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On growth and indeterminacy: some theory and evidence*

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Abstract

We overview some recent models of growth with multiple equilibria. Those models offer a potential explanation for differences in income levels across economies which stands as an alternative to theories that rely exclusively on differences in fundamentals and/or initial conditions. We discuss some of the empirical predictions generated by those models vis à vis the predictions of their deterministic counterparts. Using an approach that makes use of physical and human capital data for a cross-section of countries we show that some of the predictions of growth models with unique equilibria are hard to reconcile with the evidence.

1 Introduction

The possibility that multiple equilibria – in the sense of multiple economic outcomes consistent with some given fundamentals – may be at the heart of many macroeconomic phenomena has been given considerable attention in

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recent years. Yet, at least until recently, most of the examples of models with multiple equilibria found in the literature seemed to be motivated by the need to explain "excessive" short-term fluctuations in the level of economic activity and/or asset prices, two dimensions of economic life which have traditionally been perceived as being most susceptible to "animal spirits."[1]

Over the past few years, however, a number of researchers have explored the possibility that some of the observed long-term patterns of income, both dynamic and cross-sectional, could also be interpreted as resulting from the "selection" of alternative equilibrium outcomes, in addition to possible differences in technologies, preferences, government policies, or initial conditions. That has led economists to go back to the models that are traditionally used to explain those long-term patterns of income – perfect foresight, dynamic general equilibrium models with factor accumulation, i.e., growth models, for short – and investigate the possibility of multiple equilibria (or, equivalently, indeterminacy) in their context.

One of the goals of the present paper is to offer an (admittedly partial) overview of that literature.[2] In doing so, we restrict outselves to the infinite-lived representative consumer framework,[3] and impose throughout conditions on preferences and technology that guarantee the existence of at least one balanced growth path. We do so for the sake of analytical tractability as well as space limitaitons, but also because we want to emphasize the fact that, far from being restricted to "exotic" model economies, indeterminacy is a property that is latent in many of the growth models macroeconomists are familiar with. We start out by looking at the one-sector model with exogenous growth, for which a transformation with bounded growth exists under standard assumptions. Then we look at endogenous growth versions of the same model, in which unbounded growth is feasible even in the absence of technical progress. Then we move on to discuss two-sector models, with one and two accumulated factors. In each case we have a well-known model with a unique equilibrium as a benchmark, the predictions of which are compared with those of its multiple equilibria counterparts.

A common theme in all the models with multiple equilibria that we consider is the presence of some market imperfection, including productive externalities, market power, etc. This is not surprising for in their absence equilibrium allocations would be efficient, and would correspond to the *unique* solution of the social planner's problem. On the other hand, the introduc-

[1]See, e.g., chapter 5 of Blanchard and Fischer (1989) for a discussion of some popular models with multiple equilibria.

[2]See also Boldrin and Rustichini (1994) for a recent analysis of the conditions under which multiple equilibria may arise in growth models. The class of models considered in their paper overlaps but does not coincide with the models reviewed here.

[3]Examples of models with multiple equilibria in an OLG setting can be found in Boldrin (1992) and Galor (1992), among others.

tion of certain market imperfections makes it possible for *private* returns to the accumulation of some productive asset to be positively related to the *aggregate* stock of that asset (at least for some range of the latter). That "complementarity" leads, in some circumstances, to the possibility of more than one pair of expectations/outcomes that are consistent with individual optimization, market clearing, and rational expectations, i.e., more than one equilibrium.

The second motivation of the paper is empirical: we would want to know to what extent growth models with multiple equilibria can help us understand the observed long-term patterns of income. Unfortunately, the existing literature has largely bypassed that question. To a large extent, this is a consequence of the difficulties inherent in any attempt to test determinate vs. multiple equilibria models, for any test of determinacy is necessarily a test of a joint hypothesis, which may include the specific structure assumed to generate the data, constancy of parameter values (over time and/or across economies), absence of stochastic perturbations, etc. In other words, the alternative hypothesis is not necessarily confined to multiple equilibria. The empirical analysis contained in this paper is not immune to that general difficulty. Our contribution lies in the discussion of the differential predictions of models with a determinate equilibrium vis à vis their indeterminate counterparts, and in the assessment of their relative merits in light of the observed cross-country patterns in measures of output, and physical and human capital, as well as their changes over time. Our approach is model-based, and conditional on a number of auxiliary hypotheses. We try to minimize the latter by exploiting the information contained in measures of physical and/or human capital, which in some cases allow us to avoid using ad hoc proxies for unobserved differences in fundamentals that are common in much of the empirical growth literature.

Using an approach that makes use of capital data we find that some of the predictions of the one-sector model with a unique equilibrum are hard to reconcile with the evidence, even when we attempt to control for differences in technology levels as well as other unobserved fundamentals. Although the evidence is somewhat more tentative, we find similar results in two-sector models with physical and human capital. Some aspects of the data seem not to be in accord with the dynamics of a model with a unique equilibrum. Many explanations are possible, but the hypothesis of indeterminacy cannot be excluded on the basis of the available evidence.

The paper is organized as follows. In Section II we analyze a general one-sector model with and without endogenous growth. We then examine the empirical evidence to assess the differential predictions of versions of that model that generate single and multiple equilibria. Section III illustrates the theoretical results of a simple model of endogenous growth with endogenous labor, and discusses how indeterminacy can arise in that model. Section IV generalizes that model to include physical as well as human capital, and explores some aspects of the available data to assess the empirical plausibility of indeterminacy that can arise within models of that type.

2 Indeterminacy in the one-sector growth model

2.1 *The one-sector growth model: a general framework*

Let us introduce some notation. K denotes the capital stock. C denotes consumption. Y is output. L is employment. H is labor-augmenting technical progress or, more colorfully, human capital. We assume an aggregate production function of the form:

$$Y = \lambda(S)F(K, HL) \tag{1}$$

where F is twice continuously differentiable, concave, and homogeneous of degree one, and λ is a productivity parameter which is taken parametrically by individual agents, but which we allow to depend on a vector of economy-wide variables $S \equiv (K, H, \ldots)$. In this section we assume a constant supply of labor services $L \equiv 1$, so we can think of the remaining variables as expressed in per capita (or per worker) terms.

We focus on versions of the one-sector model of growth for which an equilibrium corresponds to a trajectory $[K(t), C(t), H(t), Y(t)]$ satisfying (1), the system of differential equations

$$\dot{K} = \lambda(S)F(K, H) - \delta K - C \tag{2}$$

$$\dot{C} = C\sigma[r(S) - (\delta + \rho)] \tag{3}$$

$$\dot{H} = H\gamma \tag{4}$$

the transversality condition $lim_{T\to\infty} e^{-\rho t}K(T)C(T)^{-\frac{1}{\sigma}} = 0$, and the initial conditions $K(0) = K_0$ and $H(0) = \phi$.

The previous laws of motion are consistent with the existence of representative consumer whose preferences are time separable with constant discount rate $\rho > 0$ and intertemporal elasticity of substitution $\sigma > 0$. γ is the exogenous rate of technical progress, which is assumed to be constant over time. r denotes the implicit rental cost of capital. In a "general" model allowing for externalities and market power in the goods market (generating a price-marginal cost markup $\mu > 1$), r is given by

$$r(S) = \frac{\lambda(S)F_1(K, H)}{\mu(S)} \equiv \theta(S)F_1(K, H) \tag{5}$$

Let $c \equiv C/H, k \equiv K/H$, and $y \equiv Y/H$ denote consumption, capital, and output, all normalized by the technology level H. Henceforth, we assume $S = k$, implying that θ and y are functions of the (normalized) capital stock k.[4] In that case we can rewrite the equilibrium conditions in terms of the transformed variables as

$$y = \lambda(k)F(k,1) \equiv \lambda(k)f(k) \equiv g(k) \tag{6}$$

$$\dot{k} = g(k) - (\delta + \gamma)k - c \tag{7}$$

$$\dot{c} = c\sigma[r(k) - (\delta + \rho + \frac{\gamma}{\sigma})] \tag{8}$$

the transversality condition

$$lim_{T\to\infty} e^{-[\rho-\gamma(1-1/\sigma)]T} k(T)c(T)^{-\frac{1}{\sigma}} = 0 \tag{9}$$

and the initial condition

$$k(0) = k_0 \tag{10}$$

where $r(k) \equiv \theta(k)f'(k)$. Notice that the dynamical system (7)-(8) is now autonomous. The previous framework is sufficiently general to encompass a number of alternative structures, including optimal growth models in both their exogenous growth (Cass (1965), Koopmans (1965)) or endogenous growth versions (Jones and Manuelli (1990), Rebelo (1991)). It also encompasses models with productive externalities (Romer (1986), Boldrin and Rustichini (1994), Zilibotti (1994)), imperfect competition (Ciccone and Matusyama (1993), Zilibotti (1993), Gali (1995), Gali and Zilibotti (1995)), as well as models with tax distortions.

For future reference it is useful at this point to define a steady state as a vector $[k^*, c^*, y^*]$ satisfying $y^* = g(k^*), c^* = g(k^*) - (\delta + \gamma)k^*$, and

$$r(k^*) = \theta(k^*)f'(k^*) = \rho + \delta + \frac{\gamma}{\sigma} \tag{11}$$

Since the nature of the equilibrium dynamics is strongly influenced by the feasibility or not of sustained endogenous growth, we choose to deal with the two cases separately in what follows.

2.2 *Indeterminacy with exogenous growth*

Throughout this subsection we rule out the possibility of long-term "endogenous" growth (i.e., sustained growth beyond that implied by the exponential

[4]That assumption excludes models in which the productivity parameter or the markup depend on a flow variable. See Gali (1994) for an example of an economy in which the equilibrium markup is a function of the savings rate, as a result of a differential in the elasticities of substitution among goods in consumption vs. production.

trend in H). A sufficient condition is the existence of a $\bar{k} > 0$, implicitly defined by $g(\bar{k}) = (\delta + \gamma)\bar{k}$, and such that $g(k) < (\delta + \gamma)k$ for $k > \bar{k}$. In addition, and in order to guarantee the existence of an interior steady state, we assume $lim_{k \to 0} r(k) = \infty$ and $lim_{k \to \bar{k}} r(k) < \rho + \delta + \gamma/\sigma$.

We take as our benchmark the optimal growth model of Cass (1965) and Koopmans (1965)–henceforth, the "neoclassical" model. That model corresponds to a particular case of the dynamical system above, with $\theta(k) = \theta$ and a strictly decreasing marginal product of capital ($f" < 0$). The implied $f(k)$ schedule, which is strictly decreasing in k, is represented in Figure 1a. Given our assumptions it is clear that there exists a unique solution to (11) in this case, i.e., a unique steady state. Furthermore, that steady state can be shown to be a saddle point of the dynamical system (6)-(10). The global equilibrium dynamics on the $[c, k]$ plane are illustrated in Figure 1b. For any initial capital stock $k_0 \in (0, \bar{k})$ there exists a unique equilibrium trajectory $[k(t), c(t)]$ satisfying (6)-(8). That trajectory moves along the stable manifold of the system and converges monotonically to $[k^*, c^*]$. Trajectories that start above the stable manifold (given the same k_0) eventually violate the Euler equation, while trajectories starting below violate the transversality condition.[5]

As long as the uniqueness of the steady state is preserved, the introduction of productive externalities or imperfect competition (with or without variable markups) does not lead to a *qualitative* change in the equilibrium dynamics of the one-sector economy.[6] In particular, the equilibrium path will still be uniquely determined by the initial conditions and it will converge monotonically to the steady state. That result can be shown to follow from the fact that $r'(k^*) < 0$ will hold at the unique steady state, which in turn guarantees that its saddle-point nature is preserved, making the uniqueness argument carry over to this case.[7]

Let us consider next the case of an economy with multiple steady states. Given the discussion above, the existence of multiple steady states is a necessary (though not sufficient) condition for multiple equilibria to emerge in the one-sector economy under consideration. Under our assumptions, multiple steady states arise if and only if there exists at least one $k* \in (0, \bar{k})$ satisfying (11) and such that $r'(k^*) > 0$. The previous condition requires, in turn, that $\theta(k)$ increases sufficiently fast over an appropriate range to more than offset the negative impact of capital's diminishing marginal product on the return to investment. Let $e_\lambda \equiv \lambda'(k)k/\lambda(k)$ (the "external" component

[5]See, e.g., Sala-i-Martin (1994) for a detailed characterization of the neoclassical model's equilibrium

[6]Obviously, the resulting equilibrium allocation will in general be suboptimal once we introduce either imperfect competition, externalities, or other distortions.

[7]See Boldrin and Rustichini (1994) for a formal proof for the discrete time case.

