign debt. A lower stock of debt relaxes subsequent participation constraints, allowing higher investment.

We show that if the government discounts at the market interest rate, then the economy asymptotes towards the first best level of capital and there is no amplification in the long-run. This result is as in Thomas and Worrall (1994) and Alburquerque and Hopenhayn (2004). We show as well that if instead the government discounts the future at a higher rate than the market, then capital converges to a unique, non-degenerate stationary distribution whose support lies strictly below the first best. Moreover, in the stationary distribution, capital is greater and debt lower following a higher shock. The model naturally generates investment and borrowing cycles whereby lower investment periods are preceded by higher levels of indebtedness.

To clarify the role of access to the sovereign debt markets in generating the investment cycles, we analyze the situation where the government is forced to run a balanced budget, and therefore cannot transfer resources across periods. In this case, distortions to investment will be independent of the current shock in an *i.i.d.* environment. Investment may be distorted, but will be constant. Further, for a discount factor lower than the market rate, it can be the case that under a balanced budget rule investment is undistorted and consumption is constant. Hence, the government's access to debt markets can increase the volatility of consumption and the distortion of investment.

Political economy frictions in developing economies are introduced in the model, albeit simplistically, by the higher discount rate of the government with respect to the domestic agents. In this case, the government will over-borrow, trading away future stability for increased current consumption. This provides a rationale for imposing a debt constraint on the government. These type of considerations have led countries such as Chile and Brazil to place budgetary restrictions on their governments. Our analysis highlights not only the circumstances under which such restrictions can be welfare improving, but also the expected effects on output volatility and investment patterns. Clearly, under a benevolent government such access to financial markets is always welfare improving.

The predictions of the model are reminiscent of emerging market crises. As predicted by the model, governments often allow foreign capital to earn large returns in booms but confiscate capital income during crises. Moreover, as documented by Calvo et al. (2005), investment remains persistently depressed following a crisis. The most recent crisis in Argentina in January 2002 is a dramatic illustration of this phenomenon. Measures of expropriation risk for Argentina as calculated by the Heritage Foundation and Fraser Institute deteriorated sharply. A similar deterioration of property rights is observed in other emerging market crises, often precipitated by a terms of trade shock or other exogenous drop in income. The oscillation between pro-growth policies and populism observed in many developing economies seems to contribute to (rather than stabilize) the volatility of output. Our paper rationalizes such behavior.

The model is also consistent with the well known debt overhang effect on investment in less developed countries. This negative effect of accumulated debt on investment has been widely explored starting with the work of Sachs (1989) and Krugman (1988). In these papers, investment by domestic agents is distorted because foreign creditors have claims on the additional output resulting from investment. The level of debt is assumed to be exogenous and debt relief is shown to enhance investment and in some cases to generate a Pareto improvement. Differently, in our model, such cyclical debt overhang effects arise endogenously due to the limited ability of the government to commit. However, at all times the optimal allocation generates payoffs on the Pareto frontier, and hence debt relief, while benefiting the government, can never generate a Pareto improvement. Furthermore, the existence of a unique long run distribution implies that debt relief programs will at most have short-lived effects.

1.1 Related Literature

This paper builds on the influential paper of Thomas and Worrall (1994), who introduce a model of foreign direct investment in an environment of limited commitment. Alburquerque and Hopenhayn (2004) develop a related model. Our analysis differs from these papers in several important respects. The first is that we analyze the case of an impatient government, which has dramatically different long run properties. Given the political economy of emerging markets, the case of an impatient government is an important generalization. Secondly, we consider more general production functions and derive our results under weaker assumptions. Lastly, we consider how the imposition of a balanced budget requirement on the host government alters the cyclicality of investment and consumption.

There is also a large literature on the question of optimal taxation by a benevolent social planner. See the survey by Chari and Kehoe (1999) and the references therein. Several papers have studied optimal fiscal policy without commitment. In a paper related to ours, Benhabib and Velasco (1996) study an open economy where the government lacks commitment and needs to finance productive investment. Their paper differs from ours by considering a deterministic economy. Therefore, there is no scope for fiscal policy to vary with shocks or to provide insurance. Our paper is also related to Phelan and Stacchetti (2001), where a policy game is analyzed for the case without uncertainty in a closed economy.

In an important paper in the international business cycle literature, Kehoe and Perri (2002) consider a model of risk sharing across two countries with limited commitment. Differently from them, we study a small open economy and emphasize the role of the government in generating amplification. Tornell and Velasco (1992) and Lane and Tornell (1999) present interesting political economy games in which a "tragedy of the commons" problem arises that may distort investment. However, these papers restrict attention to Markov equilibria, and therefore the issue of state dependent "reputation" does not arise.

Our results also relate to the "debt overhang" literature, in which outstanding debt influences borrowing for new investment projects (see, for example, Krugman (1988) and Cohen and Sachs (1986)). See Kovrijnykh and Szentes (2006) and Arslanalp and Henry (2006) for a recent discussion of this literature. Unlike our model, Kovrijnykh and Szentes (2006) present a model of strategic lending and focus on Markov equilibria.

The primary empirical study of emerging markets fiscal policy are Gavin and Perotti (1997) and Kaminksy et al. (2004). They focus on the spending side (see also Talvi and Vegh (2004)) and document that emerging markets follow fiscal policies that are more procyclical than those in developed economies. While the quarterly cyclicality of government expenditures is an important issue, our focus is on the expropriation of foreign capital during crises.

The paper is organized as follows. Section 2 describes the model environment; Section 3 characterizes the optimal policy under full commitment; Section 4 characterizes the optimal policy under limited commitment; Section 5 restricts the government to a balanced budget; and Section 6 concludes. The Appendix contains all proofs.

2 Environment

Time is discrete and runs to infinity. The economy is composed of a government and two types of agents: domestic agents and foreign capitalists. Domestic agents (or "workers") are risk averse and supply inelastically l units of labor every period for a wage w. Variables will be expressed in per capita units.

The economy receives a shock z every period. One can interpret the shock as a terms of trade shock to a developing country's exports or a productivity shock. The assumptions underlying the shock process are described below.

Assumption 1. *The shock z follows an i.i.d. process and the realizations of z lie in a finite set $Z \subset \Re$. Let the highest element of Z be $\bar{z}$ and the lowest element be $\underline{z}$. Let $q(z)$ denote the associated probability of state z.*

Let $z^t = \{z_0, z_1, ...z_t\}$ be a history of shocks up to time t. Denote by $q\left(z^t\right)$ the probability that z^t occurs.

The expected lifetime utility of workers is given by

$$\sum_{t=0}^{\infty} \sum_{z^t} \hat{\beta}^t q(z^t) U\left(c\left(z^t\right)\right),$$

where $c\left(z^t\right)$ is their consumption in history z^t and U is a standard utility function defined over non-negative consumption satisfying Inada conditions with $U' > 0, U'' < 0$. Let $U_{min} \equiv U(0)$. The parameter $\hat{\beta} \in (0, 1)$ is the discount factor. The "hat" notation on the discount rate allows us to differentiate the workers' and government's patience.

Assumption 2 (Segmented Capital Markets)**.** *Workers do not have access to financial markets. Their consumption is given by*

$$c\left(z^t\right) = w\left(z^t\right) l + T\left(z^t\right),$$

where $T\left(z^t\right)$ are transfers received from the government at history z^t and $w\left(z^t\right)$ is the competitive wage at history z^t.

As we will see below, we allow the government to borrow and lend from foreigners on behalf of workers, albeit perhaps not optimally from their perspective. If the government implements the workers' optimal plan, workers and the government can be considered a single entity. The expositional advantage of separating workers from the government is that in practice it is the government that can tax capital and not individual workers. Moreover, the government may not implement the workers' optimal plan. Note that the workers have a trivial decision problem, so we focus attention on the government's problem introduced below.

There exists a continuum of risk-neutral foreign capitalists who supply capital, but no labor. The foreign capitalists own competitive domestic firms that produce by hiring domestic labor and using foreign capital. The production function f is of the standard neoclassical form:

$$y = f(z, k, l),$$

where f is constant returns to scale with $f_k > 0$, $f_{kk} < 0$ and satisfying Inada conditions. Without loss of generality, we assume that $f(z, k, l)$ is strictly increasing in z.

The capitalists have access to financial markets. We assume a small open economy where the capitalists face the exogenous world interest rate of r^*. Capital is installed before the shock and tax rate are realized and cannot be moved until the end of the period. We denote by $k(z^{t-1})$ the capital installed at the end of period $t-1$ to be used at time t. The depreciation rate is δ. Capital profits (gross of depreciation) of the representative firm are denoted $\pi(z^t)$, where

$$\pi\left(z^t\right) = f\left(z_t, k\left(z^{t-1}\right), l\right) - w\left(z^t\right) l.$$

We make the following assumption about the government's objective function:

Assumption 3 (Government's Objective)**.** *The government's objective function is to maximize the present discounted utility of the workers, discounted at the rate $\beta \in (0,1)$:*

$$\sum_{t=0}^{\infty} \sum_{z^t} \beta^t q(z^t) U\left(c\left(z^t\right)\right),$$

A benevolent government is the case that $\beta = \hat{\beta}$. An interesting alternative is the case $\beta < \hat{\beta}$, that is, the government is more impatient than domestic agents. This may be interpreted as a reduced form of a political economy model in which the possibility of losing office makes the government discount the future at a greater rate (see, for example, Amador (2004)). Similarly, we could assume that the government maximizes the utility of a subset of agents, such as political insiders or public employees. The analysis will make clear that our results extend to these alternative objective functions as long as the favored agents are risk averse and lack access to capital markets.

The government receives an endowment income each period $g(z)$, where g is non-negative and strictly increasing in z. This captures, for example, returns to a natural resource endowment sold on the world market. The government also taxes capital profits at a linear rate $\tau\left(z^t\right)$ and transfers the proceeds to the workers $T\left(z^t\right)$. For the benchmark model, we assume the government can trade a non-contingent bond with the international financial markets. Let $b(z^t)$ denote the outstanding debt of the government borrowed at history z^t and due the next period (which is constant across potential shocks realized at $t+1$). The government's budget constraint is:

$$g(z^t) + \tau\left(z^t\right)\pi\left(z^t\right) + b(z^t) = T\left(z^t\right) + (1+r^*)b(z^{t-1}).$$

Taking as given a tax rate plan $\tau\left(z^t\right)$, firms maximize after-tax profits net of depreciation and discounted at the world interest rate,

$$E_0 \sum_t \left(\frac{1}{1+r^*}\right)^t \left(\left(1-\tau\left(z^t\right)\right)\pi\left(z^t\right) - k(z^t) + (1-\delta)k(z^{t-1})\right).$$

Profit maximization and labor market clearing imply the following two conditions:

$$w\left(z^t\right) = f_l\left(z_t, k\left(z^{t-1}\right), l\right), \tag{1}$$

and

$$r^* + \delta = \sum_{z_t \in Z} q(z_t)(1-\tau(z^t))f_k\left(z_t, k\left(z^{t-1}\right), l\right), \tag{2}$$

where f_i denotes the partial derivative of f with respect to $i = k, l$.

According to equation (2), the expected return to capitalists from investing in the domestic economy net of depreciation should equal the world interest rate r^*. Given the *i.i.d.* assumption regarding the shocks, optimal capital is a constant in a world without taxes. We denote this first best level of capital by k^*; so that $E[f_k(z, k^*, l)] = r^* + \delta$.

3 Optimal Taxation under Commitment

Before we proceed to the analysis with limited commitment, as a useful comparison we characterize the optimal fiscal policy under commitment. We show that tax policy is not distortionary and that investment will be constant at the first-best level $k^\star$.

Suppose that the government can commit at time 0 to a tax policy $\tau(z^t)$ and debt payments $(1+r^*)b(z^t)$ for every possible history of shocks z^t. This "Ramsey" plan is announced before the initial capital stock is invested. Given some initial debt, $b(-1)$, the government chooses $c(z^t)$, $b(z^t)$, $k(z^t)$, and $\tau\left(z^t\right)$ to maximize

$$\sum_{t=0}^{\infty}\sum_{z^t}\beta^t q(z^t)U\left(c\left(z^t\right)\right), \tag{3}$$

subject to the budget constraints of the domestic agents and the government as well as firm profit maximization:

$$\begin{aligned}
c\left(z^t\right) &= w\left(z^t\right)l + T\left(z^t\right), &(4)\\
T\left(z^t\right) + (1+r^*)b(z^{t-1}) &= \tau\left(z^t\right)\pi(z^t) + g(z^t) + b(z^t), &(5)\\
f_l\left(z_t, k\left(z^{t-1}\right), l\right) &= w\left(z^t\right), &(6)\\
r^* + \delta &= \sum_{z_t\in Z} q(z_t)(1-\tau(z^t))f_k\left(z_t, k\left(z^{t-1}\right), l\right). &(7)
\end{aligned}$$

Define the total output of the economy as $F(z,k,l)$:

$$F(z,k,l) \equiv f(z,k,l) + g(z). \tag{8}$$

Note that $F_i = f_i$, for $i = k, l$. Equations (4) through (6) can be combined to obtain:

$$\begin{aligned}
F\left(z_t, k\left(z^{t-1}\right), l\right) - \left(1-\tau\left(z^t\right)\right)F_k\left(z_t, k\left(z^{t-1}\right), l\right)k\left(z^{t-1}\right) + b(z^t)&\\
= c\left(z^t\right) + (1+r^*)b(z^{t-1}).& \qquad (9)
\end{aligned}$$

We have used the constant-returns-to-scale assumption (specifically, $f = f_k k + f_l l$) and $F_k = f_k$ in the derivation. Equation (9) states simply that consumption and debt payments (the right hand side) must equal total output minus equilibrium payments to capital plus new debt. The problem can be simplified once we recognize that the combination of taxes and a bond is equivalent to a complete set of state-contingent assets. In particular,

Lemma 1. *Let v equal (3) evaluated at the optimum with initial debt $b(-1)$. Then*

$$v = \max_{\{c(z^t)\},\{k(z^t)\}}\sum_{t=0}^{\infty}\sum_{z^t}\beta^t q(z^t)U\left(c\left(z^t\right)\right) \tag{10}$$

subject to,

$$\sum_{t=0}^{\infty}\sum_{z^t}\frac{q(z^t)}{(1+r^*)^{t+1}}\left(F(z_t,k(z^{t-1}),l)-(r^*+\delta)k(z^{t-1})-c(z^t)\right)\geq b(-1) \tag{11}$$

Conversely, any v that solves this problem is a solution to (3).

