Fiscal Sustainability: a New Framework for tests and indicators*

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Abstract

This paper is motivated by the importance of proposing indicators for fiscal sustainability backed by a formal definition of sustainability and coherent with some related empirical tests. First, we show that, by solving a linearized government intertemporal budget constraints, necessary and sufficient conditions for sustainability can be expressed as cross-equation restrictions on a VAR model for fiscal and macroeconomic variables. Second, we illustrate how a Bayesian approach to the evaluation of the consistency of the relevant cross-equation restrictions with the data allows to propose sustainability indicators that are naturally linked to the statistical evidence on the validity of such restrictions. Third, we use our VAR to provide evidence on how, at each moment in time, the corrections in fiscal and macroeconomic fundamentals necessary to obtain sustainability are expected to be achieved. We provide an empirical application to the case of Brazil, which we find interesting both because of the quality of the data and their time-series properties.

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1 Introduction

Fiscal sustainability is an issue of major concern for governments and supranational agencies. Sustainability judgments are important to identify spec-

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ulative bubbles, to distinguish liquidity crisis from solvency crisis and, eventually, to inform the optimal reaction to fiscal crises. Academic and non-academic research in this field have proposed a number of sustainability tests and sustainability indicators. Unfortunately, these two strands of the empirical literature have been separated and, as noted by Chalk and Hemming(2000), sustainability indicators are usually not backed by a formal definition of sustainability. Sustainability tests concentrate on the Present Value Budget Constraint (PVBC), that links actual debt to the present discounted value of future surplus, within some Vector Autoregressive specification for the relevant variables. The classic test by Hamilton and Flavin(1986) concentrates on debt stationarity as a necessary condition for sustainability. Trehan and Walsh(1988) extend this framework by arguing that a necessary and sufficient condition for the PVBC to hold can be tested by looking at cointegration between debt and primary balances. Bohn(1998) remarks that the condition tested by Trehan and Walsh is necessary and sufficient only if the cost of financing the debt is constant, the introduction of uncertainty on the cost of financing the debt leads him to argue that the sufficient condition for sustainability in an uncertain world is a positive response of the primary surplus to the debt in the government's fiscal policy reaction function. In fact, cointegration between the primary surplus and the debt stabilizing primary surplus is the necessary and sufficient condition for sustainability in presence of uncertainty on the cost of financing the debt. Finally, Roberds(1991) shows that the validity of PVBC implies cross-equation restrictions on the bivariate VAR stochastic process for debt and deficit and finds strong evidence that such restrictions are rejected for the US. The literature focusing on indicators of sustainability has developed separately from that concentrating on tests of sustainability. As argued by Chalk and Hemming (2001), this literature tends to rely on some intuitive notion of what distinguishes sustainable from unsustainable fiscal policies. Buiter(1985) derives an indicator based on the notion that sustainability implies that the ratio of the public sector net worth to output should be maintained at its current level, Blanchard(1990) modifies this indicator in the light of difficulties in measurement public sector net worth by looking at the change in policies required to maintain the current debt to GDP ratio. This paper proposes a unified framework to study the dynamics of public debt, public deficit, GDP growth and the cost of financing the deficit to test for sustainability and to propose indicators of sustainability coherent with the testing framework. We extend to the analysis of government debt-deficits dynamics the algebra of the linearized intertemporal budget constraint, that has been successfully applied to consumption and asset prices (see Lettau and Ludvigson,2001) and net foreign assets and exchange rate movements (Gourinchas and Rey, 2005). Our analysis proceeds in three steps. First, we provide a definition of sustainability based on the forward solution of the linearized government's budget constraint to define necessary and sufficient conditions for debt sustainability. Second, we propose empirical methods to analyze these conditions. In particular, we show that cointegration analysis is most suitable to analyze the necessary condition for sustainability, while the necessary and sufficient conditions are best studied via the application of Bayesian methods on the estimation of the relevant parameters in an appropriately specified VAR. The present value model describing the debt-deficits dynamics lends itself naturally to a VAR specification for macroeconomic and fiscal variables. The necessary condition for sustainability implies cointegration among the variables included in the VAR, while the necessary and sufficient condition for sustainability implies cross-equation restrictions on the VAR coefficients. The test of the validity of these restrictions constitutes a present value test of sustainability. Importantly the estimation of Present Value model has often shown that statistically rejected restrictions are too demanding for an economic point of view in the sense that time-series generated by imposing rejected restrictions fluctuate very closely to actual series[1]. This point becomes particular relevant if the sustainability of one country fiscal position has to be judged on the basis of the outcome of the relevant statistical test. To deal with this issue we consider a Bayesian approach to the restrictions by introducing a distribution on them and by analyzing how tight the posterior distribution on that restrictions is (see, for example, Del Negro and Schorfeide(2004) and Del Negro and Schorfeide, Smets and Wouters(2004)). The tightness of such distribution is controlled by an hyperparameter. We adopt a natural Bayesian criterion to evaluate the degree of debt sustainability on the basis of this hyperparameter.

Third, we show that, under the null that the restrictions are not rejected, the debt-gap, i.e. the gap between the actual debt and the sustainable one, can be exactly decomposed into three components reflecting respectively future expected changes in government's primary surplus, in future economic growth and in the expected cost of financing the debt. On the basis of this decomposition, in the presence of a gap between actual debt and sustainable debt, we can add to the information provided by our sustainability indicator the expectations on which variable is expected to move to bring the system back to equilibrium. We illustrate our methodology by considering an ap-

[1] In their famous paper Campbell-Shiller(1991) analyze the Present Value models linking long-term interest rates and short-term interest rates under the Expectations Theory. They find that the cross-equation restrictions implied by the ET are statistically rejected but a theoretical long-term rate construced by imposing on their VAR the rejected restrictions is virtually not distinguishable from the observed long-term rate.

plication to the Brazilian case, which we find relevant both because of the quality of the data, made readily available by the Banco do Brazil via webpage, and because of the interesting fluctuations in the relevant fiscal and macroeconomic variables of this country.

2 A Framework to analyze debt-deficit dynamics

The starting point of our analytical derivation of tests for fiscal sustainability and a related sustainability indicator is the debt-deficit dynamics determined by the government budget constraint:

$$b_{t+1} = \frac{1 + I_{t+1}}{1 + X_{t+1}} b_t + d_{t+1}, \tag{1}$$

where b_t is the ratio of government debt to GDP, d_t is the ratio of government primary deficit to GDP,I_t is the cost of financing the debt and X_t is the nominal GDP growth. By disaggregating the primary deficit as $d_t = g_{t+1} - t_{t+1}$ we have :

$$b_{t+1} = \frac{(1 + I_{t+1})}{(1 + X_{t+1})} b_t + (g_{t+1} - t_{t+1}). \tag{2}$$

We show in Appendix 1 that by linearizing the budget constraint, solving it forward and by imposing the appropriate transversality condition we can derive the following relation:

$$\begin{aligned} \ln(b_t) - \frac{1}{(1-\rho)} ls_{t+1} &= \frac{1}{(1-\rho)} \sum_{i=1}^{\infty} \rho^i E_t \Delta ls_{t+i+1} \\ ls_{t+1} &= -\eta \ln(g_{t+1}) + (1 - \rho + \eta) \ln(t_{t+1}) - r_{t+1} + (1-\rho) k \end{aligned} \tag{3}$$

Where $r_{t+1} = i_{t+1} - x_{t+1}$. and η, ρ, k are linearization parameters function of the unconditional means of macroeconomic and fiscal variables.

(3) can be re-written as follows:

$$
\begin{aligned}
ls_{t+1}^{*} - ls_{t+1} &= dtax_t - dg_t + dr_t \\
ls_{t+1}^{*} &= \ln(b_t)(1-\rho) \\
dtax_t &= \sum_{i=1}^{\infty} \rho^i (1-\rho+\eta)\, \Delta \ln(t_{t+1}) \\
dg_t &= \sum_{i=1}^{\infty} \rho^i \eta \Delta \ln(g_{t+1}) \\
dr_t &= \sum_{i=1}^{\infty} \rho^i \Delta r_{t+1} \\
r_t &= i_t - x_t
\end{aligned} \tag{4}
$$

(4)provides a framework for analyzing fiscal sustainability at three different levels.

