Dynamic Trading, Asset Prices, and Bubbles [*]

Giovanni Cespa [†] and Xavier Vives [‡]

First Version: October 1, 2006
This Version: March 14, 2007

Abstract

We study asset pricing bubbles within a noisy, dynamic rational expectations equilibrium model with competitive traders. Say that a "bubble" ("reverse bubble") exists if the asset price is on average farther away from (closer to) the fundamentals than investors' average expectations. In a market with long term traders, we find that bubbles do not arise provided that noise trading follows a random walk; bubbles (reverse bubbles), however appear when residual uncertainty over the liquidation value and noise trading persistence are low (high). With short-term traders there typically are two equilibria, with the stable (unstable) one displaying a bubble (reverse bubble).

Keywords: Long and short-term trading, multiple equilibria, average expectations, higher order beliefs, public information

JEL Classification Numbers: G10, G12, G14

1 Introduction

Whether asset prices are closely tied to fundamentals or are far away from them is an issue that has generated much debate among economists. In his General Theory, Keynes pioneered the vision of stock markets as beauty contests where investors try to guess not the fundamental value of an asset but the average opinion of other investors, and end up chasing the crowd. [1] This view tends to portray a stock market dominated by herding, behavioral biases, fads, booms and crashes (see, for example, Shiller (2000)), and goes against the tradition of considering market prices as aggregators of the dispersed information in the economy advocated by Hayek (1945). According to the latter view prices reflect, perhaps noisily, the collective information that each trader has about the fundamental value of the asset (see, for example, Grossman (1989)), and provide a reliable signal about assets' liquidation values. In this paper, we address the tension between the Keynesian and the Hayekian visions in a dynamic finite horizon market where investors, except noise traders, have no behavioral bias and hold a common prior on the liquidation value of the risky asset.

A bubble in an asset market is typically thought to arise when traders buy the asset only in the expectation of selling it at a higher price, which does not necessarily reflect the fundamentals. As argued by Allen, Morris, and Shin (2006), a bubble may also be understood as a situation where the asset price is on average farther away from the fundamentals than investors' consensus expectation. By the same token, whenever the price is instead *closer* to the fundamentals (than investors' average opinion), we could thus speak of a "reverse" bubble. Allen et al. (2006) argue that bubbles, in the sense described, always arise in a dynamic market with risk averse short-term traders, differential information, and independent noisy supply across periods. This gives support to Keynes' vision of the stock market since in this case prices depend on investors' higher order beliefs (i.e. the average expectation at n of the average expectation at $n+1$ of … of the average expectation of the liquidation value in the final period, as in the beauty contest allegory), and systematically depart from

---

[1] Keynes' vision of the stock market as a beauty contest or the situation in which judges are more concerned about the opinion of other judges than of the intrinsic merits of the participants in the contest: "…professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the Competitor whose choice most nearly corresponds to the average preferences of the competitor as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view." (Keynes, General Theory, 1936).

fundamentals (compared to investors' average expectations). This also implies that when higher order beliefs matter for pricing, traders display excessive reliance on public information (in relation to the optimal statistical weight). The question is thus how general is this insight or, in other words, whether Keynes always prevails over Hayek. Is it always true that prices are farther away from fundamentals than average expectations? Is there always excessive reliance on public information? [2]

There are reasons to believe that these results may depend on the properties of the noise trading process and on traders' speculative horizon. Indeed, if noise traders' demand (stock) is independent across periods, patterns of liquidity supply become more easily predictable. This, in turn, leads traders to accommodate more actively order imbalances, dampening the informational impact of speculators' positions, and potentially driving the price away from the fundamentals. [3] Furthermore, in a market with differential information the asset price relation to the fundamentals depends on traders' speculative behavior. In a *static* market agents speculate on the difference between the price and the liquidation value. As they hold private information on the latter, the price dependence from the fundamentals is positively related to the quality of such information. In a *dynamic* market, on the other hand, traders *also* speculate on short-run price differences. This makes the relation between price and fundamentals *also* depend on traders' short-term speculative behavior. Hence, bubbles should be related to speculators' trading horizons. [4]

In this paper we consider a general process for noise trading and analyze a model where traders have either a long or a short speculative horizon, and where residual uncertainty can affect the liquidation value.

We find that bubbles *do not arise* if the market is populated by long-term traders, no residual uncertainty affects the liquidation value, and noise trading follows a random walk (i.e. noise trade increments are independent). In this case in every period traders act as in a static market. When additional pay-off uncertainty and noise

---

[2]Excessive reliance on public information may have deleterious welfare consequences (see, e.g., Vives (1997), Morris and Shin (2002), and Angeletos and Pavan (2005)).

[3]Indeed, assuming that the stock of noise trade is i.i.d. implies that the *gross* position noise traders hold in a given period n completely reverts in period $n + 1$. This lowers the risk of accommodating order imbalances in any period, as speculators can always count on the possibility of unwinding their inventory of the risky asset to liquidity traders in the coming round of trade.

[4]Apart from these considerations, the assumption of a time-independent noisy supply process accords with the idea of a market where the amount of time between two consecutive trades is likely to be large. However, bubbles are typically seen as phenomena that evolve under the effect of speculators' intense and frequent trading activity, an environment which seems to be better captured if the stock of liquidity trades is autocorrelated.

trade predictability are added, traders also speculate on short-run price differences, and both bubbles and reverse bubbles can arise. Low residual uncertainty and low correlation in noise trading (stock) tend to generate bubbles; conversely, high residual uncertainty and high correlation in noise trading tend to generate reverse bubbles. This partitions the parameter space into a Keynesian region, where prices are farther away from fundamentals than average expectations, and a Hayekian region where the opposite occurs. The partition depends on risk tolerance and more risk aversion enlarges the Keynesian region. In the first region there is "excessive" reliance on public information and in the second "insufficient" reliance.

With short-term traders typically there are two equilibria ranked by traders' responsiveness to private information. In the equilibrium with low (high) signal responsiveness there is a bubble (reverse bubble). Again, the first equilibrium may be associated to Keynes and the second one to Hayek. The result holds for arbitrary small serial correlation in the noise trading process and the Hayekian equilibrium provides another instance of a situation where higher order beliefs matter but where there may be insufficient reliance on public information. In this reverse bubble equilibrium the price adjusts faster to changes in the fundamentals than the consensus expectations of investors. This equilibrium is, however, unstable. Finally, if noise trading follows a random walk and there is sufficiently large residual uncertainty in the liquidation value in equilibrium there is always a bubble.

The intuition for our results is as follows. In a dynamic market the relationship between price and fundamentals depends both on the quality of traders' information *and* on their reaction to order imbalances. Suppose a trader observes a positive signal and faces a positive order-flow. Upon the receipt of good news he increases his long position in the asset. On the other hand, his reaction to the order imbalance is either to accommodate it, counting on a future noise trade reversal (and thus acting as a "market-maker"), or to follow the market and further increase his long position anticipating an additional price rise (in this way acting as a "short-term" speculator). The more likely it is that the order imbalance reverts over time due to liquidity traders' behavior, the more actively the trader will want to accommodate it. Conversely, the more likely it is that the imbalance is due to informed speculators' activity, the more the trader will want to follow the market. [5] In the former (latter)

---

[5]In this case, indeed, the order imbalance is likely to proxy for upcoming good news that are not yet completely incorporated in the price. There is a vast empirical literature that documents the transient impact of liquidity trades on asset prices as opposed to the permanent effect due to information-driven trades. See e.g. Wang (1994), and Llorente et al. (2002).

case, the trader's speculative position is partially offset (reinforced) by his market making (short-run speculative) position. Thus, the impact of private information on the price is partially sterilized (enhanced) by traders' market-making (short-run speculative) activity. This loosens (tightens) the price from (to) the fundamentals, leading to a bubble (reverse bubble).

When long-term traders are in the market, correlation across noise trade increments helps predicting future noise trade shocks, and tilts traders towards accommodating order imbalances. The impact of residual uncertainty over the liquidation value, on the other hand, enhances the hedging properties of future positions, boosting traders' signal responsiveness and leading them to speculate more aggressively on short-run price differences. Thus, depending on the intensity of the correlation across noise trade increments, bubbles may or may not arise. Conversely, absent correlation across noise trade increments and residual uncertainty, traders act as in a static market, and no bubble arises. When traders have a short horizon, the lack of future trading opportunities eliminates intertemporal hedging possibilities. This may give rise to self-fulfilling equilibria. In particular, if traders anticipate an order imbalance due to liquidity traders (informed traders), they scale down (up) their trading aggressiveness and accommodate it (follow the market), justifying their prediction. In this case, thus, the presence of bubbles or reverse bubbles relates to the equilibrium that arises.

Our paper is linked to the recent literature that analyzes the effect of higher order expectations in asset pricing models where traders have differential information, but agree on a common prior over the liquidation value. Bacchetta and van Wincoop (2006a) develop a dynamic rational expectations equilibrium model with heterogeneous information to address the exchange rate determination puzzle, arguing that short-term departures of the exchange rate from macroeconomic fundamentals can be explained by traders' heterogeneous information. [6] Bacchetta and van Wincoop (2006b) study the role of higher order beliefs in asset prices in an infinite horizon model showing that higher order expectations add an additional term to the traditional asset pricing equation, the higher order "wedge," which captures the discrepancy between the price of the asset and the average expectations of the fundamentals. According to our results, higher order beliefs do not necessarily enter the pricing equation. In other words, for the higher order wedge to play a role in the asset price we need residual uncertainty to affect the liquidation value or noise trade increments predictability

---

[6] In related research, Vitale (2006) studies a model of central bank intervention in the FX market.

when traders have long horizons; alternatively, traders must display myopic behavior.

Other authors have analyzed the role of higher order expectations in models where traders hold different initial beliefs about the liquidation value. Biais and Bossaerts (1998) show that departures from the common prior assumption rationalize peculiar trading patterns whereby traders with low private valuations may decide to buy an asset from traders with higher private valuations in the hope to resell it later on during the trading day at an even higher price. Cao and Ou-Yang (2005) study conditions for the existence of bubbles and panics in a model where traders' opinions about the liquidation value differ. [7] In their setup, there is a bubble whenever the equilibrium price is higher than the highest price that would prevail if traders had homogeneous beliefs. Thus, a bubble denotes a situation in which traders are willing to pay more than what the most "optimistic" trader would to acquire the asset. [8]

The paper is organized as follows: in the next section we present the static benchmark, showing that in this framework no bubble occurs. In section 3 we analyze the dynamic model with long-term traders. In section 4, we then turn our attention to the model with short-term traders. The final section discusses the results and provides concluding remarks.

2   The Static Benchmark

Consider a one-period stock market where a single risky asset with liquidation value $v \sim N(\bar{v}, \tau_v^{-1})$, and a riskless asset with unitary return are traded by a continuum of risk-averse speculators in the interval $[0, 1]$ together with noise traders. Speculators have CARA preferences (denote with γ the risk-tolerance coefficient) and maximize the expected utility of their wealth: $W_{i1} = (v - p_1)x_{i1}$. Prior to the opening of the market every informed trader i receives a signal $s_{i1} = v + \epsilon_{i1}$, $\epsilon_{i1} \sim N(0, \tau_{\epsilon_1}^{-1})$, and submits a demand schedule (generalized limit order) to the market $X_1(s_{i1}, p_1)$ indicating the desired position in the risky asset for each realization of the equilibrium price. [9] Assume that v and ϵ_{i1} are independent for all i, and that error terms are also independent across traders. Noise traders submit a random demand u_1 (independent of all other random variables in the model), where $u_1 \sim N(0, \tau_u^{-1})$. Finally, assume

---

[7]Kandel and Pearson (1995) provide empirical evidence supporting the non-common prior assumption.

[8]There is a vast literature dealing with bubbles in a variety of contexts, see e.g. Abreu and Brunnermeier (2003), Brunnermeier and Nagel (2004), and Vives (2006) for a survey.

[9]The unique equilibrium in linear strategies of this model is symmetric.

that, given v, the average signal $\int_0^1 s_{i1}di$ equals v almost surely (i.e. errors cancel out in the aggregate: $\int_0^1 \epsilon_{i1}di = 0$).

Within this setup it is well known that a unique equilibrium in linear strategies exists (see e.g. Admati (1985)), where $X_1(s_{i1}, p_1) = \gamma \text{Var}[v|s_{i1}, p_1]^{-1}(E[v|s_{i1}, p_1] - p_1)$, $p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1)\bar{v}$, $z_1 = a_1 v + u_1$, $a_1 = \gamma \tau_{\epsilon_1}$, $\lambda_1 = (\gamma(\tau_1 + \tau_{\epsilon_1}))^{-1}(1 + \gamma \tau_u a_1)$, and $\tau_1 \equiv \text{Var}[v|p_1]^{-1} = \tau_v + a_1^2 \tau_u$.

In this paper we will use the above CARA-normal framework to investigate conditions under which the equilibrium price deviates systematically from the liquidation value with respect to traders' average expectations. We will say that there is a bubble (similarly as in Allen et al. (2006)) when

$$|E[p_1 - v|v]| > \left|E\left[\int_0^1 E[v|s_{i1}, p_1]di - v|v\right]\right|. \tag{1}$$

When the inequality is reversed we will say that there is a "reverse" bubble. The above definition thus captures the idea that bubbles denote departures of the price from the average opinion of the investors in a market. As argued above, with CARA utility, traders' aggregate demand is proportional to $\int_0^1 (E[v|s_{i1}, p_1] - p_1)di$. Hence, at equilibrium

$$\int_0^1 x_{i1}di + u_1 = \int_0^1 \gamma(\tau_1 + \tau_{\epsilon_1})(E[v|s_{i1}, p_1] - p_1)di + u_1 = 0,$$

and the equilibrium price can be expressed as the sum of traders' average expectations and noise (times a constant):

$$p_1 = \bar{E}_1[v] + \frac{u_1}{\gamma(\tau_1 + \tau_{\epsilon_1})}, \tag{2}$$

where $\bar{E}_1[v] = \int_0^1 E[v|s_{i1}, p_1]di$. As u_1 and v are by assumption orthogonal, we can therefore conclude that in a static setup bubbles can never arise.

Remark 1. Owing to normality one can immediately verify that

$$E[v|s_{i1}, p_1] = \alpha_{E1}s_{i1} + (1 - \alpha_{E1})E[v|p_1],$$

where $\alpha_{E1} = \tau_{\epsilon_1}/\tau_{i1}$ denotes the optimal statistical weight assigned to private information, and that in the linear equilibrium obtained above, a similar decomposition also holds for the equilibrium price: $p_1 = \alpha_{P1}\hat{z}_1 + (1 - \alpha_{P1})E[v|z_1]$, where

$$\hat{z}_1 \equiv v + \frac{1}{a_1}u_1, \tag{3}$$

7

denotes a monotone transformation of the aggregate demand intercept, and $\alpha_{P1} = a_1/(\gamma \tau_{i1})$. Specializing (1) to our setup we can thus conclude that the equilibrium price is farther away from the fundamentals with respect to traders' average expectation if and only if

$$\alpha_{P1} < \alpha_{E1}, \tag{4}$$

or, equivalently, if and only if $a_1 < \gamma \tau_{\epsilon_1}$. Given that in equilibrium $a_1 = \gamma \tau_{\epsilon_1}$, (i.e. $\alpha_{P1} = \tau_{\epsilon_1}/\tau_{i1}$), a bubble can thus *never* occur in a static setup. According to condition (4), on the other hand, bubbles arise whenever traders' private information aggressiveness falls short of the static equilibrium solution. Intuitively, the less aggressively traders speculate on private information, the lower is the weight the price assigns to the fundamentals as compared to traders' average expectation. The price is thus less "anchored" to the fundamentals, and a bubble arises. [10]

The equilibrium condition (2) also shows that for bubbles to occur an "extra" term whose expectation does not cancel conditionally on v must accrue to investors' average expectations in the pricing equation. In the following sections we will argue that the occurrence of such a term can be traced to traders' speculative activity on short-run price movements.