The complete markets equivalence results from the ability to transfer resources across states within a period using capital taxes (keeping the average tax constant) plus the ability to transfer resources across periods with the riskless bond. The combination is sufficient to transfer resources across any two histories, as also shown in Judd (1992); Zhu (1992); Chari et al. (1994). The scheme exploits the fact that capitalists are risk-neutral and that capital is immobile for one period.

We characterize some key features of the optimal policy under commitment in the following proposition,

Proposition 1. *Under commitment, for all $z_t, z'_t \in Z \times Z$, and for all $z^{t-1} \in Z^{t-1}$, the optimal fiscal policy has the following features: (i) it provides full intra-period insurance to the workers, $c\left(\{z_t, z^{t-1}\}\right) = c\left(\{z'_t, z^{t-1}\}\right)$; (ii) it smooths consumption across periods with discounting, $U'(c(z^{t-1})) = \beta(1+r^*)U'(c(z_t, z^{t-1}))$; (iii) at the begining of every period, the expected capital tax payments are zero and therefore capital is always at the first best level; and (iv) the amount of debt issued is independent of the current shock, $b(\{z_t, z^{t-1}\}) = b(\{z'_t, z^{t-1}\})$.*

Results (i) and (ii) are standard outcomes of models with complete markets and full commitment. Consumption is equalized across states of nature. Consumption trends up, down, or is constant over time, depending on whether the rate of time discount is less than, greater than, or equal to the world interest rate, respectively. Result (iii) follows from the fact that capital only enters the budget constraint, so optimality requires maximizing total output and not distorting investment; a well known result in the Ramsey taxation literature (Judd (1985); Chamley (1986), and the stochastic version in Zhu (1992)). Chari et al. (1994) obtain a similar result in a business cycle model. The small open economy assumption implies that capital is infinitely elastic ex ante and therefore that the zero-taxation of capital is optimal at all dates and not just asymptotically. See Chari and Kehoe (1999) for a related discussion. Result (iv) indicates how consumption smoothing is implemented using taxes and debt. Taxation is used to transfer resources across states, and debt to transfer resources across time. For example, when z is strictly an endowment shock or when there are only two states, the optimal plan calls for counter-cyclical taxes, with capital taxed more in low endowment states compared to high endowment states. Note that this counter-cyclical taxation does not distort investment. Whether z is an endowment or productivity shock, the amount of debt issued however, is independent of the current shock in an *i.i.d.* environment. The resulting fiscal deficit is acyclical. The results in this section show that a government with commitment will not amplify shocks through its tax policy. This holds independent of the relation between β and r^*: even if the government were more impatient relative to the market there will be no amplification as long as the government can commit.

4 Optimal Taxation with Limited Commitment

Once the investment decision by the capitalists has been made, the government has an incentive to tax capital as much as possible and redistribute the proceeds to the workers. Similarly, a government has an incentive to default on its outstanding debt obligations. Thus, the optimal tax and debt policy under commitment may not be dynamically consistent.

For the rest of this section we will assume that $\beta(1+r^*) \leq 1$, ensuring that the optimal plan does not have unbounded asset positions.

We place an upper bound on the tax rate the government can set at any time:

Assumption 4 (A Maximum Tax Rate). *The tax rate on profits cannot be higher than* $\bar{\tau} = 1$.

That is, the most the government can tax in any state is one-hundred percent of profits.

The goal is to characterize the efficient allocation of the game between the capitalists and the government. We make the standard assumption that the external financial markets can commit to deny access in case of a deviation by the government:

Assumption 5. *If the government ever deviates from the prescribed allocation on either taxes or debt payments, the country will remain in financial autarky forever; specifically the government would not be able to issue debt or hold external assets.*

As usual, efficient allocations are implemented with the threat of the worst punishment were the government to deviate on *either* taxes or debt payments. The worst outcome for the government is financial autarky coupled with a tax rate set to $\bar{\tau} = 1$, implying that no investment will take place, $k_{aut} = 0$. Note that given that after defaulting, the government has no access to financial markets forever, setting a tax rate equal to $\bar{\tau}$ is an equilibrium of the game between the capitalists and the government. Let V_{aut} denote the continuation value of the government in autarky. Specifically,

$$V_{aut} = \sum_{z \in Z} q(z) \frac{U(F(z,0))}{1-\beta}. \tag{12}$$

This assumes that the installed capital cannot be operated (or sold) by the government upon deviation. Allowing the government to sell off or immediately consume capital after deviation would not change the problem in a significant manner. We assume that $\bar{\tau}$ does not bind along the equilibrium path, but places an upper bound on seized income if the government deviates.[1]

Definition. *Given an initial debt $b(-1)$, an* ***efficient allocation*** *is a sequence of functions $c(z^t)$, $b(z^t)$, $\tau(z^t)$, and $k(z^t)$ such that (3) is maximized, constraints (4)-(7) hold, and the government at all histories prefers the continuation allocation to deviating and taxing capital at the highest possible rate $\bar{\tau}$, and/or defaulting on its debt obligations:*

$$\sum_{i=0}^{\infty} \sum_{z^{t+i}} q(z^{t+i}|z^t) \beta^i U(c(z^{t+i})) \geq U(F(z_t, k(z^{t-1}))) + \beta V_{aut}. \tag{13}$$

Note that the payoff after deviation is independent of the assets held by the government, hence if the government defaults on its tax promises while holding positive assets, $b(z^t) < 0$, it will loose them. It is not relevant for an efficient allocation to specify what happens to the seized assets as long as they are lost to the government.

4.1 A Recursive Formulation

Let us denote by v the maximal amount of utility attainable to the government in an efficient allocation, given that it has issued an amount $b \in \mathbf{b}$ to the foreign financial markets, where $\mathbf{b}$ denotes the set of possible debt levels for which the constraint set is non-empty. We also impose that the set $\mathbf{b}$ is bounded below (that is, assets have a finite upper bound). We discuss below that this bound is not restrictive.

Let us denote by B the function such that $(1 + r^*)b = B(v)$, for any $b \in \mathbf{b}$. We characterize the constrained efficient allocations recursively. Histories are summarized by promised expected discounted utility v for the government. We initially consider all v in a closed interval $[V_{aut}, V_{max}]$, where V_{max} is the value corresponding to the maximal asset level.

[1] Implicitly, the previous section assumed that $\bar{\tau}$ is greater than the maximal tax rate under the Ramsey plan (that is, the Ramsey plan is feasible without imposing negative post-tax profits on capital).

We assume $V_{max} \geq U(F(\bar{z}, k^*)) + \beta V_{aut}$. The right hand side of this inequality represents the utility obtained by deviation if capital is at its first best level and z is its maximum realization. The lower bound V_{aut} follows immediately from the government's lack of commitment. Once we have defined $B(v)$ on $[V_{aut}, V_{max}]$, we will characterize and restrict attention to the subset of $[V_{aut}, V_{max}]$ for which $B(v) \in \mathbf{b}$.

We now show that efficient allocations solve the following Bellman equation in which the state variable is v, and the choice variables are the capital stock, state-contingent flow utility (u), and a state-contingent promised utility (ω) which will be next period's state variable. Let Ω define the space of possible choices, where $\omega(z)$ is restricted to $[V_{aut}, V_{max}]$, and $u(z) \geq U_{min}$ and $k \geq 0$.

Proposition 2. *Let* $(c_0(z^t), k_0(z^t))$ *represent an efficient allocation given initial debt* $b(-1) \in \mathbf{b}$. *Let* v *represent the government's utility under this allocation. Then* $(1+r^*)b(-1) = B(v)$, *where* $B(v)$ *is the unique solution to the following recursive problem:*

$$B(v) = \max_{(u(z),\omega(z),k)\in\Omega} \sum_{z\in Z} q(z) \left[F(z,k) - c(u(z)) - (r^* + \delta)k + \frac{1}{1+r^*} B(\omega(z)) \right] \tag{14}$$

subject to

$$\begin{aligned} v &\leq \sum_{z\in Z} q(z)[u(z) + \beta\omega(z)] && (15) \\ U(F(z,k,l)) + \beta V_{aut} &\leq u(z) + \beta\omega(z), \ \forall z' \in Z. && (16) \end{aligned}$$

The function $c(u)$ *denotes the consumption required to deliver utility* u *(i.e.,* $U(c(u)) = u$*). Moreover, the sequence* $(c_0(z^t), k_0(z^t))$ *satisfies the the recursive problem's policy functions (iterating from the initial* v*).*

Conversely, let $(c_1(z^t), k_1(z^t))$ *be a sequence generated by iterating the recursive problem's policy functions starting from an initial* v *for each shock history* z^t *and* $B(v) \in \mathbf{b}$. *Then,* $(c_1(z^t), k_1(z^t))$ *is an efficient allocation starting from an initial debt* $(1 + r^*)b(-1) = B(v)$

The first constraint (15) is the promise keeping constraint that ensures the government enjoys (at least) the promised utility v. The second constraint (16) is the participation constraint. This ensures that the government never has the incentive to deviate along the equilibrium path. As discussed above, consumption during deviation is productive output plus the endowment. The continuation value post deviation is V_{aut} defined in (12). Note that constraint (15) can be treated as an inequality because more utility can be offered to the country without violating previous participation constraints.

Heuristically, an efficient allocation sits on the (constrained) Pareto frontier defined by the government's welfare and the bond holders' welfare (subject to the requirement that capital is always paid its opportunity cost). Note that the objective in the recursive problem represents payments to the bond holders. Therefore, any point on the Pareto frontier represents a solution to this problem. Conversely, as long as the promise keeping constraint (15) binds, the solution to the recursive problem represents a constrained Pareto optimal allocation and is therefore an efficient allocation.

Our problem hence collapses to the problem of finding a constrained efficient contract between a risk-neutral foreigner (who can commit) and the government (who cannot commit). Similar problems have been studied in the existing literature on dynamic contracts under lack of commitment by Thomas and Worrall (1994) and Alburquerque and Hopenhayn (2004).

The operator defined by (14) maps bounded functions into bounded functions. It also satisfies Blackwell's sufficient conditions for a contraction. The operator maps the set of continuous, non-increasing functions into itself. Standard arguments therefore imply that,

Lemma 2. *The value function $B(v)$ is non-increasing and continuous.*

The participation constraints (16) are not convex due to the presence of $U(F(z,k))$ on the left hand side of the inequality (16). So, to proceed with our characterization of the optimum we make the following additional assumption that ensures concavity of the value function.

Assumption 6. *For all $k \in [0,k^*] \times [0,k^*]$ and for all $z \in Z$, $F_{kk}(z,k)/F_k(z,k)$ is independent of z and the following inequality holds:*

$$E\left[\frac{U''(F(z,k))}{U'(F(z,k))}\frac{F_k(z,k)^3}{F_{kk}(z,k)}\right] \leq r^* + \delta$$

This states that the production function is "concave enough" relative to the utility function. The first term on the left hand side reflects the concavity of the utility function. As utility becomes linear, the left hand side approaches zero. Similarly, the second term captures the inverse of the curvature of the production function. This term becomes large as the production function becomes linear. This condition also implies that the local second order conditions for the problem are satisfied. We provide one example to show that Assumption 6 does not require extreme preferences or technology. In particular, let $U(c) = c^{1-\sigma}/(1-\sigma)$ and $F(z,k) = zk^\alpha + c_0 z$. In this case, if $\alpha > 1/2$ and c_0 sufficiently large (i.e., consumption bounded below)[2], then Assumption 6 is satisfied.

[2]Specifically, $c_0 \geq \max\langle \alpha\sigma/(1-\alpha) - 1, (1-\alpha)/(2\alpha-1)\rangle k^{*\alpha}$.

