Under-Employment and the Trickle-Down of Unemployment - Online Appendix Not for Publication

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This online Appendix contains a more comprehensive description of our static model, i.e., (i) the conditions guaranteeing the existence of under-employment in equilibrium, (ii) a more detailed analysis of the model in general equilibrium (both for the decentralized and centralized allocation), (iii) proofs of the propositions stated in the paper, (iv) a graphical representation of the model in general equilibrium, and (v) a generalization of the model to $N = 3$ islands.

1 Complements for the static model

1.1 Condition to ensure under-employment in equilibrium

The conditions guaranteeing the existence of under-employment in equilibrium boil down to ensuring that the equilibrium is not at a corner solution in which either everyone or no one is under-employed. The following condition ensures that some type 2 workers descend to island 1:

$$e^{-q_2}\varphi_{2,2} < \left(\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right) \quad (C_2^p)$$

Intuitively, when $x_2 = 0$ (no under-employment), the expected wage in island 1 is higher than the expected wage in island 2.

As will be shown in the proof of Proposition 1, this condition (coupled with super-modularity) is also sufficient to ensure that type 1 workers will always prefer island 1 in equilibrium.

In general equilibrium, we impose a similar condition. First, we set (c_1, c_2) such that, in equilibrium, $E\omega_{2,2}$ is lower than the expected wage $E\omega_{2,1}(x_2 = 0)$ when there is no under-employment. Formally, the condition can be written as follows:

$$E\omega_{2,2} < (\varphi_{2,1} - \varphi_{1,1}) + E\omega_{1,1}(x_2 = 0). \quad (C_2^g)$$

1.2 General equilibrium with Endogenous Labor Demand

In this section, we characterize and study the general equilibrium (GE) with endogenous labor demand.

Graphical representation of the model in general equilibrium The GE allocation is the triple (x_2, q_1, q_2) determined by firms' free entry conditions in islands 1 and 2, and the arbitrage equation between islands 1 and 2 for type 2 workers. In this subsection, we show that one can represent the determination of this equilibrium allocation in a graphical manner, in a similar fashion to the standard Diamond-Mortensen-Pissarides model.

We start with the characterization of the equilibrium in island 2, which is an homogeneous island only populated by type 2 workers.

With free entry, the equilibrium queue length $q_2(1 - x_2)$ is independent of the supply of type-2 workers. This result comes from the fact that the equilibrium queue length is independent of the number of job seekers, exactly as in a standard Diamond-Mortensen-Pissarides model with homogeneous workers. Recall that island 2 is an homogeneous island only populated by type 2 workers. With free entry, it is easy to see from the firm's no profit condition that the equilibrium queue length $q_2(1 - x_2)$ is independent of the supply of type-2 workers.[1] Specifically, free entry, or (L_2^D) in the main text, pins down the equilibrium queue length in island 2 –$q_2(1 - x_2)$–, regardless of the number of type 2 workers (i.e., regardless of $1 - x_2$). This is similar to what happens in standard search and matching models (Pissarides, 1985) where the supply of (homogeneous) labor has no effect on the equilibrium queue length. Even though a higher number of type 2 workers improves the matching probability of a firm, free entry ensures that more firms enter the market in order to keep profits constant.[2]

This result points to a more general property of our model in GE: our modeling of matching with ranking reduces to the canonical random matching model when

[1] Recall that the queue length is number of job seekers over the number of job openings and that the number of job seekers in island 2 is given by $L_2(1 - x_2)$.

[2] In a search and matching model, at a given vacancy level, an increase in the number of job seekers (coming from say out of the labor force, as in Pissarides (2000), Chapter 5) raises firms matching probability, i.e., reduces hiring costs, and leads more firms to enter the market, keeping profit and thus the queue length unchanged.

workers are homogeneous.[3]

Since free entry in island 2 fixes $q_2(1-x_2)$, characterizing the equilibrium allocation reduces to finding the pair (x_2, q_1) that satisfies (i) firms' free entry condition in island 1 and (ii) type 2 worker's arbitrage condition. Although one could depict the equilibrium in the (x_2, q_1) space, we prefer to depict it in the (x_2, V_1) space (recall that $V_1 = L_1/q_1$ with L_1 fixed), since it corresponds to the (U, V) space representation used in standard search and matching models.

As shown in figure A1, the equilibrium is then determined by the intersection of two curves: a "labor demand curve", (L_1^D), given by firms' free entry condition (also called job creation condition) as in search and matching models, and a "labor supply curve", (L^S), characterizing the number of type 2 workers in island 1 and given by the arbitrage condition of type 2 workers between islands 1 and 2.

The labor demand curve is upward sloping and non-linear. To understand the shape of the labor demand curve (L_1^D), it is again useful to go back to the standard Diamond-Mortensen-Pissarides (DMP) model, in which workers are homogeneous. Recall that the total number of job seekers in island 1 is given by $L_1(1 + x_2h_2)$. We can thus represent the labor demand curve, or job creation curve, in a similar fashion to DMP models by plotting the job creation curve in (U, V) space. Starting from a world with only type 1 workers and $x_2 = 0$ (i.e., being to the left of the y-axis in figure A1), all workers are homogeneous and, as in the DMP model, increasing the number of type 1 (increasing L_1) does not affect the equilibrium queue length V_1/L_1. As a result, the labor demand curve (dashed blue line) crosses the origin at 0. Now, consider the case where one adds type 2 workers and $x_2 > 0$. Because firms generate a higher profit when hiring type 2 workers than when hiring type 1 workers, an increase in x_2 generates a disproportionate increase in the number of firms in island 1, and the equilibrium queue length $\frac{V_1}{L_1(1+x_2h_2)}$ increases. In other words, the slope of the labor demand curve is initially increasing with x_2. This portion of the labor demand curve can be seen as capturing a "quality effect": as the share of type 2 workers in island 1 increases, the quality (i.e., skill level) of the average applicant improves, and this leads to a disproportionate increase in job creation. Then, as the number of type 2 workers becomes large relative to the number of type 1, the labor market in island 1 resembles more to more to that of an homogeneous market with only type 2 workers, in which the queue length is independent of the number of type

[3] A technical difference between our framework and Pissarides (1985) is that, in our set-up, an increase in the supply of workers also improves the bargaining position of the firm (as workers compete against each other when negotiated the wage). This difference has no consequence on the equilibrium queue length, because the bargaining position is also solely a function of the queue length $q_2(1-x_2)$. As a result, no matter the level of $1-x_2$, free entry ensures that the queue length adjusts to keep profits (including the fix cost) nil.

2 and the slope of (L_1^D) is again independent of x_2 (dashed red line).

The labor supply curve is capturing how x_2 depends on V_1 and is also upward slopping: the larger the number of job openings, the less competition type 2 workers will face when searching in island 1, and the higher their expected wage. As a result, an increase in V_1 raises the incentive of type 2 to move down to island 1 and increases x_2.

Expected wage curves in general equilibrium Similarly to the PE case described in the main text, figure A2 depicts the equilibrium under-employment rate as the intersection the $E\omega_{2,1}$ curve, the expected wage earned in island 1, and the $E\omega_{2,2}$ curve, the expected wage earned in island 2. The following corollary captures formally how the expected wage in island 1 or 2 depends on the share of type 2 workers searching in island 1.

Corollary A1. *Expected income of type 2 workers*

The expected income of type 2 workers searching in island 2, $E\omega_{2,2}(x_2, q_2(x_2))$, is independent of x_2. The expected income of type 2 workers searching in island 1, $E\omega_{2,1}(x_2, q_1(x_2))$, is strictly decreasing in x_2 with $\left|\frac{dE\omega_{2,1}}{dx_2}\right| < \left|\frac{\partial E\omega_{2,1}}{\partial x_2}\right|$.

Proof. Section 1.3. □

The $E\omega_{2,2}$ curve, the expected wage earned in island 2, is now flat, i.e., the expected income in island 2 is independent of the number of high-skill workers searching in island 1. This result comes from the fact that the equilibrium queue length is independent of the number of job seekers, as we discussed at length in the previous sub-section.

Turning to the $E\omega_{2,1}$ curve, an important property of the model continues to hold in GE: the wage schedule of high-skill workers looking in island 1 is decreasing in x_2. This is in stark contrast with standard search models with heterogeneous workers, in which the $E\omega_{2,1}$ curve would be upward slopping. Indeed, worker heterogeneity gives rise to a "quality effect", in that firms respond to changes in the average productivity of the unemployment pool: As more high-skill workers search in island 1, this raises firms' probability to meet high-skill applicants (who generate a higher surplus than low skill applicants), which raises firms' profits, and leads to more job creation. Thus, in random search model, the quality effect would lead to an upward slopping wage schedule. This does not happen in our framework, because hiring is not random and the wage schedule $E\omega_{2,1}$ is still decreasing in the number of high-skill workers searching in island 1.[4] However, the quality effect is still present in

[4]With non random hiring and skill-biased job competition, the wage schedule of high-skill

our model, and the wage schedule is flatter with endogenous firm entry than in PE. As under-employment increases, the average productivity of the unemployment pool increases, which fosters firm entry and limits the increase in congestion generated by the inflow of workers.

Turning to type 1 workers, figure A3 plots the expected wage curve of type 1 workers, both in PE and in GE, and the following corollary captures formally how the expected wage in island 1 depends on the share of type 2 workers searching in island 1.