3 A Dynamic Market with Long-Term Traders

Consider now a dynamic extension of the previous model where risk averse traders and noise traders exchange both the risky and the riskless asset during $N \geq 2$ periods. [11] In period $N + 1$ the risky asset is liquidated. As before, speculators have CARA preferences and maximize the expected utility of their final wealth: $W_{iN} = \sum_{n=1}^{N} \pi_{in} = \sum_{n=1}^{N-1} (p_{n+1} - p_n)x_{in} + (v - p_N)x_{iN}$. In any period n an informed trader i receives a signal $s_{in} = v + \epsilon_{in}$, where $\epsilon_{in} \sim N(0, \tau_{\epsilon_n}^{-1})$, v and ϵ_{in} are independent for all i, n and error terms are also independent both across time periods and traders.

In the general case noise trading follows an AR(1) process $\theta_n = \beta \theta_{n-1} + u_n$ with $\beta \in [0, 1]$ and $\{u_n\}_{n=1}^{N}$ an i.i.d. normally distributed random process (independent of

---

[10]In a sense it is as if the price relied too little (much) on aggregate private (public) information with respect to the average private expectation.

[11]A number of authors have studied competitive, noisy rational expectations equilibria in dynamic markets, see e.g. Brown and Jennings (1989), Grundy and McNichols (1989), He and Wang (1995), Vives (1995), and Cespa (2002).

all other random variables in the model) with $u_n \sim N(0, \tau_u^{-1})$. If $\beta = 1$, $\{\theta_n\}$ follows a random walk and we are in the usual case of independent noise trade increments $u_n = \theta_n - \theta_{n-1}$ (e.g. Kyle (1985), Brown and Jennings (1989), Vives (1995)). If $\beta = 0$, then noise trading is i.i.d. across periods (this is the case considered by Allen et al. (2006)), a plausible assumption only if the time between trading dates is very large. Finally, assume that, given v, the average signal $\int_0^1 s_{in} di$ equals v almost surely in every period n (i.e. errors cancel out in the aggregate: $\int_0^1 \epsilon_{in} di = 0$).

3.1 No Bubbles when Noise Trading follows a Random Walk

Let us first consider the classical case when noise trade increments u_n are independent across periods ($\beta = 1$). In any period $1 < n \leq N$ each informed trader has the vector of private signals $s_i^n = \{s_{i1}, s_{i2}, \ldots, s_{in}\}$ available. It follows from normal theory that the statistic $\tilde{s}_{in} = (\sum_{t=1}^n \tau_{\epsilon_t})^{-1}(\sum_{t=1}^n \tau_{\epsilon_t} s_{it})$ is sufficient for the sequence s_i^n in the estimation of v. An informed trader i in period n submits a limit order $X_{in}(\tilde{s}_{in}, p^{n-1}, \cdot)$, indicating the position desired at every price p_n, contingent on his available information. We will restrict attention to linear equilibria where in every period n a speculator trades according to $X_n(\tilde{s}_{in}, p^n) = a_n \tilde{s}_{in} - \varphi_n(p^n)$, where $\varphi_n(\cdot)$ is a linear function of the price sequence $p^n = \{p_1, \ldots, p_n\}$. [12] Let us denote with z_n the intercept of the n-th period net aggregate demand $\int_0^1 \Delta x_{in} di + u_n$, where $\Delta x_{in} = x_{in} - x_{in-1}$. The random variable $z_n \equiv \Delta a_n v + u_n$ represents the informational addition brought about by the n-th period trading round, and can thus be interpreted as the informational content of the n-th period order-flow.

Lemma 1. *In any linear equilibrium the sequences z^n and p^n are observationally equivalent.*

According to the above lemma in any period n traders form their estimation of the liquidation value using their private information, summarized by the statistic $\tilde{s}_{in}$, and the sequence of public informational additions z^n. Thus, owing to normality, we have $E[v|\tilde{s}_{in}, z^n] = (\tau_n + \sum_{t=1}^n \tau_{\epsilon_t})^{-1}(\tau_n E[v|z^n] + \sum_{t=1}^n \tau_{\epsilon_t} \tilde{s}_{in})$, where $E[v|z^n] = \tau_n^{-1}(\tau_v \bar{v} + \tau_u \sum_{t=1}^n \Delta a_t z_t)$, and $\tau_n \equiv \text{Var}[v|z^n]^{-1} = \tau_v + \tau_u \sum_{t=1}^n \Delta a_t^2$.

Proposition 1. *In the market with long-term, informed speculators there exists a unique equilibrium in linear strategies. The equilibrium is symmetric. Prices are*

---

[12]The unique equilibrium in linear strategies will be shown to be symmetric.

given by $p_o = \bar{v}$, $p_{N+1} = v$, and for $n = 1, 2, \ldots, N$, $p_n = \lambda_n z_n + (1 - \lambda_n \Delta a_n) p_{n-1}$ and strategies are given by:

$$
\begin{aligned}
X_n(\tilde{s}_{in}, p^n) &= \gamma \left(\tau_n + \sum_{t=1}^{n} \tau_{\epsilon_t} \right) (E[v|\tilde{s}_{in}, z^n] - p_n) \qquad (5) \\
&= a_n(\tilde{s}_{in} - p_n) + \gamma \tau_n (E[v|z^n] - p_n),
\end{aligned}
$$

where $a_n = \gamma(\sum_{t=1}^{n} \tau_{\epsilon_t})$, $\Delta a_n \equiv a_n - a_{n-1} = \gamma \tau_{\epsilon_n}$, $z_n = \Delta a_n v + u_n$, $z^n = \{z_t\}_{t=1}^{n}$, $\tau_n = (Var[v|z^n])^{-1}$, and $\lambda_n = (1 + \gamma \tau_u \Delta a_n)/(\gamma(\tau_n + \sum_{t=1}^{n} \tau_{\epsilon_t}))$.

Proof. See appendix A. QED

The equilibrium has a static nature: in every period n speculators trade "as if" the asset would be liquidated in the following period $n+1$, exploiting all their available information. [13] The intuition is as follows: owing to normality and CARA preferences a trader's strategy is made of two components:

$$
x_{in} = \gamma \frac{E[p_{n+1} - p_n | \tilde{s}_{in}, z^n]}{h_{22}} - \frac{h_{21} E[x_{in+1} | \tilde{s}_{in}, z^n]}{h_{22} \gamma (\tau_{n+1} + \sum_{t=1}^{n+1} \tau_{\epsilon_t})}, \qquad 1 \le n < N, \qquad (6)
$$

where $h_{21} < 0$ and $h_{22} > 0$ are constants defined in the appendix. The first component in (6) accounts for the forecasted price change, while the second component captures the anticipated future position. Were traders not to expect a change in prices, then their optimal period n position would be like the one of a static market, and the risk of holding such a position would only be due to the unpredictability of the liquidation value (taking into account risk-aversion): $(1/\gamma)Var[v|\tilde{s}_{in}, z^n]$. [14] If, on the other hand a change in prices is expected, traders optimally exploit short-run price differences. This, in turn, adds two factors to the risk of their period n position, as traders also suffer from the partial unpredictability of the price change, *and* from the impossibility of determining exactly their future position. However, the opportunity to trade again in the future also grants a hedge against potentially adverse price movements ($h_{21} < 0$). This, in equilibrium, yields a risk-reduction that exactly offsets the risk increase outlined above. As a consequence, even in this case traders' strategies have a static nature, and $x_{in} = E[x_{in+1} | \tilde{s}_{in}, z^n]$.

---

[13]Proposition 1 generalizes the equilibrium in Cespa (2002) to the case of $N \ge 2$ periods.

[14]Intuitively, if given today's information the asset price is not expected to change, no new private information is expected to arrive to the market and the model collapses to one in which traders hold for two periods the risky asset. Their position, then, naturally coincides with the one they would hold in a static market.

As one would expect, under the conditions of proposition 1, it is possible to show that the price is as close to the liquidation value as informed traders' average expectation *in every* period $n = 1, 2, \ldots, N$. Indeed, market clearing at any period n is given by

$$\int_0^1 x_{in} di + \theta_n = 0.$$

When $\beta = 1$ we have that $\theta_n = \sum_{t=1}^n u_t$ and according to proposition 1, in this case in any period n long-term traders' aggregate demand is proportional to $\int_0^1 E[v|\tilde{s}_{in}, z^n] di - p_n$. Hence, at equilibrium we obtain:

$$\int_0^1 \gamma \left(\tau_n + \sum_{t=1}^n \tau_{\epsilon t} \right) (E[v|\tilde{s}_{in}, z^n] - p_n) di + \theta_n = 0. \tag{7}$$

Using this condition, and denoting with $\bar{E}_n \equiv \int_0^1 E[v|\tilde{s}_{in}, z^n] di$ the market consensus estimation of the liquidation value at time n, we can now express the price as the sum of traders' average expectations and a risk noise adjustment:

$$p_n = \bar{E}_n[v] + \frac{\theta_n}{\gamma(\tau_n + \sum_{t=1}^n \tau_{\epsilon t})}. \tag{8}$$

As θ_n and v are by assumption orthogonal, for all $t = 1, 2, \ldots, n$, $E[p_n|v] = E[\bar{E}_n[v]|v]$, and we thus obtain the following:

Corollary 1. *In any period* $1 \leq n \leq N$, $E[p_n - v|v] = E[\bar{E}_n[v] - v|v]$.

In words: when the market is populated by long-term traders and noise traders' supply shocks follow a random walk, informed traders' demand has a *static* nature, and there is *no* bubble.

Remark 2. A different way to interpret the above corollary is the following. Owing to normality, we can express informed traders' average expectation as a linear combination of the liquidation value v and the public expectation $E[v|z^n]$:

$$\begin{aligned} \bar{E}_n[v] &= \int_0^1 E[v|\tilde{s}_{in}, z^n] di \\ &= \alpha_{En} v + (1 - \alpha_{En}) E[v|z^n], \end{aligned} \tag{9}$$

where $\alpha_{En} \equiv \sum_{t=1}^n \tau_{\epsilon t}/(\tau_n + \sum_{t=1}^n \tau_{\epsilon t})$, for $n = 1, 2, \ldots, N$. A similar decomposition can also be applied to the equilibrium price for a given responsiveness to private information a_n:

$$p_n = \alpha_{Pn} \left(v + \frac{1}{a_n} \theta_n \right) + (1 - \alpha_{Pn}) E[v|z^n] \tag{10}$$

11

where $\alpha_{Pn} = a_n/(\gamma(\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}))$. Comparing (9) and (10), in any period n the equilibrium price is therefore farther away from the fundamentals than informed traders' average expectation if and only if

$$\alpha_{Pn} < \alpha_{En}. \tag{11}$$

However, in equilibrium we have that $\alpha_{Pn} = \sum_{t=1}^n \tau_{\epsilon_t}/(\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}) = \alpha_{En}$ and no bubble occurs.

Remark 3. According to (10), in any period n the equilibrium price differs from the conditional expectation of the liquidation value given public information:

$$p_n - E[v|z^n] = \frac{\alpha_{Pn}}{a_n} E[\theta_n|z^n].$$

The intuition is as follows: suppose, without loss of generality, that in period n $E[\theta_n|z^n] > 0$, then if $p_n = E[v|z^n]$, traders would be exposed to the risk of holding an underpriced short position.[15] In order to accept holding such a position, (i.e. to offset such a risk) risk averse traders thus demand a compensation that is proportional to the expected total noisy noise trade in period n. Such a compensation is larger, the lower is their risk-tolerance and the larger is their uncertainty over the liquidation value: $\alpha_{Pn}/a_n = 1/(\gamma(\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}))$. [16]

Remark 4. Vives (1995) studies a market with the same informational structure of the one analyzed above with $\beta = 1$, but where the asset is priced by a competitive sector of risk-neutral market makers. In this case price setting agents require no premium to hold a risky position (beyond what is due to the severity of the adverse selection problem they face), thus $\alpha_{Pn} = 0$ and prices are *always* "farther away" from fundamentals than the average expectation among informed traders. The discrepancy between prices and average expectation is larger, the larger is the ratio between private and public precision. This is immediate since in this case the unique equilibrium in linear strategies shares the same informational properties of the one in proposition 1, while the period n price is given by $p_n = E[v|z^n]$. Hence

$$E[p_n - v|v] = \tau_n^{-1}\tau_v(\bar{v} - v) \text{ and } E\left[\bar{E}_n[v] - v|v\right] = \frac{\tau_v(\bar{v} - v)}{\tau_n + \sum_{t=1}^n \tau_{\epsilon_t}},$$

---

[15]In other words, they would be exposed to the risk of selling the asset for too low a price with respect to its actual liquidation value.

[16]The existence of a premium that affects the equilibrium price in models where risk averse traders price the asset is well-known in the literature (see e.g. Bhattacharya and Spiegel (1991)).

and the discrepancy increases in the ratio $(\tau_n + \sum_{t=1}^{n} \tau_{\epsilon_t})/\tau_n \equiv \text{Var}[v|\tilde{s}_{in}, z^n]^{-1}/\text{Var}[v|z^n]^{-1}$. [17]
Redefining the benchmark to which prices are compared in terms of the risk-weighted average expectation among *all* the traders in the market, we can show that the equivalence result of corollary 1 is restored. Indeed, suppose that together with the informed traders in the market there also exists a continuum of risk averse (CARA) uninformed traders. In every period n, the uninformed traders' estimation of the liquidation value is given by $E[v|z^n]$, and the risk-weighted average expectation among all traders in the market reads as follows:

$$\bar{E}_n[v] = \frac{\int_0^1 \gamma(\tau_n + \sum_{t=1}^{n} \tau_{\epsilon_t}) E[v|\tilde{s}_{in}, z^n] di + \int_0^1 \gamma^U \tau_n E[v|z^n] dj}{\gamma(\tau_n + \sum_{t=1}^{n} \tau_{\epsilon_t}) + \gamma^U \tau_n},$$

where γ^U denotes the uninformed traders' risk-tolerance coefficient. Letting $\gamma^U \to \infty$ we then obtain that when dealers are risk neutral $\bar{E}_n[v]$ converges to the equilibrium price $p_n = E[v|z^n]$, and in the limit $E[p_n - v|v] = E[\bar{E}_n[v] - v|v]$.

3.2 Residual Uncertainty and Correlated noise trade Increments: Keynes vs. Hayek

In this section we suppose *(i)* that the fundamentals is affected by the presence of residual uncertainty, and *(ii)* that noise trade shocks are serially correlated. Both the presence of residual uncertainty over the liquidation value and noise traders' predictability induce traders towards speculating on short-run price movements. This drives a wedge between the equilibrium price and average expectations about the fundamentals which, depending on parameters values, leads to the existence of bubbles and reverse bubbles.

Assume thus that the liquidation value traders receive at the terminal date is given by $v + \delta$, where $\delta \sim N(0, \tau_\delta^{-1})$ is a noise term independent from all the other random variables in the economy and about which *no* trader possesses private information, and that $\beta \in [0, 1]$. The latter assumption implies that noise trade increments are *negatively* correlated, i.e. that a positive (negative) realization of θ_1 is likely to be

---

[17]As an alternative intuition, consider the following argument: when risk-neutral dealers enter the market, they "crowd out" risk averse traders from the market making business (see Cespa (2005)). This generates two effects: first, it renders the price equal to the public expectation, eliminating the risk premium. However, given v the public expectation is a worst predictor of the liquidation value than the average private expectation. Hence, as a second effect the presence of uninformed risk neutral dealers moves the price away from the fundamentals.

followed by a negative (positive) realization of $\Delta\theta_2$.[18] Other things equal, this implies that traders find it less risky to accommodate order imbalances in the first period.