We make one more assumption about preferences to ensure that positive investment is sustainable even if promised utility is V_{aut}. Essentially, the government must be patient enough that positive investment is sustainable at the lowest admissible promised value. A sufficient condition is,

Assumption 7. $\beta(1+r^*) > \frac{U'(F(\bar{z},0))}{U'(F(\underline{z},0))}$.

This condition is similar to one used by Krueger and Perri (2005) to rule out autarky as the efficient allocation in their analysis of endowment risk sharing with limited commitment.

In the appendix we prove that Assumptions 6 and 7 imply concavity of the foreigner's value function and that optimal policies are interior:

Proposition 3. *Under the stated assumptions, (i) the value function $B(v)$ is concave and differentiable on $[V_{aut}, V_{max}]$; (ii) There exists $V_{min} > V_{aut}$, such that $B'(v) = 0$ for all $v \in [V_{aut}, V_{min}]$, promise keeping holds with strict equality for $v \geq V_{min}$, and $\mathbf{b} = [B(V_{min}), B(V_{max})]$; (iii) $B(v)$ is strictly decreasing and strictly concave for $v \in (V_{min}, V_{max}]$; and (iv) for each $v \in [V_{aut}, V_{max}]$, there exists an optimal $(k, u(z), \omega(z))$ with $k > 0$, and such that there exists non-negative multipliers $(\gamma, \lambda(z))$ that satisfy*

$$c'(u(z)) = \gamma + \frac{\lambda(z)}{q(z)} \tag{17}$$

$$B'(\omega(z)) = -\beta(1+r^*)\left(\gamma + \frac{\lambda(z)}{q(z)}\right) \tag{18}$$

$$\sum_z q(z)F_k(z,k) - (r^* + \delta) = \sum_z \lambda(z)U'(F(z,k))F_k(z,k), \tag{19}$$

with $-B'(v) = \gamma$. Moreover, the policies are unique for $v \in [V_{min}, V_{max}]$.

The first statement of the proposition is that the value function is concave, which requires Assumption 6. As discussed above, concavity (and differentiability) is not directly guaranteed as the constraint set is not convex. Nevertheless, as shown in the appendix, the value function is well behaved.

The second statement is that the promise keeping constraint does not bind in the neighborhood of V_{aut}. That is, an efficient allocation does not deliver utility below some threshold $V_{min} > V_{aut}$. This places a lower bound on v, which corresponds to the upper bound on b inherent in $\mathbf{b}$. There is no efficient equilibrium that satisfies the constraints at a debt level greater than $B(V_{min})$. The function $B(v)$ is depicted in Figure 1.

Statement (iii) strengthens statement (i) in that $B(v)$ is strictly concave over the relevant region. Despite the fact that the problem is not convex, statement (iv) indicates that there is a unique interior solution for each v.

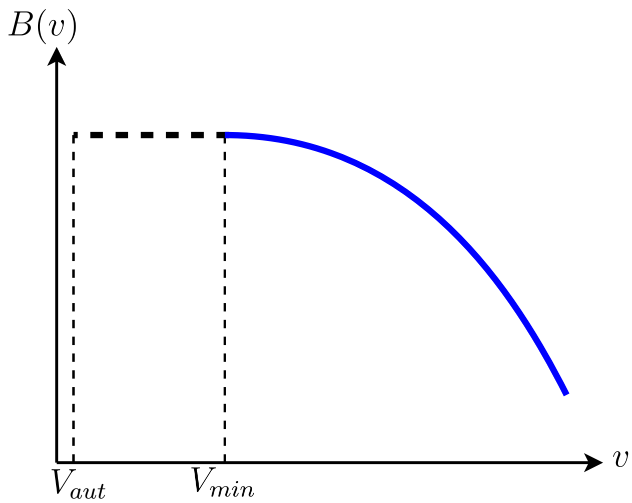

Figure 1: This figure depicts the function $B(v)$. The solid portion associated with $v \geq V_{min}$ is the Pareto frontier.

Let $g^i(v)$ denote the policies in an optimal plan for $i = u(z'), \omega(z')$, and k, at state v. We can immediately derive a number of properties of the optimal plan.

Proposition 4. *In an optimal solution,*

(i) For all $v \in [V_{aut}, V_{max}]$, $g^k(v) \leq k^*$;

(ii) For any $v \in [V_{aut}, V_{max}]$, if the participation constraints are slack for a subset $Z_o \subset Z$, then $c(g^{u(z)}(v))$ is constant for all $z \in Z_o$. Moreover, $B'(g^{\omega(z)}(v)) = \beta(1+r^*)B'(v)$ for all $z \in Z_o$;

(iii) If for some $v \in [V_{aut}, V_{max}]$ and $(z', z'') \in Z \times Z$ we have that $g^{u(z')}(v) \neq g^{u(z'')}(v)$ or $B'(g^{\omega(z')}(v)) \neq \beta(1+r^*)B'(v)$, then $g^k(v) < k^*$;

(iv) For any $v \in [V_{aut}, V_{max}]$, $g^{\omega(z)}(v) \geq V_{min}$ for all z.

The first part of the proposition states that capital never exceeds the first-best level. This can be seen from (19) and the fact that multipliers are non-negative. Benhabib and Rustichini (1997) show that in a deterministic closed economy model of capital taxation without commitment, there are situations where capital is subsidized in the long run, pushing capital above the first-best level. In our case, with an open economy, such a situation never arises. The second part states that the planner will always implement insurance across states and across time to the extent possible. If two states have unequal consumption and slack constraints, it is a strict improvement (due to risk aversion) to narrow the gap in consumption.

The third part of the proposition states that if the government fails to achieve perfect insurance, it will also distort capital. To see the intuition for this result, suppose that capital was at its first-best level but consumption was not equalized across states. The government could distort capital down slightly to relax the participation constraints. This has a second-order effect on total resources in the neighborhood of the first-best capital stock. However, the relaxation of the participation constraints allows the government to improve insurance. Starting from an allocation without perfect insurance, this generates a first-order improvement in welfare. Finally, part (iv) states that after the first period the promise keeping constraint will hold with equality.

Define $V^* = U(F(\bar{z}, k^*)) + \beta V_{aut}$. Note that for any $v \geq V^*$ we have that $g^k(v) = k^*$.[3] And that for $v < V^*$ at least one participation constraint will be binding. The next proposition further characterizes the efficient allocation,

Proposition 5. *In an optimal solution,*

(i) $g^i(v)$ *is single-valued and continuous for* $i = u(z'), \omega(z')$*, and* k*, for all* $v \in [V_{min}, V_{max}]$*;*

(ii) $g^k(v)$ *is non-decreasing in* v*, and strictly increasing for all* $v \in [V_{min}, V^*]$*;*

(iii) $g^{\omega(z')}(v)$ *and* $g^{u(z')}(v)$ *are strictly increasing in* v *for all* $v \in [V_{min}, V_{max}]$*;*

(iv) $g^{\omega(z_1)}(v) \geq g^{\omega(z_0)}(v)$ *if* $z_1 > z_0$*, and* $g^{\omega(\bar{z})}(v) > g^{\omega(\underline{z})}(v)$ *for all* $v \in [V_{min}, V^*]$*.*

Result (ii) states that capital is increasing in promised utility or is at the first best. A higher promised utility relaxes the participation constraint of the government, allowing for more capital. The fact that k is increasing in promised utility implies a higher utility and continuation value in each state to preserve government participation. That is, $g^{\omega(z')}(v)$ and $g^{u(z')}(v)$ are strictly increasing in v on $(V_{min}, V_{max}]$ for every realization $z' \in Z$, which is result (iii). Result (iv) states that future promised utility is nondecreasing in the realization of the endowment. In other words, realizations of the shock generate a "spreading out" of continuation values. If $v < V^*$, then insurance across states is not perfect and there will be at least one pair of states for which the inequality in (iv) is strict. The reason future promised utility is relatively high following a high shock is directly due to limited commitment. The strongest temptation to deviate from the optimal plan arises after receiving the highest income shock. An optimal contract then accommodates such temptation by prescribing higher domestic utility in case of a high income shock today. Consumption smoothing implies that it is optimal to increase future utility as well as current utility, a result that is achieved through a higher continuation value.

[3]If $v \geq U(F(\bar{z}, k^*, l)) + \beta V_{aut}$ then, ignoring the participation constraints, it is optimal to set $u(z) + \beta w(z) = v$ for all z and $k = k^*$, which satisfies the participation constraints.

The spreading out of continuation values and the fact that capital is increasing in promised utility implies the following:

Proposition 6 (Procyclicality)**.** *In a constrained efficient allocation,* $k(z_t, z^{t-1}) \leq k(z'_t, z^{t-1})$ *for* $z_t < z'_t$*. Also, if* $k(\underline{z}, z^{t-1}) < k^\star$ *then* $k(\underline{z}, z^{t-1}) < k(\bar{z}, z^{t-1})$*.*

This key proposition states that capital moves in a way that amplifies and prolongs shocks even in an environment in which shocks are *i.i.d.*. This pattern is reminiscent of emerging market crises, as discussed in the introduction. In many instances the increased fear of expropriation during a downturn generates a sharp drop in foreign investment, amplifying the decline in output. Optimal tax policy in the presence of limited commitment is consistent with such empirical regularities.

A common feature of models of insurance with limited commitment is that the participation constraints tend to bind when the endowment is high. This results from the fact that insurance calls for payments during booms and inflows during downturns. However, in precisely an environment that emphasizes insurance, we show that distortions of investment are greater during recessions because of the need to accumulate debt. This in turn makes low taxes in the future more difficult to sustain, depressing investment today.

4.2 Long Run Properties

If the government discounts the utility flows at the world interest rate, from (18) it follows that $-B'(\omega(z)) = -B'(v) + \lambda(z)/q(z)$. Given that the multipliers are non-negative and that B is strictly concave on $(V_{min}, V_{max}]$, this implies that v is weakly increasing over time and strictly increasing when the participation constraint binds. If the initial v lies below V^*, then $\lim_{t\to\infty} v_t = V^*$. The fact that the continuation value policies, $g^{w(z)}(v)$, are strictly increasing in v, implies that $v_t < V^*$ for all t. If the initial v lies above V^*, then no participation constraint ever binds, and v remains constant at its initial value. Monotonicity of v implies that in the long run capital monotonically approaches the first best. This result is a major feature of the models of Thomas and Worrall (1994) and Alburquerque and Hopenhayn (2004). In our environment, as in Alburquerque and Hopenhayn (2004), eventually enough collateral, in the shape of foreign assets, is built up that the participation constraint relaxes and the first best capital level obtains. Therefore, the amplification and persistence results described above only hold along the transition, but not in the steady state.

An alternate situation is one in which the government is impatient relative to the world interest rate. In this case, the government has a preference for early consumption. However, bringing consumption forward tightens the participation constraints in the future, distorting investment. In fact, if the government is relatively impatient, promised utilities and capital converge to a non-degenerate stationary distribution:

Proposition 7 (Impatience)**.** *If $\beta(1+r^*) < 1$, in a constrained optimal allocation, v and k converge to unique, non-degenerate stationary distributions with respective supports that lie strictly below V^* and k^*.*

Impatience implies that the persistence and amplification generated by limited commitment is a permanent feature of the economy. It never escapes the range in which capital is distorted. In fact, capital varies with the endowment shock in a manner that prolongs the shocks' effect on output.

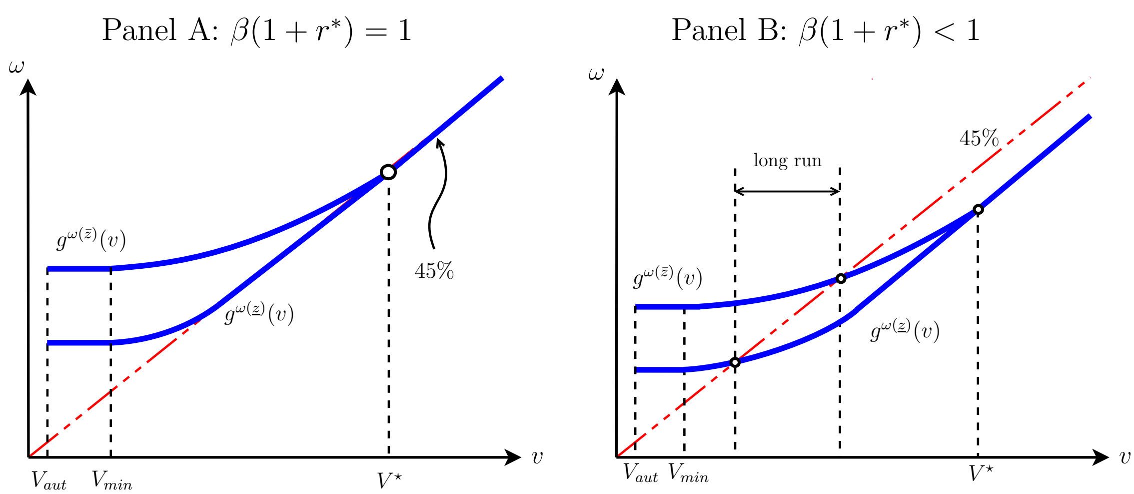

Figure 2: This figure depicts policy functions for next period′s promised utility as a function of the current promised utility. The shock takes two possible values. The top solid line represents the policy if the shock is high and the bottom solid line represents the policy if the endowment shock is low. The dashed ray is the 45 degree line. Panel A represents the case when $\beta(1+r^*) = 1$ and Panel B represents the case when $\beta(1+r^*) < 1$.