Corollary A2. *Expected income of type 1 workers*

The expected income of type 1 workers searching in island 1, $E\omega_{1,1}(x_2, q_1(x_2))$*, is a non-monotonic function of* x_2*; decreasing over* $[0, x_2^*]$ *and increasing over* $[x_2^*, 1]$ *with* $x_2^* \in [0, 1]$*.*

Proof. Section 1.3. □

As in the PE case, the expected income of low-skill workers declines with the share of high-skill workers looking in island 1, at least for x_2 low enough. This property of the model is again in stark contrast with a random search model, in which an increase in the quality of the unemployment pool leads to more job creation, which raises the job finding rate of *all* job seekers. This quality effect is present in this model, but it is dominated, at least for low values of x_2, by the effect of skill-biased job competition. Because high-skill workers are systematically hired over competing low-skill applicants, an increase in x_2 implies a lower expected income for low-skill workers. However, as x_2 increases and the pool becomes more homogeneous (i.e., becomes dominated by high-skill workers), the degree of heterogeneity in the unemployment pool diminishes and the skill-biased job competition effect becomes weaker. This explains why, for large values of x_2, an increase in the number of type 2 workers can raise low skill workers' labor market prospects, as predicted by a random search model with heterogeneous workers.

Constrained optimal allocation In this section, we discuss in more details the centralized allocation and the difference with the decentralized allocation.

To better understand the externalities at play, it is useful to contrast the worker's problem and the planner's problem. Type 2 workers allocate themselves between

workers is downward slopping, because the expected income of high skill workers is driven by their uniqueness, as it determines both their ability to find a job easily (by being preferably hired over low skill workers) and to obtain a wage premium over low skill workers. As the number of high skill workers increase, they become less unique, leading to a lower job finding rate (as they face more competition from their peers) and a lower wage premium.

islands 1 and 2 up until

$$E\omega_{2,2} = E\omega_{2,1}. \tag{A1}$$

In contrast, the planner wishes to allocate workers in order to maximize total output, while satisfying firms' zero profit condition. With free entry, we have $\pi = y - \omega = c$ so that maximizing total output is identical to maximizing the total wage bill Ω. The planner's problem is thus to maximize the wage bill while satisfying firms' free entry conditions

$$\begin{cases} \max_{x_2} \Omega \\ \Omega = (1 - x_2)h_2 E\omega_{2,2}(x_2, q_2(x_2)) + h_2 x_2 E\omega_{2,1}(x_2, q_1(x_2)) + E\omega_{1,1}(x_2, q_1(x_2)) \end{cases} \tag{A2}$$

with $q_1(x_2)$ given by fims' free entry condition in island 1: $\pi_1(x_2, q_1(x_2)) = c_1$.

When workers are homogeneous, as in island 2, the number of job seekers has no effect on the ratio of job openings to job seekers: the firm responds to the addition of one more worker by creating more vacancy to keep the queue length q constant. As a result, congestion and job creation externalities exactly cancel out for each type: the congestion generated by the addition of one more worker is exactly compensated by the posting of more vacancy such as to keep the equilibrium queue length unchanged. Mathematically, we have

$$\frac{dE\omega_{2,2}(x_2, q_2(x_2))}{dx_2} = 0.$$

so that under-employment has no effect on the decentralized allocation in island 2. In other words, a high-skill worker exerts no externality on the high-tech island.

Consider now the low-tech island (island 1). Contrasting (A1) with (A2), we can see that type 2 workers are solving the planner's problem treating $E\omega_{2,1}(x_2, q_1(x_2))$ and $E\omega_{1,1}(x_2, q_1(x_2))$ as independent of x_2. In other words, type 2 workers do not internalize how their descent affect the wages of other workers , i.e., how an increase in x_2, the fraction of type 2 workers in island 1, affects (i) competition between workers (a congestion externality), and (ii) job creation in island 1 (a job creation externality).

To see this more formally, we can contrast (A1) with (A2). The decentralized allocation is efficient if and only if

$$\frac{d\Omega_1}{dx_2} \equiv \frac{dE\omega_{1,1}(x_2, q_1(x_2))}{dx_2} + x_2 h_2 \frac{dE\omega_{2,1}(x_2, q_1(x_2))}{dx_2} = 0. \tag{A3}$$

Expression $\frac{d\Omega_1}{dx_2}$ captures the net effect of under-employment on expected wages in island 1, i.e., the effect of a marginal high-skill worker searching in the low-tech island on the labor markets of (i) low-skill workers ($\frac{dE\omega_{1,1}}{dx_2}$) and (ii) high-skill workers

($x_2 h_2 \frac{dE\omega_{2,1}}{dx_2}$) weighted by their relative population shares. In essence, it captures the congestion externality net of the compensating effect of the job creation externality.

Using the expression for $\frac{d\Omega_1}{dx_2}$, one can visualize in figure A4 how the externality evolves with the presence of high-skill workers in the low-tech island.

As x_2 increases, the congestion externality becomes stronger. Starting from a world with no high-skill workers in the low-tech island ($x_2 = 0$), the presence of one high-skill worker imposes a large cost to low-skill workers: since a high-skill is systematically ranked above competing low-skill applicants, the labor market of the low skills deteriorates with x_2. Similarly, as the number of high-skill workers increases, high-skill workers become less unique and start competing more and more against each other for jobs, leading to a downward sloping wage schedule.

As x_2 increases, the job creation externality becomes stronger (because firms are more likely to face a high-skill applicant) and compensates the increased congestion in the labor market. In fact, the income of low-skill workers starts increasing for x_2 large enough. However, this cannot fully compensate the increased congestion between the high-skill workers, and the net externality remains negative.

It is only when the number of high-skills becomes arbitrarily large compared to the number of low-skills, that job creation exactly compensates the increased congestion. Specifically, as $x_2 h_2$ increases further and becomes arbitrary large, the labor market in the low-tech island resembles that of an homogeneous labor market with only high skill workers, and as in search models with homogeneous labor, the marginal high-skill applicant has no effect on the equilibrium queue length, $\frac{d\Omega_1}{dx_2} \to 0$, and the decentralized allocation is constrained efficient.

1.3 Proofs

Proof. Proposition 1

Consider first the problem of type 2 workers. A type 2 worker has two choices, he can (i) look for a job in island 2, his "home island", or (ii) look for a job in island 1, i.e., move down the occupation ladder. We now consider these two possibilities.

When a type 2 worker looks for a job in island 2, he faces two possible outcomes: (a) with probability $e^{-q_2(1-x_2)}$, he is the only applicant and receives $\beta\varphi_{2,2}$, or (b), with probability $1 - e^{-q_2(1-x_2)}$, he is in competition with other workers and receives 0 (regardless of whether he ends up employed or unemployed). The expected payoff of a worker type 2 who searches for a job in island 2, $\omega_{2,2}$, is thus

$$E\omega_{2,2} = \beta e^{-q_2(1-x_2)}\varphi_{2,2}.$$

The expected wage is increasing in x_2. When a lot of type 2 workers descend to island 1, it becomes easier for the ones who stayed in 2 to be the only applicant to a job and receive a high wage. When a type 2 worker looks for a job in island 1, he faces three possible outcomes: (a) with probability $e^{-q_1 x_2 h_2} e^{-q_1}$, he is the only applicant and receives $\beta\varphi_{2,1}$. Note that he produces less than in his "home" island and thus receives a lower wage than would have been the case if he had been the only applicant to a type 2 firm, (b) with probability $1 - e^{-q_1 x_2 h_2}$, he is in competition with other type 2 workers and receives 0 (regardless of whether he ends up employed or unemployed), and (c) with probability $e^{-q_1 x_2 h_2}(1 - e^{-q_1})$, he is in competition with type 1 workers only and receives $\beta(\varphi_{2,1} - \varphi_{1,1})$.[5] The expected payoff of a worker type 2 who searches for a job in island 1, $\omega_{2,1}$, is thus

$$E\omega_{2,1} = \beta e^{-q_1 x_2 h_2} e^{-q_1} \varphi_{2,1} + \beta e^{-q_1 x_2 h_2} (\varphi_{2,1} - \varphi_{1,1}) \left[1 - e^{-q_1}\right].$$

The expected wage in island 1 is decreasing in x_2: when there are fewer type 2 workers in island 1, there is less competition in island 1, and type 2 workers can expect a higher wage.

In order to ensure under-employment in equilibrium, we need to assume that condition (C_2^p) is verified. A high skill worker would have a higher expected wage in the low-tech island when all high skill workers remain in the high-tech island.