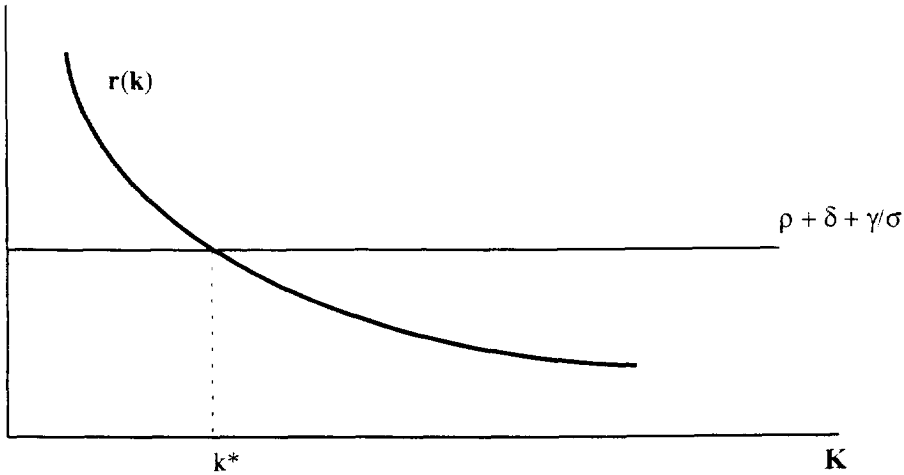


Figure 1a

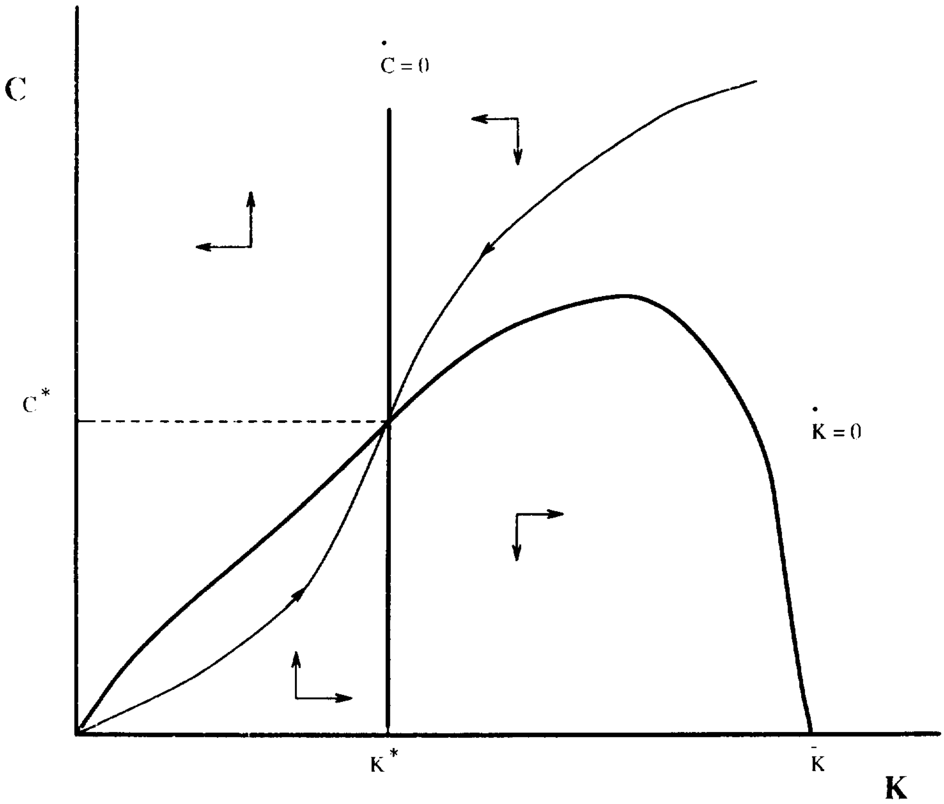


Figure 1b

of the elasticity of output with respect to the aggregate capital stock k), $e_\mu \equiv -\mu'(k)k/\mu(k)$ (minus the elasticity of the markup with respect to k), and $e_f \equiv -f''(k)k/f(k)$ (minus the elasticity of the marginal product of capital with respect to the aggregate capital stock). The above requirement can thus be formalized as $(e_\lambda + e_\mu) > e_f$. If we assume a Cobb-Douglas private technology $f(k) = k^\alpha$, the previous condition can be rewritten in turn as $(e_\lambda + e_\mu + \alpha) > 1$.

How likely is the previous condition to be satisfied, at least for some range of capital stock values? Unfortunately, recent attempts to measure the degree of increasing returns have generated too wide a range of estimates. Under the assumption that externalities are concentrated on the accumulated factor k, the size of e_λ implied by estimates in the literature range from values close to zero (Norrbin (1993)) to values around 0.5 (Caballero and Lyons (1992)).[8] On the other hand, Gali (1995) provides estimates of e_μ close to 0.1, based on cross-country regressions of markups on capital stock levels. Given the previous estimates, it seems clear that the plausibility of the condition above will very much depend on the value of α. The latter is in turn influenced by one's stand regarding what k represents: if narrowly interpreted as "physical capital," α will correspond to the observed capital income share, which is likely to be too low to make the above condition hold.[9] On the other hand, and given "reasonable" e_λ and e_μ values, the plausibility of the same condition is notably enhanced if one adopts a broad interpretation of k as being inclusive of human as well as physical capital, as in Barro, Mankiw, and Sala-i-Martin (1995).

Figure 2a depicts the $r(k)$ schedule in an economy with three steady states, which we denote by (k_L^*, k_M^*, k_H^*). It is not difficult to show that *odd* steady states (k_L^* and k_H^* in the figure) are (generically) saddle points, whereas *even* steady states (k_M^*) are (generically) unstable, though in general they can be either a node or a focus. Unfortunately, our knowledge of the local stability properties of the three steady states is not sufficient to fully characterize the *global* equilibrium dynamics, even though one can still apply the same arguments used in the neoclassical model to rule out paths that eventually enter the top-left and bottom-right regions of the phase diagram. Clearly, determining the set of equilibrium paths consistent with an arbitrary initial capital stock requires knowledge of the "full shape" of the stable manifolds associated with the two odd steady states, and given the nonlin-

[8]As discussed in Benhabib and Farmer (1995), the relevant elasticity of output with respect to inputs in aggregative models may be higher than that implied by disaggregated studies. This is due to the fact that even small externalities in disaggregated sectors will pile up, making the aggregate degree of increasing returns higher.

[9]The previous remark does not necessarily apply to many LDC's and middle-income countries, for which the capital income share is, on average, significantly higher (around 1/2), as shown in Gali (1995).

ear nature of the system that in turn requires the application of numerical solution methods to a parametrized model.

Figure 2b illustrates the kind of equilibrium dynamics that a model with three steady states can generate. One can identify three regions, on the basis of initial conditions. Economies whose initial capital stock is below k_1 have a unique equilibrium path, which converges to k_L^*. Those economies can be thought of as being in a "poverty trap." The equilibrium path for economies with an initial capital stock above k_2 is also unique, but it converges to k_H^* instead. Finally, multiple equilibria exist for economies whose initial capital stock lies in the interval (k_1, k_2); in that case some of the equilibrium paths (characterized by eventual negative saving rates) converge to k_L^*, while others (with eventual positive saving rates) converge to k_H^*. Which trajectory is "selected" depends on agents' initial expectations on the future path of the economy (expectations which will turn out to be self-fulfilling, given our assumption of perfect foresight). If each individual agent anticipates that the economy will follow a path characterized by high investment, he will find it optimal to increase his individual savings (and thus his capital holdings) since the associated private returns will be increasing along the way (at least over a certain range). The aggregation of identical private decisions will lead to the actual realization of the initial expectations, i.e., to a rapidly growing aggregate capital stock. The opposite dynamics (i.e., rapid disinvestment) will occur if agents' expectations are pessimistic.[10]

The dynamics portrayed in Figure 2b are not generic. The range of initial conditions for which multiple equilibrium paths will exist depends on the extent of the overlap of the stable manifolds associated with the high and low steady states. That overlap in turn depends on parameter values and is particularly sensitive to the elasticity of intertemporal substitution σ. If σ is high enough, the "development trap" mentioned above may disappear, in the sense that there is always an equilibrium path which converges to the high steady state (in addition to other possible equilibria), regardless of the initial capital stock. On the other hand, the overlap region tends to disappear

[10]In a number of models found in the literature, indeterminacy is associated with stable dynamics around a steady state (see, e.g., Benhabib and Farmer (1994)). In that case, there is a continuum of trajectories (all of which converge to the steady state) which satisfy all the equilibrium conditions (including the transversality condition). That type of indeterminacy is also associated with the existence of stationary sunspot equilibria (Woodford (1986)), an equilibrium concept often found in discrete-time stochastic models that generally focus on business-cycle dynamics (see, e.g., Farmer and Guo's paper in this volume). In contrast, in the model above (and in similar models) indeterminacy coexists with unstable dynamics around the middle steady state. In that case there is not a continuum of trajectories satisfying equilibrium conditions, but only two: the ones that correspond to the stable manifolds of the remaining steady states. Whenever those two trajectories overlap for a range of capital stock values (as is the case in the presence of complex roots), multiple equilibria arise.

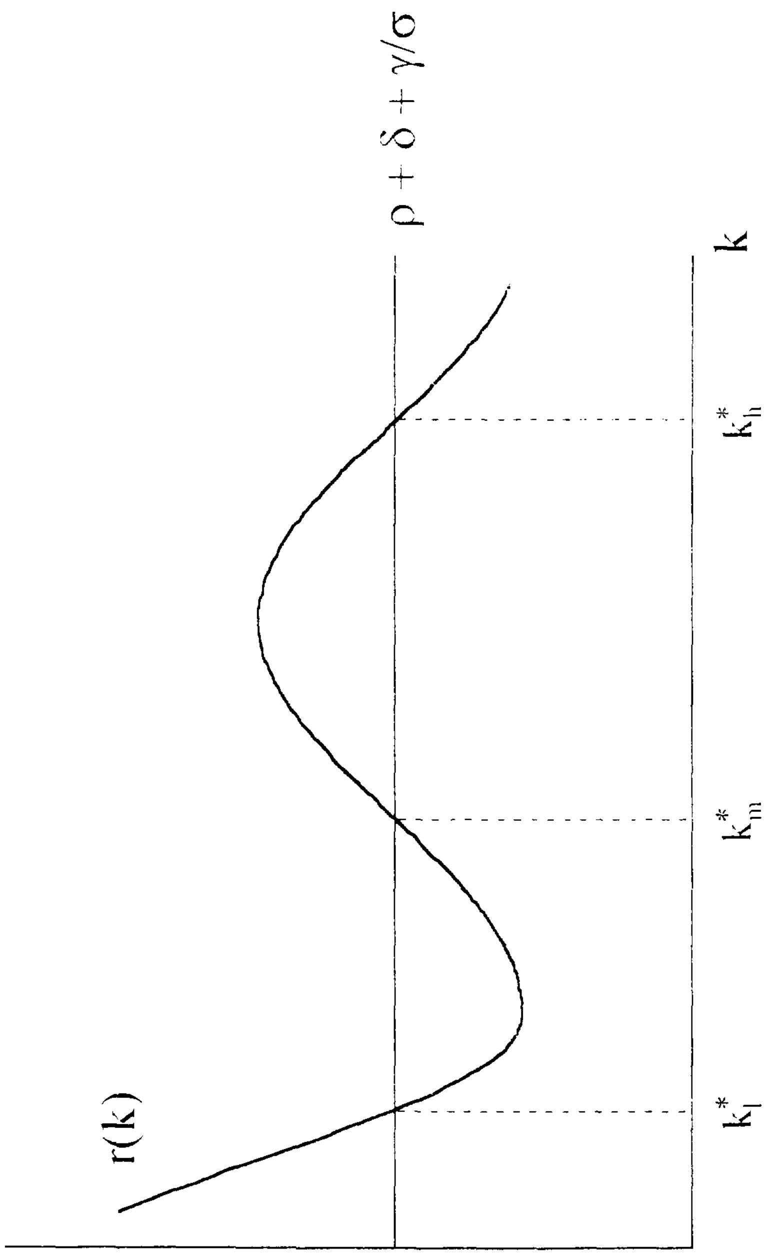


Figure 2a

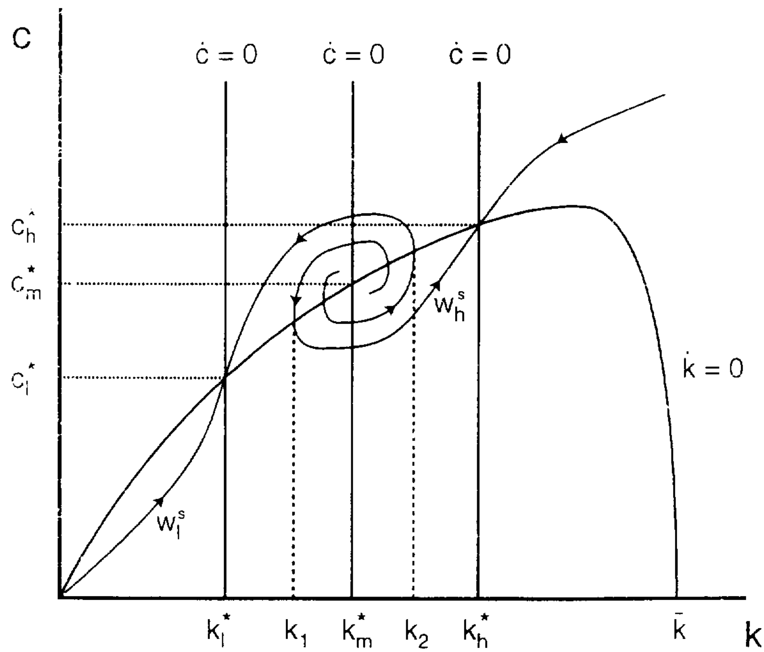


Figure 2b

as $\sigma \to 0$, in which case a model with unique equilibrium but "threshold" dynamics emerges: the steady state to which an economy converges depends exclusively on its initial capital stock conditions.[11]