Defining the net informational additions $z_1 = a_1 v + u_1$, and $z_2 = \Delta a_2 v + u_2$ ($\Delta a_2 = a_2 - \beta a_1$), in a two-period model we can prove the following result:

Proposition 2. *In the 2-period market with long-term, informed speculators, residual uncertainty over the liquidation value, and correlated noise trade increments there exists an equilibrium in linear strategies where: (1) prices are given by $p_3 = v$, $p_0 = \bar{v}$, $p_2 = \lambda_2 z_2 + (1 - \lambda_2 \Delta a_2)\hat{p}_1$, and $p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1)\bar{v}$; (2) strategies are given by:*

$$X_2\left(\tilde{s}_{i2}, p^2\right) = a_2\left(\tilde{s}_{i2} - p_2\right) + b_2\left(E[v|z^2] - p_2\right), \tag{12}$$

$$X_1(s_{i1}, p_1) = \frac{a_1(\tau_1 + \tau_{\epsilon_1})}{\tau_{\epsilon_1}}\left(E[v|s_{i1}, z_1] - p_1\right) + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1}{\gamma\left(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_2}\right)}E[\theta_1|z_1],$$

where $b_2 = \gamma\tau_2/(1 + \kappa)$, $\lambda_2 = (\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))^{-1}(1 + \gamma\tau_u\Delta a_2 + \kappa)$, $\hat{p}_1 = (\gamma\tau_1 + \beta a_1(1 + \kappa))^{-1}(\gamma\tau_1 E[v|z_1] + \beta(1 + \kappa)z_1)$, $\kappa \equiv (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})/\tau_\delta$, $\Delta a_2 = a_2 - \beta a_1$, $\rho \equiv a_1(1 + \kappa)/(\gamma\tau_{\epsilon_1})$,

$$a_1 = \frac{\gamma\tau_{\epsilon_1}(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})(1 + \kappa)(1 + \gamma\tau_u\Delta a_2)}{(\tau_{\epsilon_2} + (\tau_2 + \tau_{\epsilon_1})(1 + \kappa))(1 + \kappa + \gamma\tau_u\Delta a_2)}, \qquad a_2 = \frac{\gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})}{1 + \kappa}, \tag{13}$$

and

$$\lambda_1 = \alpha_{P1}\frac{1}{a_1} + (1 - \alpha_{P1})\frac{a_1\tau_u}{\tau_1}, \qquad \alpha_{P1} = \frac{\tau_{\epsilon_1}}{\tau_1 + \tau_{\epsilon_1}} + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})},$$

h_{21} (with $\gamma + h_{21} > 0$), $h_{22} > 0$ are constants defined in the appendix. Furthermore, in any linear equilibrium $\rho > 1$.

Proof. See appendix A. QED

Owing to residual uncertainty, a closed form solution for traders' signal responsiveness is no longer available: the expressions for a_1 and a_2 in (13) constitute a system of non-linear equations whose (potentially multiple) solution(s) must be numerically determined. According to (12) in the second period traders' strategies keep the static property. As a consequence, no bubble occurs.[19] However, owing to the higher uncertainty affecting the liquidation value, traders scale down their aggressiveness:

---

[18]To be sure: as $\theta_2 = u_2 + \beta\theta_1$, $\text{Cov}[\Delta\theta_2, \theta_1] = (\beta - 1)\tau_u^{-1} \leq 0$, since $\beta \in [0, 1]$.

[19]Using the expressions provided in the appendix it is immediate to see that in the second period no bubble can arise.

$a_2 < \gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})$. In the first period, on the other hand, strategies loose the static property and each investor's position is no longer proportional to $(E[v|s_{i1}, z_1] - p_1)$.

Indeed, according to the second of (12), the first period strategy of a trader can be decomposed in two parts. The first part captures the "static" component, while the second part captures a trader's speculative activity on short-run price movements, a term whose sign depends on the magnitude of $\beta\rho$. As argued above, for a given expected noise trade shock low values of β encourage informed traders to take the other side of the market counting on the noise trade increment future reversion. The parameter ρ, on the other hand, measures the deviation that residual uncertainty induces in traders' first period signal aggressiveness with respect to the static aggressiveness (note that ρ depends on β). [20] In any equilibrium of the market it turns out that $\rho > 1$, implying that when residual uncertainty affects the liquidation value, in the first period traders speculate more aggressively on their signal. The intuition is as follows: using the expressions provided in the appendix, we can rewrite a trader's first period signal responsiveness in the following way:

$$a_1 = \tag{14}$$
$$\frac{\gamma\tau_{\epsilon_1}}{1+\kappa}\left(\frac{\mathrm{Var}_2[v+\delta]}{\mathrm{Var}_2[v]}\frac{\hat{\lambda}_2}{\lambda_2}\frac{\gamma^2(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}{(\tau_2 + \tau_{\epsilon_1})(\gamma^2 + \mathrm{Var}_2[v+\delta]\mathrm{Var}_1[x_{i2}](1 - \rho_{1,\{x_{i2},p_2\}}^2))}\right),$$

where $\mathrm{Var}_n[Y] = \mathrm{Var}[Y|\tilde{s}_{in}, z^n]$, $\hat{\lambda}_2 = (\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))^{-1}(1 + \gamma\tau_u\Delta a_2)$ denotes the reciprocal of second period market depth in the absence of residual uncertainty, and

$$\rho_{1,\{x_{i2},p_2\}} = \frac{\mathrm{Cov}[x_{i2}, p_2|s_{i1}, z_1]}{\sqrt{\mathrm{Var}_1[x_{i2}]\mathrm{Var}_1[p_2]}}.$$

The presence of residual uncertainty over the liquidation value produces three effects on a trader's signal responsiveness. First, as $\mathrm{Var}_2[v]$ increases to $\mathrm{Var}_2[v+\delta]$, the first period signal becomes a more valuable source of information, and traders put more weight on it. However, at the same time since the liquidation value is more uncertain, traders in the second period speculate less aggressively, and p_2 becomes more reactive to the upcoming net informational addition z_2 ($\hat{\lambda}_2$ increases to λ_2). As traders in the first period also speculate on short-run price differences, this drives down a_1. Finally,

---

[20] As in the second period, when short-run price differences cannot be exploited, a trader weights his private information according to $a_2 = (1+\kappa)^{-1}\gamma(\sum_{t=1}^2 \tau_{\epsilon_t})$, in the first period a trader that intends to "buy-and-hold" should trade according to $(1+\kappa)^{-1}\gamma\tau_{\epsilon_1}$. This intuition can be proved formally by using the expressions for a trader's first period strategy provided in the appendix. Indeed, assuming $\beta = 1$, plugging $a_1 = \gamma\tau_{\epsilon_1}/(1+\kappa)$ in (38) and imposing market clearing, we obtain $\hat{p}_1 = p_1$, and $x_{i1} = E[x_{i2}|s_{i1}, p_1]$ as in the case with no residual uncertainty.

as traders scale down their second period signal aggressiveness, their second period strategy becomes less responsive to private information, and thus more correlated with the second period price. [21] From the point of view of a trader that in the first period speculates on price differences, this makes x_{i2} a better hedge against adverse price movements, and has a positive impact on a_1. The sum of the positive effects more than offset the negative one and $\rho > 1$. [22]

As first period signal responsiveness exceeds its static value, accommodating the expected noise trade shock may expose traders to the risk of clearing information-related transactions at an incorrect price. To see this consider the following example. Suppose that conditional on the observation of z_1, the market expects $\theta_1 > 0$, i.e.

$$E[\theta_1|z_1] = \theta_1 + a_1(v - E[v|z_1]) > 0.$$

According to the above expression, $E[\theta_1|z_1] > 0$ could be due either to a truly positive liquidity shock ($\theta_1 > 0$) or to a liquidation value higher than its market expectation ($v > E[v|z_1]$). In the former case, traders should accommodate the shock, counting on a reversion of the noise trade increment that lowers the risk of holding an unbalanced position in the asset in the first period. However, the higher is β, the lower are the chances that noise trade increments compensate over time and the higher is the possibility that traders sell the asset for too low a price (as $\rho > 1$). Hence, as $\beta\rho$ increases above 1, traders speculate on a further price increase in the second period, increasing their long position in the asset. [23]

Summarizing, the presence of residual uncertainty and correlated noise trade increments leads traders to speculate on short-run price differences. This, in turn,

---

[21]Suppose a trader expects $p_2 > p_1$. Then, he increases his first period long position, planning to sell in the second period, and net the short-run profits. However, if $p_2 < p_1$, second period sales are suboptimal, and a good hedge calls for holding on to the long position. This, however, is less likely to occur if traders are very reactive to their second period information. In this case, indeed, the second period strategy may unravel the hedge built in the first period, in turn reducing traders' willingness to speculate on their first period information.

[22]Higher uncertainty over the liquidation value, thus works as a commitment device for a trader's second period strategy. If in the first period the trader could be sure that his second period strategy perfectly hedged adverse price movements, he would more confidently speculate on price differences. However, traders cannot control the information they receive in the second period, and this makes x_{i2} a less reliable hedge for a short-run, speculative strategy. In this perspective higher uncertainty over the liquidation value ties a trader's hands in the second period: the trader speculates less aggressively on private information and he can also be more confident that x_{i2} will better serve his first period hedging needs.

[23]These two types of behavior are reminiscent of the "contrarian" vs. "momentum" type of strategies. Traders' behavior when $\beta\rho > 1$ is also akin to the "resale option" strategy discussed by Cao and Ou-Yang (2005).

makes traders' first period strategies depart from the static form of the previous section, implying that the equilibrium price in the first period can no longer be expressed as the sum of traders' average expectations and a risk-weighted noise component:

$$
\begin{aligned}
p_1 &= \alpha_{P1}\left(v + \frac{\theta_1}{a_1}\right) + (1 - \alpha_{P1})E[v|z_1] \\
&= \bar{E}_1[v] + \left(\frac{\alpha_{P1} - \alpha_{E1}}{a_1}\right)E[\theta_1|z_1] + \alpha_{E1}\frac{\theta_1}{a_1} \\
&= \bar{E}_1[v] + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{a_1\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t})}E[\theta_1|z_1] + \alpha_{E1}\frac{\theta_1}{a_1},
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
\alpha_{P1} &\equiv \frac{\tau_{\epsilon_1}}{\tau_1 + \tau_{\epsilon_1}} + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t})} \\
&= \alpha_{E1} + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t})}.
\end{aligned}
\tag{16}
$$

As a consequence bubbles may arise. Indeed, according to (16) the weight the price assigns to the fundamentals is the sum of the optimal statistical weight that the average expectation attributes to aggregate private information (i.e. α_{E1}), and a term that is proportional to traders' short-run speculative trading intensity. When traders deem their estimation of the noise trade shock to be biased by the presence of information driven trades (i.e. when $\beta\rho > 1$), they "side" with the market (e.g. if $E[\theta_1|z_1] > 0$, speculating on a further price increase). This reinforces the optimal statistical weight α_{E1} with the short-run speculative trading intensity, moving the price *closer* to the fundamentals. When, on the other hand, traders count on a reversion of the second period noise trade increment (i.e. when $\beta\rho < 1$), they take the other side of the market (e.g. if $E[\theta_1|z_1] > 0$, they absorb the order imbalance), in this way partially *sterilizing* the impact of informed trades on the price. This reduces the optimal statistical weight α_{E1} and widens the distance between p_1 and v. [24] A direct implication of the latter of (15) is thus:

Corollary 2. *In any linear equilibrium of the market with long-term, informed speculators when the liquidation value is affected by residual noise a bubble arises if and only if $\beta\rho < 1$.*

---

[24]To be sure, when $\beta\rho < 1$ traders on aggregate hedge their informationally driven *speculative* activity with their *market making* activity. This dampens the effect that the former has on the equilibrium price.

Corollary 2 has a straightforward graphical interpretation. To see this suppose that $E[E[\theta_1|z_1]|v] > 0$ $(E[E[\theta_1|z_1]|v] < 0)$, then if $\beta\rho < 1$ we have that $v > E[\bar{E}_1[v]|v] > E[p_1|v]$ $(v < E[\bar{E}_1[v]|v] < E[p_1|v])$. Conversely, if $\beta\rho > 1$ and $E[E[\theta_1|z_1]|v] > 0$ $(E[E[\theta_1|z_1]|v] < 0)$, we have that $v > E[p_1|v] > E[\bar{E}_1[v]|v]$ $(v < E[p_1|v] < E[\bar{E}_1[v]|v])$, the price is more firmly tightened to the fundamentals, and a "reverse" bubble occurs (see figure 1).

[Figure 1 about here.]

As it happens, when $\beta = 1$, only the latter condition can arise. Conversely, if $\beta = 0$ only positive bubbles occur:

Corollary 3. *In any linear equilibrium of the market with long-term, informed speculators when the liquidation value is affected by residual noise, there are bubbles if $\beta = 0$ and reverse bubbles if $\beta = 1$.*

Proof. When $\beta = 1$, the condition for bubble existence is that $\rho < 1$, which is impossible given proposition 2. When, on the other hand $\beta = 0$, according to (15) only bubbles can occur.   QED

Since $\rho > 1$, when noise traders' demand follows a random walk, traders speculate on price momentum anchoring more firmly the first period price to the fundamentals, and inducing a reverse bubble. When, on the other hand, noise trade increments display a strong mean reverting property, betting on momentum becomes extremely risky. Hence, traders always accommodate the shock, widening the gap between the price and the fundamentals.

On the other hand if *no* residual uncertainty affects the liquidation value reverse bubbles *never* arise.

Corollary 4. *In any linear equilibrium of the market with long-term, informed speculators when the liquidation value is not affected by residual noise (i.e. $1/\tau_\delta = 0$) there is always a bubble for any $\beta \in [0, 1)$.*

Proof. If $1/\tau_\delta = 0$, $\rho = 1$, and

$$X_1(s_{i1}, p_1) = \gamma(\tau_1 + \tau_{\epsilon_1})\left(E[v|s_{i1}, z_1] - p_1\right) + \frac{(\gamma + h_{21})(\beta - 1)\tau_1}{\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}E[\theta_1|z_1]$$

$$p_1 = \bar{E}_1[v] + \frac{(\gamma + h_{21})(\beta - 1)\tau_1\tau_{\epsilon_1}}{a_1\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}E[\theta_1|z_1] + \alpha_{E1}\frac{\theta_1}{a_1}.$$

18

and the condition for bubble existence is always satisfied for all $\beta \in [0, 1)$.   QED

Owing to the noise trade increment mean reverting property, traders in the first period absorb the expected liquidity shock at a *discount*. This loosens the grip between the price and the fundamentals, and a bubble arises. As β approaches 1, traders can decreasingly count on a noise trade increment reversal, and reduce their short-run speculative position. When $\beta = 1$, the noise process follows a random walk and the result of corollary 1 is restored.

Remark 5. Numerical simulations show that for any $1/\tau_\delta > 0$, there always exists a value of β, $\hat{\beta}$ below (above) which only positive (reverse) bubbles occur. This allows to define $\Omega \equiv \{(\beta, 1/\tau_\delta) \in [0, 1] \times \Re_+ | \beta\rho(\beta) = 1\}$ as the set of pairs $(\beta, 1/\tau_\delta)$ for which no bubble occurs. Values of $(\beta, 1/\tau_\delta)$ that fall below (above) this locus identify market conditions that yield a bubble or a reverse bubble (see figure 2).

[Figure 2 about here.]

The figure divides the parameter space $(\beta, 1/\tau_\delta)$ into a Keynesian region (below the locus) with bubbles and a Hayekian region (the rest) without bubbles or with reverse bubbles. Less residual uncertainty and less correlation in noise trading move us towards the Keynesian region. It is interesting to observe also that the Keynesian region gets larger as risk tolerance γ decreases.