Under this condition, there is some under-employment in equilibrium and a type 2 worker must be indifferent between looking for a job in island 2 or in island 1. This arbitrage condition, $A(x_2)$, determines, x_2, the equilibrium allocation of workers

$$A(x_2) = -e^{-q_2(1-x_2)} \varphi_{2,2} + e^{-q_1 x_2 h_2} e^{-q_1} \varphi_{2,1} + e^{-q_1 x_2 h_2} (\varphi_{2,1} - \varphi_{1,1}) \left[1 - e^{-q_1}\right] = 0. \quad \text{(A4)}$$

Consider now the problem of type 1 workers. When a type 1 worker looks for a job in island 1, he faces two possible outcomes: (a) with probability $e^{-q_1(1+h_2 x_2)}$, he is the only applicant and receives $\beta\varphi_{1,1}$, or (b), with probability $1 - e^{-q_1(1+h_2 x_2)}$, he is in competition with other workers and receives 0:

$$E\omega_{1,1} = \beta e^{-q_1(1+h_2 x_2)} \varphi_{1,1}.$$

Type 1 workers could choose to move up the occupation ladder and search for a job in island 2. This will not happen as long as there are type 2 workers in island 1. Given our initial assumption on the returns of type 1 to their skills in island 2, as

[5] As noted earlier, despite the presence of competing applicants, a single type 2 applicant can extract some of the surplus thanks to due to his productivity advantage over the other applicants.

long as type 2 workers are indifferent between their "home" island and the island below, type 1 workers will always prefer to remain in 1. Indeed, in island 2, type 1 workers would only receive a positive wage when not competing with any other applicants, i.e., $E\omega_{1,2} = \beta e^{-q_2(1-x_2)}\varphi_{1,2}$ and equation (A4) implies:

$$E\omega_{1,2} = E\omega_{2,2}\frac{\varphi_{1,2}}{\varphi_{2,2}} = e^{-q_1 x_2 h_2}\frac{\varphi_{1,2}}{\varphi_{2,2}}(\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}).$$

As a consequence,

$$E\omega_{1,2} = \frac{\varphi_{1,2}}{\varphi_{2,2}}\frac{\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}}{\varphi_{1,1}e^{-q_1}}E\omega_{1,1},$$

It is easy to show that $\frac{\varphi_{1,2}}{\varphi_{2,2}}\frac{\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1}}{\varphi_{1,1}e^{-q_1}} < 1$. This condition directly ensues from super-modularity, i.e., $\frac{\varphi_{1,2}}{\varphi_{2,2}} < \frac{\varphi_{1,1}}{\varphi_{2,1}}$.

Uniqueness

Under condition (C_2^p), some workers of type 2 will always apply in island 1. We have already shown that, as long as type 2 workers are indifferent between the 2 islands, there cannot be workers of type 1 looking for jobs in island 2. As a consequence, the only variable that adjusts is the number of workers of type 2 applying in island 1.

The trade-off faced by type 2 workers is monotonic, i.e. as they apply more in island 1, their relative gain of doing so is strictly decreasing. As already discussed above, under condition (C_2^p), the relative gain of applying in island 1 is initially positive (for $x_2 = 0$). The relative gain is negative when $x_2 = 1$, because $\varphi_{2,2} > \varphi_{2,1}$. It only crosses once the x-axis, and the intersection defines the unique equilibrium. □

Proof. Proposition 2

The workers no-arbitrage conditions are already derived in Proposition 1. The number of job openings is given by the free entry condition, and we only need to express the firm's expected profit as a function the the number of job openings V_1, or the "initial" queue length $q_1 = \frac{L_1}{V_1}$.

Consider first a firm that enters island 1. The firm's profit will depend on the number of applications it receives. There are 5 cases: (the outcome of the wage negotiation process in each case is described in detail in the Proof of Proposition 1)

1. The firm has no applicant. Profit is zero.

2. The firm has only one applicant. The firm gets a share $1 - \beta$ of the output, i.e. $(1-\beta)\varphi_{1,1}$ if the applicant is of type 1 (which happens with probability

$P(a_1=1,a_2=0)=q_1e^{-q_1}e^{-q_1x_2h_2})$, and $(1-\beta)\varphi_{2,1}$ if the applicant is of type 2 (which happens with probability $P(a_1=0,a_2=1)=q_1x_2h_2e^{-q_1x_2h_2}e^{-q_1})$.

3. The firm has more than one applicants of type 1 (and no applicants of type 2). The firm gets all the surplus: $\varphi_{1,1}$.

   This happens with probability $e^{-x_2h_2q_1}\left[1-e^{-q_1}-q_1e^{-q_1}\right]$.

4. The firm has more than one applicants of type 2 (and no applicants of type 1). The firm gets all the surplus: $\varphi_{2,1}$. This happens with probability $1-e^{-x_2h_2q_1}-x_2h_2q_1e^{-x_2h_2q_1}$.

5. The firm has more than one applicants *of different types.* The most productive worker is hired and gets a share β of the surplus generated over hiring the second-best applicant. The firm generates a profit $\varphi_{1,1}+(1-\beta)(\varphi_{2,1}-\varphi_{1,1})$. This happens with probability $x_2h_2q_1e^{-x_2h_2q_1}(1-e^{-q_1})$.

The expected profit for a firm with technology 1 is thus given by

$$\pi_1(x_2,q_1)=\varphi_{2,1}-e^{-x_2h_2q_1}\left[\left(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}^{-q_1}\right)\left(1+(2-\beta)x_2h_2q_1\right)+(2-\beta)\varphi_{1,1}q_1e^{-q_1}\right].$$

Proceeding in a similar fashion, one can show that the expected profit for a firm with technology 2 is

$$\pi_2(x_2,q_2)=\left(1-e^{-(1-x_2)q_2}-(1-x_2)q_2e^{-(1-x_2)q_2}\right)\varphi_{2,2}+(1-x_2)q_2e^{-(1-x_2)q_2}(1-\beta)\varphi_{2,2}.$$

Consequently, free entry imposes two no-profit conditions in addition to workers' arbitrage equations.

The unicity of the equilibrium is a direct consequence of Corollaries 1 and 2 that we prove next. □

Proof. Corollary A1

First, it is straightforward from the expression of $\pi_2(x_2,q_2)$ that the free entry condition $\pi_2=c_2$ imposes that $q_2(1-x_2)$ is constant, so that the expected wage in island 2, $E\omega_{2,2}$, is constant. We can thus restrict our analysis to the arbitrage condition coupled with the free entry condition in island 1.

$$\begin{cases} E\omega_{2,2}=\left[\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1}\right]e^{-q_1x_2h_2} & (L^S)\\ \varphi_{2,1}-c_1=\left[\left(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1}\right)\left(1+(2-\beta)q_1x_2h_2\right)+(2-\beta)\varphi_{1,1}q_1e^{-q_1}\right]e^{-q_1x_2h_2} & (L_1^D)\end{cases}$$

The (L_1^D) equation defines a job creation function $q_1(x_2)$. As before, we only consider interior solutions, i.e. we impose that:

$$\left[\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1(1)}\right] e^{-q_1(1)h_2} < E\omega_{2,1} < \left[\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1(0)}\right].$$

Under this condition, the relative gain of searching for a job in lower-tech island is positive for $x_2 = 0$ ($E\omega_{2,2} < E\omega_{2,1}$) and negative for $x_2 = 1$ ($E\omega_{2,2} > E\omega_{2,1}$).

Combining the (L_1^D) and (L^S) equations, it can be shown with a little bit of algebra that:

$$\begin{aligned} E\omega_{2,1}^{'}(x_2) &= \tfrac{\partial E\omega_{2,1}}{\partial x_2} + q_1^{'}(x_2)\tfrac{\partial E\omega_{2,1}}{\partial q_1} \\ &= \tfrac{(2-\beta)q_1(E\omega_{2,1}-E\omega_1)}{[(2-\beta)q_1x_2h_2-(1-\beta)]E\omega_{2,1}+(2-\beta)q_1E\omega_1} q_1^{'}(x_2)E\omega_1 < 0, \end{aligned}$$

and that $q_1^{'}(x_2)\frac{\partial E\omega_{2,1}}{\partial q_1} > 0$.

This proves Corollary A1. Moreover, using that $E\omega_{2,1}(x_2) < 0$ with the fact that $E\omega_{2,2}(x_2)$ is constant guarantees the uniqueness of the equilibrium. □

Proof. Corollary A2

Combining the (L_1^D) and (L^S) equations, it can be shown with a little bit of algebra that:

$$\begin{aligned} E\omega_{1,1}^{'}(x_2, q_1(x_2)) &= \tfrac{\partial E\omega_1}{\partial x_2} + q_1^{'}(x_2)\tfrac{\partial E\omega_1}{\partial q_1} \\ &= \tfrac{[(2-\beta)q_1x_2h_2-(1-\beta)](E\omega_{1,1}-E\omega_{2,1})}{[(2-\beta)q_1x_2h_2-(1-\beta)]E\omega_{2,1}+(2-\beta)q_1E\omega_{1,1}} q_1^{'}(x_2)E\omega_{1,1} \gtrless 0 \end{aligned}$$

We can see that $E\omega_{1,1}(x_2, q_1(x_2))$ is not monotonically decreasing, implying that a larger number of high-skilled workers does not necessarily imply lower expected income for low-skilled workers. For $\beta < 1$, $E\omega_{1,1}(x_2, q_1(x_2))$ is initially decreasing and then increases once $(2-\beta)q_1x_2h_2 > (1-\beta)$. This proves Corollary A2. □

Proof. Corollary 1

In addition to Corollary 1, we also prove that $\frac{dx_2}{dq} \to 0$ when $\varphi_{2,1} \to \varphi_{1,1}$.