At the first level cointegration analysis can be applied to check a necessary condition for fiscal sustainability. In fact, the present value model described by (4) implies that $\ln(b_t) - \frac{1}{(1-\rho)} ls_{t+1}$, being a discounted sum of stationary variables, is in itself stationary. So $\ln(b_t)$, should be cointegrated with $\ln(g_{t+1}), \ln(t_{t+1}), i_{t+1}, x_{t+1}$ with a cointegrating vector $\begin{bmatrix} 1 & \frac{\eta}{1-\rho} & -\frac{1-\rho+\eta}{1-\rho} & \frac{1}{1-\rho} & -\frac{1}{1-\rho} \end{bmatrix}$, where cointegrating parameters reflect the long-term means of the relevant fiscal and macroeconomic variables. The deviation from the cointegrating relation represent a "surplus-gap", which should be stationary to have fiscal sustainability. (4)tells us that a positive surplus-gap can be financed by higher growth, lower cost of financing the debt, higher taxation of lower public expenditure. While, as it is clear from the discussion in our introduction, the cointegration based analysis of fiscal sustainability is not new, our framework clearly shows that cointegration is only a necessary condition and provide the basis to investigate at a deeper level the presence of fiscal sustainability and how fiscal sustainability is achieved for any given positive debt-gap.

The further development of a test for sustainability starts from the VAR formulation for the vector of variables Z_t

$$
\begin{aligned}
Z_t &= \begin{bmatrix} ls_t^* - ls_t \\ \Delta \ln(t_t)^* \\ \Delta \ln(g_t)^* \\ \Delta i_t \\ \Delta x_t \end{bmatrix} \qquad (5) \\
\ln(s_t)^* &= \ln(b_{t-1})(1-\rho) \\
ls_t &= -\eta \ln(g_t)^* + (1-\rho+\eta)\ln(t_t)^* - (i_t - x_t) + (1-\rho)k \\
\ln(t_t)^* &= (1-\rho+\eta)\ln(t_t) \\
\ln(g_t)^* &= \eta \ln(g_t)
\end{aligned}
$$

When the necessary conditions for sustainability are satisfied this VAR is stationary and it can be interpreted as a cointegrated VAR. In fact, (5) represents an alternative but equivalent form to the traditional VECM parameterization (see, for example, Mellander et al.(1993)). The sufficient conditions for sustainability impose restrictions on the short-run dynamics of the system. To derive a test for the validity of these restrictions consider a VAR(p) representation for the vector Z_t.

$$Z_t = A(L) Z_{t-1} + \varepsilon_t$$

Importantly, given cointegration, the VAR is stationary, in fact it is an alternative representation of a VECM specification (see, for example, Mellander et al(1992)).

Appropriately stacked, this VAR has a first order companion representation:

$$\bar{Z}_{t+1} = \bar{A}\bar{Z}_t + \bar{\varepsilon}_{t+1}$$

where $\bar{Z}_t = \left(\bar{Z}_t, ..., \bar{Z}_{t-p+1}\right)$ and $\bar{\varepsilon}_t = (\varepsilon_t', 0)'$.

Define the indicator vectors $\mathbf{e}_{sgap}$,$\mathbf{e}_{\Delta \ln(t)}$, $\mathbf{e}_{\Delta \ln(g)}$, $\mathbf{e}_{\Delta x}$, $\mathbf{e}_{\Delta i}$, that are made of $(n-1)$ zero and a one, that selects the corresponding element of $\bar{Z}_t$.

Consider the sustainability condition:

$$sgap_t = \sum_{i=1}^{\infty} \rho^i \left(\Delta \ln(t_{t+1})^* - \Delta \ln(g_{t+1})^* - \Delta i_{t+1}^* + \Delta x_{t+1}^*\right)$$

which can be re-written as

$$\mathbf{e}_{sgap}'\bar{Z}_t = \left(\mathbf{e}_{\Delta \ln(t)}' + \mathbf{e}_{\Delta x}' - \mathbf{e}_{\Delta \ln(g)}' - \mathbf{e}_{\Delta i}'\right) \sum_{i=1}^{\infty} \rho^i E_t \bar{Z}_{t+i}$$

using the VAR to generate expectations we have: $E_t \bar{Z}_{t+i} = \bar{A}^i \bar{Z}_t$ and so:

$$\mathbf{e}'_{sgap}\bar{Z}_t = \left(\mathbf{e}'_{\Delta \ln(t)} + \mathbf{e}'_{\Delta x} - \mathbf{e}'_{\Delta \ln(g)} - \mathbf{e}'_{\Delta i}\right)\left(\rho \bar{A}\right)\left(I - \rho \bar{A}\right)^{-1} \bar{Z}_t$$

which implies a series of testable restriction on the coefficients of the VAR:

$$\mathbf{e}'_{sgap}\left(I - \rho \bar{A}\right) = \left(\mathbf{e}'_{\Delta \ln(t)} + \mathbf{e}'_{\Delta x} - \mathbf{e}'_{\Delta \ln(g)} - \mathbf{e}'_{\Delta i}\right)\left(\rho \bar{A}\right). \tag{6}$$

The test of the validity of these restrictions constitutes a present value test of sustainability. Importantly the estimation of Present Value model has often shown that statistically rejected restrictions are too demanding from an economic point of view. In fact it is not unusual that time-series generated by imposing rejected restrictions fluctuate very closely to actual series[2]. To deal with this issue we consider a Bayesian approach to the restrictions by introducing a distribution on them and by analyzing the tightness of such posterior distribution. We adopt a testing strategy very similar to the Bayesian analysis of DSGE models proposed by Del Negro and Schorfeide(2004) and Del Negro and Schorfeide, Smets and Wouters(2004). These authors use the Bayesian approach to compare an unrestricted VAR for macroeconomic variables with a restricted VAR, where the restrictions are derived from the DSGE theoretical model. We also compare an unrestricted VAR with a restricted VAR, however our restrictions do not represent any theory but they are immediately interpreted as conditions for debt sustainability. The tightness of the relevant posterior distribution is controlled by an hyperparameter. The analysis of the distribution of such hyperparameter provides a natural Bayesian criterion to evaluate the degree of debt sustainability. In fact, the hyperparameter that has the highest posterior probability can be interpreted as providing an assessment of the degree of debt sustainability.

To see how the Bayesian framework can be put at work for the analysis of the problem at hand consider, without loss of generality, a first order representation of our VAR:

$$Z_{t+1} = AZ_t + \varepsilon_{t+1} \tag{7}$$

with:

$$A = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} & \psi_{35} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} & \psi_{45} \\ \psi_{51} & \psi_{52} & \psi_{53} & \psi_{54} & \psi_{55} \end{bmatrix}$$

[2] See Footnote 1.