To visualize how the economy converges to the stationary distribution, we plot the policy functions for continuation utility ($g^{\omega(z')}(v)$) in figure 2. We assume two states for the endowment shock, $\bar{z}$ and $\underline{z}$. Proposition 5 states that the policy function for the high shock lies above the policy for the low shock (strictly above for any $v < V^*$), and they are both increasing in v. The policy functions lie strictly above the 45 degree line at V_{aut} because $g^z(v) \geq V_{min} > V_{aut}$ by Proposition 4 part (iv).

Panel A assumes $\beta(1+r^*) = 1$ and panel B assumes $\beta(1+r^*) < 1$. Panel A indicates that for $v \geq V^*$, the policy functions are equal (insurance across states) and on the 45 degree line (smoothing across periods). For $v < V^*$, the policy functions lie above the 45 degree line. Therefore, starting from some $v < V^*$, the promised utilities increase over time, approaching

V^* in the limit. Panel B indicates that when $\beta(1+r^*) < 1$, for any $v \geq V^*$, the policy functions lie below the 45 degree line. To see this, note that from (18) and the envelope condition,

$$-B'(\omega(z)) = \beta(1+r^*)\left(-B'(v) + \frac{\lambda(z)}{q(z)}\right).$$

When all participation constraints are slack, $\beta(1+r^*) < 1$ and strict concavity of the value function imply that $\omega(z) < v$. The policy functions intersect the 45 degree line at different points, which indicate the limits of the stationary distribution. The limiting distribution is non-degenerate from Proposition 5. Within this distribution, a low shock will lower promised utility and a high shock will raise it, reflecting that the government's utility oscillates within this range. Capital will converge to a corresponding non-degenerate distribution in which $k < k^*$, with capital oscillating one-for-one with promised utility.

The level of capital and the level of debt (the inverse of promised utility) of the economy are negatively correlated. The limited commitment of the government therefore generates a debt over-hang effect. Following high shocks the government accumulates assets. This slackens the participation constraint of the government in the future, reducing the incentive to deviate on taxes and therefore supporting higher investment. On the other hand, following low shocks, the government accumulates debt which raises the incentive to deviate and lowers investment. This mechanism differs from that described in Sachs (1989) and Krugman (1988) where a large level of debt reduces domestic investment because debt payments behave like a tax on investment. In the earlier debt over-hang literature, debt relief can not only raise investment but also be Pareto improving. In our environment, debt relief can benefit the government, but it is never a Pareto improvement, since the economy is at all points on the constrained Pareto frontier. In addition, since the long-run distribution of investment is unique, debt relief programs cannot have a long-run effect on investment. The distortions in investment arise from the lack of commitment of the government and impatience, which are issues that cannot be resolved through debt relief.

Net foreign liabilities in our model can be defined as debt plus foreign capital. The change in net foreign assets, the current account, is typically counter-cyclical in the data, particularly for emerging markets (see Aguiar and Gopinath (2007)). In our model, a positive shock generates an inflow of foreign direct investment inducing a deterioration of the current account. On the other hand, debt declines following a high shock, generating an improvement of the current account. The net effect on the current account is theoretically ambiguous, a standard outcome in a model with transitory shocks.

While the government is providing insurance to the domestic agents, this does not necessarily imply that government expenditures are higher during bad states. The model makes no distinction between private consumption and government provision of goods to the domestic agents. Nevertheless, it is the case that the sum of consumption and government expenditures is positively correlated with the shocks in the model, a fact consistent with the data (see Kaminksy et al. (2004)).

5 Balanced Budget

In this section, we explore optimal taxation of foreign capital when the government does not have access to debt. The purpose of this exercise is firstly to highlight the role of debt in generating investment cycles. In the balanced budget case, *i.i.d.* shocks imply that there is no state variable that varies over time and therefore optimal decisions are invariant to the realized history of shocks. Consequently, while investment may be distorted, there will be no cyclicality of investment. Secondly, we discuss circumstances under which budgetary restrictions can alter the volatility of consumption and be welfare improving.

To be precise, this section alters the benchmark model through the following assumption:

Assumption 8 (Balanced Budget)**.** *The government runs a balanced budget at every state:*

$$\tau\left(z^t\right)\pi\left(z^t\right)+g(z^t)=T\left(z^t\right).$$

Under the balanced budget assumption, we no longer have complete markets equivalence, and therefore cannot rewrite the government's problem as (10). However, constraints (4) through (7) can be simplified to

$$E_z\left[F\left(k\left(z^{t-1}\right),z\right)\right]-E_z\left[c\left(\left\{z,z^{t-1}\right\}\right)\right]-(r^*+\delta)k(z^{t-1})=0. \tag{20}$$

where E_z is defined to be the expectation over z.

As before, the fact that capitalists care only about the expected return to capital allows the government to use taxes to transfer resources to workers across states. However, the absence of bonds prevents inter-temporal transfers.

The full-commitment, balanced budget problem for the government can be written as maximizing:

$$\sum_{t=0}^{\infty}\sum_{z^t}\beta^t q(z^t)U\left(c\left(z^t\right)\right),$$

subject to (20). We characterize the optimal plan under commitment and a balanced budget in the following proposition:

Proposition 8. *Under commitment and a balanced budget, the optimal policy: (i) Provides full intra-period insurance to the workers,* $c\left(\{z, z^{t-1}\}\right) = c\left(\{z', z^{t-1}\}\right)$ *for all* $(z, z') \in Z \times Z$ *and* $z^{t-1} \in Z^{t-1}$*; (ii) At the beginning of every period, the expected capital tax payments are zero and investment is first best:* $E_z[\tau(\{z, z^{t-1}\})F_k(k(z^{t-1}), z)] = 0$ *and* $k(z^{t-1}) = k^*$.

This proposition differs from the complete markets case of Proposition 1 in one important respect. The optimal policy under a balanced budget insures workers across states *within* a period but does not necessarily deliver the optimal level of consumption across periods. Here consumption is equalized across periods. If $\beta(1 + r) < 1$, this is suboptimal. More generally, the optimal allocation is independent of β.

As in the benchmark model with full commitment, insurance is accomplished without distorting investment. That is, a balanced budget does not overturn the fact that the efficient policy sets expected capital taxes equal to zero. The government taxes capitalists and transfers to workers in low-endowment states while transferring from workers to capitalists in high-endowment states. However, the inability to borrow and save limits this insurance to period-by-period insurance and not insurance over the entire path.

We now consider the case without commitment under a balanced budget constraint. A balanced budget implies that payments to foreigners net of the opportunity cost of capital is zero every period. That is, in the notation of the benchmark model, foreign utility B is constrained to be zero. The punishment remains autarky, which is unaffected by the requirement of a balanced budget and is independent of z since shocks are *i.i.d.* In the absence of debt, the value function in the optimal program, V, is also independent of z.

The balanced budget problem can be written as directly maximizing the government's objective function subject to the break even constraint of the foreign capitalists. Specifically,

$$V = \max_{k, c(z')} E\left[U(c(z')) + \beta V(z')\right], \tag{21}$$

subject to

$$E[F(k, z')] - E[c(z')] - (r^* + \delta)k = 0, \tag{22}$$

and

$$U(c(z')) + \beta V \geq U(F(k, z')) + \beta V_{aut}, \forall z' \in Z. \tag{23}$$

Given that there is no state variable in this problem, all policies are constant,

Proposition 9 (No amplification)**.** *If the government is restricted to a balanced budget and shocks are i.i.d., investment will be constant.*

Investment may be distorted, but will be constant. With debt and an *i.i.d.* shock process, investment co-moved with the cycle indefinitely, as long as $\beta < 1/(1 + r^*)$. Without debt, capital is constant when shocks are *i.i.d.* regardless of the relation between β and $(1 + r^*)$. In this sense, the balanced budget case highlights the role of debt in generating cyclical investment.[4]

Another distinction the balanced budget assumption introduces is that depending on β either the first best is attainable immediately or it is never sustainable. With debt, we have that as long as $\beta \geq 1/(1 + r^*)$, the first best was achieved in the limit as time approached infinity. In the Appendix (see Lemma 14) we prove that there exists a $\beta^* \in (0, 1)$ such that for all $\beta \geq \beta^*$ the full commitment solution is sustainable under the balanced budget assumption, and it is not sustainable for $\beta \in [0, \beta^*)$. This is a version of the folk theorem. When the government is sufficiently patient, the future benefits of continuation are sufficiently important to sustain the first best capital stock.

Proposition 10. *Suppose $\beta \geq \beta^*$. (i) Restricting the government to a balanced budget achieves first best capital and constant consumption. (ii) If $\hat{\beta} = 1/(1 + r^*)$ and $\beta < \hat{\beta}$, then restricting the government to a balanced budget improves the welfare of domestic agents.*

Part (i) follows from an application of the folk theorem. The best allocation that can be attained with a balanced budget is one that delivers first best level of investment and constant consumption. When the discount factor is high enough, such allocation is sustainable. When $\hat{\beta} = 1/(1+r^*)$, this allocation is optimal from the perspective of domestic agents even when borrowing is allowed. Hence (ii) directly follows.

Access to sovereign debt markets generates cyclical and distorted investment and volatile consumption (proposition 4). If the government and the worker share the same discount factor, i.e. the government is benevolent, this increase in volatility or distortion in investment is associated with a welfare improvement. Placing a balanced budget constraint on a benevolent government can never be welfare improving. However, it can be argued, for political economy reasons as in Amador (2004), that the government discounts the future at a higher rate than the agents in the economy $(\beta < \hat{\beta})$. If $\hat{\beta} = 1/(1 + r^*) > \beta \geq \beta^*$, then imposing a balanced budget restriction will be welfare improving as well as generate greater stability in output.

This result provides a rationale for why countries such as Chile and Brazil have placed budgetary restrictions on their governments. More importantly, the analysis shows that this type of policy may stabilize investment and consumption.

[4]Debt generates persistence endogenously in the case with *i.i.d.* shocks. To see the analogy between having debt and having persistent shocks see Aguiar et al. (2006) where there is no debt, but shocks are assumed to be persistent. The persistent shocks generate a comparable investment pattern as debt in the neighborhood of the first best capital stock.

6 Conclusion

In this paper, we explored the combination of impatience, limited commitment, and access to borrowing and lending on foreign investment in a small open economy. The combination implies an optimal capital tax and debt program which generates cyclical investment relative to first best capital. Under full commitment, capital is first best regardless of the balanced budget assumption. In an environment with limited commitment, debt prolongs the impact of an *i.i.d.* endowment shock on future investment. If the government is impatient relative to the world interest rate, capital oscillates indefinitely, with low endowments associated with low investment. Conversely, capital is stable in an environment in which governments run balanced budgets and shocks are *i.i.d.* over time. Similarly, if the government is patient, the ability to save allows convergence to the first best capital stock. This highlights the role debt, impatience, and limited commitment each play in amplifying investment cycles.

While it is clear that imposing a balanced budget stabilizes capital, it does not necessarily improve welfare. Indeed, when the government is benevolent, imposing an additional constraint on the government is never welfare improving, despite the increased income stability. The government borrows because domestic agents are impatient and are willing to trade more consumption today in exchange for more volatility in the future. However, if the government is impatient relative to the workers as well as the world interest rate, the government will sub-optimally (from the workers' perspective) trade away future stability for increased current consumption. Depending on the magnitude of this distortion, this may provide a rationale for imposing a balanced budget constraint, although it leaves open the question of whether there is a more efficient mechanism that allows some inter-temporal smoothing, but not full access to debt markets.