The arbitrage equation can be written as:

$$e^{-q_2(1-x_2)}\varphi_{2,2} = e^{-q_1x_2h_2-q_1}\left[\varphi_{2,1} + (\varphi_{2,1} - \varphi_{1,1})(e^{q_1} - 1)\right].$$

We differentiate the arbitrage equation with respect to q. The left-hand side becomes:

$$e^{-q_2(1-x_2)}\varphi_{2,2}(-\frac{dq_2}{dq}(1-x_2) + q_2\frac{dx_2}{dq}),$$

and the right-hand side becomes:

$$e^{-q_1 x_2 h_2}(e^{-q_1}\varphi_{2,1}+(\varphi_{2,1}-\varphi_{1,1})\left[1-e^{-q_1}\right])(-\frac{dq_1}{dq}(x_2h_2+1)-q_1h_2\frac{dx_2}{dq})+e^{-q_1x_2h_2}\frac{dq_1}{dq}(\varphi_{2,1}-\varphi_{1,1}),$$

which is equal to:

$$e^{-q_2(1-x_2)}\varphi_{2,2}(-\frac{dq_1}{dq}(x_2h_2+1)-q_1h_2\frac{dx_2}{dq})+e^{-q_1x_2h_2}\frac{dq_1}{dq}(\varphi_{2,1}-\varphi_{1,1}).$$

As a consequence, $\frac{dx_2}{dq}$ is such that:

$$(q_1h_2+q_2)\frac{dx_2}{dq}=(\frac{dq_2}{dq}(1-x_2)-\frac{dq_1}{dq}(x_2h_2+1))+\frac{\varphi_{2,1}-\varphi_{1,1}}{\varphi_{2,2}}e^{-q_1x_2h_2+q_2(1-x_2)}\frac{dq_1}{dq},$$

and we know that $\frac{dq_2}{dq}(1-x_2)-\frac{dq_1}{dq}(x_2h_2+1)=0$.

When $\varphi_{2,1}\to\varphi_{1,1}$, the ranking advantage of high-skill workers disappears and the model collapses a random matching model. The arbitrage equation becomes

$$e^{-q_2(1-x_2)}\varphi_{2,2}=e^{-q_1x_2h_2-q_1}\varphi_{2,1},$$

and, following the same reasoning, it comes immediately that $\frac{dx_2}{dq}=0$. □

Proof. Proposition 3

The maximization program of the central planner can be written as follows (denote Y the aggregate output of the economy):

$$\max_{x_2,q_1,q_2}\{Y\},$$

subject to

$$\begin{cases}\pi_2(x_2,q_2)=c_2\\ \pi_1(x_2,q_1)=c_1\end{cases}$$

We can already simplify the program through two channels. First, with free entry, the aggregate profit of firms (net of investment costs) is zero. Consequently, the central planner equivalently maximizes the wage bill of workers. Second, free entry in island 2 imposes that q_2 is set such as to make $(1-x_2)q_2$ constant.

$$(1-x_2)q_2=f^{-1}\left(\frac{c_2}{\varphi_{2,2}}\right)$$

The program then sums up to:

$$\max_{x_2,q_1} \left\{(1-x_2)h_2 E\omega_{2,2} + h_2 x_2 e^{-q_1 h_2 x_2}\left[\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}^{-q_1}\right] + e^{-q_1 h_2 x_2 - q_1}\varphi_{1,1}\right\},$$

subject to

$$\varphi_{2,1} - e^{-x_2 h_2 q_1}\left[\left(\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}^{-q_1}\right)(1+(2-\beta)x_2 h_2 q_1) + (2-\beta)\varphi_{1,1} q_1 e^{-q_1}\right] = c_1.$$

The first-order condition in x_2 leads to:

$$A(x_2,q_1) - B_{x_2}(x_2,q_1) - \lambda C_{x_2}(x_2,q_1) = 0,$$

where

$$\begin{cases} B_{x_2}(x_2,q_1) = q_1 h_2 e^{-q_1 h_2 x_2}\left[x_2 h_2\left(\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}^{-q_1}\right) + \varphi_{1,1} e^{-q_1}\right] \\ C_{x_2}(x_2,q_1) = q_1 h_2 e^{-q_1 h_2 x_2}\left[((2-\beta)q_1 x_2 h_2 - (1-\beta))\left(\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}^{-q_1}\right) + (2-\beta)\varphi_{1,1} e^{-q_1}\right] \end{cases}$$

We can already see that profits losses are not entirely internalized by workers: the marginal gain in wages for workers of type 2 cannot fully translate in marginal profits for firms in island 1. The first-order condition in q_1 gives:

$$-B_{q_1}(x_2,q_1) - \lambda C_{q_1}(x_2,q_1) = 0,$$

where

$$\begin{cases} B_{q_1}(x_2,q_1) = x_2/q_1 B_{x_2}(x_2,q_1) + (1+x_2 h_2)\varphi_{1,1} e^{-q_1 x_2 h_2 - q_1} \\ C_{q_1}(x_2,q_1) = x_2/q_1 C_{x_2}(x_2,q_1) + \left[q_1(2-\beta)(1+x_2 h_2) - (1-\beta)\right]\varphi_{1,1} e^{-q_1 x_2 h_2 - q_1} \end{cases}$$

We can observe the symmetry between the expressions in x_2 and q_1. The basic intuition is that, since profits can be written as a function of $q_1 x_2$, the externality generated by a change in x_2 will be partly compensated by an inverse change in q_1. Indeed, combining these two equations, we have that:

$$A(x_2,q_1) = \frac{B_{x_2}(x_2,q_1)C_{q_1}(x_2,q_1) - B_{q_1}(x_2,q_1)C_{x_2}(x_2,q_1)}{C_{q_1}(x_2,q_1)}.$$

Once accounted for the expression of $B_{q_1}(x_2,q_1)$ and $C_{q_1}(x_2,q_1)$,

$$A(x_2,q_1) = \frac{(1-\beta)\,h_2 q_1 \varphi_{1,1} e^{-2q_1 x_2 h_2 - q_1}\left(\varphi_{2,1} - \varphi_{1,1}\right)}{C_{q_1}(x_2,q_1)}$$

As a consequence, $A(x_2,q_1)$ is stricity positive as long as the surplus is not entirely given to workers, i.e. $\beta < 1$, and workers are not equally productive, i.e. $\varphi_{2,1} -$

$\varphi_{1,1} > 0$. Coupled with the two free entry conditions, this equation characterizes the constrained optimum which does not coincide with the decentralized allocation. $A(x_2, q_1) > 0$ implies that wages are higher in 1 than in 2. In other words, the decentralized allocation induces a lower x_2, a higher q_1 and a lower q_2. □

1.4 A model of under-employment with three worker types and three firm types

In this section, we show that our under-employment model can be easily characterized in a $N > 2$ context. To illustrate this point, we study the equilibrium allocation in the $N = 3$ case, i.e., in an economy with three worker types and three firm types. As in $N = 2$ case, we first derive the conditions to endure the existence of under-employment in equilibrium, then characterize the partial equilibrium allocation, provide some intuition and comparative statics, then describe the general equilibrium allocation, and then study the optimal allocation.

The $N = 2$ case is a good benchmark to understand how workers decide on which island to search. However, it has no "propagation mechanism", in the sense that type 1 workers cannot respond to the competition of type 2 workers by moving further down the occupation ladder. To capture this possibility, we now study an economy with 3 islands and 3 worker types. For the sake of clarity, we limit our analysis to $N = 3$, but the mechanisms are general and would be present with more islands or worker types.

When workers can respond to the presence of higher-skill individuals, the equilibrium level of under-employment is determined by the interaction of two forces, instead of just one in the $N = 2$ case: (i) a force that "pushes" workers down the occupation ladder: as high-skill workers invade the island below, they push lower-skill individuals further down the ladder, exactly as in the $N = 2$ case, and (ii) a force that "pulls" workers down the ladder: as low-skill workers move down the ladder, they free up space in their island, which pulls the higher-skill individuals down the ladder.

Conditions to ensure under-employment in equilibrium First, and as in the $N = 2$ case, we derive here the conditions that ensure that the equilibrium we consider is an under-employment equilibrium. Our conditions boil down to ensuring that the equilibrium is not at a corner solution in which either everyone or noone is under-employed.

In the $N = 3$ case, we impose conditions guaranteeing (i) there is some under-employment of types 3 and 2, (ii) not all type 2 workers search in island 1 and (iii) type 3 workers do not search in island 1. These conditions ensure that at most 2 types co-exist in a given island.

First, a positive fraction of type 3 and type 2 workers are under-employed as long as:

$$\begin{cases} e^{-q_3}\varphi_{3,3} < (\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2}) & (C_3^p) \\ e^{-q_2}\varphi_{2,2} < (\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}) & (C_2^p) \end{cases}$$

Second, a positive fraction of type 2 workers search in island 2 as long as:

$$e^{-q_2 h_3}\varphi_{2,2} > e^{-q_1 h_2}\left(\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right) \quad (D_2^p)$$

This condition implies that, even with all type 3 workers in island 2, type 2 workers would not all descend to island 1.

Finally, we derive the condition under which type 3 workers have no incentives to search in island 1. Consider the equilibrium allocation verifying $A_3(x_3, x_2) = 0$ and $A_2(x_2, x_1) = 0$. The expected wage of a type 3 worker searching in island 1 would be

$$E\omega_{3,1} = (\varphi_{3,1} - \varphi_{2,1}) + e^{-q_1 x_2 h_2}\left[\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right]$$

Since $e^{-q_1 x_2 h_2}\left[\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right] = e^{-q_2(1-x_2) - q_2 x_3 h_3}\varphi_{2,2}$,

$$E\omega_{3,1} = (\varphi_{3,1} - \varphi_{2,1}) + e^{-q_2(1-x_2) - q_2 x_3 h_3}\varphi_{2,2}$$

In contrast, the expected wage of a type 3 worker searching in island 2 is:

$$e^{-q_2 x_3 h_3}\left[\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2(1-x_2)}\right]$$

It is then immediate that no type 3 workers have the incentives to descend to island 1 as long as:

$$e^{-h_3 q_2} > \frac{\varphi_{3,1}}{\varphi_{3,2}} \quad (D_3^p).$$

Finally, we impose the same conditions on the φ's in order to ensure that, as long as type n workers are indifferent between islands n and $n-1$, type $n-1$ workers will never move up to island n.