The possibility of multiple steady states and multipl equilibria illustrated above seems to be consistent with a number of features of the data. First, it seems to accord with the evidence of no unconditional convergence of income levels across countries, combined with the simultaneous presence of "convergence clubs" (Baumol et al. (1989)). Second, it can be reconciled with evidence of conditional convergence that relies on savings rates as conditioning variables (Mankiw, Romer, and Weil (1992)), for in the class of models with multiple equilibria discussed above the savings choice effectively determines the subsequent equilibrium path of two economies that have identical initial conditions. Third, the presence of multiple equilibria is perfectly compatible with investment returns being asymptotically equalized across economies, even in the absence of international capital mobility; most interestingly – and in contrast with the Cass and Koopmans models – such equalization of returns coexists with permanent differences in capital stock, income, and savings rates, even for countries that are endowed with identical technologies and preferences. Such a result seems consistent with the absence of persistent interest-rate differentials and with the failure of capital to flow massively from rich to poor countries (Lucas (1990)).[12]

2.3 *Indeterminacy with endogenous growth*

The recent growth literature has focused on model economies whose long-term rate of growth is endogenously determined, resulting from agents' op-

[11]Models with multiple steady states but a unique equilibrium trace back at least to Kurz (1968) and Liviatan and Samuelson (1969). Recent examples (in the context of slightly different frameworks from the one above) are reviewed in Azariadis (1993).

[12]The previous statement refers to the possible impact of liberalizing capital movements. In a small open-economy version of the standard neoclassical model, where agents face a constant, exogenously given world interest rates, and there are no adjustment costs, all transitional dynamics disappear and the economy "jumps" to the unique steady state. The same would be true were we to consider a small open-economy version of the model above with "local" productive externalities, though in that case the economy could jump to any of the three steady states and, in fact, could keep switching from one to the other over time. We do not view either case as realistic. A more appropriate modelling strategy would allow for two types of capital (showing *private* joint decreasing returns), one of which cannot be traded internationally. This is the avenue pursued by Barro, Mankiw, and Sala-i-Martin (1995) in the context of the standard neoclassical model, and for which they show a closed-economy (near) equivalence result. An analogous equivalence would obtain if we were to add an externality term $\theta(k)$, so all the (qualitative) results derived above for the closed economy would carry over to the economy with partial capital mobility, with an appropriate reinterpretation of the production function as a "reduced-form" production function with labor and nontradable capital as arguments.

timal decisions regarding the accumulation of some productive input. Well-known examples of one-sector models with endogenous growth include Romer (1986), Rebelo (1991), and Jones and Manuelli (1990). In those models the possibility of sustained endogenous growth (i.e., growth beyond that implied by the exponential trend in H) hinges critically on whether the returns to investment remain above a critical level as the capital stock grows unboundedly. More specifically, and going back to the framework and notation developed above, a necessary condition for an equilibrium with endogenous balanced growth to exist is given by

$$A \equiv lim_{k \to \infty} r(k) > \rho + \delta + \frac{\gamma}{\sigma} \tag{12}$$

a condition which we assume in the remainder of this subsection.

The possibility of multiple equilibria, illustrated for the bounded growth case in the previous section, also arises in one-sector models with endogenous growth. Examples can be found in the literature of models with multiple equilibrium paths, where at least one of those paths involves sustained "endogenous" growth. Examples that are known to us rely on the presence of either productive externalities (Zilibotti (1993)), or endogenous markups (Zilibotti (1994), Gali and Zilibotti (1995)).

Again, we find it useful to start with a well-known benchmark model with a unique equilibrium. The celebrated Ak model (Rebelo (1991)) seems the natural choice, given its simplicity. In terms of our previous notation we assume $\gamma = 0, f(k) = k, \theta(k) = A$, and $\delta = 0$. As is well-known the equilibrium of that model is unique (for any initial K_0), and it involves a constant growth rate for consumption, output, and capital given by

$$\dot{c}/c = \dot{k}/k = \dot{y}/y = \sigma(A - \rho) \tag{13}$$

Given the initial capital stock, the equilibrium path is uniquely determined by the condition

$$(c/k) = [(1 - \sigma)A + \sigma\rho] \tag{14}$$

which must hold for all t. The (trivial) $r(k)$ schedule and equilibrium trajectory for that model are shown in Figures 3a and 3b.

Gali and Zilibotti (1995) provide an example of an economy which departs from Rebelo's model by introducing Cournot competition among firms, free entry and exit, and an overhead capital requirement. In their model equilibrium markups are decreasing in the number of active firms, while the latter is in turn positively related to the aggregate capital stock. As a result, markups are a function $\mu(k)$ of the aggregate capital stock, with $\mu'(k) < 0$ and $lim_{k \to \infty} \mu(k) = 1$. The corresponding $r(k)$ schedule is shown in Figure

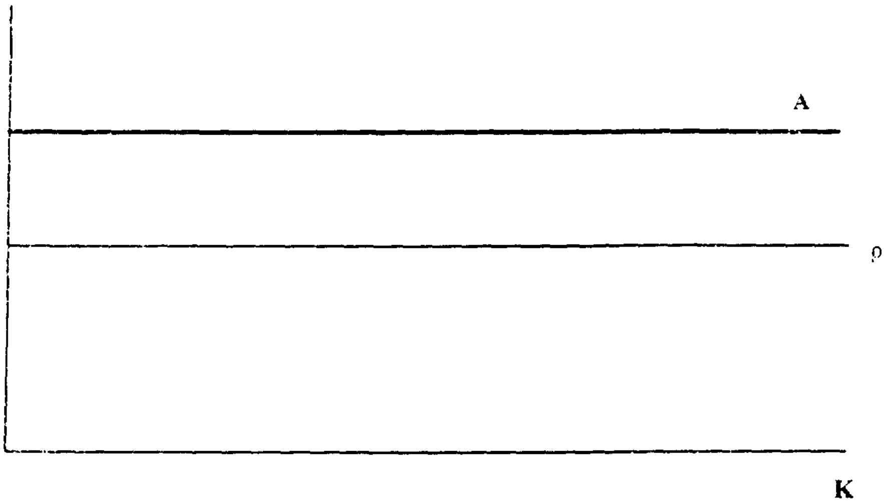


Figure 3a

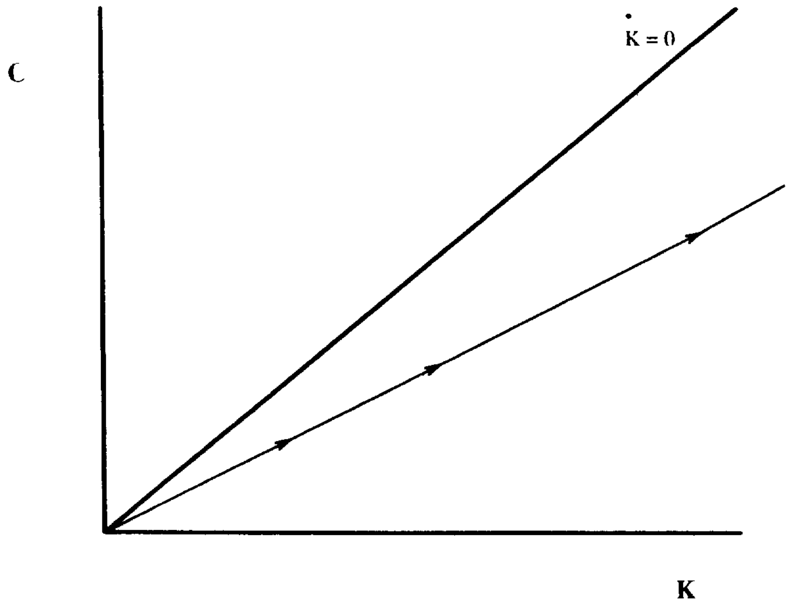


Figure 3b

4a. Notice that when the capital stock reaches a certain threshold, the effective return becomes zero. This is so because an equilibrium with positive production cannot be sustained for low levels of capital since the demand elasticity is below one (i.e., the optimal markup goes to ∞). In that case the zero return corresponds to an implicit storage technology for capital.

Figure 4b illustrates the kind of equilibrium dynamics that may emerge in the Gali-Zilibotti model. That figure corresponds to a numerical simulation of a calibrated version of that model, where parameter settings are chosen so that for an economy which manages to grow unboundedly the asymptotic growth rate and interest rate are, respectively, 2% and 6%, with the intertemporal elasticity of substitution set at $\sigma = 2.2$. The $(C/K)_\infty$ line represents the (constant) consumption/capital ratio of the corresponding model with perfect competition.

Notice that an interior steady state k^* exists, but it is unstable. We see that, for any initial capital stock above b, there exists a unique equilibrium path. That path converges in the long run to the perfectly competitive path, thus showing sustained growth. On the other hand, for initial capital stock levels below b, there exists at least one trajectory consistent with equilibrium which converges to the "autarky" steady state (i.e., a steady state with zero capital and with consumption equal to an exogenous endowment flow). Such trajectories eventually involve capital decumulation, a "collapse" of the market sector and the progressive consumption of the existing capital holdings. Notice, however that in the case depicted the autarky steady state is not a "poverty trap," since the economy would take off (and grow unboundedly) if only agents could coordinate expectations and put themselves on a high savings, high growth trajectory.

Again, the previous equilibrium dynamics are not generic. Thus, when we set $\sigma = 1$ (while keeping other parameters constant), the equilibrium path characterized by asymptotic positive growth originates in the unstable steady state (instead of the c axis). As a result a poverty trap emerges, i.e., economies with an initial capital stock below a certain threshold have a unique equilibrium involving full depletion of the capital stock.

2.4 *Some empirical evidence*

In the previous sections we briefly described some examples of one-sector growth models exhibiting multiple equilibria. In the present section we assess their empirical plausibility by discussing some of the differential predictions of determinate vs. multiple equilibria models and evaluating their empirical merits in light of the existing data.

We take as a benchmark the equilibrium dynamics of the one-sector model with exogenous growth and a determinate equilibrium. We refer to them as "neoclassical dynamics," though keeping in mind that they are qualitatively

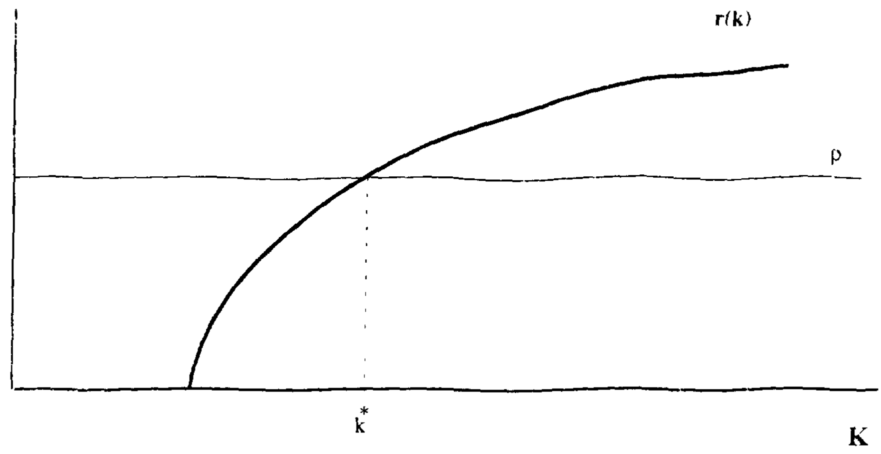


Figure 4a

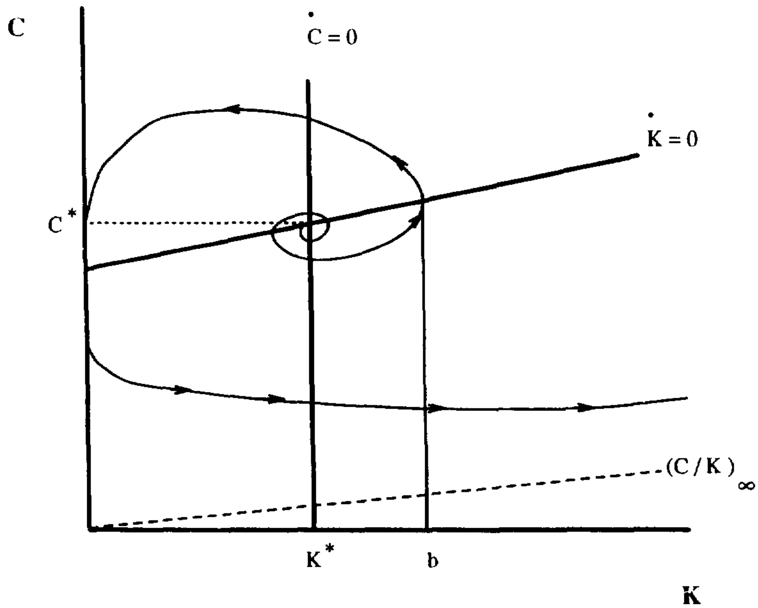


Figure 4b

not affected by the presence of distortions that are not strong enough to generate multiple steady states. We choose to focus on a key (and well-known) prediction of that model: *given the same fundamentals* – represented by identical values for the exogenous parameters – economies with different initial conditions (i.e., a different initial capital stock $K(0)$), will eventually achieve the same capital and income levels.[13] Following other researchers we refer to that prediction as *conditional convergence.* From our point of view the conditional convergence prediction is particularly relevant since it is violated by the class of one-sector models with exogenous growth and indeterminate equilibria discussed above, for which multiple "convergence clubs" will generally exist even when economies do not differ in terms of their fundamentals.