The general necessary and sufficient conditions for sustainability :

$$\mathbf{e}'_{sgap}\left(I-\rho\bar{A}\right)=\left(\mathbf{e}'_{\Delta\ln(t)}+\mathbf{e}'_{\Delta x}-\mathbf{e}'_{\Delta\ln(g)}-\mathbf{e}'_{\Delta i}\right)\left(\rho\bar{A}\right). \tag{8}$$

can be written in our case as :

$$\begin{bmatrix} \psi_{11}-\psi_{21}+\psi_{31}+\psi_{41}-\psi_{51} \\ \psi_{12}-\psi_{22}+\psi_{32}+\psi_{42}-\psi_{52} \\ \psi_{13}-\psi_{23}+\psi_{33}+\psi_{43}-\psi_{53} \\ \psi_{14}-\psi_{24}+\psi_{34}+\psi_{44}-\psi_{54} \\ \psi_{15}-\psi_{25}+\psi_{35}+\psi_{45}-\psi_{55} \end{bmatrix}=\begin{bmatrix} \frac{1}{\rho} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

where the relevant selector vectors can be written in our case as:

$$-\mathbf{e}_{\Delta\ln(g)}+\mathbf{e}_{\Delta\ln(t)}+\mathbf{e}_{\Delta x}-\mathbf{e}_{\Delta i}=\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{e}_{sgap}=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the following representation of our restrictions:

$$H\Psi=\mu_0$$

with:

$$H= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}' ; \ \Psi = \begin{bmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \\ \psi_{41} \\ \psi_{51} \\ \psi_{12} \\ \psi_{22} \\ \psi_{32} \\ \psi_{42} \\ \psi_{52} \\ \psi_{13} \\ \psi_{23} \\ \psi_{33} \\ \psi_{43} \\ \psi_{53} \\ \psi_{14} \\ \psi_{24} \\ \psi_{34} \\ \psi_{44} \\ \psi_{54} \\ \psi_{15} \\ \psi_{25} \\ \psi_{35} \\ \psi_{45} \\ \psi_{55} \end{bmatrix} ; \mu_0= \begin{bmatrix} \frac{1}{\rho} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We test the validity of these restrictions by assuming that they hold with uncertainty and by assessing how much uncertainty we need to attribute to the sustainability restrictions to make them compatible with the data. In practice, we consider :

$$H\Psi \sim N\left(\mu_{S_0}, \Sigma_{S_0}\right). \tag{9}$$

$$\mu_{S_0} = \begin{bmatrix} \frac{1}{\rho} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Sigma_{S_0} = \begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix}. \tag{10}$$

It is then natural to interpret the parameter σ as the tightness of the restrictions. A large value of σ implies that the sustainability restrictions hold with considerable uncertainty, while as σ decreases the sustainability restrictions become more binding and eventually become certain when σ approaches zero.

It is then crucial to have a criterion to evaluate the magnitude of σ.The Bayesian approach offers some interesting option.In fact, the set of restrictions (10) can be thought in a Bayesian perspective as a prior on the coefficients of the VAR (7). Therefore, we can test the sustainability by using the approach developed by Jeffreys (1935, 1961). In this approach, statistical models are introduced to represent the probability of the data according to several competing models, and Bayes's theorem is used to compute the posterior probability of each model. Models can then be compared using the Bayes factor, which is a summary of the evidence provided by the data.

In particular, we shall search across the models featuring different tightness of the sustainability of restrictions and select the value of the tightness that maximizes the Bayes factor. We shall then compare the selected model, the restricted VAR, with the mode which does not impose any restrictions on the VAR coefficients. We consider the following compact representation of the model:

$$\begin{aligned}
\mathbf{z}^{+} &= \Xi\ \Psi+\mathbf{u}^{+}, \qquad (11)\\
\mathbf{z}^{+} &= \begin{pmatrix} \mathbf{sgap} \\ \mathbf{\Delta}\ln(g)^{*}. \\ \mathbf{\Delta}\ln(t)^{*}. \\ \mathbf{\Delta x}^{*} \\ \mathbf{\Delta i}^{*} \end{pmatrix}, \Xi=[I_5\otimes \mathbf{x}]=\begin{pmatrix} \mathbf{x} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{x} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{x} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{x} \end{pmatrix},\\
\mathbf{x} &= \begin{pmatrix} \mathbf{sgap}_{-1} & \mathbf{\Delta}\ln(g)^{*}_{-1} & \mathbf{\Delta}\ln(t)^{*}_{-1} & \mathbf{\Delta x}^{*}_{-1} & \mathbf{\Delta i}^{*}_{-1} \end{pmatrix}\\
\Psi &= Vec\begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} & \psi_{35} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} & \psi_{45} \\ \psi_{51} & \psi_{52} & \psi_{53} & \psi_{54} & \psi_{55} \end{bmatrix}\\
\mathbf{u}^{+} &= \begin{pmatrix} \mathbf{u}_1 \\ . \\ . \\ \mathbf{u}_5 \end{pmatrix}. \sim N\left(0,\ \Sigma_u\ \otimes I_T\right)
\end{aligned}$$

where $\mathbf{y}^{+}$ is a $(5T\times 1)$ vector, $\mathbf{x}^{+}$ is a $(5T\times 25)$ matrix, Ψ is a (25×1), $\mathbf{u}^{+}$ is a $(5T\times 1)$ vector:

The unrestricted VAR(UVAR) is represented via a loose prior on the distribution of the vector of coefficients:

$$\Psi \sim N(\Psi_0 = \underset{25\times 1}{\mathbf{0}}, \Sigma_{\Psi_0} = \delta I_{25}). \tag{12}$$

We choose δ to be sufficiently large so that our prior does not add any information to that of the likelihood, and the posterior mean of Ψ is numerically identical to the OLS estimate. In practice we choose the smallest δ that delivers OLS estimates.

We then consider a Restricted VAR (RVAR) with the restrictions given by our sustainability conditions.

$$H\Psi \sim N\left(\underset{25x1}{\mu_{S_0}}, \Sigma_{S_0} = \sigma I_{25}\right)$$

To express the restrictions in terms of the vector of parameters we consider the following prior

$$\underset{25\times 25}{\begin{bmatrix} I_{20} & \underset{20\times 5}{\mathbf{0}} \\ \underset{5\times 25}{H} & \end{bmatrix}} \underset{25\times 1}{\Psi} \sim N\left(\mu_2 = \underset{25\times 1}{\begin{bmatrix} \underset{20\times 1}{\mathbf{0}} \\ \underset{5\times 1}{\mu_0} \end{bmatrix}}, \Sigma_2 = \underset{25\times 25}{\begin{bmatrix} \delta I_{20} & \underset{20\times 5}{\mathbf{0}} \\ \underset{5\times 20}{\mathbf{0}} & \sigma I_5 \end{bmatrix}}\right) \tag{13}$$

The matrix $H_2 = \underset{25\times 25}{\begin{bmatrix} I_{20} & \underset{20\times 5}{\mathbf{0}} \\ \underset{5\times 25}{H} & \end{bmatrix}}$ has full rank 25, by inverting it we have:

$$\Psi \sim N\left(H_2^{-1}\mu_2, H_2^{-1}\Sigma_2 H_2'^{-1}\right) \tag{14}$$

Which is a proper, conjugate, prior for a Gaussian VAR.

Thus, the RVAR is simply a linear regression model subject to a set of stochastic linear restrictions on the regression coefficients. To estimate such a model, Theil (1971) proposed the method of mixed estimation, which involves using the uncertain restrictions to supplement data. The added restrictions act as prior information on the coefficients and GLS is numerically equivalent to Bayesian estimation. More recently, Ingram and Whiteman(1994) have used a similar framework to shrink vector autoregression coefficient estimates toward a prior derived from a fully specified general equilibrium model.

We then estimate the tightness parameter σ on our restrictions using the value that maximizes the Bayes Factor of the RVAR with respect to the UVAR[3]. Finally, we assess the tightness on the sustainability restrictions by comparing the parameter σ with the parameter δ.A ratio of σ to δ close to

[3]See the Appendix for details

one implies that a very high variance must be attached to the sustainability restrictions to make them compatible with the data, a ratio close to zero implies instead that the sustainability restrictions are almost exactly not rejected by the data. Needless to say, this ratio is a natural indicator of sustainability, totally consistent with our proposed test.

Finally, under the null that the restrictions are not rejected, the debt-gap can be exactly decomposed in three components and observable counterparts for all these components can be derived:

$$
\begin{aligned}
sgap_t &= dtax_t - dg_t + dx_t - di_t && (15)\\
dtax_t - dg_t &= \sum_{i=1}^{\infty} \rho^i \left(\Delta \ln \left(t_{t+i}\right)^* - \Delta \ln \left(g_{t+i}\right)^*\right) && (16)\\
&= \left(\mathbf{e}'_{\Delta \ln(t)} - \mathbf{e}'_{\Delta \ln(g)}\right) \left(\rho \bar{A}\right) \left(I - \rho \bar{A}\right)^{-1} \bar{Z}_t && (17)\\
dx_t &= \sum_{i=1}^{\infty} \rho^i \Delta x_{t+1} && (18)\\
&= \mathbf{e}'_{\Delta x} \left(\rho \bar{A}\right) \left(I - \rho \bar{A}\right)^{-1} \bar{Z}_t && (19)\\
di_t &= \sum_{i=1}^{\infty} \rho^i \Delta i_{t+1} && (20)\\
&= \mathbf{e}'_{\Delta i} \left(\rho \bar{A}\right) \left(I - \rho \bar{A}\right)^{-1} \bar{Z}_t && (21)
\end{aligned}
$$

This is an interesting decomposition in that it can provide an assessment of how the surplus-gap is expected to be closed in any moment of time. The decomposition provides an assessment on how the burden of closing the surplus gap depends on future higher growth, future lower cost of financing the debt, and future changes in the primary surpluses.