Partial equilibrium with exogenous labor demand We now characterize the equilibrium allocation and then present some comparative statics exercises to illustrate the mechanisms underlying the equilibrium.

The equilibrium with three types of workers and firms is characterized by the following Proposition:

Proposition A1. *With $N = 3$, there is a unique equilibrium allocation of workers satisfying*

- *Type 3 workers are indifferent between islands 3 and 2, and x_3, the share of type 3 workers searching in island 2, is given by the arbitrage condition*

$$A_3(x_3, x_2) = -E\omega_{3,3} + E\omega_{3,2} = 0 \quad (A_3)$$

with

$$\begin{cases} E\omega_{3,3} = \beta e^{-q_3(1-x_3)}\varphi_{3,3} \\ E\omega_{3,2} = \beta e^{-q_2 x_3 h_3} e^{-q_2(1-x_2)}\varphi_{3,2} + \beta e^{-q_2 x_3 h_3}\left[1 - e^{-q_2(1-x_2)}\right](\varphi_{3,2} - \varphi_{2,2}) \end{cases}$$

- *Type 2 workers are indifferent between islands 2 and 1, and x_2, the share of type 2 workers searching in island 1, is given by the arbitrage condition*

$$A(x_3, x_2) = -E\omega_{2,2} + E\omega_{2,1} = 0 \quad (A_2)$$

with

$$\begin{cases} E\omega_{2,2} = \beta e^{-q_2(1-x_2) - q_2 x_3 h_3}\varphi_{2,2} \\ E\omega_{2,1} = \beta e^{-q_1 x_2 h_2} e^{-q_1}\varphi_{2,1} + \beta e^{-q_1 x_2 h_2}(\varphi_{2,1} - \varphi_{1,1})\left[1 - e^{-q_1}\right] \end{cases}$$

- *Type 1 workers only look for jobs in island 1.*

Proof. We directly consider the general problem of a worker $n \in \{1, \ldots, N\}$ who can decide to look for a job in his home island, or instead move down the occupation ladder to look for a job. In the spirit of the $N = 2$ case, we can exclude the possibility that workers look for jobs in higher technology islands or that they descend to lower levels than the one immediately below. The intuition is the same as the one developed in the 2 islands case: as long as a particular type is indifferent between two islands, the more (resp. less) skilled types will always prefer the island above (resp. below). The reason lies in the fact that the relative rent extracted between island $n - 1$ and n is increasing in the skills of agents.

A type n worker has two choices, he can (i) look for a job in island n, his "home island", or (ii) look for a job in island -1, i.e., move down the occupation ladder. As in Proposition 1, we consider these two possibilities, and the only difference with the $N = 2$ case is that workers now have to take into account the fact that some higher type workers may be looking for work in their home island.

When a type n worker looks for a job in island n, he faces two possible outcomes: (a) with probability $e^{-q_n(1-x_n)}e^{-q_n x_{n+1} h_{n+1}}$, he is the only applicant and receives $\beta\varphi_{n,n}$, or (b), with probability $1 - e^{-q_n(1-x_n)}e^{-q_n x_{n+1} h_{n+1}}$, he is in competition with other workers (either from his own island n or from island $n+1$) and receives 0 (regardless of whether he ends up employed or unemployed). The expected payoff of a worker type n who searches for a job in island n, $\omega_{n,n}$, is thus

$$E\omega_{n,n} = \beta e^{-q_n(1-x_n)}e^{-q_n x_{n+1} h_{n+1}}\varphi_{n,n}$$

Consider now the case in which worker type n moves down to island $n-1$. There are 3 possibilities: (a) with probability $e^{-q_{n-1} x_n h_n}e^{-q_{n-1}(1-x_{n-1})}$, he is the only applicant and receives $\beta\varphi_{n,n-1}$, (b) with probability $1 - e^{-q_{n-1} x_n h_n}$, he is in competition with type n workers coming, like him, from the island above, and he receives 0 (regardless of whether he ends up employed or unemployed), and (c), with probability $e^{-q_{n-1} x_n h_n}\left(1 - e^{-q_{n-1}(1-x_{n-1})}\right)$, he is in competition with type $n-1$ workers only and receives $\beta(\varphi_{n,n-1} - \varphi_{n-1,n-1})$.[6] The expected payoff of a worker type n who searches for a job in island $n-1$, $\omega_{n,n-1}$, is thus

$$E\omega_{n,n-1} = \beta e^{-q_{n-1} x_n h_n}e^{-q_{n-1}(1-x_{n-1})}\varphi_{n,n-1} + \beta e^{-q_{n-1} x_n h_n}\left[1 - e^{-q_{n-1}(1-x_{n-1})}\right](\varphi_{n,n-1} - \varphi_{n-1,n-1})$$

In equilibrium, a type n worker must be indifferent between staying in island n or moving down to island $n-1$, which implies the arbitrage equation

$$\begin{aligned} &A_n(x_{n+1}, x_n) = -e^{-q_n(1-x_n)}e^{-q_n x_{n+1} h_{n+1}}\varphi_{n,n} \\ &+e^{-q_{n-1} x_n h_n}e^{-q_{n-1}(1-x_{n-1})}\varphi_{n,n-1} + e^{-q_n x_n h_n}\left[1 - e^{-q_{n-1} x_n h_n}\right](\varphi_{n,n-1} - \varphi_{n-1,n-1}) = 0 \end{aligned}$$

These equations characterize the equilibrium allocation.

Unicity

As in the $N = 2$ case, uniqueness comes from a monotonicity argument.

Condition (C_3^p) implies that there will be some high-skilled workers descending even when all mid-skilled workers are applying in island 2. Condition (C_2^p) implies that mid-skilled workers descend even when none of the high-skilled workers are applying in their island. As in the case $N = 2$, some high skilled workers always apply in island 3 in island 3 because $\varphi_{3,3} > \varphi_{3,2}$.

[6] As noted earlier, despite the presence of competing applicants, a single type n applicant can extract some of the surplus thanks to due to his productivity advantage over the other applicants.

Under this set of conditions, we now show that there exists a unique equilibrium. With two types of actors, the relative gain depends on the others' behaviors: there is a complementarity between their choices. To see it, let us write down the conditions under which wages are equal for workers 3 in islands 2 and 3, and workers 2 in islands 1 and 2.

$$\begin{cases} \varphi_{3,3}e^{-q_3(1-x_3)} = \left[\varphi_{3,2} - \varphi_{2,2} + \varphi_{2,2}e^{-q_2(1-x_2)}\right] e^{-q_2x_3h_3} & (A_3) \\ \varphi_{2,2}e^{-q_2(1-x_2)-q_2x_3h_3} = \left[\varphi_{2,1} - \varphi_{1,1} + \varphi_{1,1}e^{-q_1}\right] e^{-q_1x_2h_2} & (A_2) \end{cases}$$

The two curves (A_3) and (A_2) both describe a positive relationship between x_3 and x_2, respectively the pull $x_3 = f_3(x_2)$ and push $x_3 = f_2(x_2)$ effects. Any interior equilibrium should be at the intersection of those two curves. It can be shown that:

$$\begin{cases} f_3'(x_2) = \frac{q_1h_2+q_2}{q_2h_3} \\ f_2'(x_2) = \frac{q_2}{q_3+q_2h_3}\frac{\varphi_{2,2}e^{-q_2(1-x_2)}}{\varphi_{3,2}-\varphi_{2,2}+\varphi_{2,2}e^{-q_2(1-x_2)}} \end{cases}$$

It can be easily verified that $\frac{q_2}{q_3+q_2h_3}\frac{\varphi_{2,2}}{\varphi_{3,2}} < \frac{q_1h_2+q_2}{q_2h_3}$. As a consequence, (A_2) is always steeper than (A_3), e.g. the push effect is always stronger than the pull effect and uniqueness derives from this observation (see figure A5).

□

As illustrated in figure A5, the two arbitrage equations (A_3) and (A_2) implicitly define a unique equilibrium allocation (x_2, x_3). Both curves are increasing, but the (A_2) curve is always steeper than the (A_3) curve.

To get some intuition, recall that the (A_2) curve captures the decision of type 2 workers to search in island 2 or 1. The (A_2) curve is increasing, because an increase in x_3, the number of type 3 workers in island 2, raises congestion in island 2, which "pushes" type 2 workers down to island 1 and increases x_2. The (A_3) curve captures the decision of type 3 workers to search in island 3 or 2. The (A_3) curve is increasing, because an increase in x_2, the number of type 2 workers in island 1, lowers congestion in island 2, which attracts, i.e., "pulls", type 3 workers down to island 2 and increases x_3. The fact that the (A_2) curve is always steeper than the (A_3) curve means that the "pushing effect" is always stronger than the "pulling effect".

Mechanisms Compared to the $N = 2$ case, under-employment is determined by the interactions of two forces: (i) a force that "pushes" workers down the ladder, captured by the (A_2) curve: as higher type workers invade the island below, they push the lower types further down the ladder, as in the $N = 2$ case, and (ii) a force that "pulls" workers down the ladder, captured by the (A_3) curve: as the lower types move down the ladder, they free up space in their islands, which pulls the higher types even further into their island.