Formally, and using our previous notation, the income path towards which all economies converge under neoclassical dynamics is given by $Y(t) = y^* \phi e^{\gamma t}$, where we recall that ϕ denotes the initial level of technology, and where $y^* = g(k^*)$ depends on a number of parameters/functions describing preferences and technology and which we represent by a vector $\Omega \equiv [\delta, \rho, \gamma, \sigma, \theta, f]$ – but is independent of either ϕ or $K(0)$.[14]

The convergence hypothesis has been elegantly formalized by Barro and Sala-i-Martin (1992) in the form of a "convergence equation". Under the assumption of a Cobb-Douglas production function $f(k) = k^a$, they derive the following (approximate) expression for the growth rate of output between periods 0 and T:

$$(1/T)log(Y_t(T)/Y_i(0)) = \gamma_i + (1/T)(1 - e^{-\beta_i T})log(y_i^*/y_i(0)) \qquad (15)$$

where subscript i is a country index and β_i is a complicated function of Ω_i. Letting $dY_i \equiv (1/T)log(Y_i(T)/Y_i(0))$, i.e., the average growth rate of per capita income in country i between 0 and T, we can rewrite the above equation as:

$$dY_i = \gamma_i + \pi_i log(y_i^*) + \pi_i log(\phi_i) - \pi_i log(Y_i(0)) \qquad (16)$$

where $\pi_i \equiv (1/T)(1 - e^{-\beta_i T})$. Since none of the variables in the first three terms are directly observable, some additional assumptions must be made to evaluate/estimate the above equation.

Under the maintained assumption that all countries share identical parameter values (including the technology level ϕ) up to i.i.d. variations unrelated to initial conditions, all i–the above equation can be rewritten in terms

[13]Unless otherwise stated all variables we refer to should be interpreted in per capita (or per worker) terms.

[14]In general, Ω will include at least $\delta, \rho, \gamma, \sigma, \theta$, and f, as well as any other factors (e.g., distortionary capital income taxes) which may affect the steady state y^*.

of observables as

$$dY_i = \kappa - \pi log(Y_i(0)) + \epsilon_i \tag{17}$$

where $\kappa = \gamma + \pi log(y*) + \pi log(\phi)$, and ϵ_i is an i.i.d. error term resulting from measurement error or random variation in the technology level orthogonal to initial income. As is well-known (Romer (1986)), the previous inverse relationship seems to be absent when one looks at data for a broad sample of countries. Figures 5a and 5b revisit that evidence. Figure 5a uses GDP/Employment (Y^n) as an income measure, while 5b uses GDP/Population (Y^p). GDP, employment, and population data are taken from the Summers-Heston data set. We take 1965–1985 as our sample period.[15] OLS estimates of (17) using the Summers-Heston data set are statistically insignificant and often have the wrong sign (Barro (1991)). That evidence is usually referred to as "lack of unconditional convergence." For completeness we reproduce the OLS estimates obtained with our data (standard errors in parentheses):

$$dY_i^n = \underset{(0.305)}{0.67} - \underset{(0.036)}{0.037}\, log(Y_i^n(0)) + \epsilon_i \tag{18}$$

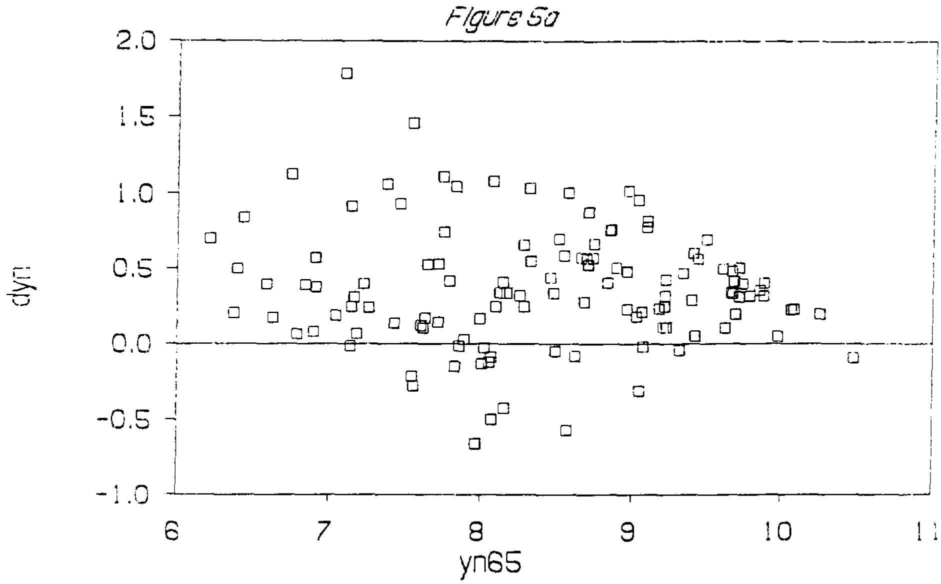
$$dY_i^p = \underset{(0.305)}{0.003} + \underset{(0.040)}{0.049}\, log(Y_i^p(0)) + \epsilon_i \tag{19}$$

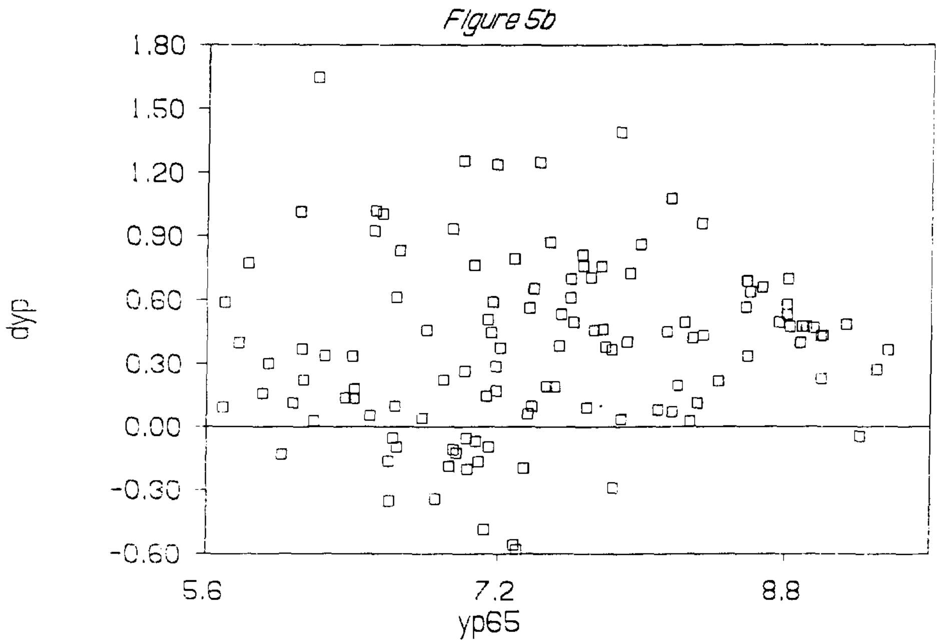
As many authors have stressed, a rejection of neoclassical dynamics on the basis of the previous evidence might be unwarranted, for the model predicts convergence in income levels only for economies with common parameters Ω and ϕ (though possibly different initial capital stocks). In other words, the model only predicts "conditional convergence." The insignificant π estimates in the regression above could potentially result from "large" variations across countries in y_i^* and/or ϕ_i that were positively correlated with initial income $Y(0)$ (as would occur if, say, all countries were "close" to their respective steady states in 1965). Unfortunately, neither y_i^* nor ϕ_i is directly observable, so one has to find some way of controlling for them. With that purpose in mind some authors have shifted their attention to regional data, under the maintained hypothesis that regions within countries (or larger homogeneous economic areas) are likely to have similar preference and technology parameters (though possible different initial conditions).[16] Other authors (e.g., Mankiw, Romer and Weil (1992), Barro (1991)) have instead reestimated the convergence equation after augmenting it with a set of additional

[15]Throughout we exclude Kuwait from our sample, whose 1965 productivity measures are a clear outlier in all our scatterplots and regressions.

[16]Well-known examples include the work of Barro and Sala-i-Martin (1991, 1992) using data on U.S. states and European regions. Much of that work has yielded unambigous evidence of convergence.

Figure 5: unconditional convergence





regressors that are meant to approximate possible variations in y^* and/or ϕ across countries. Not surprisingly, there is no widespread consensus on which observable variables should or should not be included as proxies for the unobservables, nor on how one should interpret the seeming fragility of the estimates of augmented versions of (17) to the introduction of alternative conditioning sets (Levine and Renelt (1992)).

Here we pursue a different strategy to test for conditional convergence. Our approach allows us to control for differences in fundamentals across countries without having to rely on ad hoc regression specifications. Instead, we pursue a model-based strategy, which exploits some restrictions implied by the theory, in combination with the information contained in the capital stock measures constructed by Benhabib and Spiegel (1994) for a large set of countries.[17]

Our procedure is described next. In a first stage we maintain the hypothesis that the values of all the parameters in vector Ω (i.e., those determining y^*) are the same across countries *while leaving unrestricted the range of variation of ϕ_i across countries, as well as its possible correlation with initial conditions.* Under the assumption of an isoelastic aggregate production function $g(k) = k^a, 0 < \alpha < 1$, we derive the following expression for $log\ y_i(0)$ in terms of the (observable) capital-output ratio:

$$logy_i(0) = \alpha log[k_i(0)/y_i(0)] + \alpha logy_i(0) = (\alpha/(1-\alpha))log[k_i(0)/y_i(0)] \quad (20)$$

Accordingly, we can rewrite "convergence equation" (16) as

$$dY_i = \kappa' - \psi log[K_i(0)/Y_i(0)] \quad (21)$$

where $\psi \equiv (\pi\alpha/(1-\alpha))$ and $\kappa' = \gamma + \pi log(y^*)$. We thus see that, conditional on a common parameter vector Ω, the neoclassical equilibrium dynamics imply a negative relationship between the initial capital-output ratio and the subsequent growth in income. Most importantly, *that relationship is invariant to the presence of (unobservable) differences in technology levels across countries.* We refer to the above prediction as *Ω-conditional convergence.*

Figures 6a and 6b show a scatterplot of dY_i and $[K_i(0)/Y_i(0)]$. Figure 6a uses GDP/Employment as a measure of income, while Figure 6b uses

[17]Benhabib and Spiegel have constructed capital stock time series for a large number of countries by combining time series data on investment flows with a variety of assumptions to compute the initial year capital stock. For the series we use, the initial stock was determined by regressing the capital stock on income, labor, and human capital using data for countries for which capital estimates are given in the Summers-Heston data set, and using the resulting coefficient estimates to infer the initial capital stocks of other countries, usually for some year in the 1950s, depending on the availability of investment data. Using other methods or depreciation rates made little difference to the results in Benhabib and Spiegel (1994).

GDP/Population. $[K_i(0)/Y_i(0)]$ is the capital-output ratio in 1965, using the Benhabib-Spiegel (1994) capital stock data. The corresponding OLS regression estimates of (21) are

$$dY_i^n = \underset{(0.687)}{1.199} + \underset{(0.107)}{0.130}\, log\left[\frac{K_i(0)}{Y_i^n(0)}\right] + \epsilon_i \tag{22}$$

$$dY_i^p = \underset{(0.705)}{1.475} + \underset{(0.110)}{0.172}\, log\left[\frac{K_i(0)}{Y_i^p(0)}\right] + \epsilon_i \tag{23}$$

i.e., in both regressions we get a point estimate for the capital-output coefficient that has the "wrong" sign (though insignificant).[18] Thus, we do not find evidence favorable to the hypothesis of *Ω-conditional convergence.*

The results above can hardly be interpreted as a dismissal of neoclassical equilibrium dynamics. For they could be consistent with the presence of significant cross-country differences in Ω_i – and thus in γ_i and y_i^* – which could lead to a correlation between the ϵ's and the initial capital-output ratio. In that case the estimates obtained above could result from a strong positive correlation between y_i^* and $y_i(0)$ (and thus between y_i^* and the capital-output ratio), combined with a large variance of y_i^* (relative to $y_i(0)$), a situation which we cannot rule out.