3 An Application to Brazil

Brazil is an interesting case in that fiscal data are readily available from the Banco do Brazil website (www.bcb.gov.br) and fiscal sustainability is certainly an interesting issue for this country. We use a monthly data-set covering the period 1999:1 to 2005:06. Table 1 provides a brief description of each series and the relevant code for downloading it from the website. To avoid problems related to the volatility and seasonality of the monthly series of government revenues and expenditures, we have reconstructed the debt to GDP ratio by using annual averages of the primary surplus. We report in Figure 1 the resulting series for the debt to GDP ratio. The beginning of our sample coincides with the adoption of an inflation targeting regime in Brazil, which occurred after floating the currency and a 50 per cent depreciation of the real against the dollar. Our sample includes the Argentinian crisis, a domestic energy crisis, a worldwide increase in bond yields, the political uncertainty surrounding the 2002 presidential campaign and the stabilization period beginning in 2003 (see Pastore and Pinotti(2005), for a detailed description of the Brazilian economy in this period).

3.1 Cointegration and necessary conditions for sustainability

Our baseline specification is a five variable VAR for the log of debt $\ln(b_t)$, the log of government expenditure excluding interest payment $\ln(g_{t+1})$, the log of government receipts $\ln(t_{t+1})$, and the difference between the nominal cost of financing the debt, i_t and nominal GDP growth, x_t.All fiscal variables are expressed as percentage of annual GDP. The BIC criterion gives evidence in favour of selecting one as the optimal lag length. The necessary condition for sustainability is satisfied when $\ln(b_t)$, is cointegrated with $\ln(g_{t+1}), \ln(t_{t+1}), r_{t+1}$ with a cointegrating vector $\begin{bmatrix} 1 & \frac{\eta}{1-\rho} & -\frac{1-\rho+\eta}{1-\rho} & \frac{1}{1-\rho} \end{bmatrix}$, where cointegrating parameters reflect the long-term means of the relevant fiscal and macroeconomic variables.

In particular,

$$\rho = \frac{1}{e^{i-x}} < 1, \eta = \frac{e^z}{e^{i-x}},$$

Using sample mean we calibrate $\hat{\eta} = 0.021$ and $\hat{\rho} = 0.992$. $\hat{\rho}$ is less than one as, in our selected sample, the average monthly cost of financing the debt is of about 1.6 per cent, while the average nominal GDP growth is of about 0.9 per cent. Given our chosen value for $\hat{\rho}$, the value of $\hat{\eta}$ is determined by the fact that the ratio of monthly government expenditure to debt is of about 2 per cent. Sustainability implies that $\ln(b_t)$, is cointegrated with $\ln(g_{t+1}), \ln(t_{t+1}), r_{t+1}$ with cointegrating parameter $\begin{bmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$ and the following restrictions can be imposed on the cointegrating parameters

$$\left\{ \begin{array}{c} \beta_1 = \frac{\hat{\eta}}{1-\hat{\rho}} = 2.57; \quad \beta_2 = -\frac{1-\hat{\rho}+\hat{\eta}}{1-\hat{\rho}} = -3.57 \\ \beta_3 = \frac{1}{1-\hat{\rho}} = 125; \end{array} \right. \tag{22}$$

Table 2 reports the results of the application of the Johansen(1995) procedure. The null of at most zero and at most one cointegrating vectors is rejected, while the null of at most two cointegrating vector is not rejected. We restrict the first cointegrating vector by using our sustainability conditions, while we identify the second cointegrating vector as capturing the stationarity of the difference between the cost of financing the debt and GDP growth. The null that the relevant restrictions are validly imposed on the cointegrating vectors cannot be rejected at the one per cent confidence level. In the light of this indecisive evidence, we have performed some further assessment on the validity of the necessary conditions for sustainability. Figure 2 reports the time series of the "surplus gap" delivered by our restricted cointegrating relationships, which shows a rather rapid mean reversion for this series. Figure 3 allows us to compare the level of approximation implied by the linearization by reporting the actual debt to GDP ratio and the linearized debt to GDP ratio obtained first by taking the exponential of the log of debt to GDP dynamically simulated using our calibrated values for $\hat{\rho}$ and $\hat{\eta}$.Figure 3 strengthens the evidence from the cointegrating analysis by showing that the approximation implied in the log-linearization is indeed very accurate.

3.2 Cross equation restrictions and necessary and sufficient conditions for sustainability

The results of Bayesian estimation are reported in Figures 4-5. Figures 4.1 shows that the marginal likelihood of the RVAR is maximized for a value of about 1 of σ, while a value of about 10 is needed for δ in the UVAR in order to obtain the Bayesian estimates not different from OLS estimates. This gives a ratio of the two parameters of 0.1, which gives support to the view of debt sustainability. figure 4.2 reports the value of the likelihood ratio tests for the null of sustainability as a function of σ. When $\sigma = 0$ no uncertainty is attached to the restrictions and we have the classical test which strongly rejects the null (the five per cent critical value is reported in the graph as a straight line). however, we observe that by adding very little uncertainty to the restrictions the statistical evidence against sustainability quickly disappears and for the optimal value of σ the null of interest is clearly not rejected. This evidence is further confirmed by Figure 5, which reports priors and posterior for the five restrictions in the UVAR and RVAR. In fact, despite the fact that the prior on the restrictions in the RVAR are much tighter than the priors in the UVAR, the posterior from the two models are very close to each other.

3.3 An Exact Decomposition of the debt-gap

Given the results of the previous section it is interesting to decompose the surplus-gap to see how at any moment in time the debt -gap is expected to be closed to satisfy the sustainability conditions. We use our VAR estimates on the full sample to simulate at each point in time from the beginning of 2000 onward $dtax_t - dg_t$, di_t and dx_t to compare them with the surplus gap. As we base our decomposition on the Unrestricted VAR, a comparison of the sum of these three variables with the total surplus gap provides another way of assessing the economic importance of the violation of the sustainability restrictions. Consistently with the evidence provided so far the last panel of figure 5 shows that the surplus gap constructed by adding the three simulated components cannot be virtually distinguished from the actual surplus gap. Panel 1-3 of Figure 5 allow us to provide an intuition of how sustainability is supported by the data. A steadily positive primary surplus component paired with a view of moderate but constantly positive growth compensate for a permanent component of the interest payment, while the spikes in the surplus gap observed in the occurrence of currency crisis and periods of very high long-term rates are expected to be reversed in the future. Interestingly such evidence in favour of sustainability is strongly related to the intertemporal approach to the government budget constraint. Contrary to our conclusions, a traditional analysis of the gap between the current primary surplus and the debt stabilizing primary surplus would inevitably label as unsustainable the Brazilian debt-to-GDP ratio in period of temporarily extraordinarily high cost of the debt service.

4 Conclusions

This paper was motivated by the importance of proposing indicators for fiscal sustainability backed by a formal definition of sustainability and coherent with some related empirical tests. First, we have shown that, by solving a linearized government intertemporal budget constraints, necessary and sufficient conditions for sustainability can be expressed as cross-equation restrictions on a VAR models for fiscal and macroeconomic variables. Second, we illustrate how a Bayesian approach to the evaluation of the consistency of the relevant cross-equation restrictions with the data allows to propose sustainability indicators that are naturally linked to the statistical evidence on the validity of such restrictions. Third, we use our VAR to provide evidence on how, at each moment in time, the corrections in fiscal and macroeconomic fundamentals necessary to obtain sustainability are expected to be achieved.