Remark 6. Given the discussion following proposition 2 we should expect the size of the bubble and the depth of the first period market to be related. Indeed if τ_δ grows unboundedly it is easy to show that $\lambda_1|_{\beta=0} < \lambda_1|_{\beta=1}$. To see this, suppose $1/\tau_\delta = 0$, then $\rho = 1$, and

$$\alpha_{P1} = \frac{\tau_{\epsilon_1}}{\tau_1 + \tau_{\epsilon_1}} - (1 - \beta)\frac{(\gamma + h_{21})\tau_1\tau_{\epsilon_1}}{\gamma(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t})}.$$

Thus, $\alpha_{P1}(\beta)|_{\beta=0} < \alpha_{P1}(\beta)|_{\beta=1}$, and since $\lambda_1 = a_1\tau_u/\tau_1 + \alpha_{P1}\tau_v/(a_1\tau_1)$, the result follows.

When $1/\tau_\delta = 0$, traders' responsiveness to private information does not depend on β. As a consequence the first period depth reaction to a change in β is *completely* determined by the effect that a change in the correlation across noise trade increments has on the size of the bubble. A wider discrepancy between the price and the fundamentals (a smaller α_{P1}) proxies for a stronger sterilization of informed trades

due to traders' intense market making activity. This dampens the price impact of trades, making the first period market deeper.

When $\tau_\delta < \infty$ numerical simulations confirm that the first period depth is larger (smaller) when $\beta = 0$ ($\beta = 1$). Furthermore, as β increases, both α_{P1} and λ_1 grow larger, implying that a *larger bubble occurs in a deeper market*.

Remark 7. According to Bacchetta and van Wincoop (2006b) in the presence of differential information, higher order expectations add an additional term (the "higher order wedge") to the traditional asset pricing equation. This term emerges as the difference between the equilibrium price and the asset price that obtains substituting higher order expectations with first order expectations in the pricing equation. We can apply this decomposition to (15) using the fact that $\bar{E}_1[\bar{E}_2[v]] = \bar{E}_1[v] + ((\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t}))^{-1} \tau_{\epsilon_1} \tau_1 (E[v|z_1] - v) = \bar{E}_1[v] - (a_1(\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t}))^{-1} \tau_1 \alpha_{E1}(E[\theta_1|z_1] + \theta_1)$. Denoting with p_1^* the asset price obtained susbstituting higher order expectations with first order expectations, the higher order wedge is given by

$$p_1 - p_1^* \equiv \Delta_1 = \bar{E}_1[\bar{E}_2[v] - v] = -\frac{\tau_1 \alpha_{E1}}{a_1(\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t})} \left(E[\theta_1|z_1] + \theta_1\right),$$

and captures the expectational error about the liquidation value due to the presence of noise in the public signal (z_1). Using this definition, the equilibrium price can be written as

$$p_1 = \bar{E}_1[v] - \frac{(\gamma + h_{21})(\beta\rho - 1)}{\gamma} \Delta_1 + \left(1 + \frac{(\gamma + h_{21})(\beta\rho - 1)\tau_1}{\gamma(\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t})}\right) \frac{\alpha_{E1}}{a_1} \theta_1.$$

According to section 3.1 when $\beta = 1$, as soon as residual uncertainty vanishes, $\rho \to 1$ and the higher order wedge ceases to affect the pricing equation.

4   The Market with Short-Term Traders

In the previous section we have argued that a stronger focus on short-run price differences allows bubbles to arise in the CARA-normal model with long-term traders. In this section we thus turn our attention to the analysis of a dynamic model in which traders are *exclusively* concentrated on short-run profit opportunities and $\beta \in [0, 1]$. As it happens, in the absence of residual uncertainty over the liquidation value, lack of a second period hedge has a dramatic effect on traders' first period strategies: as

long as $\beta > 0$ multiple equilibria arise. In one equilibrium there is a bubble and in another a reverse bubble. The bubble equilibrium is the stable one. When $\beta = 0$ or when $\beta = 1$ *and* considerable residual uncertainty affects the liquidation value, a bubble always arises in the first period price, restoring the result of Allen et al. (2006).

Suppose thus that in the dynamic market analyzed in the previous section traders have short-term horizons (i.e. they take a position in period n and unwind it in period $n+1$) and that the private information each trader i receives in every period n is transmitted to the corresponding trader in period $n+1$. [25] Traders may have a short horizon for incentive reasons, for example. Thus, in every period n every trader i maximizes the expected utility of his short-term profits $\pi_{in} = (p_{n+1} - p_n)x_{in}$, $E[U(\pi_{in})|\tilde{s}_{in}, p^n] = -E[\exp\{-\pi_{in}/\gamma\}|\tilde{s}_{in}, p^n]$.

4.1 A General Formula

Before turning attention to the linear equilibria of the market, it is worth analyzing the general pricing formula. Assume that $N = 2$. Owing to CARA preferences and normality, in the second period a trader's optimal position is given by $X_2(\tilde{s}_{i2}, p^2) = \gamma(\text{Var}[v|\tilde{s}_{i2}, z^2])^{-1}(E[v|\tilde{s}_{i2}, z^2] - p_2)$, while the corresponding market clearing equation reads as follows: $\int_0^1 x_{i2}di + \theta_2 = 0$. Let $\text{Var}_n[Y] = \text{Var}[Y|\tilde{s}_{in}, z^n]$, and $\bar{E}_n[Y] = \int_0^1 E[Y|\tilde{s}_{in}, z^n]di$, then the second period equilibrium price is given by

$$p_2 = \bar{E}_2[v] + \frac{\theta_2}{\gamma}\text{Var}_2[v]. \tag{17}$$

Owing to short-term trading horizons, in the first period a trader's optimal position is given by $X_1(s_{i1}, p_1) = \gamma(\text{Var}[p_2|s_{i1}, p_1])^{-1}(E[p_2|s_{i1}, p_1] - p_1)$, and the corresponding market clearing equation reads as follows: $\int_0^1 x_{i1}di + \theta_1 = 0$. Solving for the first period equilibrium price yields

$$p_1 = \bar{E}_1[p_2] + \frac{\theta_1}{\gamma}\text{Var}_1[p_2].$$

Substituting (17) in the latter equation and using the fact that $\theta_2 = \beta\theta_1 + u_2$ with u_2 serially uncorrelated, yields

$$p_1 = \bar{E}_1\left[\bar{E}_2[v] + \frac{\beta\theta_1}{\gamma}\text{Var}_2[v]\right] + \frac{\theta_1}{\gamma}\text{Var}_1[p_2]. \tag{18}$$

---

[25]This is without loss of generality. The same qualitative results would arise if traders in period n did not inherit the information held by their previous period peers.

According to (17) and (18), in any period the price of the asset depends on two components: the market average expectation of the market average expected liquidation value and the risk associated with holding a position in the asset (due to the presence of noise traders). [26]

Extending this line of reasoning to the N-period case we obtain

$$p_n =$$

$$\bar{E}_n \left[\bar{E}_{n+1} \left[\cdots \bar{E}_{N-1} \left[\bar{E}_N[v] + \frac{\mathrm{Var}_N[v]}{\gamma} \beta^{N-n} \theta_n \right] + \frac{\mathrm{Var}_{N-1}[p_N]}{\gamma} \beta^{N-(n+1)} \theta_n \cdots \right] + \frac{\mathrm{Var}_{n+1}[p_{n+2}]}{\gamma} \beta \theta_n \right]$$

$$+ \frac{\mathrm{Var}_n[p_{n+1}]}{\gamma} \theta_n, \tag{19}$$

for $1 \leq n \leq N$. [27] Recall that when $\beta = 1$ we have that $\theta_n = \sum_{t=1}^{n} u_t$. The above pricing formula coincides with Allen et al. (2006) when $\beta = 0$. Indeed, in their framework additional noise shocks do not cumulate over the trading periods. Hence, the price in period N is simply the average expectation of investors about the fundamental value plus the period N risk-adjusted noise shock:

$$p_N = \bar{E}_N[v] + \frac{\theta_N}{\gamma} \mathrm{Var}_N[v].$$

In period $N - 1$ the price is then given by the average expectation of investors about the price in period N (because investors have a one period horizon) plus the corresponding period, risk-adjusted noise shock:

$$p_{N-1} = \bar{E}_{N-1}[p_N] + \frac{\theta_{N-1}}{\gamma} \mathrm{Var}_{N-1}[p_N].$$

When $\beta = 0$, $\bar{E}_{N-1}[\theta_N] = 0$. Hence, we obtain

$$p_{N-1} = \bar{E}_{N-1} \left[\bar{E}_N[v] \right] + \frac{\theta_{N-1}}{\gamma} \mathrm{Var}_{N-1}[p_N].$$

This recursive relationship can be iterated backwards to obtain that in any period n

$$p_n = \bar{E}_n \left[\bar{E}_{n+1} \left[\cdots \bar{E}_{N-1} \left[\bar{E}_N[v] \right] \cdots \right] \right] + \frac{\theta_n}{\gamma} \mathrm{Var}_n[p_{n+1}].$$

Thus, p_n is the average expectation at n of the average expectation at $n + 1$ of the average expectation at $n + 2$ of... the liquidation value in period $N + 1$, plus the

---

[26] This is generally true in both periods since in the first period traders anticipate liquidating their position at p_2, whereas in the second period traders hold the asset until uncertainty is resolved.

[27] See appendix B.

corresponding period, risk-adjusted noise shock. This is reminiscent of Keynes' vision of the stock market as a beauty contest.

An interesting observation by Allen et al. (2006) is that, when averaging over the realizations of noise trading, the price at date n – the average expectation of the average expectation of the... – will in general not coincide with the period n average expectation of the fundamental value (the price at $N + 1$). In this sense the consensus value of the fundamentals $\bar{E}_n[v]$ does not coincide with the price p_n, with the exception of the last period $n = N$. The mean price path p_n gives a higher weight to history – relies more on public information – than the mean consensus path $\bar{E}_n[v]$. This is because of the bias towards public information when a Bayesian agent has to forecast the average market opinion knowing that it is formed also on the public information observed by other agents. This also implies that the current price will be always farther away from fundamentals than the average of investors' expectations and that it will be more sluggish to adjust.

However, according to the previous section we know that the weight the price assigns to public information depends on traders' reaction to order imbalances. Indeed, whenever traders deem the order-flow to be mostly liquidity driven, they take the other side of the market, in this way partially sterilizing the effect of informed trades on the price. This reinforces the effect of public information on the price, driving the latter away from the fundamentals. If, on the other hand, traders estimate the order-flow to be mostly information driven, they speculate on price momentum reinforcing the impact of informed trades on the price. This weakens the weight on public information, tying the price more firmly to the fundamentals. Intuitively, the same effects should be at work in a market where traders have short-term horizons. Indeed, in the coming section we formalize this intuition showing that the results in Allen et al. (2006) can be overturned when noise trading is not independent across periods, i.e. examining the case $\beta \in (0, 1]$ with no residual uncertainty. We also present an example with large residual uncertainty and $\beta = 1$ where a bubble reappears.

4.2   Short-Term Trading and Bubbles

Suppose $N = 2$, and let $\beta \in (0, 1]$, then we can prove the following result: [28]

Proposition 3. *In the 2-period market with short-term traders when $\beta \in (0, 1]$*

---

[28]Proposition 3 extends the multiplicity result in Cespa (2002) to the case of general patterns of noise trading and private information arrival.

there exist two symmetric equilibria in linear strategies where: (1) prices are given by
$p_3 = v$, $p_o = \bar{v}$, $p_2 = \lambda_2 z_2 + (1 - \lambda_2 \Delta a_2)\hat{p}_1$, *and* $p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1)\bar{v}$, *(2) strategies are given by:* $X_2(\tilde{s}_{i2}, p^2) = a_2(\tilde{s}_{i2} - p_2) + \gamma\tau_2(E[v|z^2] - p_2)$,

$$X_1(s_{i1}, p_1) = \gamma\rho(\tau_1 + \tau_{\epsilon_1})\left(E[v|s_{i1}, z_1] - p_1\right) + \frac{(\beta\rho - 1)\tau_1}{(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}E[\theta_1|z_1], \qquad (20)$$

where $a_2 = \gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})$, a_1 *is given by the (two) real solutions to the quartic equation* $f(a_1) \equiv \lambda_2 a_1(\tau_2 + \tau_{\epsilon_1}) - \gamma\Delta a_2 \tau_u \tau_{\epsilon_1} = 0$, *and satisfies* $0 < a_1^* < \gamma\tau_{\epsilon_1}$, $a_2/\beta < a_1^{**} < \infty$, *and*

$$\lambda_1 = \alpha_{P1}\frac{1}{a_1} + (1 - \alpha_{P1})\frac{a_1\tau_u}{\tau_1}, \qquad \alpha_{P1} = \frac{\tau_{\epsilon_1}}{\tau_1 + \tau_{\epsilon_1}} + \frac{(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})}.$$

In any equilibrium $\hat{p}_1 = (\beta a_1 + \gamma\tau_1)^{-1}(\beta z_1 + \gamma\tau_1 E[v|z_1])$, $\lambda_2 = (\gamma(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}))^{-1}(1 + \gamma\Delta a_2\tau_u)$, *and* $\rho = (a_1/\gamma\tau_{\epsilon_1})$. *Therefore,* $a_1^* < a_2/\beta < a_1^{**}$ *and* $\lambda_2(a_1^{**}) < 0 < \lambda_2(a_1^*)$.

Proof. See appendix A. QED

Owing to short horizons traders speculate on short-run price movements. However, differently from section 3.2, they cannot count on a second period hedge to protect them from adverse price swings. Hence, equilibria must be self-fulfilling. In particular, along the high trading intensity equilibrium (i.e. when $a_1 = a_1^{**} > a_2/\beta$) agents escalate their signal aggressiveness and thus anticipate their estimation of the noise trade shock to be biased by the presence of informed trades. As a consequence, they side with the market ($\beta\rho > 1$), and speculate on price momentum. The magnitude of the first period signal aggressiveness ($a_1^{**} > a_2/\beta$), in turn induces a negative second period depth ($\lambda_2(a_1^{**}) < 0$) which limits adverse price movements when traders unwind their position in the second period, justifying their first period speculative behavior. Along the low trading intensity equilibrium (i.e. when $a_1 = a_1^* < \gamma\tau_{\epsilon_1}$) traders scale down their first period signal aggressiveness, and thus anticipate their estimation of the noise trade shock to be due to liquidity trades. As a consequence, they take the other side of the market ($\beta\rho < 1$), and speculate on a noise trade increment reversion. The magnitude of the first period signal aggressiveness ($a_1^* < a_2/\beta$) induces a positive second period depth ($\lambda_2(a_1^*) > 0$), which again justifies informed traders' first period behavior. Thus, in the absence of a risk neutral market making sector (as in Vives (1995)) short-term trading delivers equilibrium multiplicity.