Notice, however, that as long as γ_i and $log(y_i^*)$ remain constant over time (for each i), we can treat them as "fixed effects" which can be differenced away by splitting the sample period into two subsample periods, and rewriting (21) as

$$dY_i(t,T) - dY_i(0,t) = -\psi_i[dK_i(0,t) - dY_i(0,t)] \tag{24}$$

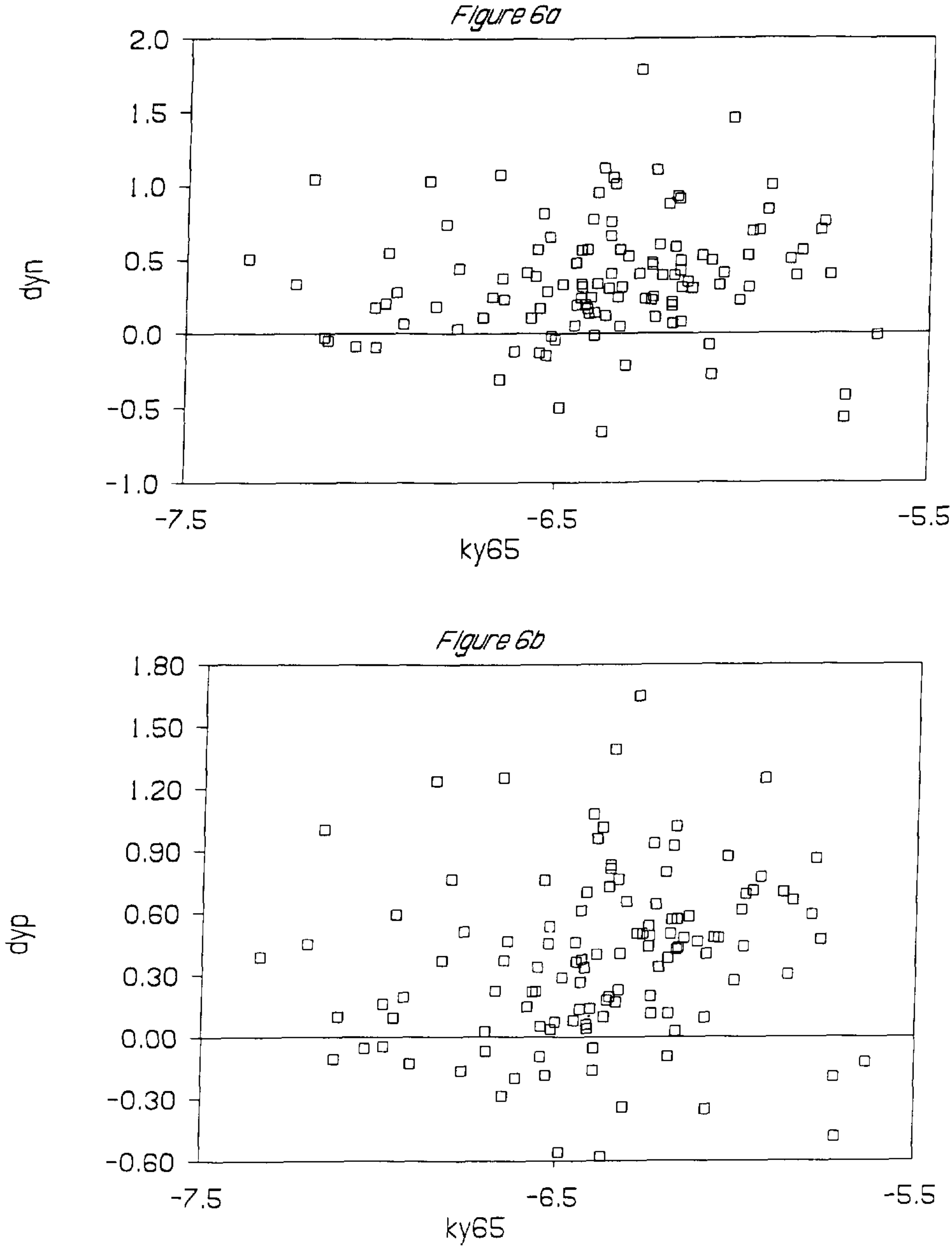
where $\psi_i = (\pi_i\alpha_i/(1-\alpha_i)) > 0, 0 < t < T$, and where we define $dX_i(a,b) \equiv (1/(b-a))[log(X_i(b)) - log(X_i(a))]$. Thus, we see that differencing allows us to eliminate the country-specific intercepts in the convergence equation, though it still leaves us with possible cross-country variations in the slope ψ_i. Yet, even in the presence of the latter, we can derive the following observable restriction on the data:

$$\Sigma_i \equiv [dY_i(t,T) - dY_i(0,t)][dK_i(0,t) - dY_i(0,t)] < 0, all \;\; i \tag{25}$$

Intuitively, (25) says that economies that experience an increase in their capital-output ratio between period 0 and period t should also experience a slowdown in the rate of output growth in period (t,T) relative to $(0,t)$.

[18]The positive coefficient estimates make it hard to explain the lack of evidence of convergence as resulting from classical measurement error in our capital stock measures, since the latter would tend to bias the coefficient estimator toward zero but preserving the true correlation sign.

Figure 6: omega-conditional convergence



We refer to the previous prediction as *weak conditional convergence.* Notice that (25) should hold under neoclassical dynamics even in the presence of arbitrary differences in fundamentals (i.e., in parameters Ω and ϕ) across countries, as long as the latter are constant over time (country fixed-effects).

Figures 7a and 7b permit us to assess the empirical validity of (25). The horizontal axis represents $dK_i(65,75) - dY_i(65,75)$, i.e., the average growth rate of the capital-output ratio over the period 1965-1975. The vertical axis measures $dY_i(75,85) - dY_i(65,75)$, i.e., the difference between average income growth rate over the period 1975-1985 and the corresponding growth rate over the period 1965-1975. Again, Figure 7a uses per worker measures of income, while Figure 7b uses per capita measures. Notice that (25) implies that all the observations should lie in the northwest and southeast quadrants (though not necessarily on a straight line). A glance at either figure points to a clear violation of such a prediction for a large number of countries. In fact, for more than 50% of the countries in our sample (in both cases), the observations lie in the wrong quadrants!

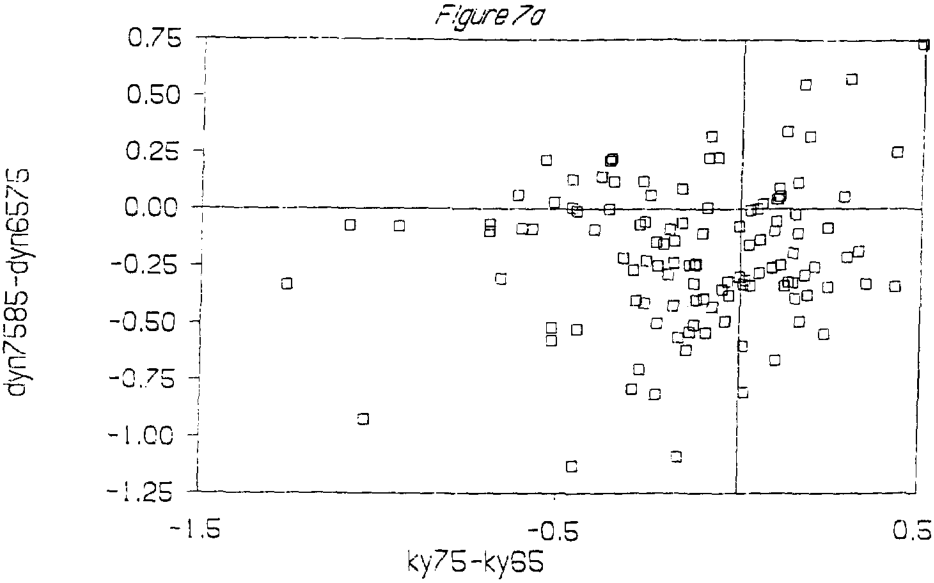
The previous evidence seems to reject the prediction of conditional convergence implied by the neoclassical model with exogenous growth and, more generally, by versions of the one-sector model with a unique steady state and globally determinate equilibrium dynamics. That evidence, though clearly consistent with models with multiple equilibria discussed above, seems also consistent with one-sector models with determinate equilibria, including models with endogenous growth (e.g., the Ak model discussed above), as well as models with multiple steady states and threshold effects (e.g., Azariadis and Drazen (1990)).

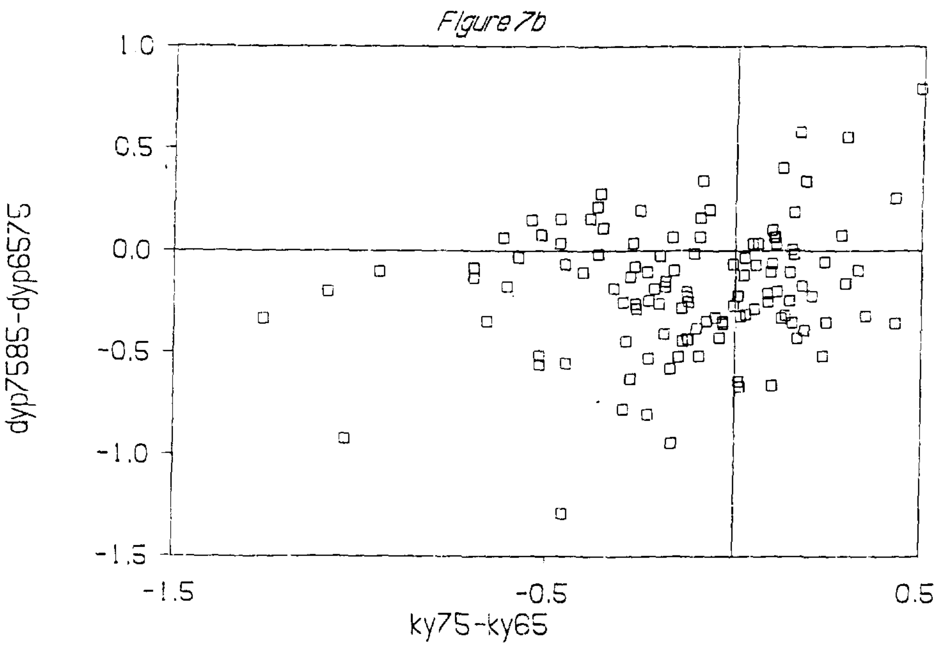
Let us first consider the plausibility of threshold models. As is well-known, those models predict convergence in capital and income levels among economies with identical parameters *and initial capital stock* $k(0)$. More precisely (and restricting ourselves to the case of one accumulated factor, capital), the domain of $k(0)$ is partitioned into a number of nonoverlapping intervals $Q_1, Q_2, \ldots$ each of which is associated with a steady state k_j^*, in the sense that if $k(0) \in Q_j$ then $lim_{T\to\infty} k(T) = k_j^*$. That sort of equilibrium dynamics is predicted by models with nonconvex technologies, where the nonconvexity results from the existence of capital accumulation "thresholds" that induce a shift in productivity (e.g., Azariadis and Drazen (1990)), but also in models with convex technologies (e.g., Azariadis (1993)).

Durlauf and Johnson (1994) have attempted to test for the presence of threshold effects and multiple steady states by estimating a "regime-dependent" version of the Mankiw, Romer and Weil (1992) convergence equation after splitting the set of countries into several groups on the basis of their initial income and literacy rates.[19] They test the cross-subsample restrictions

[19]The MRW convergence equation corresponds, essentially, to a version of (17) aug-

Figure 7: weak-conditional convergence





implied by the single steady state neoclassical model using alternative groupings. They find that for most splits (including those generated endogenously by means of a regression-tree procedure), the null of constancy of coefficients across subsamples is easily rejected. That rejection appears robust to the inclusion of additional control variables. Durlauf and Johnson interpret their rejection of the neoclassical equilibrium dynamics as being compatible with models with multiple steady states and determinate dynamics, but do not attempt to relate it to models with multiple equilibria. Even though their "alternative hypothesis" does not embed models with multiple equilibria, we view some of their results as hinting that possibility. In particular, the estimates of the coefficients associated with initial income for some of the groupings generated by the Durlauf and Johnson regression-tree procedure are not significant, suggesting lack of convergence for *some* countries with similar initial conditions, a result which one could interpret as consistent with the presence of multiple equilibria (for some range of initial conditions).

Perhaps more enlightening for our purposes is the analysis in Quah (1993, 1994). Quah attempts to characterize the dynamics of the cross-sectional distribution of income (relative to the mean) by estimating the Markov chain transition matrix associated with an underlying (evolving) relative income distribution. In particular, Quah's estimated 23-year transition matrix is characterized by large estimated probabilities along the main diagonal (suggesting strong persistence in relative income), but also includes far from negligible estimated probabilities on the adjacent minor diagonals (implying a substantial "reshuffling" or mobility over time in cross-country relative income distributions). The latter observation appears to be at odds with determinate equilibrium dynamics (either neoclassical or of the threshold type). Most interestingly, the ergodic distribution of relative income implied by Quah's estimates shows a thinning in the middle, and an accumulation in both low and high tails (suggesting the presence of convergence clubs). On that basis, we view Quah's results as being consistent with some of the models with multiple equilibria discussed above.