In order to understand how these two forces interact, consider the thought experiment in which island $n = 1$ was closed for agents of type $n = 2, 3$. This initial point corresponds to the point E_0 in figure A5 and is identical to the $N = 2$ case previously discussed: type 2 agents are stuck in island 2 and $x_2 = 0$. Imagine that island 1 suddenly opens up, allowing anyone to look for a job in island 1.

1. Given x_3^0, the initial fraction of type 3 workers in island 2, workers in island 2 have an incentive to look for a job in island 1, because E_0 is above the (A_2) curve, so that $(A_2(0, x_3)) > 0$ and $E\omega_{2,1} > E\omega_{2,2}$. As a result a fraction x_2^1 of type 2 workers moves down to island 1, up until the point where $(A_2(x_2^1, x_3)) = 0$ (point E_1 in figure A5). In effect, type 2 workers are "pushed down" the ladder by type 3 workers, and this "pushing" effect is captured by the curve (A_2).

2. Following the downward movement of type 2 workers, island 2 is less congested than when type 3 agents initially made their island choice, and E_1 is below the (A_3) curve, so that $E\omega_{3,3} < E\omega_{3,2}$. As a result, more type 3 workers will descend to island 2 up until the point where $(A_3(x_2^1, x_3^2)) = 0$ with x_3^2 the new number of type 3 workers in island 2 (point E_2 in figure A5). In effect, type 3 workers are "pulled down" the ladder by type 2 workers leaving their island, and this "pulling" effect is captured by the curve (A_3).

3. Again, type 2 workers respond to the increased number of type 3 workers by further descending down to island 1, which triggers a response from type 3 workers and so on. This cascade ends at the equilibrium point E.

Comparative Statics: the Effect of Job Polarization We now discuss one comparative statics exercise to illustrate how the interactions between agents' decisions across islands (when $N > 2$) play out in equilibrium, and how a local shock can end up affecting all workers.

Consider an adverse labor demand shock hitting the middle productivity island, i.e., an increase in the queue length q_2. This thought experiment can be seen as studying the effect of job polarization and the disappearance of jobs in middle-skill occupations (the "hollowing out" of the skill distribution, Autor (2010)) on the allocation of workers.

Job polarization has two effects (see figure A6). On the one hand, the (A_2) curve shifts down, because of fewer job opportunities for type 2 workers in island 2, which would increase x_2, i.e., under-employment. On the other hand, the (A_3) curve also shifts down, because of fewer job opportunities for type 3 workers in island 2. This decreases x_2, because there are fewer type 3 workers pushing type 2 workers down the ladder. Overall, the effect of job polarization on the under-employment rate of middle-skill workers is thus ambiguous. However, under-employment amongst high-skill workers will unambiguously decrease.

In terms of expected income, it is easy to show that job polarization leads the expected income of type 3 (high-skill) workers to decrease and the expected income of type 2 (middle-skill) workers to decrease. However, the expected income of type 1 (low-skill) workers can either increase or decrease, depending on the effect of job polarization on the under-employment rate of middle-skill workers.

1.5 General equilibrium with Endogenous Labor Demand

The equilibrium with three types of workers and firms is characterized by the following Proposition:

Proposition A2. *With $N = 3$, there is a unique equilibrium allocation satisfying*

- *The arbitrage conditions characterizing the allocation of workers*
  - *Type 3 workers are indifferent between islands 2 and 3, and x_3, the share of type 3 workers searching in island 2, is given by the arbitrage condition*

$$A(x_3, x_2, q_2) = -E\omega_{3,3} + E\omega_{3,2}(x_3, x_2, q_2) = 0$$

  - *Type 2 workers are indifferent between islands 1 and 2, and x_2, the share of type 2 workers searching in island 1, is given by the arbitrage condition*

$$A(x_3, x_2, q_2, q_1) = -E\omega_{2,2}(x_3, x_2, q_2) + E\omega_{2,1}(x_2, q_1) = 0$$

  - *Type 1 workers only look for jobs in island 1: $x_1 = 0$.*
- *Firms' free entry conditions (market clearing) in islands 1, 2 and 3*

$$\begin{cases} \pi_1(x_2, q_1) = c_1 \\ \pi_2(x_3, x_2, q_2) = c_2 \\ \pi_3(x_3, q_3) = c_3 \end{cases}$$

Proof. The proof for the $N = 3$ case is very similar to the $N = 2$ case. The equilibrium is characterized by the allocation of workers of type 3 and 2, and the free-entry conditions in islands 1, 2 and 3. First, the free entry condition imposes that $q_3(1-x_3)$ is constant, and thus the expected wage in island 3, $E\omega_{3,3}$, is constant. We can thus restrict our analysis to the arbitrage conditions for workers of type 2 and 3 coupled with the free entry conditions in island 1 and 2. $\square$

The general equilibrium allocation with $N = 3$ is thus the vector $(x_3, x_2, q_3, q_2, q_1)$ determined by firms' free entry conditions in islands 1, 2 and 3, and the arbitrage equations for type 2 and type 3 workers.

1.6 Efficiency with $N = 3$: three worker types and three firm types

The following proposition states that the decentralized allocation is generally also inefficient when $N = 3$. In the constrained allocation, there is less under-employment of type 2 workers (lower x_2) and less under-employment of type 3 workers (lower x_3) than in the decentralized allocation.

Proposition A3. *When $N = 3$, the constrained optimal allocation $(x_2^*, x_3^*, q_1^*, q_2^*, q_3^*)$ does not coincide with the decentralized allocation. It is characterized by the same free entry conditions in islands 1, 2 and 3 but the difference in expected income between two islands for type 3 and type 2 workers is now respectively*

$$\begin{aligned} A_3(x_2^*, x_3^*, q_1^*, q_2^*, q_3^*) &= -E\omega_{3,3} + E\omega_{3,2} \\ &= \frac{(1-\beta)\, h_3 h_2 (1-x_2^*)^2 q_2^* \varphi_{2,2} e^{-2q_2^* x_3^* h_3 - q_2^*(1-x_2^*)} \left(\varphi_{3,2} - \varphi_{2,2}\right)}{\frac{\partial \pi_2(x_3^*, x_2^*, q_2^*)}{\partial q_2}} \\ &\geq 0 \end{aligned} \tag{A5}$$

and

$$\begin{aligned} A_2(x_2^*, x_3^*, q_1^*, q_2^*) &= -E\omega_{2,2} + E\omega_{2,1} \\ &= \frac{(1-\beta)\, h_2 q_1^* \varphi_{1,1} e^{-2q_1^* x_2^* h_2 - q_1^*} \left(\varphi_{2,1} - \varphi_{1,1}\right)}{\frac{\partial \pi_1(x_2^*, q_1^*)}{\partial q_1}} \\ &+ \frac{(1-\beta)(1-x_2^*)(\varphi_{3,2} - \varphi_{2,2})\varphi_{2,2} h_2 q_2^* x_3^* h_3 e^{-2q_2^* x_3^* h_3 - q_2^*(1-x_2^*)}}{\frac{\partial \pi_2(x_3^*, x_2^*, q_2^*)}{\partial q_2}} \\ &\geq 0 \end{aligned} \tag{A6}$$

with the expression for $\frac{\partial \pi_2(x_3, x_2, q_2)}{\partial q_2} > 0$ and $\frac{\partial \pi_1(x_2, q_1)}{\partial q_1} > 0$ given in the proof.

Proof. We proceed here exactly as we did for Proposition 4. The maximization program of the central planner can be written as follows (denote Y the aggregate output of the economy):

$$\max_{x_2,x_3,q_1,q_2,q_3} \{Y\}$$

subject to

$$\begin{cases} \pi_3(x_3,q_3) = c_3 \\ \pi_2(x_3,x_2,q_2) = c_2 \\ \pi_1(x_2,q_1) = c_1 \end{cases}$$

As before, two remarks help us simplify the program. First, with free entry, the aggregate profit of firms (net of investment costs) is zero: the central planner maximizes the wage bill of workers. Second, free entry in island 3 imposes that q_3 is set such as to make $(1-x_3)q_3$ constant.

$$(1-x_3)q_3 = f^{-1}\left(\frac{c_3}{\varphi_{3,3}}\right)$$

The program then sums up to (where each line represents wages earn by agents of different types):

$$\max_{x_2,x_3,q_1,q_2} \left\{ \begin{array}{l} h_2h_3(1-x_3)E\omega_{3,3} + h_2h_3x_3e^{-q_2h_3x_3}\left[\varphi_{3,2}-\varphi_{2,2}+\varphi_{2,2}e^{-q_2(1-x_2)}\right] \\ +h_2(1-x_2)\varphi_{2,2}e^{-q_2x_3h_3-q_2(1-x_2)} + h_2x_2\left(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1}\right)e^{-q_1x_2h_2} \\ +\varphi_{1,1}e^{-q_1x_2h_2-q_1} \end{array} \right\}$$

subject to

$$\begin{cases} \varphi_{3,2} - e^{-x_3h_3q_2}\left[\left(\varphi_{3,2}-\varphi_{2,2}+\varphi_{2,2}e^{-q_2(1-x_2)}\right)(1+(2-\beta)x_3h_3q_2) + (2-\beta)\varphi_{2,2}(1-x_2)e^{-q_2(1-x_2)}\right] = c_2 \\ \varphi_{2,1} - e^{-x_2h_2q_1}\left[\left(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1}\right)(1+(2-\beta)x_2h_2q_1) + (2-\beta)\varphi_{1,1}e^{-q_1}\right] = c_1 \end{cases}$$