In proposition 3 we denote with

$$\rho = \frac{a_1}{\gamma\tau_{\epsilon_1}},$$

a measure of the deviation that short-term horizons induce in traders' first period signal aggressiveness with respect to the static aggressiveness. Depending on the equilibrium that arises $\beta\rho > (<)1$, and, as in the previous section it is possible to relate the presence of a bubble in the first period to the magnitude of $\beta\rho$. [29] Indeed, we have:

$$
\begin{aligned}
p_1 &= \alpha_{P1}\left(v + \frac{\theta_1}{a_1}\right) + (1 - \alpha_{P1})E[v|z_1] \\
&= \bar{E}_1[v] + \left(\frac{\alpha_{P1} - \alpha_{E1}}{a_1}\right)E[\theta_1|z_1] + \alpha_{E1}\frac{\theta_1}{a_1} \\
&= \bar{E}_1[v] + \frac{(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{a_1(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t})}E[\theta_1|z_1] + \alpha_{E1}\frac{\theta_1}{a_1},
\end{aligned}
\tag{21}
$$

where

$$
\alpha_{P1} \equiv \alpha_{E1} + \frac{(\beta\rho - 1)\tau_1\tau_{\epsilon_1}}{(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t})}.
$$

Using the latter of (21) we readily realize that the existence of a discrepancy between prices and investors' average expectations depends on traders' signal responsiveness. In the low trading intensity equilibrium we have $a_1^* < \gamma\tau_{\epsilon_1}/\beta \Leftrightarrow \rho < 1/\beta$, traders thus accommodate the expected noise trade shock and a bubble occurs. In the high trading intensity equilibrium we have $a_1^{**} > \gamma\tau_{\epsilon_1}/\beta \Leftrightarrow \rho > 1/\beta$, traders side with the market and a reverse bubble arises. Hence, we can conclude that:

Corollary 5. *In the market with short-term traders along the low (high) trading intensity equilibrium the first period equilibrium price is always farther away (closer) to the fundamentals than traders' first period average expectation.*

It is again an immediate consequence of (21) that

Corollary 6. *In the market with short-term traders when $\beta = 0$ there exists a unique equilibrium in linear strategies. In this equilibrium there is a bubble.*

As β approaches 0, along the high trading intensity equilibrium traders need to speculate increasingly more aggressively to avoid an adverse price swing in the second period, up to the point that if $\beta = 0$, a_1^{**} should grow unboundedly. However, in this case the equilibrium becomes fully revealing, $p_2 = p_1 = v$, and traders earn no return from their private information. Hence, they concentrate on the low trading intensity equilibrium. Our model then coincides with Allen et al. (2006) and the first period equilibrium price is always farther away from fundamentals (see figure 3).

---

[29]In the second period traders act as in a static market and, as argued in the previous section, in this case no bubble arises.

Remark 8. Differently from what happens in the market with long-term traders, when traders have short horizons a bubble (reverse bubble) occurs in a thin (thick) first period market. The intuition is as follows. As argued above a bubble arises along the low trading intensity equilibrium where, given traders' low aggressiveness the degree of adverse selection is high. Thus, in this equilibrium the market is thin. Conversely, a reverse bubble occurs in the high trading intensity equilibrium where traders' speculative aggressiveness yields a faster resolution of the underlying uncertainty, reducing the degree of adverse selection. This, in turn, yields a thick market (see figure 4).

Remark 9. In the market with short-term traders it is again possible to apply Bacchetta and van Wincoop (2006b)'s decomposition to the equilibrium price. As one would expect, along the high (low) trading intensity equilibrium a positive wedge (integrating out the effect of the noise term) ties the price more firmly to the fundamentals. Conversely, along the low trading intensity equilibrium the opposite occurs.

Remark 10. Summarizing, as soon as additional noise traders' demand shocks become correlated, both positive and reverse bubble equilibria arise. In a model with $\beta = 1$, Cespa (2002) proves that along the high trading intensity equilibrium since $a_1^{**} > a_2$, the slope of the aggregate excess demand function is always positive, arguing that as a consequence, the equilibrium in this case is unstable. Using our results we can build a similar argument for a market in which $\beta \in (0, 1)$. In particular, given the realization of the first period informational shock z_1, we can define the second period aggregate excess demand function as follows:

$$XD \equiv z_2 + \lambda_2^{-1}(1 - \lambda_2 \Delta a_2)\hat{p}_1 - \lambda_2^{-1}p_2, \tag{22}$$

where $XD = 0$ when the market is in equilibrium and $XD \neq 0$ otherwise. Notice that the slope of (22) depends on the sign of λ_2. This can be determined using the equation that defines the first period signal responsiveness:

$$f(a_1) \equiv \lambda_2 a_1(\tau_2 + \tau_{\epsilon_1}) - \gamma \Delta a_2 \tau_u \tau_{\epsilon_1} = 0.$$

In the high trading intensity equilibrium, we know that $\Delta a_2 < 0$, hence for $f(a_1) = 0$ to be satisfied it must be that $\lambda_2 < 0$ too. Conversely, along the low trading intensity equilibrium we have $\Delta a_2 > 0$, hence for a solution to $f(a_1) = 0$ to exist, λ_2 must also be positive. [30] In the unstable case notice that a price decline (e.g. spurred by a selling pressure) drives the market away from equilibrium. In the low trading intensity equilibrium, the aggregate excess demand function slopes downwards, and the associated equilibrium is stable. Hence, restricting attention to stable equilibria we can conclude that the positive bubble equilibrium in Allen et al. (2006) is robust.

Remark 11. A further feature of the equilibrium obtained in proposition 3 is that prices do not necessarily display inertia. In particular, along the high (low) trading intensity equilibrium, the first period price rapidly (slowly) adjusts to the fundamentals. Indeed, as traders "overreact" to their signal, after the first period the price jumps close to v and then fully adjusts in the following two periods. Thus, differently from Allen et al. (2006) the equilibrium price exhibits inertia if and only if $\rho < 1$.

4.3   The Effect of Residual Uncertainty

If the liquidation value is affected by residual uncertainty and $\beta = 1$ we can prove that a bubble occurs in the first period whenever such uncertainty is sufficiently large. To see this, consider that in the second period the equilibrium strategy of a trader coincides with the one of the model with long-term traders. In the first period

$$X_1(s_{i1}, z_1) = \gamma \frac{E[p_2|s_{i1}, z_1] - p_1}{\mathrm{Var}[p_2|s_{i1}, z_1]},$$

where $E[p_2|s_{i1}, z_1] = \lambda_2 \Delta a_2 E[v|s_{i1}, z_1] + (1 - \lambda_2 \Delta a_2)\hat{p}_1$, $\mathrm{Var}[p_2|s_{i1}, z_1] = \lambda_2^2(\tau_2 + \tau_{\epsilon_1})/(\tau_u(\tau_1 + \tau_{\epsilon_1}))$, and λ_2, $\hat{p}_1$ are defined in proposition 2. Thus, identifying the first period trading aggressiveness:

$$a_1 = \frac{\gamma \tau_{\epsilon_1} \tau_u \Delta a_2}{\lambda_2(\tau_2 + \tau_{\epsilon_1})}.$$

---

[30]Incidentally, the above equation also shows that $a_1 > 0$ in equilibrium since if $a_1 < 0$ both $\Delta a_2 > 0$ and $\lambda_2 > 0$, preventing the existence of a solution to $f(a_1) = 0$.

Imposing market clearing and simplifying yields

$$
\begin{aligned}
p_1 &= \alpha_{P1}\left(v + \frac{\theta_1}{a_1}\right) + (1 - \alpha_{P1})E[v|z_1] \\
&= \bar{E}_1[v] + \left(\frac{\alpha_{P1} - \alpha_{E1}}{a_1}\right)E[\theta_1|z_1] + \alpha_{E1}\frac{\theta_1}{a_1} \qquad (23) \\
&= \bar{E}_1[v] + \frac{(\rho - 1)\tau_1\tau_{\epsilon_1}}{a_1(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t})}E[\theta_1|z_1] + \alpha_{E1}\frac{\theta_1}{a_1},
\end{aligned}
$$

where

$$
\alpha_{P1} \equiv \alpha_{E1} + \frac{(\rho - 1)\tau_1\tau_{\epsilon_1}}{(\tau_1 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t})},
$$

and $\rho = a_1(\tau_\delta + \tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t})/(\gamma\tau_\delta\tau_{\epsilon_1})$, denotes a measure of the deviation that the joint effect of residual uncertainty and short-term horizons induce on traders' first period signal aggressiveness with respect to the static aggressiveness. We can thus conclude the following:

Corollary 7. *In any linear equilibrium of the market with short-term, informed speculators when the liquidation value is affected by residual noise, there is a bubble if and only if $\rho < 1$.*

Intuitively, the above condition is more likely to be satisfied whenever the residual uncertainty affecting the liquidation value is sufficiently large. Indeed, as in the market with long-term traders, higher uncertainty over the liquidation value exacerbates the reaction of the second period price to the upcoming net informational addition z_2. However, owing to short-term horizons, traders cannot count on the improved hedging properties of their second period strategies. Hence, in the face of the increased second period price unpredictability, traders in the first period should scale down their aggressiveness yielding a price which is less anchored to the fundamentals. [31] This intuition can be formalized by the following

Proposition 4. *In any linear equilibrium of the market with short-term traders, when the liquidation value is affected by the presence of residual uncertainty, for sufficiently large residual uncertainty a bubble always arises in the first period.*

Proof. See appendix A. QED

---

[31] The increased uncertainty over the second period price in a way "crowds-out" informed traders in the first period. This effect is reminiscent of De Long et al. (1998). Note, however, that in our context all traders – except liquidity traders – are "rational."

Numerical simulations show that for low values of τ_δ the equilibrium that features a *positive* bubble is unique. [32]

5   Discussion and Concluding Remarks

In this paper we have studied under what conditions asset prices systematically depart from fundamentals compared to the average market consensus in a variety of market contexts. We proved that when long-term traders populate the market and noise trading follows a random walk, equilibrium prices are as close to fundamentals as investors' average opinion. Furthermore, when residual uncertainty on the liquidation value and noise trade predictability are accounted for, the price can either be farther away or closer to the fundamentals than investors' average opinion. Finally, when traders are myopic, multiple equilibria arise, delivering a richer set of possibilities. Depending on the equilibrium that obtains the price can either be closer or farther away from the fundamentals compared to the market consensus but the bubble equilibrium is the stable one.

Our paper thus shows that in contrast to the results put forth by Allen et al. (2006), there are situations where equilibrium prices may work as *better* aggregators of traders' private information than the average consensus opinion, yielding a more reliable forecast of the relevant liquidation value. Indeed, in our setup Keynes' beauty contest allegory represents only one of the possible outcomes of the information aggregation process that arises as traders focus their short-term activities on the exploitation of noise trade predictability. An alternative result which is more in line with Hayek (1945)'s view of the market is possible. Indeed, to the extent that with autocorrelated noise trading – in the presence of residual uncertainty when traders have long horizons or along the high trading intensity equilibrium when traders have short horizons – informed trades *drive* the order-flow, speculators short-term trading activity reinforces the weight the price assigns to fundamental information. This, in turn, may draw the price closer to the fundamentals.

Which of the two scenarios is more realistic is largely an empirical issue. In this respect our analysis naturally yields an empirically testable implication. Indeed, according to our results bubbles should be more likely to occur whenever the patterns of liquidity trading are more easily predictable. Conversely, whenever no systematic pattern drives noise trade reverse bubbles should arise.

---

[32]Conversely, as τ_δ increases, the multiplicity result of the previous section is restored.

References

Abreu, D. and M. K. Brunnermeier (2003). Bubbles and crashes. *Econometrica 71*(1), 173–204.

Admati, A. R. (1985). A noisy rational expectations equilibrium for multi-asset securities markets. *Econometrica 53*(3), 629–657.

Allen, F., S. Morris, and H. S. Shin (2006). Beauty contests and iterated expectations in asset markets. *Review of Financial Studies 19*(3), 719–752.

Angeletos, G. M. and A. Pavan (2005). Efficiency and welfare in economies with incomplete information. *Working Paper: MIT and Northwestern University.*

Bacchetta, P. and E. van Wincoop (2006a). Can information heterogeneity explain the exchange rate determination puzzle? *American Economic Review 96*(3), 552–576.

Bacchetta, P. and E. van Wincoop (2006b). Higher order expectations in asset pricing. *Working Paper*.

Bhattacharya, U. and M. Spiegel (1991). Insiders, outsiders, and market breakdowns. *Review of Financial Studies 4*(2), 255–282.

Biais, B. and P. Bossaerts (1998, April). Asset prices and trading volume in a beauty contest. *Review of Economic Studies 65*(2), 307–340.

Brown, D. P. and R. H. Jennings (1989). On technical analysis. *Review of Financial Studies 2*(4), 527–551.

Brunnermeier, M. K. and S. Nagel (2004). Hedge funds and the technology bubble. *Journal of Finance 59*(5), 2013–2040.

Cao, H. H. and H. Ou-Yang (2005, October). Bubbles and panics in a frictionless market with heterogeneous expectations. *Working Paper*.

Cespa, G. (2002). Short-term investment and equilibrium multiplicity. *European Economic Review 46*(9), 1645–1670.

Cespa, G. (2005). Giffen goods and market making. *Economic Theory 25*(4), 883–897.

De Long, B., Shleifer, A., Summers, L. H. and R. J. Waldmann (1990). Noise trader risk in financial markets. *Journal of Political Economy 98*(4), 703–738.

Grossman, S. J. (1989). *The Informational Role of Prices*. Cambridge: MIT Press.

Grundy, B. D. and M. McNichols (1989). Trade and the revelation of information through prices and direct disclosure. *Review of Financial Studies 2*(4), 495–526.

Hayek, F. A. (1945). The use of knowledge in society. *American Economic Review 35*(4), 519–530.

He, H. and J. Wang (1995). Differential information and dynamic behavior of stock trading volume. *Review of Financial Studies 8*(4), 919–972.

Hellwig, M. F. (1980). On the aggregation of information in competitive markets. *Journal of Economic Theory 22*(3), 477–498.

Kandel, E. and N. D. Pearson (1995). Differential interpretation of public signals and trade in speculative markets. *The Journal of Political Economy 103*(4), 831–872.

Kyle, A. (1985). Continuous auctions and insider trading. *Econometrica 53*(6), 1315–1336.

Llorente, G. Michaely R., Saar, G. and J. Wang (2002). Dynamic volume-return relation of individual stocks. *Review of Financial Studies 15*(4), 1005–1047.

Morris, S. and H.S. Shin (2002). The social value of public information. *American Economic Review 92*, 1521–1534.

Shiller, R. J. (2005). *Irrational Exuberance*. Princeton, New Jersey: Princeton University Press.

Vitale, P. (2006). A market microstructure analysis of foreign exchange intervention. *CEPR Working Paper # 5468*.

Vives, X. (1995). Short-term investment and the informational efficiency of the market. *Review of Financial Studies 8*(1), 125–160.

Vives, X. (1997). Learning from others: a welfare analysis. *Games and Economic Behavior 20*(2), 177–200.

Vives, X. (2006). *Information and Learning in Markets. The Impact of Market Microstructure*. Mimeo.

Wang, J. (1994). A model of competitive stock trading volume. *Journal of Political Economy 102*(1), 127–168.