We provide an empirical application to the case of Brazil, which we find interesting both because of the quality of the data and their time-series properties. Our evidence is in favour of sustainability of the Brazilian debt as in June 2005. Given a range of our sustainability indicator between zero and one, where zero indicates sustainable debt and one indicates not sustainable debt, we observe a value of 0.1 in our illustrative case study.

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Table 1. Brazilian Macroeconomic and Fiscal variables

All quantities are expressed in R$ million			
Series	Source	Code	Description of the original series
X_t	Banco do Brasil	4382	GDP accumulated in the last 12 months - current prices
I_t	Banco do Brasil	6046	PBSR with exchange devaluation - Nominal interest
t_t	Banco do Brasil	7639	Tax revenues - Accrual basis - Total
g_t	Banco do Brasil	4649+7639	4649 - PSBR - Current monthly flows - Primary result
b_t	Banco do Brasil	10823	Fiscal net debt with exchange devaluation

Table 2: The Cointegrated System

This panel contains the results of the application of the Johansen (1995) procedure allowing an intercept in the cointegrating vector and in the VAR. The panel below reports the result of the cointegration test over different samples and the tests for the validity of the relevant long-run restrictions .

Cointegration Rank Test (Maximum Eigenvalue) for 1999:4 2005:06				
Hypotesized No. of CE(s)	Eigenvalue	Max - Eigen Statistic	0.05 Critical Value	Prob.**
None*	0.387	36.75	28.58	0.004
At most 1*	0.292	26.00	22.29	0.0145
At most 2	0.075	5.90	15.89	0.79

Max-eigenvalue and trace tests indicates 1 cointegrating eqn(s) at the 0.01 level.
**MacKinnon-Haug-Michelis (1999) p-values

The Cointegrating Relation					
Sample	Restricted Cointegrating Parameters				
1999:4-2005:06	b_{t-1}	g_t	t_t	x_t	i_t
	1	2.75	-3.57	-125	125
	0	0	0	1	-1
Tests for the validity of long-run restrictions					
$\chi^2(4) = 12.84(0.012)$					

Appendix 1: the linearization of the Government Budget Constraint

Consider the representation of the static budget constraint introduced in the main text:

$$b_{t+1} = \frac{(1+I_{t+1})}{(1+X_{t+1})} b_t + (g_{t+1} - t_{t+1}). \tag{23}$$

Dividing by b_t and taking logs :

$$i_{t+1} - x_{t+1} = \ln(e^{\ln(b_{t+1}) - \ln(b_t)} - e^{\ln(g_{t+1}) - \ln(b_t)} + e^{\ln(t_{t+1}) - \ln(b_t)})$$

Where $i_{t+1} = \ln(1+I_{t+1}), x_{t+1} = \ln(1+X_{t+1})$ .Assuming that $\ln(b_{t+1}) - \ln(b_t)$, $\ln(g_{t+1}) - \ln(b_t)$, $\ln(t_{t+1}) - \ln(b_t)$ are stationary and taking a Taylor expansion around the long-term means $\ln(b_{t+1}) - \ln(b_t) = 0$, $\ln(g_{t+1}) - \ln(b_t) = z$, $\ln(t_{t+1}) - \ln(b_t) = y = \ln(e^{i-x} - 1 + e^z)$ we obtain:

$$\begin{aligned} i_{t+1} - x_{t+1} &= \rho[\ln(b_{t+1}) - \ln(b_t)] - \eta[\ln(g_{t+1}) - \ln(b_t)] \\ &\quad + (1-\rho+\eta)[\ln(t_{t+1}) - \ln(b_t)] + (1-\rho)k, \end{aligned}$$

where

$$\begin{aligned} \rho &= \frac{1}{(1+e^y - e^z)} = \frac{1}{e^{i-x}} < 1, \\ \eta &= \frac{e^z}{(1+e^y - e^z)} = \frac{e^z}{e^{i-x}}, \end{aligned}$$

and

$$(1-\rho)k = \eta(z-y) - (1-\rho)y.$$

The equation above can be re-written as follows:

$$\begin{aligned} -\rho\ln(b_{t+1}) + \ln(b_t) &= ls_{t+1} \\ ls_{t+1} &= -\eta\ln(g_{t+1}) + (1-\rho+\eta)\ln(t_{t+1}) - r_{t+1} + (1-\rho)k. \end{aligned}$$

Where $r_{t+1} = i_{t+1} - x_{t+1}$. The coefficients on logs of expenditure and log of revenue are in general different from one another. They tend to converge as i converges to x, and ρ converges to 1. In this case constancy of the debt implies equality of the two coefficients, that is, for i converging towards x the debt ratio is stable when the primary surplus, as a function of GDP, is zero.

Appendix 2: Derivation of the Bayes factor for the RVAR and the UVAR

To compute posterior and marginal likelihood, consider the general model at our hands:

$$
\begin{aligned}
\mathbf{z}^{+} &= \Xi\ \Psi + \mathbf{u}^{+}, \qquad (24)\\
\underset{MT\times 1}{\mathbf{z}^{+}} &= \begin{pmatrix} \mathbf{sgap} \\ \mathbf{\Delta}\ln(g)^{*}. \\ \mathbf{\Delta}\ln(t)^{*}. \\ \mathbf{\Delta x} \\ \mathbf{\Delta i} \end{pmatrix}, \Xi = \left[\underset{MxM}{I_5} \otimes \underset{Txk}{\mathbf{x}}\right] = \begin{pmatrix} \mathbf{x} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{x} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{x} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{x} \end{pmatrix},\\
\mathbf{x} &= \begin{pmatrix} \mathbf{sgap}_{-1} & \mathbf{\Delta}\ln(g)^{*}_{-1} & \mathbf{\Delta}\ln(t)^{*}_{-1} & \mathbf{\Delta x}_{-1} & \mathbf{\Delta i}_{-1} \end{pmatrix}\\
\Psi &= Vec\begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} & \psi_{25} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} & \psi_{35} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} & \psi_{45} \\ \psi_{51} & \psi_{52} & \psi_{53} & \psi_{54} & \psi_{55} \end{bmatrix}\\
\mathbf{u}^{+} &= \begin{pmatrix} \mathbf{u}_1 \\ . \\ . \\ \mathbf{u}_5 \end{pmatrix}. \sim N\left(0,\ \Sigma_u\ \otimes I_T\right)
\end{aligned}
$$

where $M = 5$ is the number of equations, $p = 1$ is the number of lags included, $k = pM + 1$ and T is the sample size. The prior density is:

$$p(\Psi) = (2\pi)^{-Mk/2}\left|\Sigma_{\Psi_0}\right|^{-1/2}\exp\left\{-1/2(\Psi-\Psi_0)'\Sigma_{\Psi_0}^{-1}(\Psi-\Psi_0)\right\},$$

the likelihood is:

$$p(y|\Psi) = (2\pi)^{-MT/2}\left|\Omega\right|^{-1/2}\exp\left\{-1/2(y-\Xi\Psi)'\Omega^{-1}(y-\Xi\Psi)\right\},$$

a posterior density kernel is:

$$
\begin{aligned}
p(y|\Psi)p(\Psi) &= (2\pi)^{-M(T+k)/2}\left|\Omega\right|^{-1/2}\left|\Sigma_{\Psi_0}\right|^{-1/2}\\
&\quad \exp\left\{-1/2\begin{bmatrix} (y-\Xi\Psi)'\Omega^{-1}(y-\Xi\Psi) \\ +(\Psi-\Psi_0)'\Sigma_{\Psi_0}^{-1}(\Psi-\Psi_0) \end{bmatrix}\right\}.
\end{aligned}
$$

Now define:

$$\Sigma_{\Psi} = \left[\Sigma_{\Psi_0}^{-1} + \Xi'\Omega^{-1}\Xi\right]^{-1},$$

$$\Psi = \Sigma_{\Psi} * \left[\Sigma_{\Psi_0}^{-1}\Psi_0 + \Xi'\Omega^{-1}y\right].$$