We need now to write the four first-order conditions:

$$\begin{cases} A_3(x_3,x_2,q_2) - B_{x_3}(x_3,x_2,q_2) - \lambda_2C_{x_3}(x_3,x_2,q_2) = 0 & [x_3] \\ -B_{q_2}(x_3,x_2,q_2) + \lambda_2C_{q_2}(x_3,x_2,q_2) = 0 & [q_2] \\ A_2(x_3,x_2,q_2,q_1) - B_{x_2}(x_3,x_2,q_2,q_1) - \lambda_2C_{x_2}(x_3,x_2,q_2) - \lambda_1D_{x_2}(x_3,x_2,q_1) = 0 & [x_2] \\ -B_{q_1}(x_2,q_1) - \lambda_2D_{q_1}(x_2,q_1) = 0 & [q_1] \end{cases}$$

Let us detail the notations, A_3 (resp. A_2) denotes the difference between wages earned in level 2 and 3 (resp. 1 and 2) for workers of type 3 (resp. 2). B_{x_3} and B_{q_2} represents the additional terms deriving from differentiating W with respect to x_3 and q_2. We report their exact expression below.

$$\begin{cases} B_{x_3}(x_3,x_2,q_2) = & h_3q_2e^{-x_3q_2h_3}\left[\left(\varphi_{3,2}-\varphi_{2,2}+\varphi_{2,2}e^{-q_2(1-x_2)}\right)(1+x_3h_3q_2) + \varphi_{2,2}q_2(1-x_2)e^{-q_2(1-x_2)}\right] \\ B_{q_2}(x_3,x_2,q_2) = & \frac{x_3}{q_2}B_{x_3}(x_3,x_2,q_2) + (x_3h_3+1-x_2)\,h_2\,(1-x_2)\,\varphi_{2,2}e^{-q_2h_3x_3-q_2(1-x_2)} \end{cases}$$

B_{x_2} and B_{q_2} represents the additional terms deriving from differentiating W with respect to x_3 and x_2. We report their exact expression below.

$$\begin{cases} B_{x_2}(x_3,x_2,q_2,q_1) = & -q_2h_2\varphi_{2,2}e^{-q_2h_3x_3-q_2(1-x_2)}\left(x_3h_3+1-x_2\right) \\ & +q_1h_2e^{-q_1h_2x_2}\left[h_2x_2(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1})+\varphi_{1,1}e^{-q_1}\right] \\ B_{q_1}(x_2,q_1) = & x_2h_2e^{-q_1h_2x_2}\left[h_2x_2(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1})+\varphi_{1,1}e^{-q_1}\right] \\ & +(1+x_2h_2)\varphi_{1,1}e^{-q_1h_2x_2-q_1} \end{cases}$$

C_l represents the additional terms deriving from differentiating the profits in island 2 with respect to l. D_l represents the additional terms deriving from differentiating the profits in island 1 with respect to l. We report their exact expression below.

$$\begin{cases} C_{x_3}(x_3,x_2,q_2) = & h_3q_2e^{-x_3h_3q_2}\left[\left(\varphi_{3,2}-\varphi_{2,2}+\varphi_{2,2}e^{-q_2(1-x_2)}\right)((2-\beta)x_3h_3q_2-(1-\beta))\right. \\ & \left.+(2-\beta)\varphi_{2,2}q_2(1-x_2)e^{-q_2(1-x_2)}\right] \\ C_{q_2}(x_3,x_2,q_2) = & \frac{x_3}{q_2}C_{x_3}(x_3,x_2,q_2)+\varphi_{2,2}(1-x_2)e^{-x_3h_3q_2-q_2(1-x_2)}\left[(x_3h_3+1-x_2)(2-\beta)q_2-(1-\beta)\right] \\ D_{x_2}(x_3,x_2,q_1) = & h_2q_1e^{-x_2h_2q_1}\left[(\varphi_{2,1}-\varphi_{1,1}+\varphi_{1,1}e^{-q_1})((2-\beta)x_2h_2q_1-(1-\beta))\right. \\ & \left.+(2-\beta)\varphi_{1,1}q_1e^{-q_1}\right] \\ D_{q_1}(x_2,q_1) = & \frac{x_2}{q_1}D_{x_2}(x_3,x_2,q_1)+\varphi_{1,1}e^{-x_2h_2q_1-q_1}\left[(x_2h_2+1)(2-\beta)q_1-(1-\beta)\right] \\ C_{x_2}(x_3,x_2,q_2) = & -q_2\varphi_{2,2}\left[(x_3h_3+1-x_2)(2-\beta)q_2-(1-\beta)\right]e^{-x_3h_3q_2-q_2(1-x_2)} \end{cases}$$

The main difference with the $N=2$ case comes from an additional interaction term between workers of type 2 and 3. Workers of type 2 influences the profits that firms can make in island 2. C_{x_2} represents this gain in profits.

We eliminate the shadow prices in the first-order conditions:

$$\begin{cases} A_3(x_3,x_2,q_2) = B_{x_3}(x_3,x_2,q_2)-\frac{C_{x_3}(x_3,x_2,q_2)}{C_{q_2}(x_3,x_2,q_2)}B_{q_2}(x_3,x_2,q_2) \\ A_2(x_3,x_2,q_2,q_1) = B_{x_2}(x_3,x_2,q_2,q_1)-\frac{C_{x_2}(x_3,x_2,q_2)}{C_{q_2}(x_3,x_2,q_2)}B_{q_2}(x_3,x_2,q_2)-\frac{D_{x_2}(x_3,x_2,q_1)}{D_{q_1}(x_2,q_1)}B_{q_1}(x_2,q_1) \end{cases}$$

Let us focus on the first equation:

$$A_3(x_3,x_2,q_2) = \frac{B_{x_3}(x_3,x_2,q_2)C_{q_2}(x_3,x_2,q_2)-C_{x_3}(x_3,x_2,q_2)B_{q_2}(x_3,x_2,q_2)}{C_{q_2}(x_3,x_2,q_2)}$$

As in proposition 4,

$$A_3(x_3,x_2,q_2) = \frac{(1-\beta)\,h_3h_2(1-x_2)^2q_2\varphi_{2,2}e^{-2q_2x_3h_3-q_2(1-x_2)}\left(\varphi_{3,2}-\varphi_{2,2}\right)}{C_{q_2}(x_3,x_2,q_2)}$$

It is easy to see that $A_3(x_3,x_2,q_2)>0$. The centralized allocation gives a higher wage to agents 3 in island 2 than what they would receive in island 3. x_3 is lower than in the decentralized allocation, q_3 is higher.

We now turn to the second equation.

$$A_2(x_3,x_2,q_2,q_1) = \frac{B_{x_2}D_{q_1}-D_{x_2}B_{q_1}}{D_{q_1}}-\frac{C_{x_2}}{C_{q_2}}B_{q_2}$$

$$A(x_2,q_1) = \frac{(1-\beta)h_2 q_1 \varphi_{1,1} e^{-2q_1 x_2 h_2 - q_1}(\varphi_{2,1}-\varphi_{1,1})}{D_{q_1}(x_2,q_1)} + \frac{(1-\beta)(1-x_2)(\varphi_{3,2}-\varphi_{2,2})\varphi_{2,2} h_2 q_2 x_3 h_3 e^{-2q_2 x_3 h_3 - q_2(1-x_2)}}{C_{q_2}}$$

□

Compared to the $N = 2$ case, interactions across agents' decision introduces an additional effect. Comparing with the $N = 2$ case, we can notice an additional (positive) term in A_2 in the $N = 3$ case, which brings the constrained allocation further away from the decentralized one. This additional term captures the fact that, when deciding to search in island 1, type 2 workers affect not only the ratio of type 1 to type 2 workers in island 1, which affects the job creation decision of firms in island 1 (as is the $N = 2$ case), but also the ratio of type 2 to type 3 workers in island 2, which affects the job creation decision of firms in island 2.

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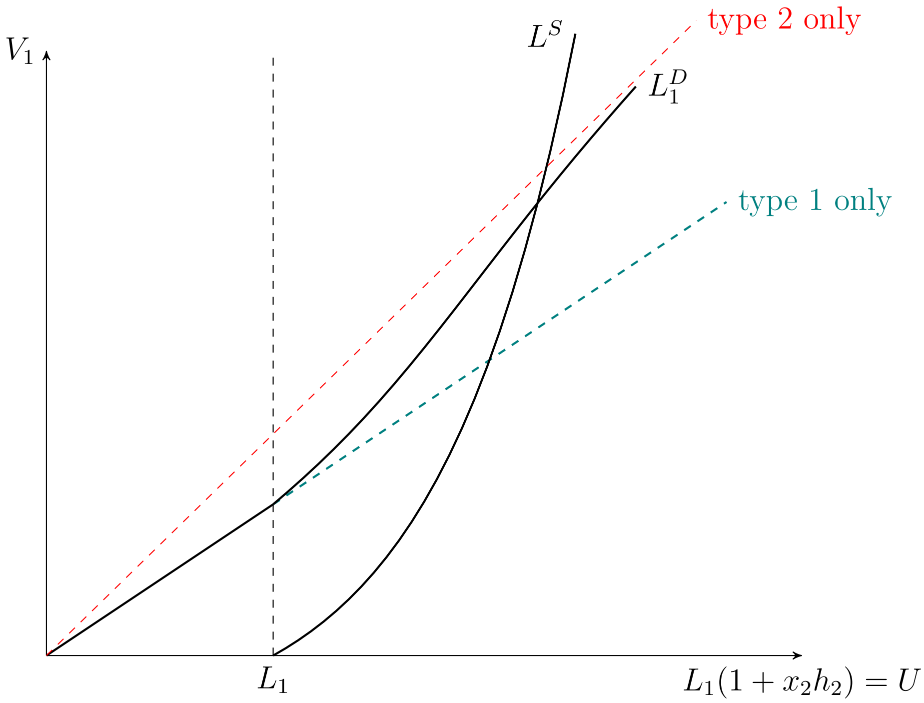

Figure A1. Labor market equilibrium – N=2.