A Appendix

A.1 Proofs of Lemma 1 and Proposition 1

Proof of Lemma 1:

Proof. As the objective functions are the same, we need to show that the constraint sets are equivalent. Suppose that $c(z^t)$ and $k(z^t)$ satisfy constraints (4) through (7). Taking expectations of both sides of equation (9) from the initial information set, we have (suppressing labor in the production function):

$$E\left[F(z_t, k(z^{t-1})) - c(z^t) - (r^* + \delta)k(z^{t-1})\right] = E\left[(1 + r^*)b(z^{t-1}) - b(z^t)\right], \tag{24}$$

where we have used $E\left[(1 - \tau(z^t))F_k(z_t, k(z^{t-1})k(z^{t-1})\right] = E\left[(r^* + \delta)k(z^{t-1})\right]$. We can solve this first order difference equation forward, applying the No Ponzi condition, to obtain:

$$E\left[\sum_{t=0}^{\infty} \frac{1}{(1 + r^*)^t}\left(F(z_t, k(z^{t-1})) - c(z^t) - (r^* + \delta)k(z^{t-1})\right)\right] \geq (1 + r^*)b(-1). \tag{25}$$

To go the other way, suppose that $c(z^t)$ and $k(z^t)$ satisfy (11). Starting from $b(-1)$, construct a sequence of $b(z^t)$ from the law of motion:

$$b(z^t) = E[c\left(z^t\right) - F\left(z_t, k\left(z^{t-1}\right), l\right)|z^{t-1}] + (r^* + \delta)k(z^{t-1}) + (1 + r^*)b(z^{t-1}). \tag{26}$$

Given $b(z^t)$, $c(z^t)$ and $k(z^t)$, the $\tau(z^t)$ solve equation (9) at each history. From (26), these taxes satisfy (2). The derivation of (9) also verifies that this choice is consistent with conditions (4) through (6). □

Proof of Proposition 1:

Proof. The proof of (i) through (iii) follows straightforwardly from the solution to (10). Let λ be the multiplier on the single budget constraint. The first order conditions from the optimization are

$$\beta^t(1 + r^*)^t U'(c(z^t)) = \lambda, \tag{27}$$

$$F_k(k(z^t), l) = r^* + \delta. \tag{28}$$

To prove (iv) we use the budget constraint, which holds with equality as $\lambda > 0$. Let c_t be consumption at time t, which is independent of the history of shocks by (ii). The budget constraint (11) implies that at any history z^t, we have

$$(1+r^*)b(z^{t-1}) = \sum_{s=t}^{\infty} \frac{E\left[F(z_s,k^*)\right]-(r^*+\delta)k^*}{(1+r^*)^{s-t}} - \sum_{s=t}^{\infty} \frac{c_t}{(1+r^*)^{s-t}}. \tag{29}$$

Note that the expectation of $F(z_s,k^*)$ is independent of history given that capital is constant at k^* and z is *iid*. Therefore, debt does not depend on the particular path of shocks. □

A.2 Proof of Proposition 2

Proof. The proposition holds by inverting the Pareto frontier. Specifically, the government's problem in recursive form with state variable b is

$$V(b) = \max_{c(z),b(z),k} E\left[u(c(z))+\beta V(b(z))\right] \tag{30}$$

subject to

$$\begin{aligned} E\left[c(z)+(1+r^*)b\right] &= \left[F(z,k)-(r^*+\delta)k+b(z)\right] \\ u(c(z))+\beta V(b(z)) &\geq U(F(z,k))+\beta V_{aut},\ \forall z\in Z \\ V(b(z)) &\geq V_{aut},\ \forall z \in Z. \end{aligned}$$

The first constraint is the expected budget constraint, derived from (9), where we have substituted in the first order condition for foreign direct investment. The second and third constraint ensure participation. Note that optimality ensures the budget constraint binds with equality. Therefore, $V(b)$ is strictly decreasing and has an inverse. By definition, $B(v)/(1+r^*)$ is this inverse, i.e. $V(B(v)/(1+r^*))=v$. Therefore, an allocation that solves (30) must also solve (14). The converse is true as long as the promise keeping constraint (15) binds. When (15) does not bind, $B(v)$ is flat as we increase v (see Figure 1). The domain of v on which $B(v)$ is flat cannot be part of the Pareto frontier and $B(v)$ does not solve (30). This implies that if promise keeping does not bind at v, then there are no b such that $V(b)=v$. That is, the constraint set of (30) is empty and $b\notin \mathbf{b}$. □

A.3 Proof of Proposition 3

This subsection characterizes the solution to the Bellman equation. The main technical challenge stems from the fact that the constraint set is not convex. Nevertheless, under Assumptions 6 and 7, we prove in this appendix that the value function is concave and differentiable and the policy functions are interior, unique, and continuous. With these results in hand, Proposition 3 follows immediately.

Let T denote the operator associated with the Bellman equation (14):

$$TB^n(v) = \max_{(u(z),\omega(z),k)\in\Omega} \sum_{z\in Z} q(z) \left[F(z,k) - c(u(z)) - (r^* + \delta)k + \frac{1}{1+r^*} B^n(\omega(z)) \right] \tag{31}$$

subject to

$$\begin{aligned} v &\le \sum_{z\in Z} q(z)[u(z) + \beta\omega(z)] \\ U(F(z,k,l)) + \beta V_{aut} &\le u(z) + \beta\omega(z), \ \forall z' \in Z. \end{aligned}$$

Note that the operator defined by (31) is a contraction. The value function is therefore the unique fixed point of this operator.

We begin by finding conditions under which the optimal k is strictly greater than zero. Define $B^{n+1}(v) = TB^n(v)$ and define $k^n(v)$ as the corresponding optimal capital at promised utility v. When we are operating on the fixed point $B(v) = TB(v)$, we denote the optimal capital as $k(v)$.

Lemma 3. *If $B^{n\prime}(V_{aut}) = 0$, then $k^n(v) > 0$ for all $v \in [V_{aut}, V_{max}]$. Conversely, if $k^n(v) > 0$ is optimal for all v, then $B^{n+1\prime}(V_{aut}) = 0$.*

Proof. Suppose $B^{n\prime}(V_{aut}) = 0$, and, to generate a contradiction, $k^n(v) = 0$ is optimal at some $v \in [V_{aut}, V_{max}]$. We show this cannot be optimal using a variational argument. Consider increasing k^n from zero by a small amount Δk. This requires increasing $u(z) + \beta\omega(z)$ in all states for which the participation constraint binds. Let Z' be the set of states for which the participation constraint binds. For any $z \in Z'$, $u(z) + \beta\omega(z)$ must be increased by $U'(F(z,0))F_k(z,0)\Delta k$. If for some $z \in Z'$ we have $\omega(z) = V_{aut}$, then by the premise it is costless at the margin to increase $\omega(z)$ to satisfy the participation constraint. Let Z'' be the set of $z \in Z'$ such that $\omega(z) > V_{aut}$. The binding participation constraint implies $u(z) < U(F(z,0))$, $\forall z \in Z''$. The marginal cost of increasing utility in state z by the required amount is $c'(u(z))U'(F(z,0))F_k(z,0)\Delta k$. As $c'(U(F(z,0))) = 1/U'(F(z,0))$, this equals $F_k(z,0)\Delta k(c'(u(z))/c'(U(F(z,0))) < 1$, where the last inequality follows from the facts $u(z) < U(F(z,0))$ and the strict convexity of $c(u)$. Therefore the net benefit of increasing capital at the margin is at least

$$\sum_{z\in Z-Z''} q(z)F_k(z,0) + \sum_{z\in Z''} q(z) \left(1 - \frac{c'(u(z))}{c'(U(F(z,0)))}\right) F_k(z,0) - (r*+\delta). \tag{32}$$

As $F_k(z,0)$ becomes arbitrarily big, this term is strictly positive. Therefore, $k = 0$ cannot be optimal.

To go the other way, assume that $k^n(V_{aut}) > 0$. Then $u(z) + \beta\omega(z) \geq U(F(z,k)) + \beta V_{aut} > U(F(z,0)) + \beta V_{aut}$, where the last inequality follows from $k > 0$. Taking expectation over z, this implies $E\left[u(z) + \beta\omega(z)\right] > V_{aut}$ and promise keeping does not bind at V_{aut}. Therefore, $B^{n+1}(v)$ is constant at V_{aut} for small increases in v, implying that $B^{n+1}(v)$ is differentiable at V_{aut} from the right and the derivative is zero. □

We now characterize the solution to (31) more fully. We begin by characterizing the first order conditions for a particular subset of functions. Define $\mathbf{B}$ as the set of bounded, concave, differentiable functions defined on $[V_{aut}, V_{max}]$ that satisfy $B'(V_{aut}) = 0$ and $B'(V_{max}) \leq -\beta(1+r^*)c'((1-\beta)V_{max})$. This set is not empty.[5] We start with this set as the solution to (31) is always interior when $B(v) \in \mathbf{B}$, as the next lemma demonstrates. Moreover, we show below that the fixed point of the operator lies in this set.

The next lemma states that if $B(v) \in \mathbf{B}$, the solution is interior.

Lemma 4. *For any v and $B^n(v) \in \mathbf{B}$, there exist non-negative multipliers $(\gamma, \lambda(z)))$ such that an optimal solution $(k^n, u(z), \omega(z))$ of problem (31) has $k^* \geq k^n > 0$ and satisfies:*

$$\begin{aligned} c'(u(z)) &= \gamma + \frac{\lambda(z)}{q(z)} && (33)\\ B^{n\prime}(\omega(z)) &= -\beta(1+r^*)\left(\gamma + \frac{\lambda(z)}{q(z)}\right) && (34)\\ \sum_z q(z)F_k(z,k^n) - (r^* + \delta) &= \sum_z \lambda(z)U'(F(z,k^n))F_k(z,k^n). && (35) \end{aligned}$$

Proof. To simplify notation, in the proof we omit the n superscript on optimal capital. Our problem satisfies the conditions stated in Luenberger (1969) Chapter 9 Theorem 1 (the Generalized Kuhn-Tucker Theorem). The only technical issue is that any solution with $k = 0$ does not satisfy the required "regularity condition", as $F_k \to \infty$ as $k \to 0$. However, Lemma 3 ensures that $k = 0$ is never optimal given that $B^n(v) \in \mathbf{B}$. The Lagrangian is

$$\begin{aligned} \mathcal{L} = &-\sum_{z\in Z} q(z)\left[F(z,k,l) - c(u(z)) - (r^*+\delta)k + \frac{1}{1+r^*}B^n(\omega(z))\right] \\ + \quad &\gamma\left(v - \sum_{z\in Z} q(z)(u(z)+\beta\omega(z))\right) + \sum_{z\in Z}\lambda(z)\Big(U(F(z,k)) + \beta V_{aut} - u(z) - \beta\omega(z)\Big) + \\ &\sum_{z\in Z}\left[\bar{\mu}(z)\left(\omega(z) - V_{max}\right) + \underline{\mu}(z)\left(V_{aut} - \omega(z)\right) + \eta(z)\left(U_{min} - u(z)\right)\right], \end{aligned}$$

[5] For example, consider $B(v) = \frac{-A}{2}(V_{aut} - v)^2$ with $A = \frac{\beta(1+r^*)c'((1-\beta)V_{max})}{V_{max} - V_{aut}}$.

where $(\gamma, \lambda(z), \bar{\mu}(z), \underline{\mu}(z), \eta(z))$ are non-negative multipliers. The first order conditions are:

$$c'(u(z)) = \gamma + \frac{\lambda(z) + \eta(z)}{q(z)} \tag{36}$$

$$B^{n\prime}(\omega(z)) = -\beta(1+r^*)\left(\gamma + \frac{\lambda(z)}{q(z)}\right) + (1+r*)(\bar{\mu}(z) - \underline{\mu}(z)) \tag{37}$$

$$\sum_z q(z)F_k(z,k) - (r^* + \delta) = \sum_z \lambda(z)U'(F(z,k))F_k(z,k). \tag{38}$$

The lemma is proved once we show that $\eta(z) = \bar{\mu}(z) = \underline{\mu}(z) = 0, \forall z \in Z$ and that $k \le k^*$, $\forall v \in [V_{aut}, V_{max}]$.

The fact that $k \le k^*$ follows immediately from the first order condition for capital and the fact that right hand side of (38) is non-negative.

To see that $\eta(z) = 0$, note that if $\eta(z) > 0$, then $u(z) = U_{min}$. However, the Inada condition, $c'(U_{min}) = 0$, and the first order condition (36), imply that $\gamma = \lambda(z) = \eta(z) = 0$, which contradicts $\eta(z) > 0$.

To see that $\underline{\mu} = 0$, recall that $B^{n\prime}(V_{aut}) = 0$. Equation (37) then requires $\underline{\mu} = 0$ (this follows because $\underline{\mu} > 0$ implies $\overline{\mu} = 0$).

To show that $\bar{\mu} = 0$, suppose that instead $\bar{\mu}(z) > 0$ for some $z \in Z$. Then the constraint binds and we have $\omega(z) = V_{max}$. The first order condition for $\omega(z)$ implies $-B^{n\prime}(\omega(z)) < \beta(1+r^*)c'(u(z))$. As $B^n \in \mathbf{B}$, $-B^{n\prime}(\omega(z)) = -B^{n\prime}(V_{max}) \ge \beta(1+r^*)c'((1-\beta)V_{max})$. Therefore, $c'(u(z)) > c'((1-\beta)V_{max})$. Convexity of $c(.)$ implies

$$u(z) > (1-\beta)V_{max}. \tag{39}$$

Therefore, $u(z) + \beta\omega(z) > (1-\beta)V_{max} + \beta V_{max} = V_{max}$. By definition, $V_{max} > U(F(\bar{z}, k^*)) + \beta V_{aut}$. Therefore, $\lambda(z) = 0$. This shows that $\bar{\mu}(z) > 0$ implies that $\lambda(z) = 0$.