Using the above definitions and completing the square yields:

$$(y - \Xi\Psi)'\Omega^{-1}(y - \Xi\Psi) + (\Psi - \Psi_0)'\Sigma_{\Psi_0}^{-1}(\Psi - \Psi_0)$$

$$\begin{aligned} = \quad & y'\Omega^{-1}y - [y'\Omega^{-1}\Xi + \Psi_0{}'\Sigma_{\Psi_0}^{-1}]\Psi - \Psi'[\Xi'\Omega^{-1}y + \Sigma_{\Psi_0}^{-1}\Psi_0] \\ & +\Psi'[\Xi'\Omega^{-1}\Xi + \Sigma_{\Psi_0}^{-1}]\Psi + \Psi_0{}'\Sigma_{\Psi_0}^{-1}\Psi_0 \end{aligned}$$

$$= y'\Omega^{-1}y - \Psi'\Sigma_{\Psi}^{-1}\Psi - \Psi'\Sigma_{\Psi}^{-1}\Psi + \Psi'\Sigma_{\Psi}^{-1}\Psi + \Psi_0{}'\Sigma_{\Psi_0}^{-1}\Psi_0.$$

This can be rewritten as:

$$\begin{aligned} & (y - \Xi\Psi)'\Omega^{-1}(y - \Xi\Psi) + (\Psi - \Psi_0)'\Sigma_{\Psi_0}^{-1}(\Psi - \Psi_0) \\ = \quad & y'\Omega^{-1}y - \Psi'\Sigma_{\Psi}^{-1}\Psi + (\Psi - \Psi)'\Sigma_{\Psi}^{-1}(\Psi - \Psi) + \Psi_0{}'\Sigma_{\Psi_0}^{-1}\Psi_0, \end{aligned}$$

so a posterior density kernel can be also written as follows:

$$\begin{aligned} p(y|\Psi)p(\Psi) \quad = \quad & (2\pi)^{-M(T+k)/2} \left|\Omega\right|^{-1/2} \left|\Sigma_{\Psi_0}\right|^{-1/2} \\ & \exp\left\{-1/2[(\Psi - \Psi)'\Sigma_{\Psi}^{-1}(\Psi - \Psi) + Q]\right\}, \end{aligned}$$

where:

$$Q = y'\Omega^{-1}y - \Psi'\Sigma_{\Psi}^{-1}\Psi + \Psi_0{}'\Sigma_{\Psi_0}^{-1}\Psi_0.$$

Forgetting constants:

$$p(y|\Psi)p(\Psi) \propto \exp\left\{-1/2[(\Psi - \Psi)'\Sigma_{\Psi}^{-1}(\Psi - \Psi)]\right\} \Longrightarrow p(\Psi|y) \sim N(\Psi, \Sigma_{\Psi}),$$

which shows that Ψ, Σ_{Ψ} are the moments of the posterior. The posterior properly normalized density is:

$$p(\Psi|y) = (2\pi)^{-Mk/2} \left|\Sigma_{\Psi}\right|^{-1/2} \exp\left\{-1/2\,(\Psi - \Psi)'\,\Sigma_{\Psi}^{-1}\,(\Psi - \Psi)\right\}.$$

The marginal likelihood is given by integral over the $M \times k$ dimensional space of the product of the properly normalized prior and data densities:

$$ML = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} p(y|\Psi)p(\Psi)\ d\Psi_1...d\Psi_{Mk} = \int_{\Re^{Mk}} p(y|\Psi)p(\Psi)d\Psi$$

$$= \int_{\Re^{Mk}} (2\pi)^{-M(T+k)/2} \left|\Omega\right|^{-1/2} \left|\Sigma_{\Psi_0}\right|^{-1/2} \exp\left\{-1/2\left[\begin{array}{c} (y - \Xi\Psi)'\Omega^{-1}(y - \Xi\Psi)+ \\ (\Psi - \Psi_0)'\Sigma_{\Psi_0}^{-1}(\Psi - \Psi_0)] \end{array}\right]\right\} d\Psi$$

$$\begin{aligned} = \quad & \int_{\Re^{Mk}} (2\pi)^{-M(T+k)/2} \left|\Omega\right|^{-1/2} \left|\Sigma_{\Psi_0}\right|^{-1/2} \\ & \exp\left\{-1/2[(\Psi - \Psi)'\Sigma_{\Psi}^{-1}(\Psi - \Psi) + Q]\right\} d\Psi \\ = \quad & (2\pi)^{-M(T+k)/2} \left|\Omega\right|^{-1/2} \left|\Sigma_{\Psi_0}\right|^{-1/2} \exp\left\{-Q/2\right\} \\ & \exp \int_{\Re^{Mk}} \left\{-1/2[(\Psi - \Psi)'\Sigma_{\Psi}^{-1}(\Psi - \Psi)]\right\} d\Psi. \end{aligned}$$

Note that it is important that the properly normalized prior and properly normalized likelihood, and not arbitrary kernels of these densities, be used in forming the marginal likelihood.

Now recognize a posterior kernel in the above expression and exploit the fact that the posterior properly normalized density integrates to one:

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} p(\Psi|y)\ d\Psi_1...d\Psi_{Mk} = \int_{\Re^{Mk}} p(\Psi|y)d\Psi = 1 \Longrightarrow$$

$$\begin{aligned} 1 &= \int_{\Re^{Mk}} (2\pi)^{-Mk/2} \left|\Sigma_\Psi\right|^{-1/2} \exp\left\{-1/2\left(\Psi-\Psi\right)'\Sigma_\Psi^{-1}\left(\Psi-\Psi\right)\right\} d\Psi \\ \Longrightarrow & \frac{1}{(2\pi)^{-Mk/2}\left|\Sigma_\Psi\right|^{-1/2}} = \int \exp\left\{-1/2\left(\Psi-\Psi\right)'\Sigma_\Psi^{-1}\left(\Psi-\Psi\right)\right\}. \end{aligned}$$

The marginal likelihood is thus:

$$\int_{\Re^{Mk}} p(y|\Psi)p(\Psi)d\Psi = (2\pi)^{-MT/2} \left|\Omega\right|^{-1/2} \frac{\left|\Sigma_{\Psi_0}\right|^{-1/2}}{\left|\Sigma_\Psi\right|^{-1/2}} \exp\left\{-Q/2\right\},$$

where:

$$Q = y'\Omega^{-1}y - \Psi'\Sigma_\Psi^{-1}\Psi + \Psi_0{}'\Sigma_{\Psi_0}^{-1}\Psi_0.$$

From this it is immediate to derive the Bayes factor of the RVAR against the UVAR:

$$BF = \left[\frac{\frac{\left|\Sigma_\Psi^{priorRVAR}\right|}{\left|\Sigma_\Psi^{postRVAR}\right|}}{\frac{\left|\Sigma_\Psi^{priorUVAR}\right|}{\left|\Sigma_\Psi^{posUVAR}\right|}}\right]^{-1/2} \exp\left\{\frac{Q^{UVAR}-Q^{RVAR}}{2}\right\}. \tag{25}$$