Appendix A

Proof of lemma 1

Consider the first period. In any linear equilibrium market clearing yields $\int_0^1 a_1 s_{i1} - \varphi_1(p_1)di + \theta_1 = a_1 v - \varphi_1(p_1) + \theta_1 = 0$ or, denoting with $z_1 = a_1 v + \theta_1$ the informational content of the first period order-flow, $z_1 = \varphi_1(p_1)$, where $\varphi_1(\cdot)$ is a linear function. Hence, z_1 and p_1 are observationally equivalent. Suppose now that $z^{n-1} = \{z_1, z_2, \ldots, z_{n-1}\}$ and $p^{n-1} = \{p_1, p_2, \ldots, p_{n-1}\}$ are observationally equivalent and consider the n-th period market clearing condition: $\int_0^1 X_n(\tilde{s}_{in}, p^{n-1}, p_n)di + \theta_n = 0$. Adding and subtracting $\sum_{t=1}^{n-1} a_t v$, the latter condition can be rewritten as follows:

$$\sum_{t=1}^{n} z_t - \varphi_n(p^n) = 0,$$

where $\varphi_n(\cdot)$ is a linear function, and $z_t = \Delta a_t v + u_t$ denotes the informational content of the t-th period order-flow. As by assumption p^{n-1} and z^{n-1} are observationally equivalent, it follows that observing p_n is equivalent to observing z_n.   QED

Proof of proposition 1

Given the past trading history, in the last trading period (N) each informed trader maximizes the expected utility of his last period profit $\pi_{iN} = (v - p_N)x_{iN}$. Owing to CARA and normality of the random variables his optimal strategy is:

$$X_N\left(\tilde{s}_{iN}, p^N\right) = a_N\left(\tilde{s}_{iN} - p_N\right) + \gamma\tau_N\left(E\left[v|z^N\right] - p_N\right), \tag{24}$$

where $a_N = \gamma(\sum_{t=1}^{N} \tau_{\epsilon_t})$. Imposing market clearing yields

$$\int_0^1 a_N(\tilde{s}_{iN} - p_N) + \gamma\tau_N\left(E\left[v|z^N\right] - p_N\right)di + \theta_N = 0.$$

Adding and subtracting $\sum_{n=1}^{N-1} a_n v$ to the above, and rearranging:

$$(1 + \gamma\tau_u\Delta a_N)z_N + \sum_{n=1}^{N-1}(1 + \gamma\tau_u\Delta a_n)z_n + \gamma\tau_v\bar{v} = (a_N + \gamma\tau_N)p_N.$$

Denoting with $\lambda_N = (1 + \gamma\tau_u\Delta a_N)/(a_N + \gamma\tau_N)$ the N-th period reciprocal of market depth we can solve for p_N obtaining

$$p_N = \lambda_N z_N + (1 - \lambda_N\Delta a_N)\hat{p}_{N-1}, \tag{25}$$

where $\hat{p}_{N-k} = \lambda_{N-kl}z_{N-k} + (1-\lambda_{N-kl}\Delta a_{N-1})\hat{p}_{N-(k+1)l}$, and $\hat{\lambda}_{N-k} = (1+\gamma\tau_u\Delta a_{N-k})/(a_{N-k} + \gamma\tau_{N-k})$, for $k = 1,\ldots N-1,$. Substituting (24) into the trader's objective function yields:

$$E\left[U\left(\pi_{iN}\right)|\tilde{s}_{iN}, z^N\right] = -\exp\left\{-\frac{x_{iN}^2}{2\gamma^2(\tau_N + \sum_{t=1}^N \tau_{\epsilon_t})}\right\}.$$

In period $N-1$ the trader then maximizes

$$E\left[U\left(\pi_{iN-1} + \pi_{iN}\right)|\tilde{s}_{iN-1}, z^{N-1}\right] = \tag{26}$$

$$-E\left[\exp\left\{-\frac{1}{\gamma}\left((p_N - p_{N-1})x_{iN-1} + \frac{x_{iN}^2}{2\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N})}\right)\right\}|\tilde{s}_{iN-1}, z^{N-1}\right].$$

Denote with ϕ_{iN-1} the term in parenthesis in the above equation: $\phi_{iN-1} = (p_N - p_{N-1})x_{iN-1} + x_{iN}^2/(2\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N}))$. As one can easily check ϕ_{iN-1} is a quadratic form of the random vector $Z = (x_{iN} - \mu_1, p_N - \mu_2)'$, which is normally distributed (conditionally on $\tilde{s}_{iN-1}, p^{N-1}$) with zero mean and variance covariance matrix

$$\Sigma = \frac{1}{(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_t})\tau_u} \times$$
$$\left(\begin{array}{cc} (\Delta a_N - \gamma\tau_{\epsilon_N})^2\tau_u + (\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_t})(1 + \gamma^2\tau_u\tau_{\epsilon_N}) & \cdot \\ \lambda_N((\gamma\tau_{\epsilon_N} - \Delta a_N)\Delta a_N\tau_u - (\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})) & \lambda_N^2(\tau_N + \sum_{t=1}^{N-1} \tau_{\epsilon_t}) \end{array}\right),$$

(i.e. $\mu_1 = E[x_{iN}|\tilde{s}_{iN-1}, z^{N-1}] = \gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N})(1 - \lambda_N\Delta a_N)(E[v|\tilde{s}_{iN-1}, z^{N-1}] - \hat{p}_{N-1})$ and $\mu_2 = E[p_N|\tilde{s}_{iN-1}, z^{N-1}] = \lambda_N\Delta a_N E[v|\tilde{s}_{iN-1}, z^{N-1}] + (1 - \lambda_N\Delta a_N)\hat{p}_{N-1})$:

$$\phi_{iN-1} = c + b'Z + Z'AZ,$$

where $c = (\mu_2 - p_{N-1})x_{iN-1} + \mu_1^2/(2\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N}))$, $b = (\mu_1/(\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N})), x_{iN-1})'$, and A is a 2×2 matrix with $a_{11} = 1/(2\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N}))$ and the rest zeroes. Owing to a well-known property of multivariate normal random variables the objective function (26) can then be rewritten as

$$E\left[U\left(\pi_{iN-1} + \pi_{iN}\right)|\tilde{s}_{iN-1}, z^{N-1}\right] = \tag{27}$$
$$-|\Sigma|^{-1/2}\left|\Sigma^{-1} + 2/\gamma A\right|^{-1/2} \times \exp\left\{-1/\gamma\left(c - \frac{1}{2\gamma}b'\left(\Sigma^{-1} + (2/\gamma)A\right)^{-1}b\right)\right\}.$$

Maximizing the above function with respect to x_{iN-1} yields

$$x_{iN-1} = \gamma\frac{\mu_2 - p_{N-1}}{h_{22}} - \frac{h_{21}\mu_1}{h_{22}\gamma(\tau_N + \sum_{t=1}^N \tau_{\epsilon_N})}, \tag{28}$$

33

where h_{ij} denotes the ij-th term of the symmetric matrix $H = (\Sigma^{-1} + (2/\gamma)A)^{-1}$:

$$h_{11} = \frac{\gamma^2 \text{Var}_{N-1}[x_{iN}]}{D},$$

$$h_{12} = \frac{\gamma^2 \text{Cov}_{N-1}[x_{iN}, p_N]}{D},$$

$$h_{22} = \frac{\gamma^2 \text{Var}_{N-1}[p_N] + \text{Var}_N[v](\text{Var}_{N-1}[x_{iN}]\text{Var}_{N-1}[p_N] - \text{Cov}_{N-1}[x_{iN}, p_N]^2)}{D},$$

and $\text{Var}_n[Y] = \text{Var}[Y|\tilde{s}_{in}, z^n]$, $\text{Cov}_{N-1}[X, Y] = \text{Cov}[X, Y|\tilde{s}_{N-1}, z^{N-1}]$, $D = \gamma^2 + \text{Var}_{N-1}[x_{iN}]\text{Var}_N[v]$. Identifying the $N-1$-th period trading aggressiveness:

$$a_{N-1} = \frac{\sum_{t=1}^{N-1} \tau_{\epsilon_t}}{h_{22}(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})} (\gamma\lambda_N \Delta a_N - h_{21}(1 - \lambda_N \Delta a_N))$$

$$= \gamma \left(\sum_{t=1}^{N-1} \tau_{\epsilon_t} \right),$$

since $(\gamma\lambda_N \Delta a_N - h_{21}(1 - \lambda_N \Delta a_N))/h_{22} = \gamma(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})$. Substituting a_{N-1} in (28) and rearranging

$$X_{N-1}(\tilde{s}_{iN-1}, z^{N-1})$$
$$= \left(\frac{\gamma\lambda_N \Delta a_N - h_{21}(1 - \lambda_N \Delta a_N)}{h_{22}} \right) \left(E\left[v|\tilde{s}_{iN-1}, z^{N-1}\right] - \hat{p}_{N-1} \right) - \frac{\gamma}{h_{22}}(p_{N-1} - \hat{p}_{N-1})$$
$$= \gamma \left(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N} \right) \left(E\left[v|\tilde{s}_{iN-1}, z^{N-1}\right] - \hat{p}_{N-1} \right) - \frac{\gamma}{h_{22}}(p_{N-1} - \hat{p}_{N-1}). \qquad (29)$$

Interpreting $\hat{p}_{N-1}$ as the "static" period $N-1$ equilibrium price (i.e. the price that would arise if the asset was liquidated in period N), a trader i's strategy is thus the sum of his static position and the adjustment he makes to exploit short-run price movements. For example, suppose that $E[v|\tilde{s}_{iN-1}, z^{N-1}] - \hat{p}_{N-1} > 0$, but that $p_{N-1} > \hat{p}_{N-1}$, then the trader scales down his period $N-1$ static position to avoid buying the asset for too high a price. Imposing market clearing yields:

$$\int_0^1 \gamma \text{Var}\left[v|\tilde{s}_{iN-1}, z^{N-1}\right]^{-1} \left(E\left[v|\tilde{s}_{iN-1}, z^{N-1}\right] - \hat{p}_{N-1} \right) di - \frac{\gamma}{h_{22}}(p_{N-1} - \hat{p}_{N-1}) + \theta_{N-1} = 0. \qquad (30)$$

Notice that

$$\int_0^1 \gamma \mathrm{Var}\left[v|\tilde{s}_{iN-1}, z^{N-1}\right]^{-1}\left(E\left[v|\tilde{s}_{iN-1}, z^{N-1}\right] - \hat{p}_{N-1}\right) di =$$

$$= \int_0^1 \gamma \left(\tau_v \bar{v} + \tau_u \sum_{n=1}^{N-1} \Delta a_n z_n + \sum_{n=1}^{N-1} \tau_{\epsilon_n} s_{in}\right) - \left(\gamma \tau_v \bar{v} + \sum_{n=1}^{N-1} (1 + \gamma \tau_u \Delta a_n) z_n\right) di$$

$$= a_{N-1} v - \sum_{n=1}^{N-1} z_n$$

$$= -\theta_{N-1}.$$

Hence, since $\gamma/h_{22} \neq 0$, (30) is satisfied if and only if $\hat{p}_{N-1} = \hat{\lambda}_{N-1} z_{N-1} + (1 - \hat{\lambda}_{N-1}\Delta a_{N-1})\hat{p}_{N-2} = p_{N-1}$: traders' aggregate static position is just enough to hold the total noisy supply that has accumulated up to period $N-1$. Therefore, $\hat{\lambda}_{N-1} = \lambda_{N-1}$, and

$$X_{N-1}(\tilde{s}_{iN-1}, z^{N-1}) = \gamma \left(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_t}\right)(E[v|\tilde{s}_{iN-1}, z^{n-1}] - p_{N-1})$$

$$= a_{N-1}(\tilde{s}_{iN-1} - p_{N-1}) + \gamma \tau_{N-1}\left(E\left[v|z^{N-1}\right] - p_{N-1}\right). \ (31)$$

Plugging (31) into the term in parenthesis in the exponential of (27) we can now evaluate the $N-1$-th period objective function of a trader i at the optimum:

$$c - \frac{1}{2\gamma}b'\left(\Sigma^{-1} + 2/\gamma A\right)^{-1} b = \frac{x_{iN-1}^2}{2\gamma(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})},$$

given that, as one can check, in equilibrium

$$\mu_2 - p_{N-1} = \frac{\lambda_N \Delta a_N}{\gamma(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})} x_{iN-1},$$

and $\mu_1^2 = x_{iN-1}^2$. The $N-2$-th objective function of a trader is then given by

$$E\left[U\left(\pi_{iN-2} + \pi_{iN-1}\right)|\tilde{s}_{iN-2}, z^{N-2}\right] = \tag{32}$$

$$-E\left[\exp\left\{-\frac{1}{\gamma}\left((p_{N-1} - p_{N-2})x_{iN-2} + \frac{x_{iN-1}^2}{2\gamma(\tau_{N-1} + \sum_{t=1}^{N-1} \tau_{\epsilon_N})}\right)\right\} |\tilde{s}_{iN-2}, z^{N-2}\right].$$

Comparing (32) with (26) the form of the objective function in the recursion between the $N-1$-th and the $N-2$-th period looks exactly as the one in the recursion between the N-th and the $N-1$-th period. Thus, the trader's optimal strategy in

period $N-2$ is given by $x_{iN-2} = a_{N-2}(\tilde{s}_{iN-2} - p_{N-2}) + \gamma\tau_{N-2}(E[v|\tilde{s}_{iN-2}, z^{N-2}] - p_{N-2})$, where $a_{N-2} = \gamma(\sum_{t=1}^{N-2} \tau_{\epsilon_t})$, and the corresponding equilibrium price is given by:

$$p_{N-2} = \lambda_{N-2}z_{N-2} + (1 - \lambda_{N-2}\Delta a_{N-2})\hat{p}_{N-3},$$

where $\lambda_{N-2} = (1 + \gamma\tau_u\Delta a_{N-2})/(a_{N-2} + \gamma\tau_{N-2})$. Iterating this procedure until the first period, we obtain that $x_{in} = a_n(\tilde{s}_{in} - p_n) + \gamma\tau_n(E[v|z^n] - p_n)$, $a_n = \gamma\sum_{t=1}^{n} \tau_{\epsilon_t}$, and $\hat{p}_n = p_n$ for all $1 \le n \le N$. Finally, in any period n the response to private information is equal across traders. Hence, the equilibrium is symmetric.   QED

Proof of proposition 2

To find the equilibria of the model we proceed as illustrated in the proof of proposition 1. In the second period, a trader behaves like in a static model and trades according to

$$\begin{aligned} X_2\left(\tilde{s}_{i2}, z^2\right) &= \gamma\frac{E[v|\tilde{s}_{i2}, z^2] - p_2}{\text{Var}[v + \delta|\tilde{s}_{i2}, z^2]} \\ &= a_2(\tilde{s}_{i2} - p_2) + b_2(E[v|z^2] - p_2), \end{aligned} \quad (33)$$

where

$$a_2 = \frac{\gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})}{1 + \kappa}, \quad (34)$$

$b_2 = \gamma\tau_2/(1+\kappa)$, and $\kappa = (\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t})/\tau_\delta$. Imposing market clearing, we find that the second period equilibrium price is given by

$$p_2 = \lambda_2 z_2 + (1 - \lambda_2\Delta a_2)\hat{p}_1,$$

where $\lambda_2 = (\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t}))^{-1}(1 + \gamma\tau_u\Delta a_2 + \kappa)$, $\Delta a_2 = a_2 - \beta a_1$, and

$$\hat{p}_1 = \frac{\gamma\tau_1 E[v|z_1] + \beta(1 + \kappa)z_1}{\gamma\tau_1 + \beta a_1(1 + \kappa)}.$$

Notice that $\alpha_{P2} = (\gamma\tau_\delta(\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t}))^{-1}a_2(1 + \kappa) = \alpha_{E2}$. Hence, no bubble occurs in the second period. Substituting (33) in the second period objective function, a trader in the first period maximizes

$$E\left[U\left(\pi_{i1} + \pi_{i2}\right)|s_{i1}, z_1\right] = \quad (35)$$
$$-E\left[\exp\left\{-\frac{1}{\gamma}\left((p_2 - p_1)x_{i1} + \frac{x_{i2}^2\text{Var}[v + \delta|\tilde{s}_{i2}, z^2]}{2\gamma}\right)\right\}|s_{i1}, z_1\right].$$