The previous interpretation of Quah's work is, however, subject to a caveat. The evidence of reshuffling and convergence clubs that emerges from Quah's results has a potential explanation consistent with determinate equilibria if cross-country differences in parameters and technology levels are important.[20] We can at least control for possible differences in technology levels by replacing per capita income in Quah's analysis with capital-output

mented with a number of regressors that are meant to proxy exogenous differences in rates of investment in physical and human capital. See Durlauf and Johnson (1994) for details.

[20]To see this, consider two countries whose ranking in terms of initial capital stocks is different from their ranking in terms of technology levels: the neoclassical model would predict "overtaking" in finite time.

ratios for, under the null hypothesis of determinate dynamics, the latter are a monotonic transformation of the normalized capital stock k. The outcome of that exercise is a 20-year transition frequency matrix for (K/Y) (relative to its mean value), which is displayed in Table 1. The grid $(0, 0.8, 1.2, \infty)$ was used in both periods. Clearly, we see that the use of the capital/labor ratio (instead of per capita income) does not alter Quah's basic results, in particular, the presence of non-negligible off-diagonal terms, indicating significant changes in countries' position in the distribution of capital/labor ratios, a finding difficult to reconcile with models with determinate equilibrium dynamics. We also observe a substantial thinning over the period considered of the distribution in the middle, suggesting a trend towards increased inequality in the capital-output ratio for countries that enjoyed similar initial conditions (i.e., middle-income countries in 1965).

Table 1:

	1985		
1965	<0.8	(0.8, 1.2)	> 1.2
< 0.8	0.69	0.24	0.06
(0.8, 1.2)	0.31	0.34	0.34
> 1.2	0.09	0.21	0.68
1965 dist.	0.24	0.48	0.26
1985 dist.	0.35	0.28	0.37

Could some of the evidence of nonconvergence presented above be accounted for by the one-sector model with endogenous growth and a determinate equilibrium, as in Rebelo (1991) or Jones and Manuelli (1990) or indeterminacy (as in Gali and Zilibotti (1995))? As is well-known, even under the assumption of common fundamentals (other than initial conditions), those models imply the absence of convergence in income levels, at least for countries that do not get stuck in a poverty trap. Yet, we can think of a number of reasons why those models may be difficult to reconcile with the data. First, to the extent that the underlying parameters are the same across countries, those models will have a hard time explaining the observed dynamics of cross-sectional distributions of income or capital-output ratios discussed above and, in particular, the evidence of reshuffling. The kind of parameter differences that would be necessary to account for those dynamics would most likely imply permanent differences in growth rates (and thus "explosive" inequalities) and in private returns to capital across countries (with the persistent incentives for capital to move to richer countries), implications which we find implausible on a priori grounds. Second, there seems to be some evidence for the existence of a "convergence club" of industrialized economies (see, e.g., Baumol et al. (1989)), a phenomenon that suggests the presence of eventual diminishing returns to capital, as in the models with multiple equilibria and exogenous growth discussed above. Finally, we view the standard versions of the endogenous growth version of the one-sector model as a useful shortcut with notable pedagogical value, but little realism, for among other things it implies a zero labor income share (at least asymptotically).[21] The latter observation motivates our turning our attention to two-sector models of endogenous growth, with and without indeterminacy.

3 Indeterminacy and growth with endogenous labor

Indeterminacy also occurs in models with a single accumulated factor and an endogenous labor supply in the presence of increasing returns, introduced through externalities or sustained by imperfect competition (Benhabib and Farmer (1994)). Since the model with externalities is simpler we choose it for expository purposes.[22]

A representative consumer maximizes discounted lifetime utility with respect to consumption and labor-supply trajectories, where consumption is c and labor supply is L:

$$Max \int_0^\infty (\ell nc - (1+\epsilon)^{-1}(L^{1+\epsilon} - 1))e^{-\rho t}dt \tag{26}$$

where ϵ is the reciprocal of the labor-supply elasticity. The production function for consumption and investment in the single accumulated factor h are given respectively by:

$$c = h^\alpha h_a^v u^\omega \tag{27}$$

$$\dot{h} = \delta h^\theta h_a^{1-\theta}(L-u)^\phi (L-u)_a^\zeta \tag{28}$$

$$h(0) = h_0 \tag{29}$$

The subscript "a" indicates that the maximizing agent takes the path of the associated variable as exogenous and represents an externality in the usual way. $\alpha, v, \omega, \theta, \varphi$ and ζ are exponents of the production function and δ is a constant. The amounts of labor input allocated to consumption and investment are, respectively, u and $(L-u)$.

[21] This is not necessarily true in the presence of externalities, though in that case a balanced growth path will exist only in knife-edge cases.

[22] For an analysis of the corresponding two-sector model with inelastic labor supply, see Boldrin and Rustichini (1994). The corresponding model with two accumulated factors is discussed in the following section.

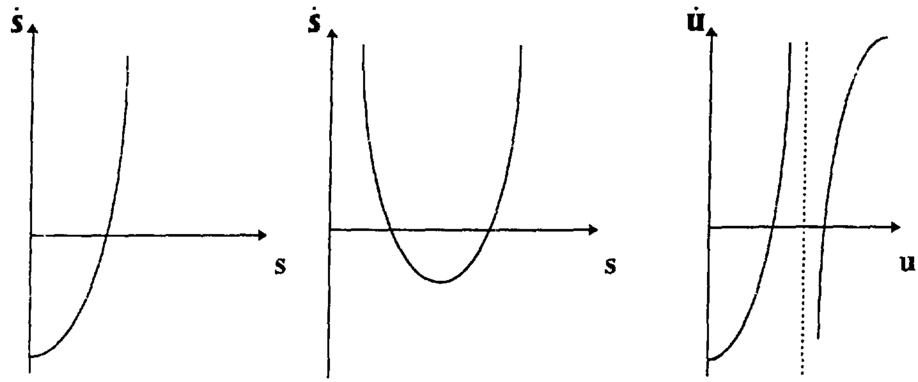
Under the assumption that $\epsilon = 0$ (perfectly elastic labor supply), the analysis of the model is considerably simplified. In that case it is easy to show that $u = \omega\varphi$, a constant. Furthermore, if λ is the costate variable associated with the accumulation equation for h, the dynamics of the system can be entirely described by $s \equiv \lambda h$. The classification of the equilibrium dynamics is quite simple in that case. If the labor externality in the investment sector is small so that $\phi + \zeta < 1$, we have a unique balanced growth path (BGP). Just as in the Ak model, the growth rate is constant from the start, so the BDP effectively corresponds to the unique equilibrium path of the economy. The previous case is illustrated in Figure 8a and corresponds to the intersection of the curve $\dot{s} \equiv f(s)$ and the horizontal axis. If a BGP exists when $\phi + \zeta > 1$, there will generically be two of them, one determinate and one indeterminate. In that case the growth rate is not pinned down by the parameters of the model. Given an initial value for h, labor supply can be high or low since the initial choice of s (or λ) is not determined by that initial condition. There is a whole range of values of s that are consistent with equilibrium, all of them involving long-term convergence to the low BGP (see Figure 8b). It is easily shown that in the indeterminate case the labor supply L and $(L - u)$ are inversely related to s, so that a low s implies higher L and a faster growth rate for the economy. Numerical examples (not reported) suggest that welfare levels are higher on the fast-growing BGP.

The possibility of indeterminacy reflects a tradeoff between the labor supply elasticity and the size of the labor externality. If $\epsilon \neq 0, u$ is no longer a constant in equilibrium and the dynamics must now be formulated in terms of u instead of s. When $\phi + \zeta > 1$ and ϵ is increased while keeping the other parameters fixed, we do not revert to the determinate case with a unique BGP. The two BGPs persist, but both become locally determinate. From a global perspective, however, we still have indeterminacy: given an initial $h(0)$ the economy can be placed at either of the locally determinate BGPs (see Figure 8c). That situation illustrates that the tradeoff between the labor-supply elasticity and the labor externality is only relevant for local but not global indeterminacy and expands the set of parameter values for which indeterminacy obtains. It can be shown that the BGP with the lower u and therefore with the higher growth rate yields higher overall welfare for reasonable parameter value configurations.

4 Indeterminacy in a model with physical and human capital

4.1 *A brief description of the model*

In the previous sections we focused on models in which a single factor of production was accumulated endogenously. In this section we turn our attention to versions of the Lucas (1988) model, which allows for endogenous accumu-



(a) (b) (c)

Figure 8

lation of two factors, physical and human capital. As shown by Benhabib and Perli (1994), Chamley (1993), and Zie (1994), there exists a range of parameter values for which the equilibrium of that model is not determinate. In that case, the restrictions on the transitional dynamics derived under assumptions that guarantee the uniqueness of equilibrium may break down.[23] We next discuss the nature of those restrictions in the context of a variant of the Lucas model, and then use our data on physical and human capital to explore their empirical validity.

We sketch out the variant of the Lucas model as developed in Benhabib and Perli (1994). Briefly, an infinite-lived agent maximizes lifetime utility (26), subject to the constraints:

$$\dot{k} = Ak^{\beta}h^{\alpha}u^{\omega} - c \tag{30}$$

$$\dot{h} = \delta h^{\theta}h_a^{1-\theta}(L-u)^{\phi}(L-u)_a^{\zeta} \tag{31}$$

$$k(0) = k_0, h(0) = h_0 \tag{32}$$

where k is physical, h is human capital and, as above, the subscript a indicates that h also enters as an external effect in the production of human capital. There are a few differences between the model above and the Lucas (1988) model. The most important is the inclusion of a labor-leisure choice. Allowing for an endogenous choice of labor supply requires that the production function for human capital be concave from the agent's perspective in order to preserve the overall concavity of the maximization problem.[24] In the Lucas model we have $\theta = 1, \phi = 1, \zeta = 0$, and $\omega = \alpha = 1 - \beta$. In our setup with an endogenous labor choice, concavity requires $\phi + \theta \leq 1$, which we assume. Furthermore, unlike the Lucas model we have eliminated externalities from the production function in physical goods. Restoring such externalities would not change our analysis but we leave them out for simplicity.

The maximization problem of the agent results in a four-dimensional differential equation system with two state variables, k and h, and the two associated costate variables, λ_1, and λ_2, respectively. The previous variables can be transformed into ratios and the dimension of the model can be reduced by one. The dynamics and the balanced growth paths can be analyzed in terms of the following three transformed variables.[25]

$$s = \lambda_2 h, \quad m = k^{\beta-1}h^{\alpha}, \quad q = \frac{c}{k}. \tag{33}$$

In the Lucas model with externalities in the production of goods and inelastic labor supply, the balanced growth path (BGP) is, except for hairline

[23]See Caballe and Santos (1993) and Mulligan and Sala-i-Martin (1993) for a discussion of the transitional dynamics in the case of a unique equilibrium.

[24]See Benhabib and Perli (1994) for a further discussion of this point.

[25]See Benhabib and Perli (1994) for details.

cases, unique, but it may be determinate or indeterminate (see Benhabib and Perli (1994)). In the modified case considered above there may be one or two BGPs depending on the parameter value configurations. When the BGP is unique, it is also determinate. When there are two BGPs, one is locally determinate while the other may be locally determinate or indeterminate. The case of two BGPs arises when $\phi + \zeta > 1$, while for $\phi + \zeta < 1$ there is a unique BGP. This characterization holds for a wide range of values of the other parameters. Basically, indeterminacy or a continuum of equilibria can be heuristically explained as follows. Starting from an arbitrary equilibrium trajectory, constructing another equilibrium path with a higher rate of accumulation of human capital requires a higher rate of return. This higher rate of return may obtain if the externality on labor is high enough (in fact, only if the "social" marginal product of labor is increasing), resulting in a sustainable reallocation of labor to the production of human capital and thereby increasing the return to human capital. Clearly this cannot happen in a concave model without externalities. Two critical issues arise. The initial reallocation of labor away from the production of goods must decrease consumption initially, which cannot happen with too low an intertemporal elasticity of consumption. Thus, in the original Lucas (1988) model indeterminacy requires a low curvature of the utility function. When the labor supply is elastic however, the increase in labor allocated to the production of human capital can come from leisure, so a high intertemporal elasticity of consumption is not necessary. In the latter case there is still a tradeoff between the size of the labor externality and the labor supply elasticity $1/\epsilon$. The table below, taken from Benhabib and Perli (1994) illustrates this tradeoff. Note that we are considering a case for which $\phi + \zeta > 1$, with $\phi = 0.9$ and the externality $\zeta = 0.12$, a reasonably small number, especially since the externality is only confined to the production of human capital and knowledge.[26]

Table 2 reports the local stability properties of the two BGPs for alternative ϵ values.[27] The first one is always determinate, with the linearized dynamics showing two positive and one negative root: initial jump variables $s(0)$ and $q(0)$ must be chosen to place the economy on the stable manifold

[26]It may be thought that the small externality required for indeterminacy is due to the special two-sector structure of the model. Benhabib and Farmer (1995) have shown that only very small externalities are required for indeterminacy in a model where the consumption and investment goods have identical technologies. The difference from the earlier Benhabib and Farmer (1994a) paper is that externalities do not arise from aggregate inputs or the aggregate output, but each sector benefits from externalities that arise from its own output or use of inputs. If interpreted as markups, the markup ratio required for indeterminacy in this model can be as low as 1.12 to 1.16, coupled with high values of ϵ, representing an inelastic labor supply.