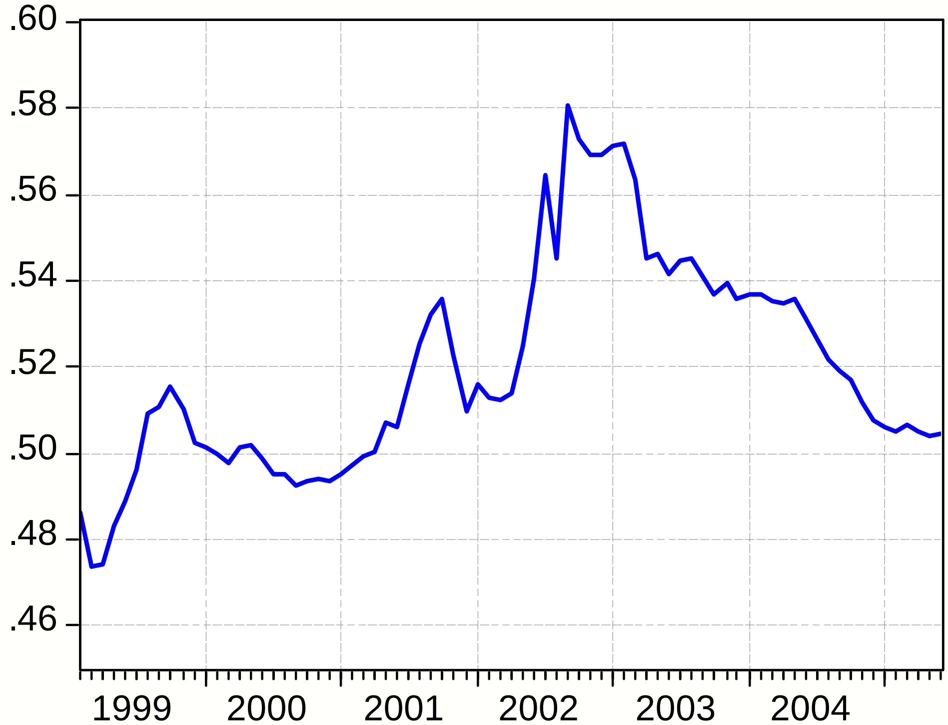

Figure 1: The Brazilian debt to GDP ratio reconstructed using expenditures and revenues cumulated in the last twelve months

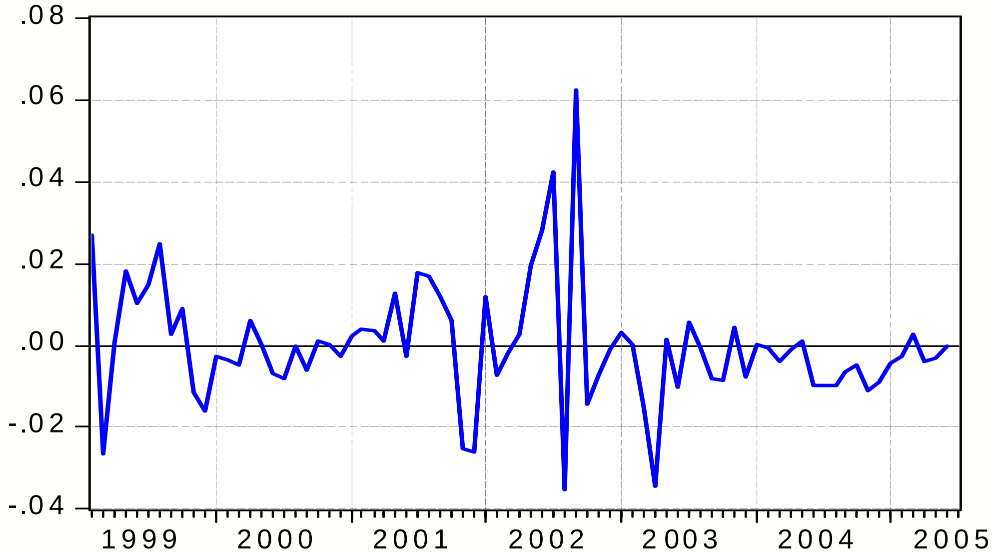

Figure 2: The "surplus-gap" in Brazil

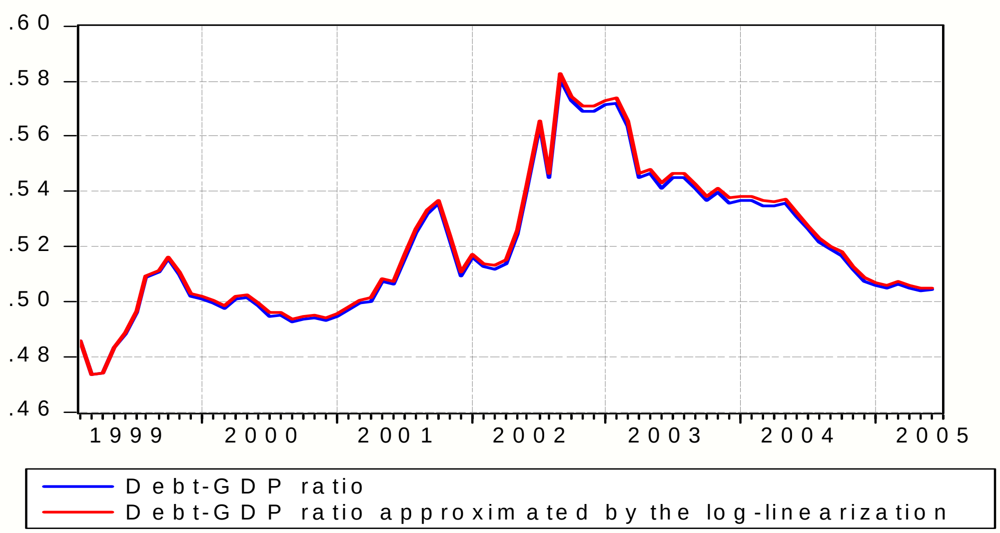

Figure 3: Assessing the log-linearization

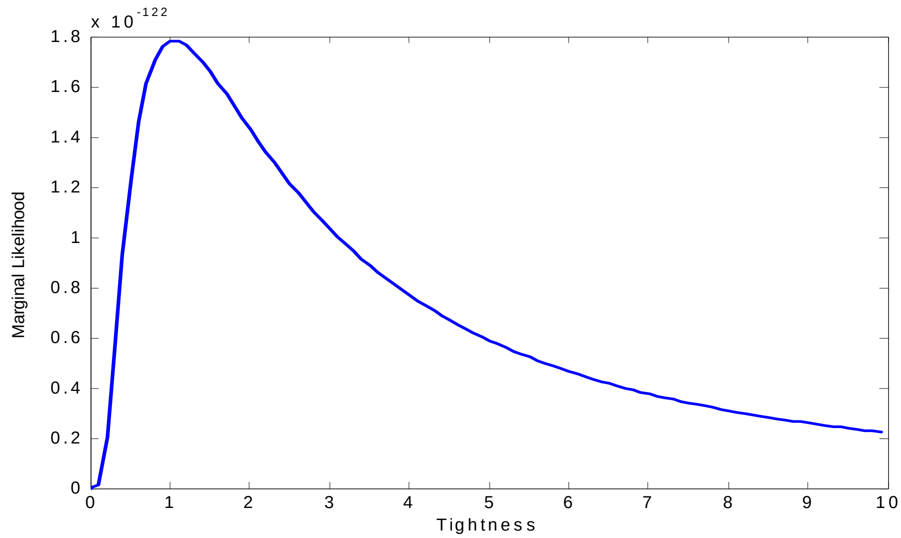

Figure 4.1: Value of the marginal likelihood as a function of the parameter determining the tighness of the distribution of the restrictions necessary and sufficient for sustainability.

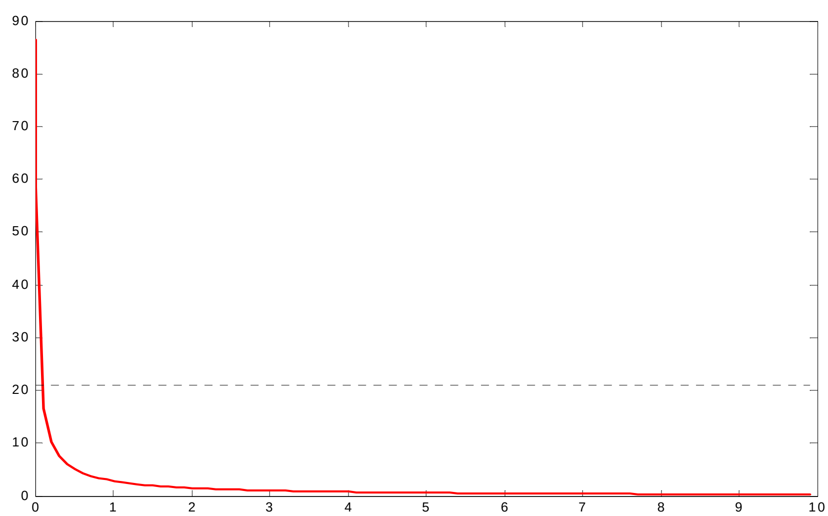

Figure 4.2: Likelihood ratio tests on the null of debt sustainability as a function of sigma.