Let $\phi_{i1} = (p_2 - p_1)x_{i1} + x_{i2}^2 \text{Var}[v + \delta | \tilde{s}_{i2}, z^2]/(2\gamma)$. The term ϕ_{i1} is a quadratic form of the random vector $Z = (x_{i2} - \mu_1, p_2 - \mu_2)'$, which is normally distributed (conditionally on s_{i1}, p_1) with zero mean and variance covariance matrix

$$\Sigma = \begin{pmatrix} \text{Var}[x_{i2}|s_{i1}, z_1] & \text{Cov}[x_{i2}, p_2|s_{i1}, z_1] \\ \text{Cov}[x_{i2}, p_2|s_{i1}, z_1] & \text{Var}[p_2|s_{i1}, z_1] \end{pmatrix}, \tag{36}$$

where

$$\text{Var}[x_{i2}|s_{i1}, z_1] = \frac{(\Delta a_2(1+\kappa) - \gamma\tau_{\epsilon_2})^2\tau_u + (\tau_1 + \tau_{\epsilon_1})((1+\kappa)^2 + \gamma^2\tau_u\tau_{\epsilon_2})}{(\tau_1 + \tau_{\epsilon_1})\tau_u(1+\kappa)^2},$$

$$\text{Cov}[x_{i2}, p_2|s_{i1}, z_1] = \lambda_2 \left(\frac{(\gamma\tau_{\epsilon_2} - (1+\kappa)\Delta a_2)\Delta a_2\tau_u - (1+\kappa)(\tau_1 + \tau_{\epsilon_1})}{(\tau_1 + \tau_{\epsilon_1})\tau_u(1+\kappa)} \right),$$

$$\text{Var}[p_2|s_{i1}, z_1] = \lambda_2^2 \left(\frac{\tau_2 + \tau_{\epsilon_1}}{(\tau_1 + \tau_{\epsilon_1})\tau_u} \right),$$

(i.e. $\mu_1 = E[x_{i2}|s_{i1}, z_1] = (a_2 + b_2)(1 - \lambda_2\Delta a_2)(E[v|s_{i1}, z_1] - \hat{p}_1)$ and $\mu_2 = E[p_2|s_{i1}, z_1] = \lambda_2\Delta a_2 E[v|s_{i1}, z_1] + (1 - \lambda_2\Delta a_2)\hat{p}_1$):

$$\phi_{i1} = c + b'Z + Z'AZ,$$

where $c = (\mu_2 - p_1)x_{i1} + \mu_1^2 \text{Var}[v + \delta | \tilde{s}_{i2}, z^2]/(2\gamma)$, $b = (\mu_1 \text{Var}[v + \delta | \tilde{s}_{i2}, z^2]/\gamma, X_{i1})'$, and A is a 2×2 matrix with $a_{11} = \text{Var}[v + \delta | \tilde{s}_{i2}, z^2]/(2\gamma)$ and the rest zeroes. We can now rewrite the objective function (35) as

$$E\left[U\left(\pi_{i1} + \pi_{i2}\right) | s_{i1}, z_1\right] = \tag{37}$$
$$- |\Sigma|^{-1/2} \left|\Sigma^{-1} + 2/\gamma A\right|^{-1/2} \times \exp\left\{-1/\gamma \left(c - \frac{1}{2\gamma}b' \left(\Sigma^{-1} + 2/\gamma A\right)^{-1} b\right)\right\}.$$

Maximizing the above function with respect to x_{i1} yields

$$x_{i1} = \gamma\frac{\mu_2 - p_{N-1}}{h_{22}} - \frac{h_{21}\mu_1\text{Var}[v + \delta | \tilde{s}_{i2}, z^2]}{\gamma h_{22}}, \tag{38}$$

where h_{ij} denotes the ij-th term of the symmetric matrix $H = (\Sigma^{-1} + 2/\gamma A)^{-1}$:

$$h_{11} = \frac{\gamma^2\text{Var}_1[x_{i2}]}{D},$$

$$h_{12} = \frac{\gamma^2\text{Cov}_1[x_{i2}, p_2]}{D},$$

$$h_{22} = \frac{\text{Var}_1[p_2](\gamma^2 + \text{Var}_2[v + \delta]\text{Var}_1[x_{i2}](1 - \rho_{1,\{x_{i2}, p_2\}}^2))}{D},$$

and $\text{Var}_n[Y] = \text{Var}[Y | \tilde{s}_{in}, z^n]$, $\text{Cov}_1[X, Y] = \text{Cov}[X, Y | s_{i1}, z_1]$,

$$\rho_{1,\{x_{i2}, p_2\}}^2 = \frac{\text{Cov}_1[x_{i2}, p_2]^2}{\text{Var}_1[p_2]\text{Var}_1[x_{i2}]},$$

$D = \gamma^2 + \mathrm{Var}_1[x_{i2}]\mathrm{Var}_2[v + \delta]$. Substituting in (38) the expressions for μ_1 and μ_2 and rearranging we obtain

$$
\begin{aligned}
X_1(s_{i1}, p_1) &= \frac{a_1(\tau_1 + \tau_{\epsilon_1})}{\tau_{\epsilon_1}}\left(E[v|s_{i1}, z_1] - \hat{p}_1\right) - \frac{\gamma}{h_{22}}(p_1 - \hat{p}_1) \qquad (39) \\
&= \frac{a_1(\tau_1 + \tau_{\epsilon_1})}{\tau_{\epsilon_1}}\left(E[v|s_{i1}, z_1] - p_1\right) + \frac{(\gamma + h_{21})(1 - \lambda_2\Delta a_2)}{h_{22}}(\hat{p}_1 - p_1),
\end{aligned}
$$

where a_1 denotes the 1st period trading aggressiveness:

$$
\begin{aligned}
a_1 &= \frac{\tau_{\epsilon_1}\left(\gamma\lambda_2\Delta a_2 - h_{21}(1 - \lambda_2\Delta a_2)\right)}{h_{22}(\tau_1 + \tau_{\epsilon_1})} \\
&= \frac{\gamma\tau_{\epsilon_1}(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})(1 + \kappa)(1 + \gamma\tau_u\Delta a_2)}{(1 + \kappa + \gamma\tau_u\Delta a_2)(\tau_{\epsilon_2} + (\tau_2 + \tau_{\epsilon_1})(1 + \kappa))}. \qquad (40)
\end{aligned}
$$

Both (34) and (40) are implicit solutions for a_1 and a_2. Thus, equilibria must be determined via numerical methods. [33] Rearranging the first period equilibrium strategy yields the second of (12). Using this expression and imposing market clearing yields $p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1)\bar{v}$, where

$$
\lambda_1 = \frac{\alpha_{P1}}{a_1} + \frac{(1 - \alpha_{P1})a_1\tau_u}{\tau_1},
$$

and the expression for α_{P1} is given by (16). In order to obtain the expression for the first period equilibrium price given in (15), we impose market clearing on (39) obtaining $\tau_{\epsilon_1}^{-1}a_1(\tau_1 + \tau_{\epsilon_1})\int_0^1(E_1[v] - \hat{p}_1)di - (\gamma/h_{22})(p_1 - \hat{p}_1) + \theta_1 = 0$. Simplifying the latter condition yields

$$
\frac{\gamma\tau_1(\beta\rho - 1)}{(\gamma\tau_1 + \beta a_1(1 + \kappa))}E[\theta_1|z_1] - \frac{\gamma}{h_{22}}(p_1 - \hat{p}_1) = 0, \qquad (41)
$$

which can be easily rearranged to obtain (15). Notice also that according to the above market clearing equation the first period equilibrium price will differ from the static solution as long as $\rho \neq 1/\beta$.

To show existence note that (13) defines a system of non-linear equations. Let us denote with $f(a_1, a_2) = 0$ the equation defining a_2, and with $g(a_1, a_2) = 0$ the equation defining a_1. Both $f(\cdot)$ and $g(\cdot)$ are continuous. In particular, it is easy to check that $f(a_1, a_2) = (\tau_\delta + \sum_{t=1}^2 \tau_{\epsilon_t})(a_2^3\tau_u - 2a_2^2 a_1\tau_u + a_2(\tau_1 + a_1^2\tau_u)) - \gamma\tau_\delta(\sum_{t=1}^2 \tau_{\epsilon_t}) = 0$ is a nondegenerate cubic in a_2, given that $(\tau_\delta + \sum_{t=1}^2 \tau_{\epsilon_t})\tau_u > 0$, and always admits a real solution for any a_1. Furthermore, since $\partial f/\partial a_2 = (\tau_\delta + \sum_{t=1}^2 \tau_{\epsilon_t})(3a_2^2\tau_u -$

---

[33] If $\tau_\delta \to \infty$, residual noise vanishes, $\kappa \to 0$ and $a_n \to \gamma(\sum_{t=1}^n \tau_{\epsilon_t})$ as in proposition 1.

$4a_2a_1\tau_u + \tau_1 + a_1^2\tau_u)$ and the discriminant associated to this quadratic equation in a_1 can be shown to be negative, we have that $\partial f/\partial a_2 \neq 0$ and the solutions to the cubic equation are continuous in a_1.[34] Hence, denoting by $a_2(a_1)$ a (real) solution to the cubic we have that

$$\lim_{a_1 \to 0} a_2(a_1) = \bar{a}_2 > 0, \qquad \lim_{a_1 \to \infty} a_2(a_1) = 0.$$

We can now verify that a real solution always exists to the equation $g(a_1, a_2(a_1)) = 0$. Indeed,

$$\lim_{a_1 \to 0} g(a_1, a_2(a_1)) = \gamma\tau_\delta\tau_{\epsilon_1}(1 + \bar{a}_2\gamma\tau_u)\left(\tau_v + \bar{a}_2^2\tau_u + \sum_{t=1}^{2}\tau_{\epsilon_t} + \tau_\delta\right) > 0,$$

$$\lim_{a_1 \to \infty} g(a_1, a_2(a_1)) = -\infty,$$

and the result follows.

We are now left with the task of proving that in any linear equilibrium $\rho > 1$. Notice that in any equilibrium $a_1 > 0$, hence if $1 + \gamma\tau_u\Delta a_2 > (<)0$, then also $1 + \kappa + \gamma\tau_u\Delta a_2 > (<)0$.[35] Notice also that if $\Delta a_2 < 0$ then $1 + \kappa + \gamma\tau_u\Delta a_2 < 0$. To see this last point, compute Δa_2 using (34) and (40):

$$\Delta a_2 = \tag{42}$$
$$\frac{\gamma}{D}\left(\tau_{\epsilon_2}(1 + \kappa + \gamma\tau_u\Delta a_2)(\tau_{\epsilon_2} + \tau_{\epsilon_1} + \tau_2(1 + \kappa)) - \tau_{\epsilon_1}\kappa(1 + \kappa)\gamma\tau_u\Delta a_2\left(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t}\right)\right),$$

where $D = (1 + \kappa + \gamma\tau_u\Delta a_2)(\tau_{\epsilon_2} + (\tau_2 + \tau_{\epsilon_1})(1 + \kappa))(1 + \kappa)$. Suppose that $\Delta a_2 < 0$ but that $(1 + \kappa + \gamma\tau_u\Delta a_2) > 0$, then given (42) this is impossible.

To prove our claim start by assuming that $\Delta a_2 > 0$, then using (40) we can directly check whether $\rho < 1$ since as one can see

$$a_1 < \frac{\gamma\tau_{\epsilon_1}}{1 + \kappa} \Leftrightarrow \tau_{\epsilon_2}(1 + \gamma\tau_u\Delta a_2 + \kappa) + (1 + \kappa)\left(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t}\right)\gamma\tau_u\Delta a_2 < 0,$$

which is clearly impossible. If, on the other hand $\Delta a_2 < 0$, given what we have said above for $\rho < 1$ we need

$$\tau_{\epsilon_2}(1 + \gamma\tau_u\Delta a_2 + \kappa) + (1 + \kappa)\left(\tau_2 + \sum_{t=1}^{2}\tau_{\epsilon_t}\right)\gamma\tau_u\Delta a_2 > 0,$$

---

[34]Indeed, as one can check $\Delta \equiv 16a_1^2\tau_u^2 - 12\tau_u(\tau_1 + a_1^2\tau_u) = -(8a_1^2\tau_u + 12\tau_v)\tau_u < 0$.

[35]For suppose $a_1 < 0$, then $\Delta a_2 > 0$ and both $1 + \gamma\tau_u\Delta a_2 > 0$ and $1 + \gamma\tau_u\Delta a_2 + \kappa > 0$, implying $a_1 > 0$, a contradiction.

which is again impossible. [36] Therefore, in any linear equilibrium $\rho > 1$.

$$\text{QED}$$

Proof of proposition 3

In the second period a trader speculates according to $X_2(\tilde{s}_{i2}, p^2) = (\gamma/\text{Var}[v|\tilde{s}_{i2}, z^2]) \times (E[v|\tilde{s}_{i2}, z^2] - p_2)$. Imposing market clearing yields

$$\int_0^1 x_{i2} di + \theta_2 =$$
$$= a_2(v - p_2) + \gamma\tau_2 \left(E\left[v|z^2\right] - p_2\right) + \theta_2 \tag{43}$$
$$= (a_2 - \beta a_1)v + \beta a_1 v + u_2 + \beta\theta_1 - (a_2 + \gamma\tau_2)p_2 + \gamma\tau_2 E\left[v|z^2\right] = 0,$$

where $a_2 = \gamma(\tau_{\epsilon_1} + \tau_{\epsilon_2})$, and $\theta_2 = u_2 + \beta\theta_1$. Let $\Delta a_2 = a_2 - \beta a_1$, then the above market clearing condition can be rewritten as

$$z_2 + \beta z_1 + \gamma\tau_2 E[v|z^2] = (a_2 + \gamma\tau_2)p_2,$$

where $z_2 = \Delta a_2 v + u_2$. Rearranging it yields

$$p_2 = \lambda_2 z_2 + (1 - \lambda_2 \Delta a_2)\hat{p}_1,$$

where $\lambda_2 = (a_2 + \gamma\tau_2)^{-1}(1 + \gamma\tau_2\Delta a_2)$, and

$$\hat{p}_1 = \frac{\gamma\tau_v \bar{v} + (\beta + \gamma\tau_u a_1)z_1}{\beta a_1 + \gamma\tau_1}.$$

In the first period owing to short-term horizons $X_1(s_{i1}, p_1) = (\gamma/\text{Var}[p_2|s_{i1}, z_1])(E[p_2|s_{i1}, z_1] - p_1)$, where $E[p_2|s_{i1}, z_1] = \lambda_2\Delta a_2 E[v|s_{i1}, z_1] + (1 - \lambda_2\Delta a_2)\hat{p}_1$, and $\text{Var}[p_2|s_{i1}, z_1] = \lambda_2^2(\tau_2 + \tau_{\epsilon_1})/((\tau_1 + \tau_{\epsilon_1})\tau_u)$. Identifying the first period signal responsiveness yields

$$a_1 = \gamma\frac{\Delta a_2 \tau_u \tau_{\epsilon_1}}{\lambda_2(\tau_2 + \tau_{\epsilon_1})}. \tag{44}$$

Let $f(a_1) \equiv a_1\lambda_2(\tau_2 + \tau_{\epsilon_1}) - \gamma\Delta a_2\tau_u\tau_{\epsilon_1}$. The solutions to the quartic $f(a_1) = 0$ clearly identify the equilibria of the model. Notice that $f(0) = -a_2\gamma\tau_{\epsilon_1}\tau_u < 0$, while

$$f(\gamma\tau_{\epsilon_1}) = \frac{\tau_{\epsilon_1}((2(1-\beta) + \beta^2)\gamma^2\tau_{\epsilon_1}^2\tau_u + \tau_{\epsilon_1}(1 + (1-\beta)\gamma^2\tau_{\epsilon_2}\tau_u) + \tau_v)}{\tau_{\epsilon_2} + (2(1-\beta) + \beta^2)\gamma^2\tau_{\epsilon_1}^2\tau_u + \gamma^2\tau_{\epsilon_2}^2\tau_u + \tau_{\epsilon_1}(1 + 2(1-\beta)\gamma^2\tau_{\epsilon_2}\tau_u) + \tau_v} > 0.$$

---

[36]Suppose $\tau_{\epsilon_2} = 0$, then a_1 must satisfy $(a_1(1 + \kappa) - \gamma\tau_{\epsilon_1}) + \gamma\tau_u\Delta a_2(a_1 - \gamma\tau_{\epsilon_1}) = 0$. It is easy to see that a solution to this equation is $a_1 = a_2 = \gamma\tau_{\epsilon_1}/(1 + \kappa)$. Then, $\alpha_{P1} = \alpha_{E1}$ and no bubble occurs in the first period. The intuition is straightforward: if $\tau_{\epsilon_2} = 0$ the second period price is just a noisy version of p_1, traders do not expect any price variation that justifies a rebalancing of their speculative position in the second period, and the absence of a bubble when $n = 2$ also extends to the first period.