[27]For additional tables with other parameter values, see Benhabib and Perli (1994).

of the BGP. This BGP exhibits very little growth, since L_1^* almost equals u_1^* and almost no labor is allocated to the production of human capital. The second BGP is locally indeterminate for $\epsilon = 1.1$ and 1.2, since the linearized dynamics have two negative roots: given the predetermined variable $m(0)$ it is possible to choose $s(0)$ and $q(0)$ from a whole continuum and still converge to the BGP to satisfy the standard transversality conditions. It is important to stress here that such an indeterminacy result arises for very plausible values for both the intertemporal elasticity of substitution of consumption (equal to one, as implied by the log utility assumption) and the elasticity of labor supply (less than one).

Table 2:

ϵ	L_1^*	u_1^*	Roots	L_2^*	u_2^*	Roots
1.1	0.629	0.625	- + +	0.636	0.619	- - +
1.2	0.641	0.640	- + +	0.656	0.626	- - +
1.3	0.653	0.653	- + +	0.673	0.633	- + +
2.0	0.721	0.721	- + +	0.756	0.669	- + +
10.0	0.844	0.844	- + +	0.955	0.755	- + +

Note: A = 1, $\beta = 0.25$, $\omega = 0.375$, $\alpha = 0.375$, $\delta = 0.05$, $\rho = 0.025$, θ=0.1, ϕ = 0.9, $\zeta = 0.12$

Note also that for ϵ values of 1.3, 2.0, 10.0 (i.e., for substantially lower labor-supply elasticities) both BGPs are locally determinate, but a global indeterminacy still persists: given initial conditions it is possible to choose initial values of s and q to place the economy on the stable manifold of either BGP (see Benhabib and Perli (1994)).

4.2 *Empirical evidence*

In the absence of indeterminacy, the model above imposes restrictions on the transitional dynamics of the physical and human capital stocks. Given the initial conditions on k and h, the investment rates on the equilibrium path are uniquely determined. Caballe and Santos (1993) show that in the Lucas Model without externalities, (k/h) ratios above its value for the balanced growth path are associated with subsequent physical capital growth rates below the asymptotic BGP growth rate.[28] More generally, Bond, Wang and,

[28]This is also true in a version of the Lucas model for which the production of human capital utilizes physical capital more intensively than the production of goods

Yip (1993) have shown that the transitional dynamics of a model with both human and physical capital but without externalities, where both physical and human capital enter the production function of human capital, produce a negatively sloped projection of the saddle-path in the (c/k) and (k/h) plane, with both ratios monotonically converging to their BGP value. Those results suggest that the investment rate in physical capital minus the investment rate in human capital should be negatively correlated with the initial (k/h) ratio, while the growth of consumption minus the rate of growth in physical capital should be negatively related to the (c/k) ratio. At least locally, similar results should hold for models that allow for externalities, provided there is no indeterminacy. Thus, in the model described in the previous section, determinacy requires the Jacobian matrix of the linearized dynamics to have two roots with positive real parts and one negative real root. The initial values of s and q (where $q = c/k$) are chosen to neutralize the effects of the positive real roots on the local dynamics. The dynamics are then governed by the single negative real root and generate a monotonic convergence of those variables to the BGP, i.e., c/k and $m \equiv k^{(\beta-1)}h^{\alpha}$ converge monotonically to their balanced growth values.[29] Thus, at least locally, the weighted difference $\alpha(\dot{h}/h) - (1-\beta)(\dot{k}/k)$ must be negatively related to initial m and the growth rate in consumption minus the growth rate in physical capital must be negatively correlated to (c/k).

Whenever there is local indeterminacy, the Jacobian of the linearized dynamics around one of the balanced growth paths has two negative and one positive root. Accordingly, given initial values of k and h (such that m and q are in the neighborhood of their steady-state values), the initial values of s and q can be chosen from a two-dimensional manifold–in (s, q, m) space– which neutralizes the effect of the positive root on the local dynamics. Any choice of initial values of s and q on this manifold generates a path that converges to the BGP and satisfies the transversality conditions. Since the initial s and q can be chosen arbitrarily on the appropriate manifold, consumption and investment rates in h and k are not uniquely determined by the initial k, h, or the ratio m. Given a cross-section of countries at different levels of m, an arbitrary selection of investment rates from the locally stable manifold would break the simple relation between m and the rates of investment that obtains in the determinate case. Since the dynamics of the variables are now governed by two negative roots instead of one, the monotonic convergence property of m and (c/k) to the balanced growth path

[29]Strictly speaking, it is the variable $m = k^{(\beta-1)}h^{\alpha}$ that uniquely determines s and q in the model above. Note from Benhabib and Perli (1994) that when there is a human capital externality in the production of physical goods, say parametrized by γ, then $m = k^{(\beta-1)}h^{(\alpha+\gamma)}$. In the Lucas (1988) model without endogenous labor and $\alpha = \omega, m$ reduces to (k/h) if $\gamma = 0$.

need not hold, even locally. Furthermore, there is no reason for a country not to suddenly jump to another convergent path[30] with lower (c/k), even if (c/k) were to begin with below its balanced growth value.[31]More intuitively, under indeterminacy countries with similar endowments of resources may coordinate on high or low consumption rates or investment rates, and thereby grow at different rates in equilibrium. These considerations can break the prediction of a negative relationship between the growth rates of (c/k) and m, and their respective initial levels.[32]

The following step consists in using cross-country data on k and h to investigate the empirical relevance of the predictions of the two-sector model with a determinate equilibrium. The physical capital data, constructed in Benhabib and Spiegel (1994), was described above. Our measure of human capital corresponds to the average years of education of the labor force multiplied by the number of workers, and is taken from Kyriacou (1991).

The results of Caballe and Santos (1993) for the Lucas model without externalities imply that investment rates in physical capital will be negatively (positively) related to the initial (k/h) ratio. The first set of plots (Figure 9) explore this prediction by plotting the average growth of k for the sequence of five-year periods 65-70, 70-75, 75-80, and 80-85, against (respectively) the log of (k/h) in the years 1965, 1970, 1975, and 1980. We also look at a similar relationship over ten-year periods (with initial years of 1965 and 1975), as well as over the twenty-year period (starting in 1965). To test the predictions obtained by Bond, Wang, and Yip (1993), we follow the same procedure as above, but now we plot the growth rate of (k/h) against its initial level (in logs). Their monotonic convergence results imply that there should be a negative relationship between those two variables. The corresponding results are displayed in the second set of plots (Figure 10). We also examine the corresponding prediction for the model with externalities and determinacy of an inverse relationship between the growth rate of m and its initial level. In order to do so we must assign values to α and β, the shares of capital and labor in the production of goods. Remember that in the model above, this sector has no externalities and that $m = k^{\beta-1}h^{\alpha}$. As a rough guide

[30]We admit, though, that this argument is not fully consistent with a strict interpretation of our model, which assumes perfect foresight.

[31]If a country does not jump across convergent equilibrium paths, variables like c/k, k/h, or m will still not necessarily converge to the BGP in a monotonic fashion. Therefore under indeterminacy, we still cannot expect the growth in these variables to be negatively related to their initial values over shorter periods like five or ten years. If, however, we relate growth over a long period (20 years?) to initial income and there are no jumps across convergent paths, the negative relation between growth rates and initial values can reappear despite the nonmonotonicity over shorter intervals of five or ten years even under indeterminacy.

[32]A formal treatment of this point in an abstract context and an illustration in the context of a simple game is given by Jovanovic (1989).

we take the estimates of Mankiw, Romer, and Weil (1992) which assign an approximate share of 1/3 to each of the three factors. Thus, $\alpha = 1/3$ and $\beta = 1/3$. The results are presented in the third set of plots (Figure 11). Finally, the model with determinacy also predicts a local inverse relation between the growth rates in (c/k) and the initial levels of (c/k). The results are in the fourth and last set of plots (Figure 12). The regression estimates associated with all these plots are presented in Tables 3-6.

Table 3:
Growth Rates of k/H (Dependent Variables in Columns) Regressed on Logs of Initial Year Ratios of K/H. Constant Term not Reported. Standard Errors in Parentheses.

	DKH6570	DKH7075	DKH7580	DKH8085	DKH6575	DKH7585	DKH6585
LKH65	-0.124 (0.067)						
LKH70		-0.105 (0.072)					
LKH75			0.019 (0.037)				
LKH80				0.055 (0.026)			
LKH65					-0.636 (0.201)		
LKH75						0.019 (0.090)	
LKH65							-2.697 (0.754)

Note: DKHxxyy is the growth rate of k/h from 19xx to 19yy. LKHxx is the natural log of k/h for the year 19xx.

It seems clear that no systematic negative correlation pattern between initial values and subsequent growth rates emerges from these figures. The observations are widely dispersed and very often slopes are positive rather than negative, a result which is in conflict with determinate equilibrium dynamics. Nevertheless, these results should only be taken as suggestive and subject to many caveats. Part of the dispersion in the behavior of investment rates at similar endowment ratios could be caused by fixed effects (e.g., differences in policies, technologies, or preferences), which we do not attempt to control in this section. Yet, in order for those differences to explain our results, one would have to argue that they are systematically correlated with the initial conditions. Second, a model with more than two accumulated factors can have much more complex transitional dynamics, even in the neighborhood of a determinate BGP, so that looking at the behavior of just two factors (physical and human capital) may be quite misleading. Third, measurement error in physical and human capital stocks may in some circumstances have biased some of our results. Finally, the predictions of the model presented above are, strictly speaking, of a local nature. While restricting the range of

GROWTH OF k AGAINST LOG OF INITIAL k/h. DK2xxyy ON VERTICAL AXIS INDICATES GROWTH FROM 19xx TO 19yy. LKHxx ON HORIZONTAL AXIS INDICATES INITIAL YEAR 19xx.

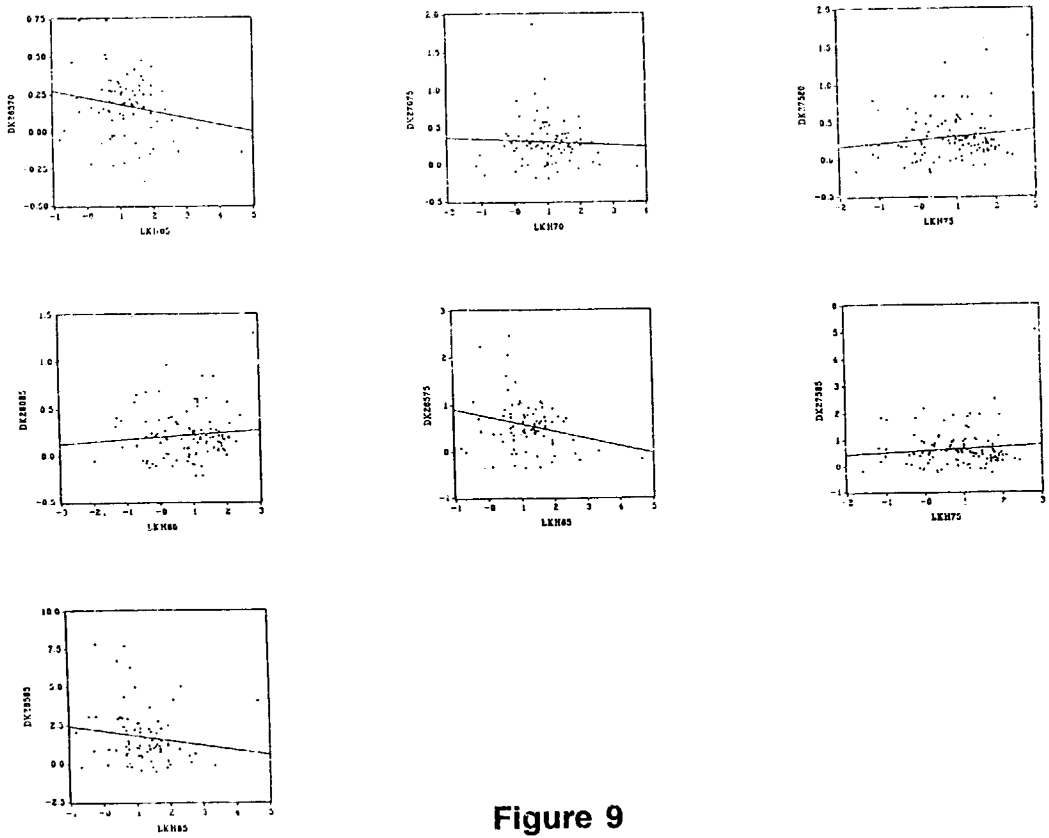


Figure 9

GROWTH OF k/h AGAINST LOG OF k/h. DKHxxyyON VERTICAL AXIS INDICATES GROWTH FROM 19xx TO 19yy. LKHxx ON HORIZONTAL AXIS INDICATES INITIAL YEAR 19xx