As $\lambda(z) = 0$ and $\omega(z) = V_{max}$, (37) implies that

$$B^{n\prime}(V_{max}) = -\beta(1+r^*)\gamma + (1+r^*)\bar{\mu}(z). \tag{40}$$

Now consider some $z' \in Z$ such that $z' \ne z$. Concavity and the fact that $\omega(z') \le V_{max}$, we have $B^{n\prime}(\omega(z')) \ge B^{n\prime}(V_{max})$. Using (40), the first order condition (37) evaluated at $\omega(z')$ and the fact that $\lambda(z') \ge 0$ we have

$$B^{n\prime}(\omega(z')) \le B^{n\prime}(V_{max}) + (1+r^*)(\bar{\mu}(z') - \bar{\mu}(z)).$$

As $\bar{\mu}(z) > 0$, we have $B^{n\prime}(\omega(z')) - (1+r^*)\bar{\mu}(z') < B^{n\prime}(V_{max})$. Together, this implies $\bar{\mu}(z') > 0$.

As z' was arbitrary, then if $\bar{\mu}(z) > 0$ for one $z \in Z$, we have $\bar{\mu}(z) > 0$ for all $z \in Z$. Therefore, $\omega(z) = V_{max}$, $\forall z \in Z$. Moreover, we have $\lambda(z) = 0$ for all $z \in Z$, and so $u(z) = \bar{u}$, $\forall z \in Z$ for some constant $\bar{u}$. Now consider delivered utility: $E\left[u(z) + \beta\omega(z)\right] = \bar{u} + \beta V_{max}$. For this to be less than or equal to V_{max}, we need $\bar{u} \leq (1-\beta)V_{max}$, which contradicts (39). Therefore, $\overline{\mu}(z) = 0$, $\forall z \in Z$. $\square$

As the participation constraint is not convex, we can not directly infer how the multipliers respond to changes in v by appealing to global concavity. Nevertheless, the next lemma uses Assumption 6 to show how the multipliers on participation change as we vary v.

Lemma 5. *Suppose that $k^n(v_1) > k^n(v_0)$ for some (v_0, v_1), where we do not exclude the case of $v_0 = v_1$. Then there exists at least one $z \in Z$ such that $\lambda(z|v_1) < \lambda(z|v_0)$. Moreover, if $k^n(v_0) = k^n(v_1)$, then $\lambda(z|v_1) = \lambda(z|v_0)$ for all $z \in Z$.*

Proof. Define

$$H(k, \lambda(z)) \equiv E\left[\left(1 - \frac{\lambda(z)}{q(z)}U'(F(z,k))\right)F_k(z,k)\right] - (r^* + \delta). \tag{41}$$

The first order condition for capital implies $H(k_i, \lambda_i(z)) = 0$, if $(k_i, \lambda_i(z))$ constitute an optimum.

Differentiating $H(k, \lambda(z))$ with respect to k, we have:

$$\begin{aligned} H_k(k, \lambda(z)) &= E\left[\left(1 - \frac{\lambda(z)}{q(z)}U'(F(z,k))\right)F_{kk}(z,k)\right] \\ &- E\left[\frac{\lambda(z)}{q(z)}U''(F(z,k))(F_k(z,k))^2\right]. \end{aligned}$$

By Assumption 6, $\frac{F_{kk}}{F_k}$ is independent of z. Therefore, dividing both sides and using (41),

$$\frac{F_k H_k(k, \lambda(z))}{F_{kk}} = H(k, \lambda(z)) + (r^* + \delta) - E\left[\frac{\lambda(z)}{q(z)}U''(F(z,k))\frac{(F_k(z,k))^3}{F_{kk}}\right]. \tag{42}$$

Now consider any pair $(k_i, \lambda_i(z))$ which constitute an optimum for some v, hence $k_i > 0$. If $\lambda_i(z) > 0$, then from the binding participation constraint, $u_i(z) < U(F(z,k_i))$ as $\omega_i(z) > V_{aut}$ (this last inequality is strict as $B^{n\prime}(V_{aut}) = 0$). The first order conditions for $u_i(z)$ imply that $c'(u_i(z)) \geq \lambda_i(z)/q(z)$. Therefore $1/U'(F(z,k_i)) > c'(u_i(z)) \geq \lambda_i(z)/q(z)$. Note that $1/U'(F(z,k_i)) > \lambda_i(z)/q(z)$ if $\lambda_i(z) = 0$ given $k_i > 0$. Therefore, for all $k \geq k_i$, we have $1/U'(F(z,k_i)) > \lambda_i(z)/q(z)$. Substituting into the last expression on the right of (42), we have

$$\frac{F_k H_k(k, \lambda_i(z))}{F_{kk}} - H(k, \lambda_i(z)) > (r^* + \delta) - E\left[\frac{U''(F(z,k))}{U'(F(z,k))}\frac{(F_k(z,k))^3}{F_{kk}}\right],$$

for $k \geq k_i$. By Assumption 6, the right hand side is positive, or

$$\frac{F_k H_k(k, \lambda_i(z))}{-F_{kk}} < -H(k, \lambda_i(z)),$$

for $k \geq k_i$. Therefore, as long as $H(k, \lambda_i) \geq 0$, $H_k(k, \lambda_i) < 0$. Starting from $H(k_i, \lambda_i) = 0$, we therefore have $H(k_j, \lambda_i) < 0$ for all $k_j > k_i$.

Now consider $k_1 \equiv k^n(v_1) > k^n(v_0) \equiv k_0$. From the above, we have $H(k_1, \lambda_0) < H(k_0, \lambda_0) = 0$. Now suppose, to generate a contradiction and prove our first claim, that $\lambda(z|v_0) \leq \lambda(z|v_1)$, for all $z \in Z$. The first order condition at k_1 is

$$\begin{aligned} 0 = H(k_1, \lambda_1) &= E\left[\left(1 - \frac{\lambda(z|v_1)}{q(z)} U'(F(z, k_1))\right) F_k(z, k_1)\right] - (r^* + \delta) \\ &\leq E\left[\left(1 - \frac{\lambda(z|v_0)}{q(z)} U'(F(z, k_1))\right) F_k(z, k_1)\right] - (r^* + \delta) \\ &= H(k_1, \lambda_0), \end{aligned}$$

where the middle inequality follows from the premise that $\lambda(z|v_0) \leq \lambda(z|v_1)$. But this contradicts $H(k_1, \lambda_0) < H(k_0, \lambda_0) = 0$.

Now we show that if $k_0 = k_1 = k$, then $\lambda_0(z) = \lambda_1(z)$ for all $z \in Z$. The two first order conditions can be differenced to give

$$E\left[(\lambda_0(z) - \lambda_1(z))\, g(z)\right] = 0, \tag{43}$$

where $g(z) = 1/q(z)U'(F(z, k))F_k(z, k) > 0$. Now suppose, to generate a contradiction, there is some z' such that $\lambda_0(z') > \lambda_1(z')$. Equation (43) and $g(z) > 0, \forall z \in Z$ implies that there exists a z'' such that $\lambda_0(z'') < \lambda_1(z'')$. As $\lambda_0(z') > 0$, participation binds at z' for $u_0(z')$ and therefore (given equal k) we have $u_1(z') \geq u_0(z')$. The first order conditions for $u(z')$ then imply $\gamma_0 + \lambda_0(z')/q(z') \leq \gamma_1 + \lambda_1(z')/q(z')$. Given $\lambda_0(z') > \lambda_1(z')$, we require $\gamma_0 < \gamma_1$. Now consider z'', for which $\lambda_0(z'') < \lambda_1(z'')$. This implies $\lambda_0(z'')/q(z'') + \gamma_0 < \lambda_1(z'')/q(z'') + \gamma_1$, or $u_0(z'') < u_1(z'')$. However, this implies participation is slack at $u_1(z'')$, contradicting $\lambda_1(z'') > \lambda_0(z'')$. □

The following lemma defines an useful property of the new value function.

Lemma 6. *Define V_{min}^{n+1} to be the highest value such that $B^{n+1}(v) = B^{n+1}(V_{aut})$ for all $v < V_{min}^{n+1}$. Then the promise keeping constraint will be binding for all $v \geq V_{min}^{n+1}$.*

Proof. Note that if the promise keeping constraint is slack for some v_0, then $B^{n+1}(v) = B^{n+1}(v_0)$ for all $v < v_0$ (lowering the promised value cannot be worse). Then it follows that the promise keeping is binding for all $v > V_{min}^{n+1}$. It is also binding at V_{min}^{n+1} because if it were not, that is the promise keeping constraint is slack at V_{min}^{n+1}, then a small increase in the promised value has to deliver the same utility $B(V_{min}^{n+1})$, as the same allocation satisfy all the constraints, a contradiction of the definition of V_{min}^{n+1}. $\square$

We can show that the policies are unique and continuous for all $v \in [V_{min}^{n+1}, V_{max}]$.

Lemma 7. *The optimal policy $(k^n(v), u(z|v), \omega(z|v))$ and multipliers $(\gamma(v), \lambda(z|v))$ of problem (31) for $B^n(v) \in \mathbf{B}$ are unique and continuous in v for $v \in [V_{min}^{n+1}, V_{max}]$.*

Proof. Note that we are considering the range of v such that the promise keeping binds.

For uniqueness, suppose that $(k_0^n, u_0(z), \omega_0(z))$ and $(k_1^n, u_1(z), \omega_1(z))$ both are optimal allocations given some v. Let the associated multipliers be denoted by corresponding subscripts. Note that for fixed k, the constraint set is convex and the objective strictly concave, in which case the optimal policy is unique. Therefore, if $k_0^n = k_1^n$, the remaining policies are unique. So suppose $k_1^n > k_0^n$. Let $Z' \subseteq Z$ be the set of z such that $\lambda_0(z) > 0$. For these z, participation binds at k_0^n and so $u_1(z) + \beta\omega_1(z) \geq U(F(z, k_1)) + \beta V_{aut} > U(F(z, k_0)) + \beta V_{aut} = u_0(z) + \beta\omega_0(z)$. As $c'(u(z)) = -\beta(1 + r^*)B^{n\prime}(\omega(z))$ and both sides are strictly convex on the relevant domain, this implies $u_1(z) > u_0(z)\ \forall z \in Z'$. For equality of delivered utility (i.e. Pareto optimality), we must have at least one $\tilde{z} \notin Z'$ such that $u_1(\tilde{z}) + \beta\omega_1(\tilde{z}) < u_0(\tilde{z}) + \beta\omega_0(\tilde{z})$. This implies $\gamma_1 \leq c'(u_1(\tilde{z})) < c'(u_0(\tilde{z})) = \gamma_0$, where the last equality follows from $\lambda(\tilde{z}) = 0$ for $\tilde{z} \notin Z'$. As $c'(u_1(z)) > c'(u_0(z))$ for $z \in Z'$, $\gamma_1 < \gamma_0$ and the first order condition for $u(z)$ imply that $\lambda_1(z) > \lambda_0(z)\ \forall z \in Z'$. As $\lambda_0(z) = 0$ for $z \notin Z'$, we have $\lambda_1(z) \geq \lambda_0(z)$ for all $z \in Z$ and strictly greater for at least one z. By Lemma 5, this is inconsistent with $k_1^n > k_0^n$. Therefore the optimal policy is unique. From Lemma 5, uniqueness of capital implies that $\lambda(z)$ are unique. The uniqueness of γ then follows from the first order conditions and the uniqueness of policies.

For continuity, consider a small change in v. As the promise keeping constraint binds on the stated domain, this induces a small change in $E\left[u(z|v) + \beta\omega(z|v)\right]$. If $u(z|v)$ jumps up or down discretely, then there must be an offsetting discrete change in $\omega(z|v)$ or a discrete change in $u(z'|v) + \beta\omega(z'|v)$ for $z' \neq z$. We can rule out a discrete offsetting change in $\omega(z|v)$ as the first order conditions imply $c'(u(z|v)) = -\beta(1 + r^*)B^{n\prime}(\omega(z|v))$, and both sides are convex. The same reasoning (and the fact that $B^{n\prime}(v)$ is continuous) implies that if $\omega(z'|v)$ changes discretely, then so does $u(z'|v)$, and vice versa. We therefore can focus on a discrete change in $u(z'|v)$ as the off-setting response to a discrete change in $u(z|v)$. Suppose that in response to the small change in v, we label z and z' so that $u(z|v)$ increases discretely and $u(z'|v)$ falls discretely. From the first order conditions, this requires a discrete increase in $\lambda(z|v)$ and a discrete decrease in $\lambda(z'|v)$. A discrete decrease in $\lambda(z'|v)$ requires that before the change in v we had $\lambda(z'|v) > 0$ and the participation constraint was binding. The discrete decline in $u(z'|v)$ (and the associated decline in $\omega(z'|v)$) requires a discrete decline in capital to maintain participation. However, the increase in $u(z|v)$ means that after the change, $\lambda(z|v) = 0$. This rules out that $\lambda(z|v)$ increased discretely. This implies that the optimal policy for $u(z|v)$ is continuous for all $z \in Z$, and by extension the optimal policy for $\omega(z|v)$ is continuous.

To show that this implies continuity of $k^n(v)$, consider a discontinuous movement of $k^n(v)$ for a small change in v. If $k^n(v)$ falls discretely, then continuity of $u(z|v) + \beta\omega(z|v)$ implies that all participation constraints are slack, which is only consistent with the new $k^n = k^*$, which contradicts a discrete drop in capital as it is never optimal for $k^n(v) > k^*$. If capital increases discontinuously, then all participation constraints must have been slack at the initial capital. This implies the initial capital was k^*, which contradicts the optimality of a discrete increase.