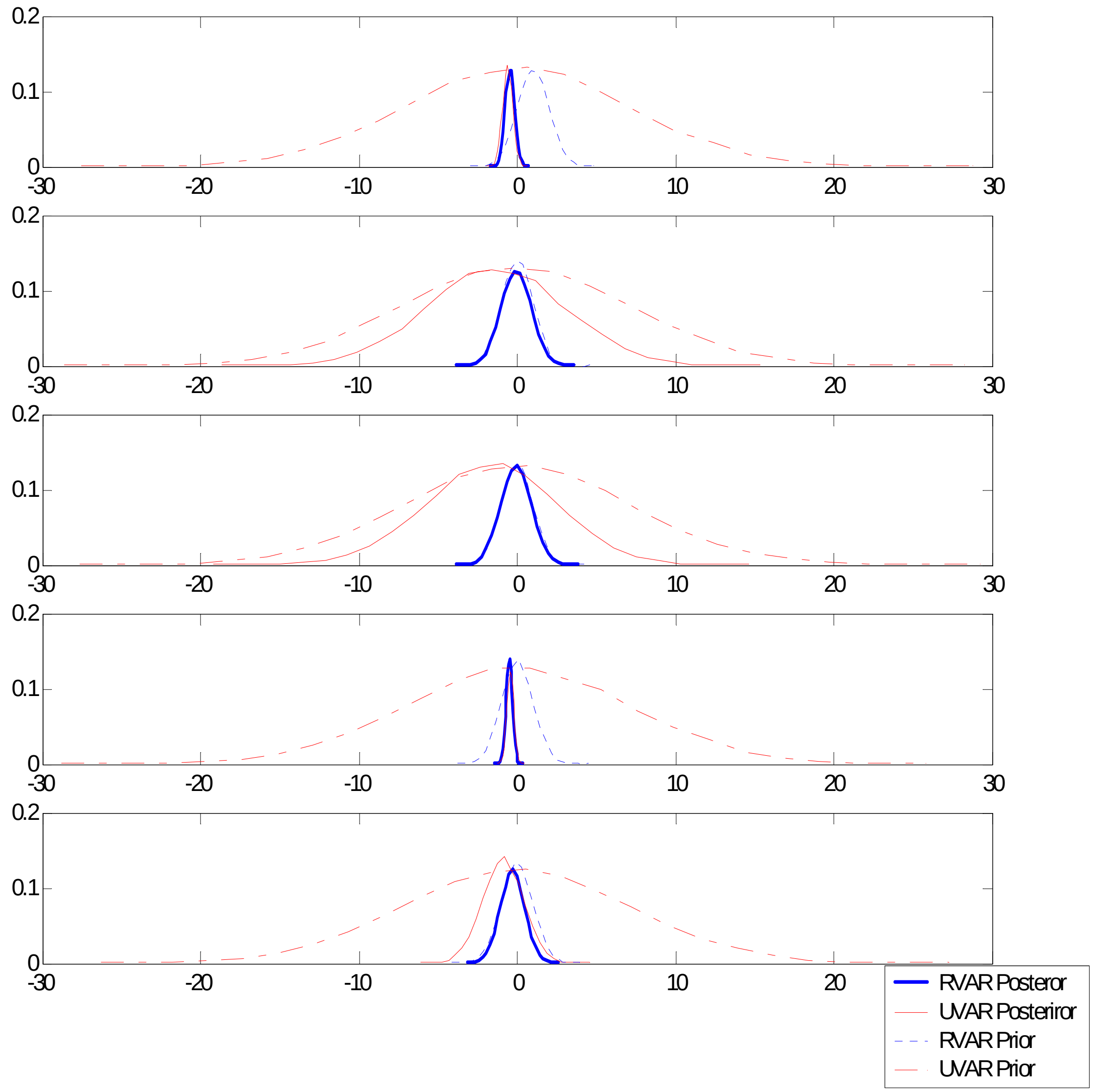

Figure 5:Prior and Posterior distribution for the five sutainability restrictions in the UVAR and in the RVAR.

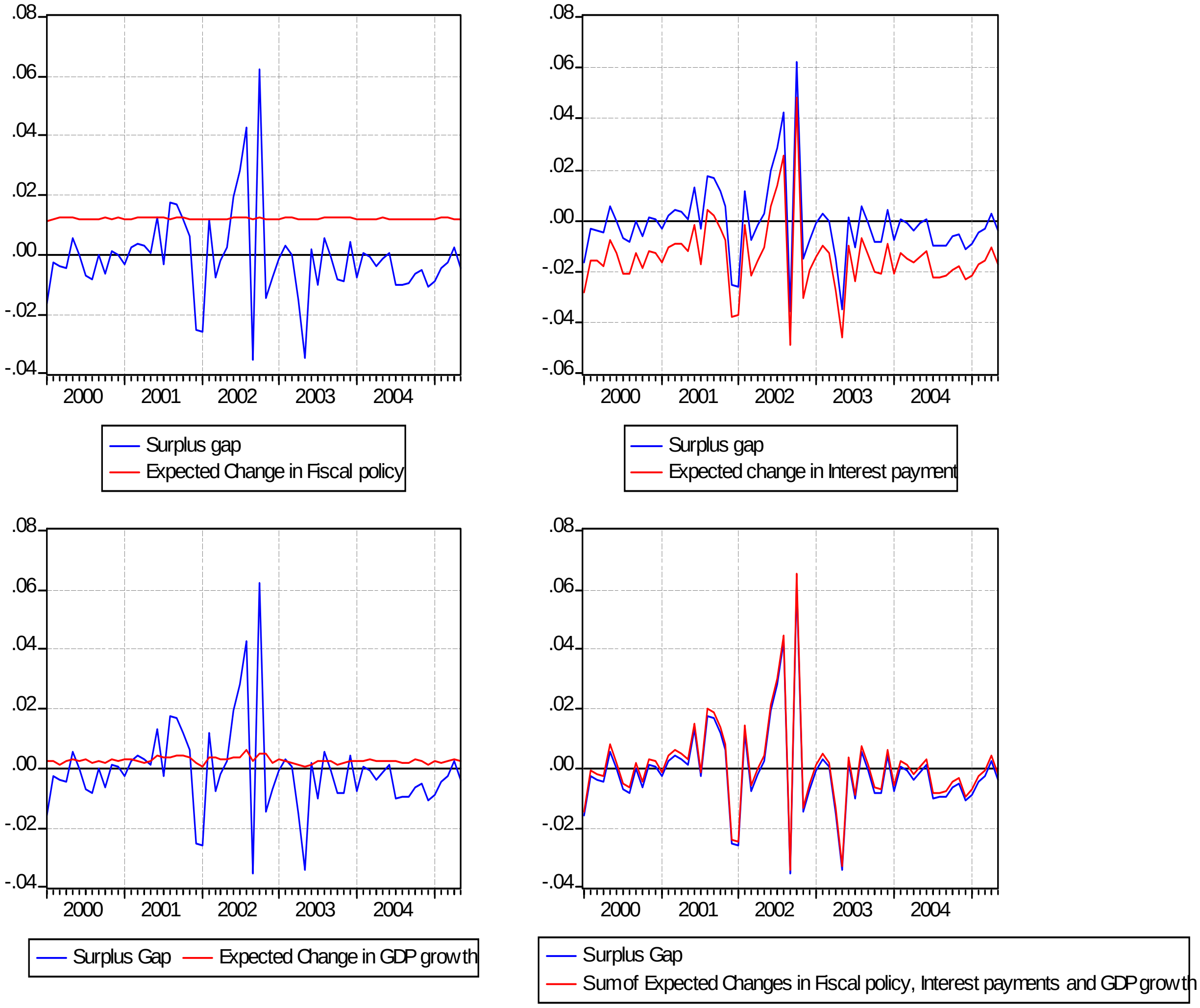

Figure 6: A forward-looking decomposition of the debt-gap