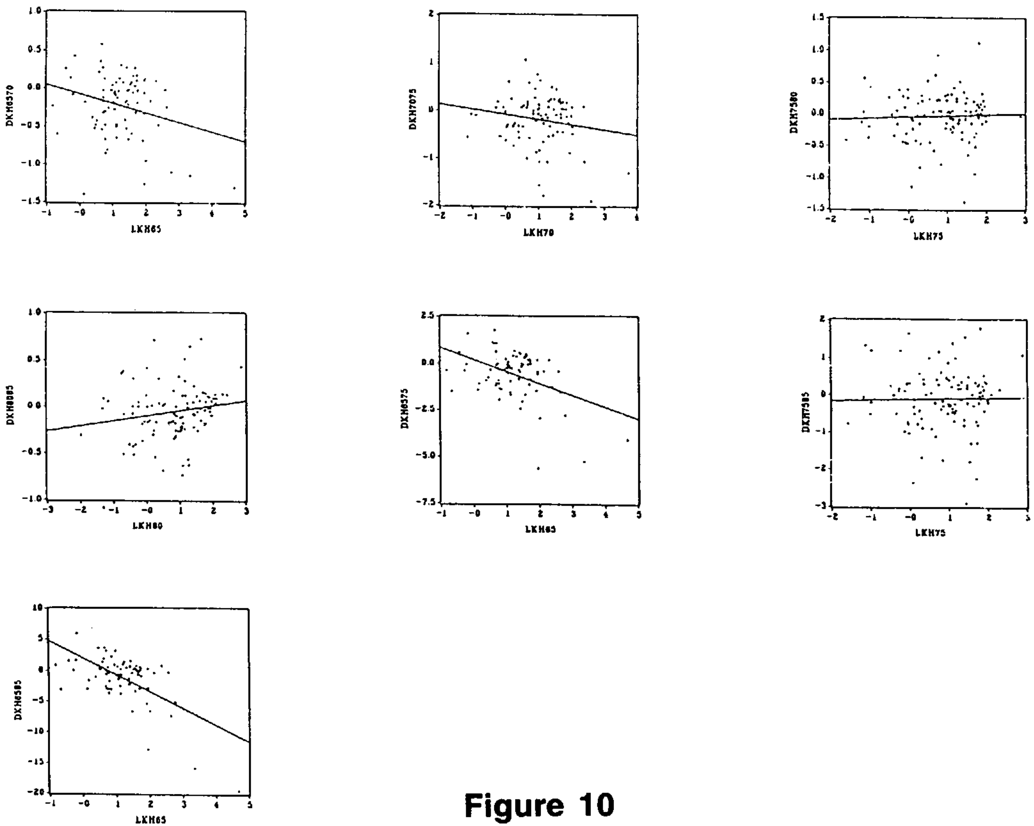


Figure 10

GROWTH OF m AGAINST LOG OF INITIAL m. DMxxyy ON VERTICAL AXIS INDICATES GROWTH FROM 19xx TO 19yy. LMxx ON HORIZONTAL AXIS INDICATES INITIAL YEAR 19xx.

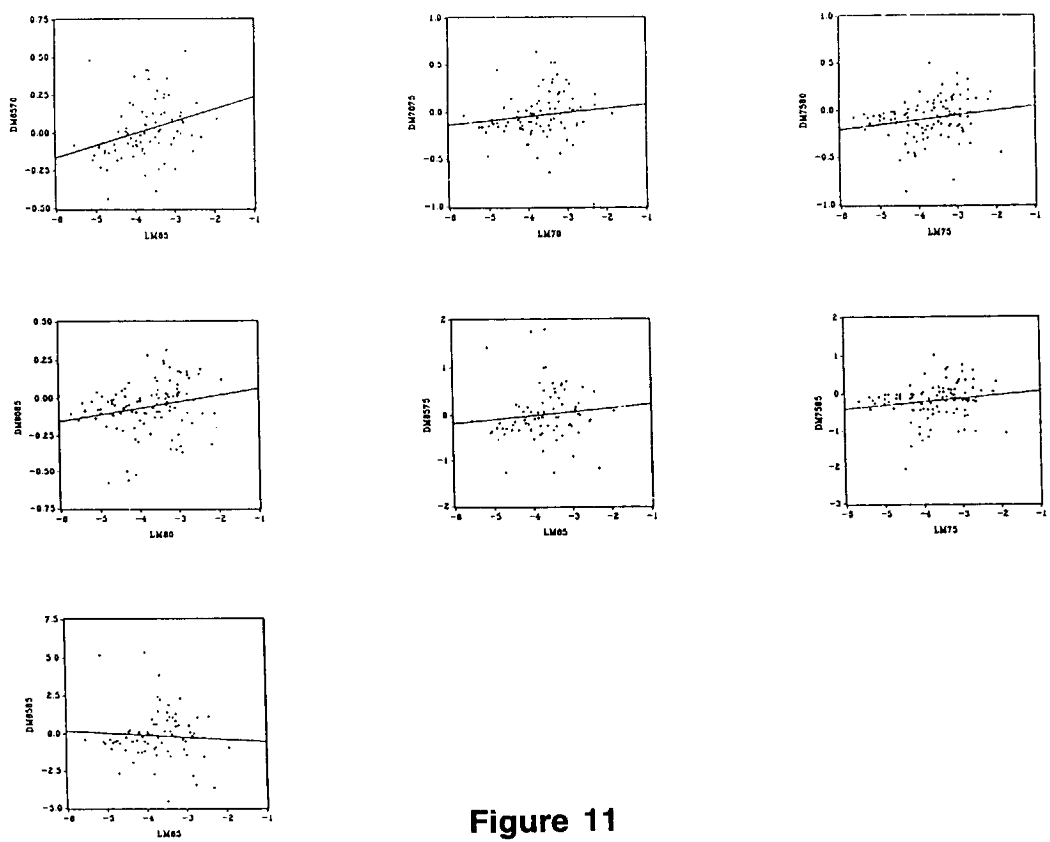


Figure 11

GROWTH OF INITIAL c/k AGAINST LOG OF INITIAL c/k. DCK2xxyy ON VERTICAL AXIS INDICATES GROWTH FROM 19xx TO 19yy. LCKxx ON HORIZONTAL AXIS INDICATES INITIAL YEAR 19xx.

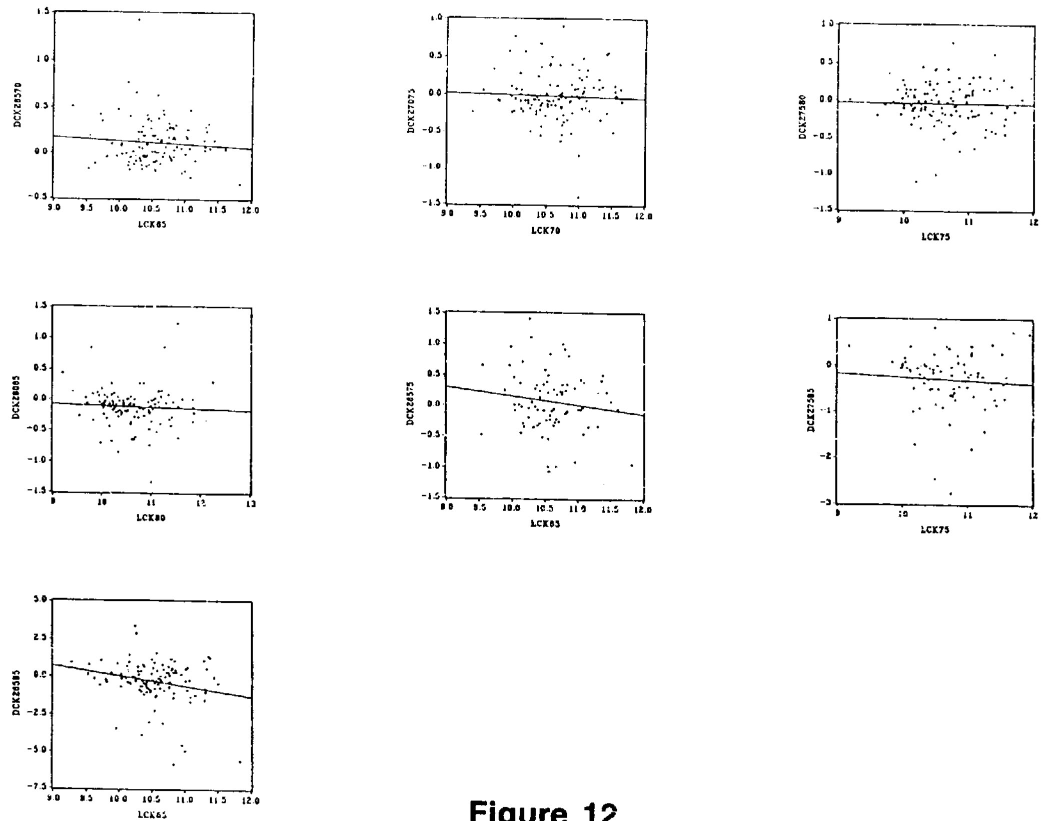


Figure 12

Table 4:
Growth Rates of k (Dependent Variables in Columns) Regressed on Logs of Initial Year Ratios of k/h. Constant Term not Reported. Standard Errors in Parentheses.

	DK6570	DK7-75	DK7580	DK8085	DK6575	DK7585	DK6585
LKH65	-0.045						
	(0.027)						
LKH70		-0.020					
		(0.035)					
LKH75			0.044				
			(0.042)				
LKH80				0.023			
				(0.031)			
LKH65					-0.154		
					(0.071)		
LKH75						0.067	
						(0.121)	
LKH65							-0.304
							(0.303)

Note: DKxxyy is the growth rate of k from 19xx to 19yy. LKHxx is the natural log of k/h for the year 19xx.

Table 5:
Growth Rates of m (Dependent Variables in Columns) Regressed on Logs of Initial Year Ratios of m. Constant Term not Reported. Standard Errors in Parentheses. $m = k^{\beta-1}h^{\alpha}$

	DM6570	DM7075	DM7580	DM8085	DM6575	DM7585	DM6585
LM65	0.079						
	(0.028)						
LM70		0.044					
		(0.035)					
LM75			0.050				
			(0.023)				
LM80				0.042			
				(0.017)			
LM65					0.081		
					(0.085)		
LM75						0.093	
						(0.053)	
LM65							-0.142
							(0.267)

Note: Mxxyy is the growth rate of m from 19xx to 19yy. LMxx is the natural log of m for the year 19xx.

Table 6:
Growth Rates of c/k (Dependent Variables in Columns) Regressed on Logs of Initial Year Ratios of c/k. Constant Term not Reported. Standard Errors in Parentheses.

	DCK6570	DCK7075	DCK7580	DCK8085	DCK6575	DCK7585	DCK6585
LCK65	-0.039 (0.048)						
LCK70		-0.021 (0.055)					
LCK75			0.007 (0.043)				
LCK80				-0.024 (0.059)			
LCK65					-0.144 (0.124		
LCK75						-0.069 (0.131)	
LCK65							-0.670 (0.329)

Note: DKCxxyy is the growth rate of c/k from 19xx to 19yy. LCKxx is the natural log of c/k for the year 19xx.

variables $(c/k), m$, or (k/h) around possible BGP values would not change the nature of the results and isolate possible local behavior, nevertheless we should be careful not to draw definite conclusions at this point. All those caveats notwithstanding, we view much of the evidence above as compatible with the presence of indeterminacy.

5 Conclusion

In this paper we have overviewed some recent models of growth that are capable of generating multiple equilibria for given initial conditions. Those models offer a potential explanation for differences in income levels across economies which stands as an alternative to theories that rely exclusively on differences in fundamentals and/or initial conditions. We have discussed some of the empirical predictions generated by those models vis à vis the predictions of their deterministic counterparts.

Using an approach that made use of physical capital data for a large number of countries, we have shown that some of the predictions of one-sector models with unique equilibria are hard to reconcile with the evidence, even when we attempt to control for differences in technology levels as well as other fundamentals. Although the evidence is somewhat more tentative, we find similar results in two-sector models with physical and human capital. Some aspects of the data seem not to be in accord with the dynamics of a model with a unique equilibruim. Many explanations are possible, but the hypothesis of indeterminacy cannot be excluded on the basis of the available evidence.

The cross-sectional evidence on the dynamics of growth, although imperfect, seems to be compatible with the predictions of growth models that generate multiple equilibria. While other explanations are possible, the evidence so far suggests that we should take seriously the possibility that the differences in economic performance among countries with similar endowments may be partly the result of self-fulfilling expectations coordinated on different equilibrium paths. We view further empirical work aimed at sorting out the different explanations (including multiple equilibria) for the observed dynamics of cross-sectional income distributions as a fruitful avenue for future research.

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