The continuity of the policy functions implies continuity of the multipliers. To see this, a discrete change in $\gamma(v)$ requires a discontinuous change in all $\lambda(z|v)$ of the opposite sign. However, this is not consistent with a continuous change in capital. Therefore, $\gamma(v)$ is continuous. The continuity of $\lambda(z|v)$ then follows from the continuity of $u(z|v)$, $\gamma(v)$ and the first order condition $c'(u(z|v)) = \gamma(v) + \frac{\lambda(z|v)}{q(z)}$. □

The next lemma states that if $B^n(v) \in \mathbf{B}$, then B^{n+1} is differentiable.

Lemma 8. *Suppose $B^n \in \mathbf{B}$. Then $B^{n+1} = TB^n$ is differentiable and the envelope condition applies, that is $B^{n+1\prime}(v) = -\gamma(v)$.*

Proof. For any point $v \in [V_{aut}, V_{max}]$ let $k(v)$ be the optimal capital given B^n. Consider maximizing the objective function subject to promise keeping but with the following alternative set of participation constraints:

$$u(z) + \beta\omega(z) \geq U(F(z, k_0)) + U'(F(z, k_0))F_k(z, k_0)(k - k_0) + \beta V_{aut}. \tag{44}$$

Note that concavity of U and F imply that

$$U(F(z, k)) \leq U(F(z, k_0)) + U'(F(z, k_0))F_k(z, k_0)(k - k_0), \tag{45}$$

so this participation constraint is "tighter" than the original when $k_0 = k(v)$. Let $\tilde{B}^{n+1}(v|k_0)$ be the solution to this problem and $B^{n+1}(v)$ the solution to our original problem. Note that when $k_0 = k(v)$, $\tilde{B}^{n+1}(v|k(v)) = B^{n+1}(v)$ and the allocations and multipliers are the same in the two problems.[6] Note also that this alternative problem maximizes a strictly concave objective function subject to convex constraints. Standard arguments imply that $\tilde{B}^{n+1}(v|k_0)$ is concave and differentiable with respect to v, and $\tilde{B}^{n+1\prime}(v|k(v)) = -\gamma(v)$. We also have that for any given k_0, $\tilde{B}^{n+1}(v|k_0) \leq B^{n+1}(v)$, $\forall v$, where the inequality follows from the fact that the alternative participation constraints are tighter. Hence, $B^{n+1}(v) = \max_{k_0} \tilde{B}^{n+1}(v|k_0)$. Note that $\tilde{B}^{n+1\prime}(v|k(v)) = -\gamma(v)$ is continuous in v by Lemma 7. By Theorem 2 of Milgrom and Segal (2002), we can express $B^{n+1}(v)$ as $B^{n+1}(v) = B^{n+1}(v_0) + \int_{v_0}^{v} \tilde{B}^{n+1\prime}(\tilde{v}|k(\tilde{v}))d\tilde{v}$. Hence continuity of $\gamma(v)$ implies that $B^{n+1}(v)$ is differentiable and $B^{n+1\prime}(v) = \tilde{B}^{n+1\prime}(v, k(v)) = -\gamma(v)$. To see that Theorem 2 applies, let $f(x,t) = \tilde{B}^{n+1}(t|k(x))$. Note then that $B^{n+1}(t) = f(t,t) = \max_x f(x,t)$. Now, as argued above, $f(x,t)$ is differentiable with respect to t for all x. The derivative $f_t(x,t)$ is bounded above by 0, and for any particular z, we have that $\tilde{B}^{n+1\prime}(v, k_0) \geq -c'(\tilde{u}(z|v,k_0))$ where $\tilde{u}(z|v,k_0)$ is the optimum $u(z)$ given the modified participation constraints. We can then show that $\tilde{u}(z|v,k_0)$ is bounded above for all admissible (k_0, v) and hence $\tilde{B}^{n+1\prime}$ is bounded. So $|f_t(x,t)| \leq M$ for positive M, and the assumptions of Milgrom and Segal (2002), Theorem 2 are satisfied. $\square$

The next two lemmas state that we can use the operator T to iterate on the set $\mathbf{B}$ to converge to a concave fixed point.

Lemma 9. *Suppose $B^n \in \mathbf{B}$. Then $B^{n+1}(v) = TB^n(v)$ is concave for all $[V_{aut}, V_{max}]$, and strictly concave on $[V_{min}^{n+1}, V_{max}]$.*

Proof. First note that if $B^{n+1\prime}(v_0) = 0$ for some v_0, then $B^{n+1\prime}(v) = 0$ for all $v < v_0$ (if the participation constraint is not binding at v_0, it will not be binding for lower promises). Hence $B^{n+1}(v)$ is strictly decreasing for all $v \in [V_{min}^{n+1}, V_{max}]$, by definition of V_{min}^{n+1}. Note also that $B^{n+1}(v)$ is constant for all $v \in [V_{aut}, V_{min}^{n+1}]$. The fact that $B^{n+1}(v)$ is concave when it is flat is trivially true. Consider $v_1 > v_0 \geq V_{min}^{n+1}$. Let γ_1 and γ_0 be the corresponding multipliers on the respective promise keeping constraints. From Lemma 8, strict concavity follows if $\gamma_1 > \gamma_0$. To generate a contradiction, suppose $\gamma_1 \leq \gamma_0$. If $k^n(v_1) > k^n(v_0)$, then from Lemma 5, there exists $z' \in Z$ such that $\lambda(z'|v_1) < \lambda(z'|v_0)$. The first order conditions then imply $u(z'|v_1) < u(z'|v_0)$ and $\omega(z'|v_1) < \omega(z'|v_0)$. As $k^n(v_1) > k^n(v_0)$, this implies $\lambda(z'|v_0) = 0$, contradicting the premise that $\lambda(z'|v_1) < \lambda(z'|v_0)$. Now suppose that $k^n(v_1) \leq k^n(v_0)$. As $v_1 > v_0$ and promise keeping binds at v_0, there exists a $z'' \in Z$ such that $u(z''|v_1) > u(z''|v_0)$ and $\omega(z''|v_1) > \omega(z''|v_0)$. As $k^n(v_1) \leq k^n(v_0)$, this implies $\lambda(z''|v_1) = 0$.

[6] The allocations and multipliers must be the same as the first order conditions are the same at k_0 and Lemma 7 implies these conditions have a unique solution.

However, $u(z''|v_1) > u(z''|v_0)$ then implies that $\gamma_1 = c'(u(z''|v_1)) > c'(u(z''|v_0)) \geq \gamma_0$, a contradiction of the premise that $\gamma_1 \leq \gamma_0$. $\square$

Lemma 10. *For any bounded $B^n(v)$, not necessarily in $\mathbf{B}$, with $B^{n\prime}(V_{aut}) = 0$, and $B^{n+1}(v) \equiv TB^n(v)$ concave, then $B^{n+1}(v) \in \mathbf{B}$.*

Proof. Note that $B^{n\prime}(V_{aut}) = 0$ implies that capital is interior by Lemma 3. Concavity and the Benveniste-Scheinkman Theorem implies that $B^{n+1}(v)$ is differentiable on the interior. The fact that $B^{n+1\prime}(V_{aut}) = 0$ follows from Lemma 3. So for B^{n+1} to lie in $\mathbf{B}$ we just need to show that $B^{n+1\prime}(V_{max}) \leq -\beta(1+r^*)c'((1-\beta)V_{max})$. At $v = V_{max}$, we have that $V_{max} \leq E\left[u(z) + \beta\omega(z)\right] \leq E[u(z)] + \beta V_{max}$, where the second inequality follows from $\omega(z) \leq V_{max}$. Therefore, there exists at least one $z^0 \in Z$ such that $u(z^0) \geq (1-\beta)V_{max}$. Now we have that $B^{n+1\prime}(V_{max}) \leq -c'(u(z))$ for all z, hence $B^{n+1\prime}(V_{max}) \leq -c'((1-\beta)V_{max}) \leq -\beta(1+r^*)c'((1-\beta)V_{max})$ given that $\beta(1+r^*) \leq 1$. $\square$

Corollary 1. *If $B^n(v) \in \mathbf{B}$, then $B^{n+1} = TB^n \in \mathbf{B}$.*

Proof. From Lemma 9, B^{n+1} is concave. Then Lemma 10 implies that $B^{n+1} \in \mathbf{B}$. $\square$

Lemma 11. *The (unique) fixed point of T is concave.*

Proof. Let $B^{n+1}(v) = TB^n = T^nB^0$. Start with $B^0(v) \in \mathbf{B}$. From the Corollary 1, $B^{n+1}(v) \in \mathbf{B}$ if $B^n(v) \in \mathbf{B}$. Therefore, B^n, $n = 1, 2, ..$, is a sequence in $\mathbf{B}$. As T is a contraction, this sequence converges to a unique limit. Moreover, as $\mathbf{B}$ is a subset of the set of concave, bounded functions, which is closed, the limit is concave. $\square$

The next lemma uses Assumption 7, to show that the optimal k will always be strictly greater than zero.

Lemma 12. *At the optimum, $k(v) > 0$ for all $v \in [V_{aut}, V_{max}]$ and $B'(V_{aut}) = 0$.*

Proof. If $k(V_{aut}) > 0$, then $k(v) > 0$ for all $v \in [V_{aut}, V_{max}]$. To see this, note that if $k > 0$ is optimal at V_{aut}, then the same argument as in the proof of Lemma 3 implies that the promise keeping is slack at V_{aut} and $B'(V_{aut}) = 0$. The same lemma applied to the fixed point would then imply that $k > 0$ is optimal at all $v \in [V_{aut}, V_{max}]$. Therefore it suffices to show that $k = 0$ is not optimal at V_{aut}.

Suppose $k = 0$ were optimal at V_{aut}. From Lemma 3, this implies that promise keeping binds at V_{aut}, that is $E\left[u(z) + \beta\omega(z)\right] = V_{aut} \equiv E\left[U(F(z,0)) + \beta V_{aut}\right]$. As $u(z) + \beta\omega(z) \geq U(F(z,0)) + \beta V_{aut}$, $\forall z \in Z$, this implies that $u(z) + \beta\omega(z) = U(F(z,0)) + \beta V_{aut}$, $\forall z \in Z$. As $\omega(z) \geq V_{aut}$, this requires $u(z) \leq U(F(z,0))$ $\forall z \in Z$. Now suppose that $u(z) < U(F(z,0))$ for some z. Then, along the lines of the proof of Lemma 3, a slight increase in k requires an increase of $Ec'(u(z'))U'(F(z',0))F_k(z',0) < EF_k(z',0)$ in outlays. This implies that a small increase in k is optimal as $F_k(z,0)$ becomes large. Therefore, optimal $k = 0$ requires $u(z) = U(F(z,0))$, $\forall z \in Z$, which in turn implies $\omega(z) = V_{aut}$ and V_{aut} is a stationary point (i.e. if $v = V_{aut}$ today, $v = V_{aut}$ next period regardless of the shock).

Now consider reducing $u(\bar{z})$ at the margin by a small amount Δ. This saves $c'(U(F(\bar{z},0)))\Delta$ in current costs if $\bar{z}$ is realized. To maintain participation, $\omega(\bar{z})$ must be increased by $\beta^{-1}\Delta$. One way (although perhaps not the optimal way) of delivering on this increase in promised utility is to increase the next period's $u(\underline{z})$ by $\frac{\Delta}{\beta q(\underline{z})}$. This increases expected utility by $\beta^{-1}\Delta$. This has an expected cost of $\frac{c'(U(F(\underline{z},0)))\Delta}{\beta(1+r*)}$ in current units. By Assumption 7, $\frac{c'(U(F(\underline{z},0)))}{\beta(1+r*)} < c'(U(F(\bar{z},0)))$, making such a variation an improvement over the original allocation. Therefore, $k = 0$ was not optimal.

From Lemma 3, $k(v) > 0$ implies that $B'(V_{aut}) = 0$. □

Next we show that any fixed point of T is a member of $\mathbf{B}$.

Lemma 13. *The fixed point of T is a member of $\mathbf{B}$.*

Proof. Let $B(v)$ denote the fixed point. From Lemma 11, we know $B(v)$ is concave. From Lemma 12 we have $B'(V_{aut}) = 0$. Lemma 10 then implies that $B(v) \in \mathbf{B}$. □

We now are ready to prove Proposition 3.

Proof. Part (i) follows from the fact that $B(v) \in \mathbf{B}$.