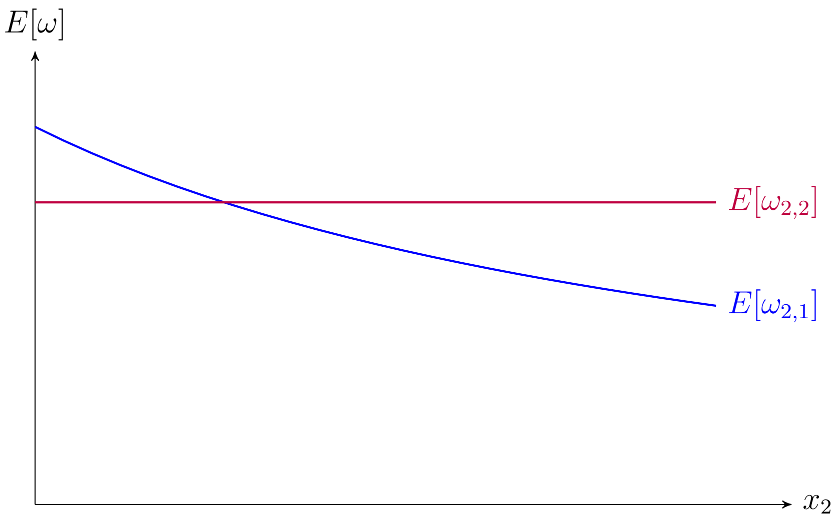

Figure A2. General Equilibrium – wages for type-2 workers.

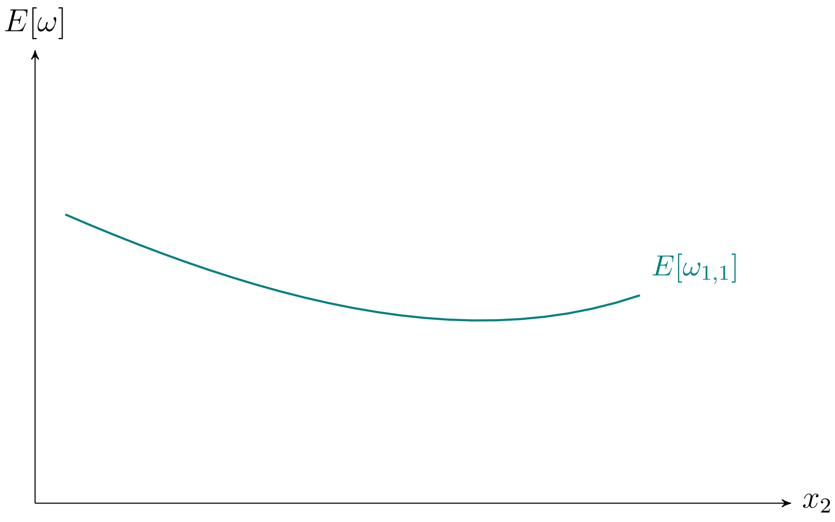

Figure A3. General Equilibrium – wages for type-1 workers.

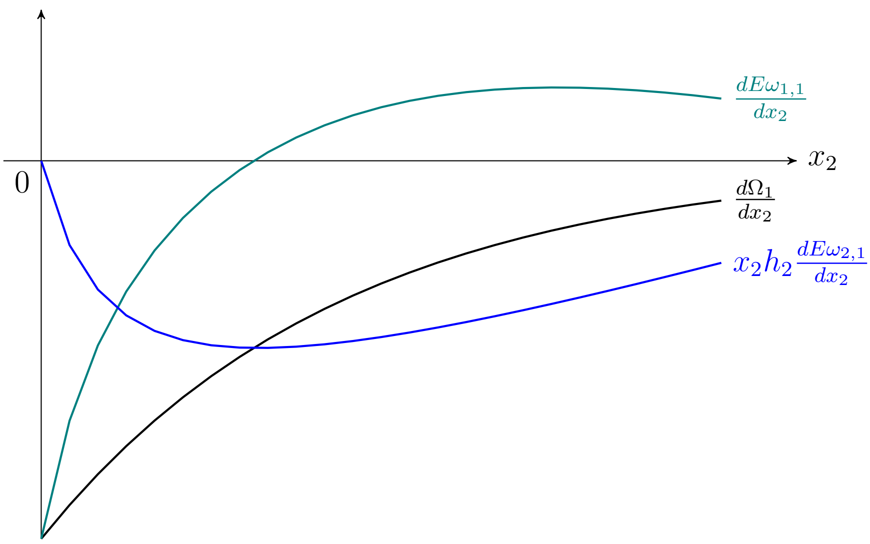

Figure A4. Inefficiency – ranking externality $\frac{d\Omega_1}{dx_2} = \frac{dE\omega_{1,1}}{dx_2} + x_2 h_2 \frac{dE\omega_{2,1}}{dx_2}$.

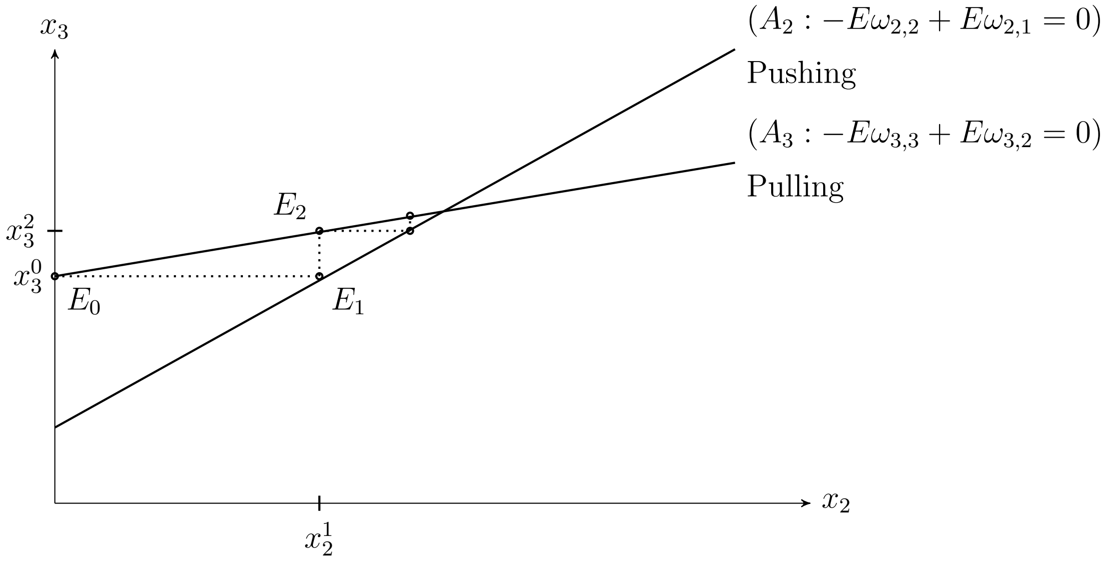

Figure A5. Partial Equilibrium – N=3.

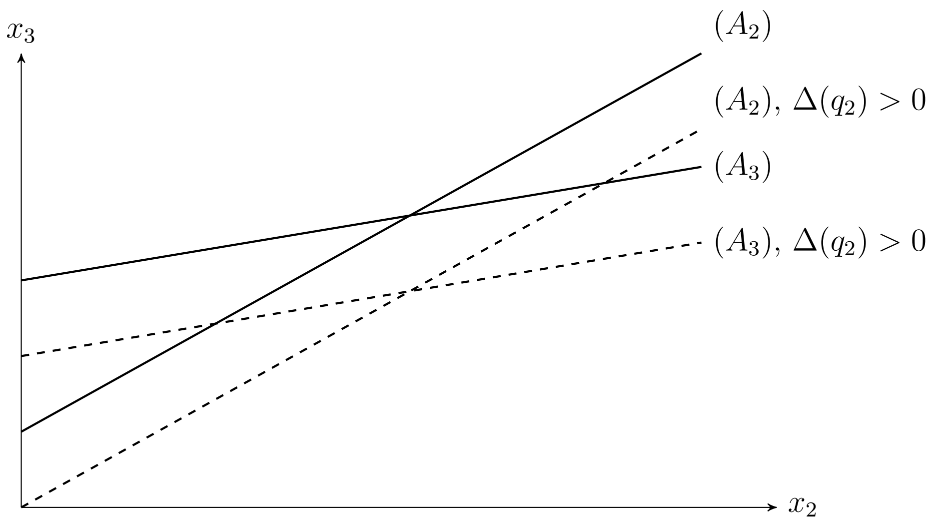

Figure A6. Partial Equilibrium – Effect of job polarization – N=3.