Hence, a first solution a_1^* to the equation $f(a_1) = 0$ belongs to the interval $(0, \gamma\tau_{\epsilon_1})$. Next, since

$$f(a_2/\beta) = \frac{a_2(\tau_2 + \tau_{\epsilon_1})}{\beta\gamma \left(\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t}\right)} > 0,$$

and $\lim_{a_1 \to \infty} f(a_1) = -\infty$, a further solution a_1^{**} to $f(a_1) = 0$ belongs to the interval $(a_2/\beta, \infty)$. To see that these are the only two real solutions (i.e. that the remaining two roots must be complex), notice that the cubic equation $f'(a_1) = 0$ has a unique real root (its discriminant is positive). Hence, the graph of $f(a_1)$ changes slope only once (between a_1^* and a_1^{**}). Rearranging the first period market clearing equation yields (21), where $\alpha_{P1} \equiv \alpha_{E1}(1 + (\beta\rho - 1)\tau_1/(\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t}))$, and $\rho = a_1/(\gamma\tau_{\epsilon_1})$. According to the latter of (21), a bubble occurs whenever $\alpha_{P1} < \alpha_{E1} \Leftrightarrow a_1 < (\gamma\tau_{\epsilon_1}/\beta)$. which in the first equilibrium is always satisfied since we have $a_1^* < \gamma\tau_{\epsilon_1} < \gamma\tau_{\epsilon_1}/\beta$. Next, a reverse bubble arises if and only if $\alpha_{P1} > \alpha_{E1} \Leftrightarrow a_1 > (\gamma\tau_{\epsilon_1}/\beta)$, which in the second equilibrium is again always satisfied since $a_1^{**} > a_2/\beta > \gamma\tau_{\epsilon_1}/\beta$. Using the definition of α_{P1} and rearranging the pricing equation yields: $p_1 = \lambda_1 z_1 + (1 - \lambda_1 a_1)\bar{v}$, where

$$\lambda_1 = \frac{\alpha_{P1}}{a_1} + \frac{(1 - \alpha_{P1})a_1\tau_u}{\tau_1}.$$

Finally, note that for $f(a_1) \equiv a_1\lambda_2(\tau_2 + \tau_{\epsilon_1}) - \gamma\Delta a_2\tau_u\tau_{\epsilon_1} = 0$ to have a real solution it must be the case that λ_2 and Δa_2 have the same sign. In the high trading intensity equilibrium $a_1^{**} > a_2/\beta$, and $\Delta a_2 < 0$. Therefore, $\lambda_2(a_1^{**}) < 0$.

QED

Proof of corollary 6

Imposing $\beta = 0$ in (44) one can see that the linear equilibria of the model are given by the solutions to the following cubic equation $f(a_1) \equiv a_1\lambda_2(\tau_2 + \tau_{\epsilon_1}) - \gamma a_2\tau_u\tau_{\epsilon_1} = 0$. Now, it is immediate to check that $f(0) < 0$, $f(\gamma\tau_{\epsilon_1}) > 0$, and that

$$f'(a_1) = \frac{(1 + \gamma a_2\tau_u)(2a_1^2\tau_u\tau_{\epsilon_2} + (\tau_2 + \tau_{\epsilon_1})(\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t}))}{\gamma \left(\tau_2 + \sum_{t=1}^{2} \tau_{\epsilon_t}\right)^2} > 0,$$

showing that the equilibrium is unique.

QED

Proof of proposition 4

To prove the claim we check what happens to $\rho = a_1(1+\kappa)/(\gamma\tau_{\epsilon_1})$ as $\tau_\delta \to 0$. Notice that

$$
\begin{aligned}
\lim_{\tau_\delta \to 0} a_1 &= \lim_{\tau_\delta \to 0} \frac{\gamma^2 \tau_\delta \tau_{\epsilon_1} \tau_u \Delta a_2 \left(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}\right)}{(\tau_2 + \tau_{\epsilon_1})\left(\tau_\delta(1 + \gamma \Delta a_2 \tau_u) + \left(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}\right)\right)} \\
&= 0,
\end{aligned}
$$

and $\lim_{\tau_\delta \to 0} a_2 = \lim_{\tau_\delta \to 0} \gamma\tau_\delta(\tau_{\epsilon_1} + \tau_{\epsilon_2})/(\tau_\delta + (\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t})) = 0$. Then,

$$
\begin{aligned}
\lim_{\tau_\delta \to 0} \rho &= \frac{\gamma\tau_u \Delta a_2 \left(\tau_2 + \sum_{t=1}^2 \tau_{\epsilon_t}\right)}{\tau_2 + \tau_{\epsilon_1}} \\
&= 0 < 1,
\end{aligned}
$$

since, as argued above, in the limit $a_1 = a_2 = 0$. By continuity, there must then exist an open interval $[0, \tau_\delta^*)$, such that for all $\tau_\delta \in [0, \tau_\delta^*)$ we have $\rho < 1$.   QED

Appendix B

In this appendix we generalize the pricing formula obtained in section 4.1 to the $N \geq 2$-period case. To fix notation, let $\bar{E}_n[Y] = \int_0^1 E[Y|\tilde{s}_{in}, p^n]di$, and $\mathrm{Var}_n[Y] = \mathrm{Var}[Y|\tilde{s}_{in}, p^n]$. In the N-th period the market clearing equation reads as follows:

$$
\gamma \frac{\bar{E}_N[v] - p_N}{\mathrm{Var}_N[v]} + \theta_N = 0.
$$

Therefore, the price of the asset in period N is given by

$$
p_N = \bar{E}_N[v] + \frac{\mathrm{Var}_N[v]}{\gamma}\theta_N, \tag{45}
$$

In period $N - 1$, optimality and market clearing require that

$$
\gamma \frac{\bar{E}_{N-1}[p_N] - p_{N-1}}{\mathrm{Var}_{N-1}[p_N]} + \theta_{N-1} = 0,
$$

and using (45) the corresponding market clearing price is given by:

$$
\begin{aligned}
p_{N-1} &= \bar{E}_{N-1}[p_N] + \frac{\mathrm{Var}_{N-1}[p_N]}{\gamma}\theta_{N-1} \\
&= \bar{E}_{N-1}\left[\bar{E}_N[v] + \frac{\mathrm{Var}_N[v]}{\gamma}\beta\theta_{N-1}\right] + \frac{\mathrm{Var}_{N-1}[p_N]}{\gamma}\theta_{N-1}.
\end{aligned}
$$

Iterating this procedure, and using the fact that

$$\theta_n = \beta \left(\beta^{n-1} \theta_1 + \sum_{t=1}^{n-1} \beta^{n+1-t} u_t \right) + u_n,$$

recursive substitution yields

$$p_n =$$
$$\bar{E}_n \left[\bar{E}_{n+1} \left[\cdots \bar{E}_{N-1} \left[\bar{E}_N[v] + \frac{\text{Var}_N[v]}{\gamma} \beta^{N-n} \theta_n \right] + \frac{\text{Var}_{N-1}[p_N]}{\gamma} \beta^{N-(n+1)} \theta_n \cdots \right] + \frac{\text{Var}_{n+1}[p_{n+2}]}{\gamma} \beta \theta_n \right]$$
$$+ \frac{\text{Var}_n[p_{n+1}]}{\gamma} \theta_n,$$

for $1 \le n \le N$.
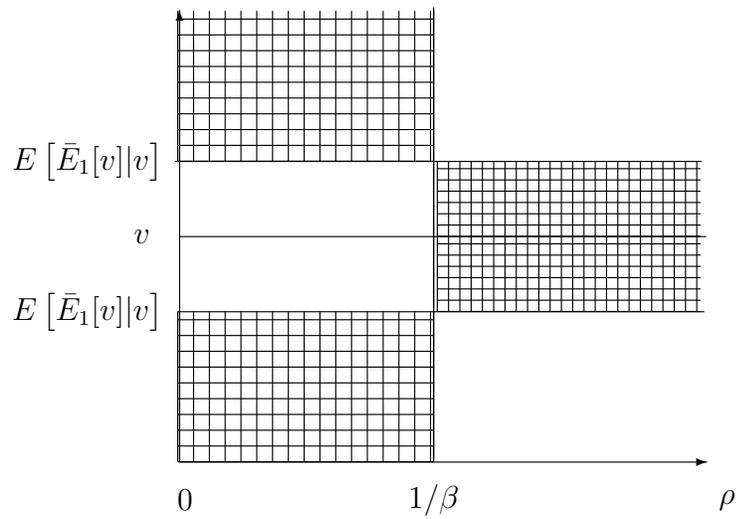
Figure 1: If $\beta\rho < 1$ $(\beta\rho > 1)$ $E[p_1|v]$ lays in the thickly (thinly) meshed area, and a bubble (reverse bubble) occurs.
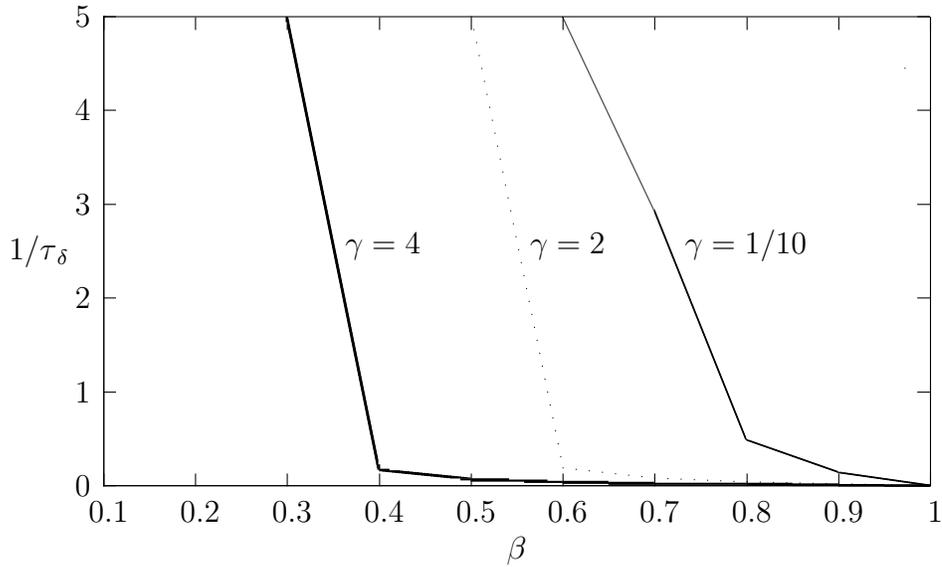
Figure 2: The figure plots the set $\Omega \equiv \{(\beta, 1/\tau_\delta) \in [0,1] \times \Re_+ | \beta\rho(\beta) = 1\}$ for $\gamma \in \{1/10, 2, 4\}$ (parameters' values: $\tau_v = \tau_u = \tau_{\epsilon_n} = 1$). Values of $(\beta, 1/\tau_\delta)$ that fall below (above) this set identify market conditions leading to a bubble (reverse bubble). As γ increases, traders speculate more and more aggressively on their first period information. This increases the relevance of informed trades in the aggregate order-flow, widening the parameter space for which a reverse bubble occurs.
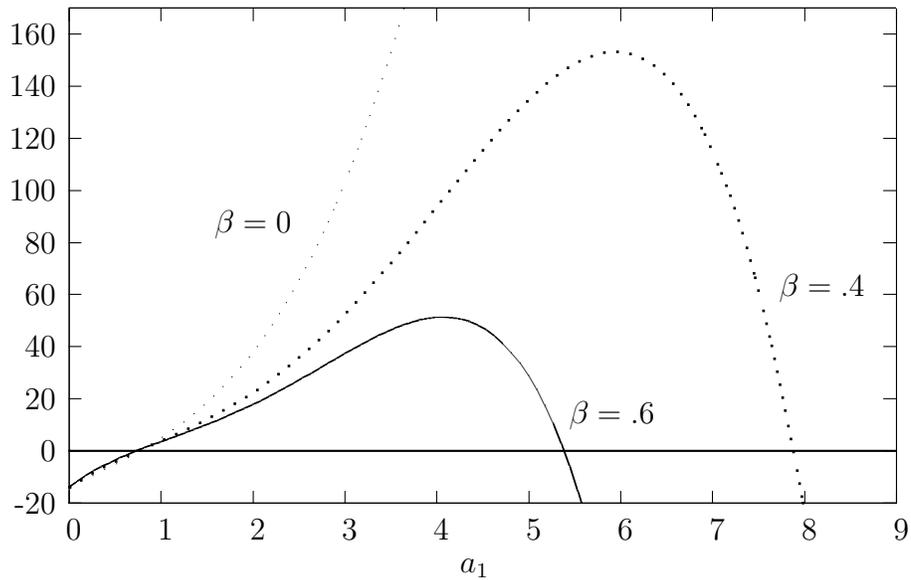
Figure 3: The continuous (heavily, lightly dotted) curve graphs the equation that determines the equilibria when $\beta = .6$ ($\beta = .4$, $\beta = 0$). Since $a_1^* < \gamma\tau_{\epsilon_1}/\beta < a_1^{**}$, as $\beta \to 0$, $\gamma\tau_{\epsilon_1}/\beta \to \infty$, the high trading intensity equilibrium disappears and only the low trading intensity equilibrium survives (parameter values: $\tau_v = \tau_u = \tau_{\epsilon_n} = \gamma = 1$ and $\beta \in \{0, .4, .6\}$).
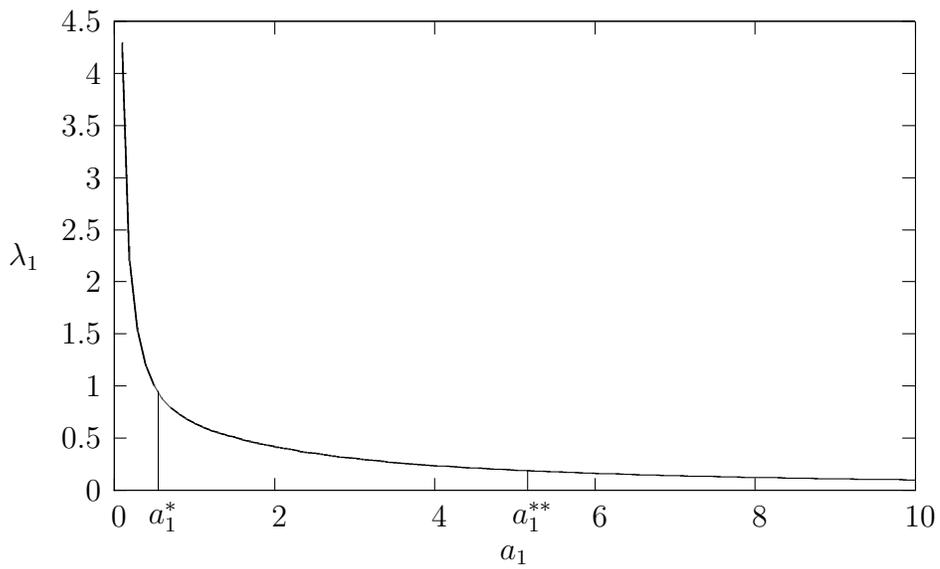
Figure 4: Differently from the market with long-term traders, with short-term traders a positive (reverse) bubble occurs in a thin (thick) market (parameter values: $\tau_v = \tau_u = \tau_{\epsilon_n} = \gamma = 1$ and $\beta = .6$; $a_1^* = 0.732$, $a_1^{**} = 5.393$).