For part (ii) and (iii), recall that $k > 0$ given that $B(v) \in \mathbf{B}$ and Lemma 3. Hence at $v = V_{aut}$, we have

$$u(z) + \beta\omega(z) \geq U(F(z,k)) + \beta V_{aut} > U(F(z,0)) + \beta V_{aut}. \tag{46}$$

Adding up, we obtain

$$V_{min} \equiv \sum_{z \in Z} q(z)(u(z) + \beta\omega(z)) > \sum_{z \in Z} q(z)(U(F(z,0)) + \beta V_{aut}) = V_{aut} \tag{47}$$

So $B(v) = B(V_{aut})$ for all $v \in [V_{aut}, V_{min}]$ and hence $B'(v) = 0$ in that domain. Note that this implies that $v < V_{min}$ is not on the Pareto frontier. Lemma 6 implies that the promise keeping is holding with equality for any $v \geq V_{min}$. Concavity follows from Lemma 9. Therefore, $B(v)$ is strictly decreasing for $v \in [V_{min}, V_{max}]$, generating the Pareto frontier. This implies that $\mathbf{b} = [B(V_{min}), B(V_{max})]$. Part (iv) follows from Lemmas 4 and 7. □

A.4 Proof of Propositions 4 and 5

Proof of Proposition 4:

Proof. Part (i) follows from the first order conditions for capital. Part (ii) follows from the first order conditions and the envelope condition, as well as the strict convexity of $c(u)$. For part (iii), note that the first order conditions imply that if utility varies across states or if $B'(\omega(z))$ varies across states, then at least one $\lambda(z) > 0$, hence the first order condition for capital implies that capital is below the first best. Part (iv) then follows from the first order condition and the definition of V_{min}. □

Proof of Proposition 5:

Proof. Recall that a constrained efficient equilibrium requires $v \in [V_{min}, V_{max}]$, as $v < V_{min}$ are not part of the Pareto frontier. Part (i) follows directly from Lemma 7. For (ii), let $v_0 > v_1$. For the remainder of the proof, let $k^i = g^k(v_i)$, $i = 0, 1$. To generate a contradiction, suppose $k^0 < k^1$. From Lemma 5, there exists at least one $z \in Z$, call it z_1, such that

$$\lambda(z_1|v_0) > \lambda(z_1|v_1). \tag{48}$$

Concavity of $B(v)$ implies $\gamma(v_0) \geq \gamma(v_1)$. The first order conditions then require $g^{\omega(z_1)}(v_0) \geq g^{\omega(z_1)}(v_1)$ and $g^{u(z_1)}(v_0) > g^{u(z_1)}(v_1)$. This implies that the total utility delivered in z_1 (that is, $u(z_1) + \beta\omega(z_1)$) is greater following v_0 than v_1. The premise of the contradiction is that $k^0 < k^1$, which implies that the participation constraint is easier to satisfy. Therefore, the participation constraint in state z_1 following (v_0) must be slack. This implies that $0 = \lambda(z_1|v_0) \leq \lambda(z_1|v_1)$, which is a contradiction of (48).

We now rule out $k^0 = k^1$ if $k^1 < k^*$. From Lemma 5, $k^1 = k^0$ requires $\lambda(z|v_0) = \lambda(z|v_1)$, for all $z \in Z$. By concavity, $\gamma_1 > \gamma_0$. Therefore, $u(z)$ and $\omega(z)$ is strictly greater in all $z \in Z$ following v_1 than following v_0. This implies that $\lambda(z|v_1) = 0$, $\forall z \in Z$, which implies $k^1 = k^*$, a contradiction.

For (iii), the fact that $-B'(g^{\omega(z)}(v)) = \beta(1 + r^*)c'(g^{u(z)})$ and concavity ensures that $g^{\omega(z)}$ moves one-for-one with $g^{u(z)}$. (17) and the envelope condition imply $c'(g^{u(z)}) = -B'(v) + \frac{\lambda(z)}{q(z}$. Strict concavity of B implies that $B'(v)$ is strictly increasing. If $\lambda(z) = 0$, this proves the claim as $\lambda(z)$ cannot fall below zero. If $\lambda(z) > 0$, then the binding participation constraint and the fact that k is strictly increasing in v requires that $u(z)+\beta\omega(z)$ increases a v increases.

For (iv), we first show that promised continuation values are non-decreasing in z. The fact that $U(F(z, k))$ is strictly increasing in z implies that either $u(z_1)+\beta\omega(z_1) > u(z_0)+\beta\omega(z_0)$, or that $\lambda(z_0) = 0$. If the former, the fact that $c'(u(z)) = \beta(1+r)B'(\omega(z))$ and strict convexity of $c(u)$, imply that $u(z_1) + \beta\omega(z_1) > u(z_0) + \beta\omega(z_0)$ requires $\omega(z_1) \geq \omega(z_0)$. In the case that $\lambda(z_0) = 0$, we have that $\lambda(z_1) \geq \lambda(z_0)$ and (18) plus the concavity of B gives the result.

We now show that $g^{\omega(\bar{z})}(v) > g^{\omega(\underline{z})}(v)$ for $v < V^*$. Consider the set $Z' \subseteq Z$ such that $\lambda(z) > 0$ if $z \in Z'$. As $v < V^*$ there exists at least one z such that $\lambda(z) > 0$, and therefore Z' is not empty. As $U(F(z,k)) + \beta V_{aut}$ is strictly increasing in z, then so is $u(z) + \beta\omega(z)$ for $z \in Z'$. As $c'(u(z)) = -B'(\omega(z))$ and the strict concavity of $B(v)$ on the relevant domain, we have that $\omega(z)$ is strictly increasing in z for $z \in Z'$. Moreover, the first order condition for $\omega(z)$ implies that $\omega(z') > \omega(z)$ if $z' \in Z$ and $z \in Z - Z'$. This implies that the distribution of $\omega(z)$ is not a single point. The fact that $\omega(z)$ is non-decreasing in z over the entire set Z then implies the result. □

A.5 Proof of Proposition 7

Proof. We focus on the invariant distribution of v. The invariant distribution of k follows immediately from the policy function $g^k(v)$. The policy function $g^{\omega(z')}(v)$ and the transition function for z induce a first-order Markov process for v. As $g^{\omega(z')}(v)$ is continuous, the transition function has the Feller property (see Stokey and Lucas (1989), Exercise 8.10). Theorem 12.10 of Stokey and Lucas (1989) implies that there exists an invariant distribution.

To show that any invariant distribution is bounded above by V^*: As $\beta(1 + r^*) < 1$ and the participation constraints are slack at V^* by definition, we have $g^{\omega(z')}(V^*) < V^*, \forall z' \in Z$. As $g^{\omega(z')}(v)$ is monotonic in v, then $g^{\omega(z')}(v) < V^*, \forall v \in [V_{aut}, V^*]$. This proves that the invariant distribution lies below V^*. As $g^k(v)$ is a function of v, the invariant distribution of v generates a corresponding distribution for k. As $g^k(v)$ is increasing in v for $v < V^*$ and $g^k(V^*) = k^*$, this implies that invariant distribution of k lies below k^*.

The fact that the invariant distribution is non-degenerate follows from the above fact that elements of the invariant distribution are less than V^* and Proposition 5.

To show that the invariant distribution is unique, we prove that there exists a "mixing point" $\tilde{v}$ and an $N \geq 1$ such that there is strictly positive probability that $v \geq \tilde{v}$ and strictly positive probability $v \leq \tilde{v}$ after N periods starting from any point in $[V_{min}, V_{max}]$. The result then follows from Theorem 12.12 of Stokey and Lucas (1989). Define $\underline{v}$ to be the highest $v \in [V_{aut}, V_{max}]$ such that $g^{\omega(\underline{z})}(\underline{v}) = \underline{v}$. That is, $\underline{v}$ is the highest v at which the policy function for $\omega(\underline{z})$ crosses the 45 degree line. Such a $\underline{v}$ exists as the policy function is continuous and maps $[V_{aut}, V_{max}]$ into itself and from (ii) we know $\underline{v} < V^*$. Define $\bar{v}$ to be the smallest v such that $g^{\omega(\bar{z})}(v) = v$, that is the smallest fixed point of the policy function for $\bar{z}$. We now show that $\bar{v} > \underline{v}$: any fixed point of the policy function for the highest shock is to the right of $\underline{v}$. Suppose not. From Proposition 5, we know that $\bar{v} \neq \underline{v}$. Therefore, the premise implies that $\bar{v} < \underline{v}$. The fact that $k(v)$ is strictly increasing in v for $v < V^*$ implies that $k(\underline{v}) > k(\bar{v})$. From Lemma (5), this implies that for at least one $z' \in Z$ we have $\lambda(z'|\bar{v}) > \lambda(z'|\underline{v})$. Now at the fixed points of the policy functions, the first order conditions and the envelope conditions imply

$$
\begin{aligned}
-B'(\underline{v}) &= \frac{\beta(1+r^*)}{1-\beta(1+r^*)} \frac{\lambda(\underline{z}|\underline{v})}{q(\underline{z})} \\
-B'(\bar{v}) &= \frac{\beta(1+r^*)}{1-\beta(1+r^*)} \frac{\lambda(\bar{z}|\bar{v})}{q(\bar{z})}.
\end{aligned}
$$

By concavity and $\underline{v} > \bar{v}$, this implies $\lambda(\underline{z}|\underline{v})/q(\underline{z}) \geq \lambda(\bar{z}|\bar{v})/q(\bar{z})$. The fact that $u(z)$ is increasing in z (Proposition 5) then implies that $\lambda(z|v)/q(z)$ are increasing in z and therefore $\lambda(z|\underline{v})/q(z) \geq \lambda(\underline{z}|\underline{v})/q(\underline{z})$ and $\lambda(\bar{z}|\bar{v})q(\bar{z}) \geq \lambda(z|\bar{v})/q(z)$, for all $z \in Z$. This implies that $\lambda(z|\underline{v}) \geq \lambda(z|\bar{v})$, $\forall z \in Z$, which contradicts the existence of a z' such that $\lambda(z'|\bar{v}) < \lambda(z'|\underline{v})$. Therefore, $\bar{v} > \underline{v}$. Select $\tilde{v}$ to the midpoint of the interval $[\underline{v}, \bar{v}]$. Iterating on the highest shock policy function starting from any v, a long enough but finite sequence of high shocks will result in $v \geq \tilde{v}$. Similarly, using the lowest shock policy function and starting from any v, a finite sequence of low shocks will bring v below $\tilde{v}$. Therefore, $\tilde{v}$ is a mixing point, and a unique stationary distribution follows. □

A.6 Section 5 Proofs

Proof of Proposition 8:

Proof. The Lagrangian of the problem is

$$
\sum_{t=0}^{\infty} \beta^t \left[\sum_{z^t} q(z^t) U\left(c\left(z^t\right)\right) + \sum_{z^{t-1}} q(z^{t-1}) \gamma\left(z^{t-1}\right) \left\{ \begin{array}{c} E\left[z_t|z^{t-1}\right] + F\left(k\left(z^{t-1}\right), l\right) \\ -E\left[c\left(z^t\right)|z^{t-1}\right] - \left(r^* + \delta\right) k(z^{t-1}) \end{array} \right\} \right],
$$

where z^{t-1} evaluated at $t=0$ refers to the initial information set. The first-order conditions for the maximization of the Lagrangian are

$$
\begin{aligned}
U'\left(c\left(z^t\right)\right) &= \gamma\left(z^{t-1}\right), \text{ and} \\
F_k\left(k\left(z^{t-1}\right), l\right) &= r^* + \delta,
\end{aligned}
$$

where the first condition implies that $c\left(\{z_t, z^{t-1}\}\right) = c\left(\{z'_t, z^{t-1}\}\right)$ for all $(z_t, z'_t) \in Z_t \times Z_t$ and the second condition implies that $E\left[\tau\left(z^t\right)|z^{t-1}\right] = 0$. □

Lemma 14. *[A folk theorem] There exists a $\beta^* \in (0,1)$ such that for all $\beta \geq \beta^*$ the full commitment, balanced budget solution is sustainable, and it is not sustainable for $\beta \in [0, \beta^*)$.*

Proof. Note that the full commitment and the deviation allocations are independent of the value of β. Let $c^*(z_{t-1})$ denote consumption at time t under commitment, conditional on z_{t-1}. Define the difference in the present discounted value of utility under the commitment allocation and autarky conditional on $z_{t-1} = z$ as $\Delta(z, \beta)$:

$$\Delta(z, \beta) \equiv E \sum_{s=t}^{\infty} \beta^{s-t} [U(c^*(z_{s-1})) - U(z_s)].$$

Note that $c^*(z_{t-1}) > E[z_t | z_{t-1}]$. This is so because $k^* > 0$ and the fact that c^* is the optimal plan. Therefore, the expected value of the term in brackets is strictly positive. Therefore, $\Delta(z, \beta)$ is strictly increase in β, is equal to zero when $\beta = 0$ and approaches infinity as β approaches one. We can write the participation constraints at the commitment allocation as

$$U(c^*(z_{t-1})) - U(F(k^*, l, z_t)) \geq -\Delta(z_t, \beta). \tag{49}$$

As the right-hand side of (49) is strictly increasing in β, and the left-hand side does not vary with β, if this constraint is satisfied at β, then it is satisfied at any $\beta' > \beta$. When $\beta = 0$, the right-hand side of (49) is zero and the constraint will not hold for some z. When $\beta \to 1$, the right-hand side of (49) approaches minus infinity, implying there is a $\beta^* < 1$ for which all the participation constraints are satisfied at the full commitment allocation for $\beta \geq \beta^*$, and at least one constraint is violated at the full commitment allocation for $\beta < \beta^*$. ☐

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